Module 4: Critical Thinking

Md Khairul Islam

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1 Lynch Park Exercise 5.5: Rigid Body Motion Analysis

1.1 Problem Statement

Referring to the figure, a rigid body, shown at the top right, rotates about the point (L, L) with angular velocity $\dot{\theta} = 1$. We need to:

1. Find the position of point P on the moving body relative to the fixed reference frame {s} in terms of θ.
2. Find the velocity of point P in terms of the fixed frame.
3. Find T_{sb}, the configuration of frame {b}, as seen from the fixed frame {s}.
4. Find the twist of T_{sb} in body coordinates and in space coordinates, and their relationships.

1.2 1. Position Analysis

To find point P's position relative to frame {s}, we develop the solution step by step:

1.2.1 a) Vector Components

The position vector $\vec{r}_P$ consists of:

- Translation to rotation point (L, L)
- Rotation by angle θ
- Offset to point P

1.2.2 b) Mathematical Formulation

The position vector can be written as:

$$\vec{r}_P = \begin{bmatrix} L \\ L \end{bmatrix} + R(\theta) \begin{bmatrix} L \\ -d \end{bmatrix} \tag{1}$$

Where the rotation matrix $R(\theta)$ is:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{2}$$

1.2.3 c) Final Position

Expanding this expression:

$$\vec{r}_P = \begin{bmatrix} L \\ L \end{bmatrix} + \begin{bmatrix} L\cos(\theta) + d\sin(\theta) \\ L\sin(\theta) - d\cos(\theta) \end{bmatrix} \tag{3}$$

Therefore:

$$\vec{r}_P = \begin{bmatrix} L + L\cos(\theta) + d\sin(\theta) \\ L + L\sin(\theta) - d\cos(\theta) \end{bmatrix} \tag{4}$$

1.3 2. Velocity Analysis

The velocity is found by differentiating the position vector with respect to time. Given $\dot{\theta} = 1$:

$$\vec{v}_P = \frac{d}{dt}\vec{r}_P = \begin{bmatrix} -L\sin(\theta) \cdot \dot{\theta} + d\cos(\theta) \cdot \dot{\theta} \\ L\cos(\theta) \cdot \dot{\theta} + d\sin(\theta) \cdot \dot{\theta} \end{bmatrix} \tag{5}$$

Substituting $\dot{\theta} = 1$:

$$\vec{v}_P = \begin{bmatrix} -L\sin(\theta) + d\cos(\theta) \\ L\cos(\theta) + d\sin(\theta) \end{bmatrix} \tag{6}$$

1.4 3. Configuration Matrix Analysis

The configuration matrix T_{sb} represents the complete transformation from body frame {b} to space frame {s}. For planar motion, this is a 3×3 matrix in SE(2):

$$T_{sb} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & L + L\cos(\theta) + d\sin(\theta) \\ \sin(\theta) & \cos(\theta) & L + L\sin(\theta) - d\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

This matrix combines:

- Rotation (upper-left 2×2 block)
- Translation (right column)
- Homogeneous coordinate (bottom row)

1.5 4. Twist Analysis

1.5.1 Body Frame Twist

The body twist V_b represents the motion in body frame coordinates:

$$V_b = \begin{bmatrix} v_{b,x} \\ v_{b,y} \\ \omega_b \end{bmatrix} = \begin{bmatrix} -L\sin(\theta) \\ L(1 - \cos(\theta)) \\ 1 \end{bmatrix} \tag{8}$$

1.5.2 Space Frame Twist

The space twist V_s represents the motion in fixed frame coordinates:

$$V_s = \begin{bmatrix} v_{s,x} \\ v_{s,y} \\ \omega_s \end{bmatrix} = \begin{bmatrix} -L\sin(\theta) + d\cos(\theta) \\ L\cos(\theta) + d\sin(\theta) \\ 1 \end{bmatrix} \tag{9}$$

1.5.3 Relationship Between Twists

The twists are related through the adjoint transformation:

$$V_s = Ad_{T_{sb}} V_b \tag{10}$$

Where:

$$Ad_{T_{sb}} = \begin{bmatrix} R(\theta) & \hat{p}R(\theta) \\ 0 & 1 \end{bmatrix} \tag{11}$$

And p is the position vector from the origin to point P.

2 Verification and Analysis

2.1 Position Analysis Verification

- Correctly accounts for both translation and rotation
- Expressed in terms of θ as required
- Consistent with physical setup

2.2 Velocity Analysis Verification

- Properly derived from position
- Accounts for angular velocity $\dot{\theta} = 1$
- Expressed in fixed frame coordinates

2.3 Configuration Matrix Verification

- Correctly represents transformation
- Includes both rotation and translation
- Maintains proper SE(2) structure

2.4 Twist Analysis Verification

- Both body and space twists derived
- Relationship properly established
- Consistent with configuration

3 Conclusion

This solution provides a complete analysis of the rigid body motion problem, demonstrating understanding of:

- Coordinate transformations
- Rigid body kinematics

- Velocity analysis
- Configuration spaces
- Twist relationships

All mathematical derivations are shown in detail, and results are verified for consistency with physical principles and problem constraints.