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HW-1

Introduction to Quantum Information

1. Consider the normalized states in $\mathbb{C}^2$

$$\begin{pmatrix}\cos(\theta_1)\\ \sin(\theta_1)\end{pmatrix}, \begin{pmatrix}\cos(\theta_2)\\ \sin(\theta_2)\end{pmatrix}.$$

where $\theta_1, \theta_2 \in [0, 2\pi)$, Find the condition on θ_1 and θ_2 such that the vector below is normalized.

$$\begin{pmatrix}\cos(\theta_1)+\cos(\theta_2)\\ \sin(\theta_1)+\sin(\theta_2)\end{pmatrix}.$$

Sol. Given the normalized states in $\mathbb{C}^2$ is:

$$\begin{pmatrix}\cos(\theta_1)\\ \sin(\theta_1)\end{pmatrix}, \begin{pmatrix}\cos(\theta_2)\\ \sin(\theta_2)\end{pmatrix}.$$

Given to find the condition on θ_1 & θ_2 such that vector below is normalized:-

Consider $\alpha = \cos(\theta_1) + \cos\theta_2$
$\beta = \sin\theta_1 + \sin\theta_2$

for the vector to be normalized the summation of squares should be equal to 1

$\Rightarrow (\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 + \sin\theta_2)^2 = 1$

$\Rightarrow \cos^2\theta_1 + \cos^2\theta_2 + 2\cos\theta_1\cos\theta_2 + \sin^2\theta_1 + \sin^2\theta_2 + 2\sin\theta_1 \sin\theta_2 = 1$

$[\because \cos^2\theta + \sin^2\theta = 1]$

$\Rightarrow 1 + 1 + 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = 1$

$\Rightarrow 2 + 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = 1$

$\Rightarrow \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = -1/2$

$= \cos(\theta_1 - \theta_2) = -1/2$ $\quad [\because \cos(A-B) = \cos A\cos B - \sin A \sin B]$

$$\boxed{\theta_1 - \theta_2 = 2\pi/3 \text{ (or) } \theta_1 - \theta_2 = 4\pi/3} \quad - (1)$$

The Condition in which vector is normalized is specied in (1).

2. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis in the Hilbert Space $\mathbb{C}^2$. The NOT operation is defined as

$|0\rangle \Rightarrow |1\rangle$, $\quad |1\rangle \longrightarrow |0\rangle$.

i) find the unitary operator U_{NOT} that implements the NOT operation w.r.t basis $\{|0\rangle, |1\rangle\}$.

ii) Consider the Standard basis

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

find the matrix representation of U_{NOT} for basis.

iii) Consider the Hadamard basis

$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\quad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

Find the matrix representation of U_{NOT} for this basis.

Sol: Obviously

(i) $U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$

Since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

(ii) for the standard basis as per the NOT operation is defined in the problem

$|0\rangle \rightarrow |1\rangle$ & $|1\rangle \rightarrow |0\rangle$

then

$$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) As for the hadamard basis

$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

As per the Hadamard basis

$$U_{NOT} = H U_{NOT} H$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Consider the hadamard matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and normalized state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix},$$

Calculate the variance

$V_H(|\psi\rangle) = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$,

and find its maximum Value and the minimum Value.

Sol: Given that

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the H^2 is given by

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$V_H(|\psi\rangle) = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$$

$$\langle\psi|H^2|\psi\rangle = (\cos\theta \ \ \sin\theta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$\left[\because \text{Given } |\psi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \ \langle\psi| = (\cos\theta \ \ \sin\theta)\right]$

$$\Rightarrow (\cos\theta \ \ \sin\theta)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\Rightarrow \cos^2\theta + \sin^2\theta$$

$$= 1 \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$\langle\psi|H|\psi\rangle = (\cos\theta \ \ \sin\theta)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}}(\cos\theta \ \ \sin\theta)\begin{pmatrix} \cos\theta + \sin\theta \\ \cos\theta - \sin\theta \end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}}(\cos^2\theta + \cos\theta\sin\theta + \cos\theta\sin\theta - \sin^2\theta)$$

$$\Rightarrow \frac{1}{\sqrt{2}}(\cos^2\theta - \sin^2\theta + 2\cos\theta\sin\theta)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(\cos 2\theta + 2\cos\theta\sin\theta\right)$$

$$\left[\because \cos^2\theta - \sin^2\theta = \cos 2\theta\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos 2\theta + \sin 2\theta\right)$$

$$\left[\because \sin 2\theta = 2\sin\theta\cos\theta\right]$$

$$\langle\psi|H|\psi\rangle^2 = \frac{1}{2}\left(\cos 2\theta + \sin 2\theta\right)^2$$

$$= \frac{1}{2}\left(\cos^2 2\theta + \sin^2 2\theta + 2\sin 2\theta\cos 2\theta\right)$$

$$= \frac{1}{2}\left(1 + \sin 4\theta\right)$$

$$\left[\because \begin{array}{l} \cos^2\alpha + \sin^2\alpha = 1 \\ 2\sin\alpha\cos\alpha = \sin 2\alpha \end{array}\right]$$

$$V_H(|\psi\rangle) = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$$

$$= 1 - \frac{1}{2}\left(1 + \sin 4\theta\right)$$

$$= 1 - \frac{1}{2} - \frac{\sin 4\theta}{2}$$

$$= \frac{1 - \sin 4\theta}{2}$$

$\Rightarrow$ for the maximum value $\sin 4\theta$ should be $\boxed{-1}$ then

$$V_H(|\psi\rangle) = \frac{1+1}{2} = 1$$

$\Rightarrow$ for the minimum value $\sin 4\theta$ should be $\boxed{+1}$ the

$$V_H(|\psi\rangle) = \frac{1-1}{2} = 0$$

4. The one bit Walsh-hadamard transform is the Unitary transform W given by

$$W|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad W|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

The n-bit Walsh-Hadamard transform W_n is defined as

$$W_n = W \otimes W \otimes \dots \otimes W \quad (n \text{ times})$$

Consider $n=2$, find the normalized state $W_2(|0\rangle \otimes |0\rangle)$.

Sol:

We have the

$$\Rightarrow W_2(|0\rangle \otimes |0\rangle) = (W \otimes W)(|0\rangle \otimes |0\rangle)$$

$$= W|0\rangle \otimes W|0\rangle$$

then the above expression is converted as:-

$$\Rightarrow W_2(|0\rangle \otimes |0\rangle) = \frac{1}{2}((|0\rangle + |1\rangle) \otimes (|1\rangle + |0\rangle).$$

and then finally

$$\Rightarrow W_2(|0\rangle \otimes |0\rangle) = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle$$

$$+ |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

W_2 generates the linear combination of states and this will be appliable to W_n also.

5. In the product Hilbert Space $C^4 \cong C^2 \otimes C^2$

the Bell States are given by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle),$$

which ~~foorm~~ form an orthonormal basis in C^4.

$$|0\rangle = \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} -e^{i\phi}\sin(\theta) \\ \cos\theta \end{pmatrix}$$

(i) find $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi\rangle$ for basis.

(ii) find the above states when $\phi = 0$ & $\theta = 0$.

Sol: (i) $|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} \otimes \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix} \otimes \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i2\phi}(\cos^2\theta + \sin^2\theta) \\ e^{i\phi}(\cos\theta\sin\theta - \cos\theta\sin\theta) \\ e^{i\phi}(\cos\theta\sin\theta - \cos\theta\sin\theta) \\ \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i2\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(ii) $|\Phi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} \otimes \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} - \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix} \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i2\phi}(\cos^2\theta - \sin^2\theta) \\ e^{i\phi}(2\sin\theta\cos\theta) \\ e^{i\phi}(2\sin\theta\cos\theta) \\ \sin^2\theta - \cos^2\theta \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i2\phi} \cos 2\theta \\ e^{i\phi} \sin 2\theta \\ e^{i\phi} \sin 2\theta \\ -\cos 2\theta \end{pmatrix}$$

iii) $|\Psi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} \cos\theta \\ \sin\theta \end{pmatrix} \otimes \begin{pmatrix} -e^{i\phi} \sin\theta \\ \cos\theta \end{pmatrix} + \begin{pmatrix} -e^{i\phi} \sin\theta \\ \cos\theta \end{pmatrix} \otimes \begin{pmatrix} e^{i\phi} \cos\theta \\ \sin\theta \end{pmatrix}$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{i2\phi}(2\sin\theta\cos\theta) \\ e^{i\phi}(\cos^2\theta - \sin^2\theta) \\ e^{i\phi}(\cos^2\theta - \sin^2\theta) \\ 2\sin\theta\cos\theta \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{i2\phi} \sin 2\theta \\ e^{i\phi} \cos 2\theta \\ e^{i\phi} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$

iv) $|\Psi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} \cos\theta \\ \sin\theta \end{pmatrix} \otimes \begin{pmatrix} -e^{+i\phi} \sin\theta \\ \cos\theta \end{pmatrix} - \begin{pmatrix} -e^{+i\phi} \sin\theta \\ \cos\theta \end{pmatrix} \otimes \begin{pmatrix} e^{i\phi} \cos\theta \\ \sin\theta \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i2\phi}(-\cos\theta\sin\theta + \sin\theta\cos\theta) \\ e^{i\phi}(\cos^2\theta + \sin^2\theta) \\ -e^{i\phi}(\sin^2\theta + \cos^2\theta) \\ \sin\theta\cos\theta - \sin\theta\cos\theta \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ e^{i\phi} \\ -e^{i\phi} \\ 0 \end{bmatrix}$$

2nd part)

i) [redacted] Given $\phi = 0$ & $\theta = 0$ $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

ii)

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

iii) $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

iv) $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] \Rightarrow$

$$\Rightarrow \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

6. An $n\times n$ Circulant matrix C is given by

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for example, the permutation matrix, P

is a circulant matrix. Let

$f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1}$, Show that

$C = f(P)$.

Sol. Given from the above

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$ then the P^2 is given

as

$$P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

given to Show that $C = f(P)$

$f(P) = c_0 + c_1 P + \cdots + c_{n-1}P^{n-1}$

$$= c_0\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1\end{bmatrix} + c_1\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 0\end{bmatrix} + c_2\begin{bmatrix}0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & & & & & \\ 0 & 1 & 0 & 0 & \cdots & 0\end{bmatrix} + \cdots$$

$$= \begin{bmatrix}c_0 & 0 & 0 & \cdots & 0\\ 0 & c_0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & c_0\end{bmatrix} + \begin{bmatrix}0 & c_1 & 0 & \cdots & 0\\ 0 & 0 & c_1 & \cdots & 0\\ \vdots & \vdots & & & \\ c_1 & 0 & 0 & \cdots & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & c_2 & 0 & \cdots & 0\\ 0 & 0 & 0 & c_2 & \cdots & 0\\ \vdots & & & & & \\ 0 & c_2 & 0 & 0 & \cdots & 0\end{bmatrix} + \cdots$$

$$= \begin{bmatrix}c_0 & c_1 & 0 & \cdots & c_{n-1}\\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2}\\ \vdots & & & & \\ c_1 & c_2 & c_3 & \cdots & c_0\end{bmatrix} = C$$

$$\boxed{\therefore F(P) = C}$$

(7) Let $|\psi\rangle = \begin{pmatrix}\cos(\theta)\\ e^{i\phi}\sin\theta\end{pmatrix}$ be a normalized state in $\mathbb{C}^2$. Is $\rho = |\psi\rangle\langle\psi|$ a density matrix?

<u>Sol:</u> Given $|\psi\rangle = \begin{pmatrix}\cos\theta\\ e^{i\phi}\sin\theta\end{pmatrix}$

For ρ to be density matrix it should be

$\rho^2 = \rho$ & $tr(\rho) = 1$

$\rho = |\psi\rangle\langle\psi| \Rightarrow |\psi\rangle = \begin{pmatrix}\cos\theta\\ e^{i\phi}\sin\theta\end{pmatrix}$, $\langle\psi| = \begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\end{pmatrix}$

$$\rho = \begin{pmatrix}\cos\theta\\ e^{i\phi}\sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\theta & e^{-i\phi}\sin\theta\cos\theta\\ e^{i\phi}\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$$

$$\rho^2 = \begin{pmatrix}\cos^2\theta & e^{-i\phi}\sin\theta\cos\theta\\ e^{i\phi}\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}\cos^2\theta & e^{-i\phi}\sin\theta\cos\theta\\ e^{i\phi}\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$$

$$= \begin{pmatrix} \cos^4\theta + e^{i\phi}e^{-i\phi}\sin^2\theta\cos^2\theta & e^{-i\phi}\sin\theta\cos^3\theta + e^{-i\phi}\sin^3\theta\cos\theta \\ e^{i\phi}\sin\theta\cos^3\theta + e^{i\phi}\sin^3\theta\cos\theta & e^{i\phi}e^{-i\phi}\sin^2\theta\cos^2\theta + \sin^4\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta(\cos^2\theta + \sin^2\theta) & e^{-i\phi}\sin\theta\cos\theta(\cos^2\theta + \sin^2\theta) \\ e^{i\phi}\sin\theta\cos\theta(\cos^2\theta + \sin^2\theta) & \sin^2\theta(\cos^2\theta + \sin^2\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

$\therefore \rho^2 = \rho$

$\therefore$ Given ρ is a density matrix.

(8) Find the trace distance

$$D_T(\rho_1, \rho_2) = \frac{1}{2} tr\left(\sqrt{(\rho_1-\rho_2)(\rho_1-\rho_2)^{\dagger}}\right)$$

between two density matrices.

$$\rho_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Sol: Consider

$$\rho_1 - \rho_2 = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$$

$$(\rho_1-\rho_2)(\rho_1-\rho_2)^{t} = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

$$\sqrt{(e_1-e_2)(e_1-e_2)^t} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$1/2 \sqrt{(e_1-e_2)(e_1-e_2)^t} = \frac{1}{2} I$$
$$= 1/2$$

$$\boxed{D_T (e_1, e_2) = 1/2}$$

(9) Consider the following matrix representation of N qubits in terms of tensor products of Pauli matrices

$$\rho = \frac{1}{2^N} \sum_{j_0=0}^{3} \sum_{j_1=0}^{3} \cdots \sum_{j_{N-1}=0}^{3} c_{j_0 j_1 \cdots j_{N-1}} \sigma_{j_0} \otimes \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_{N-1}}$$

where $\sigma_0 = I_2$. Calculate

$$\mathrm{tr}\left(\rho\, \sigma_{k_0} \otimes \sigma_{k_1} \otimes \cdots \otimes \sigma_{k_{N-1}}\right).$$

Sol: Given

$$\rho = \frac{1}{2^N} \sum_{j_0=0}^{3} \sum_{j_1=0}^{3} \cdots \sum_{j_{N-1}=0}^{3} c_{j_0 j_1 \cdots j_{N-1}} \sigma_{j_0} \otimes \sigma_{j_1} \otimes \sigma_{j_{N-1}}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as Per the terms given in lecture notes

now we Calculate $\sigma_1 \times \sigma_2$

$$\sigma_1 \times \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$tr(\sigma_1 \times \sigma_2) = 0$$

The trace is zero.

$$\sigma_2 \times \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$tr(\sigma_2 \times \sigma_3) = 0$$

$$\sigma_1 \times \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$tr(\sigma_1 \times \sigma_3) = 0$$

Hence, $tr(\sigma_1 \otimes \sigma_2) = tr(\sigma_2 \otimes \sigma_3) = tr(\sigma_1 \otimes \sigma_3) = 0$

So, $\boxed{tr(\rho\, \sigma_{k_0} \otimes \sigma_{k_1} \otimes \ldots \otimes \sigma_{k_{N-1}}) = C_{k_0 k_1 \ldots k_{N-1}}}$

(10) Let $\sigma_1, \sigma_2; \sigma_3$ be the Pauli spin matrices. Is the 4×4 matrix

$$\rho = \frac{1}{4}\left(I_4 - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3\right)$$ a density matrix?

Sol: Given from the lecture slides we know that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho = \frac{1}{4}\left(I_4 - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3\right)$$

we Know that

$$I_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho = \frac{1}{4}\left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{tr(\rho) = \frac{4}{4} = 1}$$

$$\rho^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & +1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \rho$$

$\therefore \rho^2 = \rho$ & $tr(\rho) = 1$

$\therefore$ Given ρ is the density matrix.

(11) Consider the GHZ entangled State in the Hilbert Space $\mathbb{C}^8 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$;

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)$$

with the density matrix being $\rho = |\psi\rangle\langle\psi|$.
Calculate the Partial trace $\rho_A = tr_B(\rho_{AB})$ w.r.t basis $I_2 \otimes \begin{pmatrix}1\\0\end{pmatrix}$, $I_2 \otimes \begin{pmatrix}0\\1\end{pmatrix}$,
where ρ_{AB} has been obtained in (182) of Lec notes.

Sol By the Page 182 we get:-

$$\rho_{AB} = \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

$$tr_B(\rho_{AB}) = \frac{1}{2}(2)$$
$$= 1$$

$tr_B(\rho_{AB}) = 1$

$$\text{P. Trace} = \left(I_2 \otimes \begin{pmatrix}1\\0\end{pmatrix}\right)^T \rho \left(I_2 \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) + \left(I_2 \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)^T \rho \left[I_2 \otimes \begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\Rightarrow \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right)^T \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] +$$

$$\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\right)^T \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\Rightarrow \frac{1}{2}\begin{pmatrix}1&0\\0&0\\0&1\\0&0\end{pmatrix}^T\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}^T\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0\\0&0\\0&0\\0&1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1&0\\0&0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$\boxed{\rho_A = \frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}}$$