④

a) $a = 1$ and $b = -1$

b) $n = -1$

c) $n = 8$, then $n^2+1 = 65$ is not a prime number.

d) $n = 1$

⑩ We have $n \in \{4, 6, 8, 10, 12\}$ and:

$$4 = 2+2$$
$$6 = 3+3$$
$$8 = 3+5$$
$$10 = 5+5$$
$$12 = 5+7$$

⑮ Let x, y are 2 odd numbers.

Because x is odd, therefore, there exists an interger k such that: $x = 2k+1$

Simillarly, $y = 2j+1$

$$\Rightarrow x+y = (2k+1) + (2j+1)$$
$$= 2k + 2j + 2$$
$$= 2(k+j+1)$$

$\Rightarrow x+y$ is an even number $\Rightarrow$ q.e.d

(19) We have 2 cases:

- If the smaller number is an odd number, then it can ~~reprent~~ represented as: $2K+1$. (K is an integer)

$\Rightarrow$ the larger is represented as: $2K+2$

$\Rightarrow$ their product is: $(2K+1).(2K+2)$

$= 2.(2K+1)(K+1)$

$\Rightarrow$ their product is even in this case.

- If the smaller number is an even number, then it can be represented as: $2K$ (K is an integer)

$\Rightarrow$ the larger is represented as: $2K+1$

$\Rightarrow$ their product is: $2K.(2K+1)$ is an even number

$\Rightarrow$ their product is even in this case

From both cases $\Rightarrow$ q.e.d

(20) Let x is an integer.

$\Rightarrow$ Sum of x and its square is:
$x + x^2 = x(x+1)$

$x(x+1)$ is a product of 2 consecutive numbers. Therefore, apply the proof from question 19 $\Rightarrow$ q.e.d

(32) Let 2 integers are a, b.

~~Need to proof: if $(n \nmid ab)$, then $((n \nmid a) \wedge (n \nmid b))$~~

We can prove by contradictions.

(*) If at least a number of a, b is divisible by n, assume that number is a. (without ~~the~~ loss of generality).

Then, we have $n|a \Rightarrow$ there exists an integer K such that:

$$a = Kn$$

$$\Rightarrow ab = (Kn).b = n.(Kb)$$

$\Rightarrow$ ab is divisible by n $\rightarrow$ which conflicts with the given condition

$\Rightarrow$ q.e.d

(44) Assume that, there are 2 integers p and q such that:

$$\begin{cases} (p,q) = 1 \\ \sqrt{5} = \dfrac{p}{q} \end{cases} \quad \text{and } p>0,\ q>0$$

We have: $\sqrt{5} = \dfrac{p}{q} \Leftrightarrow 5 = \dfrac{p^2}{q^2}$

$$\Leftrightarrow 5q^2 = p^2$$

Because LHS is divisible by 5 $\Rightarrow p^2$ is divisible by 5. However, 5 is a prime, then p is divisible by 5.

$\Rightarrow$ exists k such that: $p = 5K$ (K is an integer)

$$\Rightarrow 5q^2 = (5K)^2 \Leftrightarrow q^2 = 5K^2$$

Because RHS is divisible by 5 $\Rightarrow$ $5|q^2$. However, 5 is a prime

$\Rightarrow 5|q$

$\Rightarrow$ p and q have at least a common factor (5), which conflicts with $(p,q) = 1$

$\Rightarrow$ There are not any pairs p and q satisfy the assumption

$\Rightarrow$ q.e.d

61 We have: $n > 0 \Rightarrow \sqrt{n} > 0$

We have: $n + \frac{1}{n} = (\sqrt{n})^2 + \left(\frac{1}{\sqrt{n}}\right)^2$

$$= \left[(\sqrt{n})^2 - 2.\sqrt{n}.\frac{1}{\sqrt{n}} + \left(\frac{1}{\sqrt{n}}\right)^2\right] + 2.\sqrt{n}.\frac{1}{\sqrt{n}}$$

$$= \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^2 + 2$$

Since $\left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^2 \geq 0 \ \forall n > 0$

$\Rightarrow n + \frac{1}{n} \geq 0 + 2 = 2 \Rightarrow$ q.e.d

"=" happens when $\Leftrightarrow \begin{cases} \sqrt{n} - \frac{1}{\sqrt{n}} = 0 \\ n > 0 \end{cases} \Leftrightarrow n = 1$

66 A counter example is choosing $n=6$, then: $2^6+1=65$, which is not a prime.

71 $\begin{cases} \sqrt{2} \text{ is an irrational number} \\ \sqrt{2} \text{ is an irrational number} \end{cases}$

However, $\sqrt{2}.\sqrt{2}=2$ is ~~a rational n~~ not an irrational number.

74 We notate $\hat{a}$ is the measure of angle a.

We have:

$$\begin{cases} \hat{1}+\hat{2}+\hat{3}=180^\circ & (\text{fact } 1) \\ \hat{3}=\hat{6} & (\text{fact } 2) \end{cases}$$

$$\Rightarrow \quad \hat{1}+\hat{2}+\hat{6}=180^\circ \quad (1)$$

$$\begin{cases} \hat{4}+\hat{6}=180^\circ & (\text{fact } 3) \\ \hat{1}+\hat{2}+\hat{6}=180^\circ & (\text{from } (1)) \end{cases}$$

$$\Rightarrow \hat{4}+\hat{6}=\hat{1}+\hat{2}+\hat{6}$$

$$\Leftrightarrow \hat{4}=\hat{1}+\hat{2}$$

$\Rightarrow$ q.e.d