Homework-3 CS 4331/5331

Introduction to Quantum Information

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1. Calculate the detailed Calculation that led to information Capacity of $c(e)=2$ in eq (355) Of Lecture Notes.

Sol: Let ρ_{AB} be a density matrix on $N \times N$ dimensional Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$

The information Capacity is defined as

$$c(\rho) = \ln(N) + S(\rho_B) - S(\rho_{AB})$$

where ρ_B = reduced density matrix

$\rho_{AB} = tr_A(\rho_{AB})$

S = Von Neumann entropy

Consider the example of Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right).$$

The density matrix of pure state is fund by

$$\rho_{AB} = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

By using partial trace formula we obtain

$$\rho_B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

with $N=2$ and $S(\rho_{AB})=0$, it follows

that

The Von Neumann entropy $S(\rho)$ for density matrix ρ is defined as:

$$S(\rho) = -\mathrm{tr}(\rho \log \rho)$$

For the reduced density matrix ρ_B, we have:

$$\rho_B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

So, the Von Neumann entropy $S(\rho_B)$ is:

$$S(\rho_B) = -\mathrm{tr}(\rho_B \log \rho_B)$$

Substituting the values we get,

$$S(\rho_B) = -\left(\frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2}\right) \Rightarrow -\left(\frac{1}{2}(-1) + \frac{1}{2}(-1)\right)$$

$$\Rightarrow -\left(-\frac{1}{2} - \frac{1}{2}\right)$$

$$\Rightarrow \log 2$$

Now we have $S(\rho_{AB}) = 0$

Substituting in formula for $C(\rho)$:-

$$C(\rho) = \ln(N) + S(\rho_B) - S(\rho_{AB})$$

$$= \ln(2) + \log(2) - 0$$

$$= 2$$

So, $C(\rho) = 2$ for the given Bell state.

2. For the mutual information Computation, Complete the detailed Calculation that led to results in Equations (361)-(362) of Lecture Notes.

Sol Given $|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $|\psi_1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

and the decomposition of I_2,

$P_0 = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$, $P_1 = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$

$P_{00} = \langle\psi_0|P_0|\psi_0\rangle$

$\Rightarrow \frac{1}{\sqrt{2}}(1\ -1)\cdot\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$= \frac{1}{2\times 2}(1-1 \quad 1-1)\begin{pmatrix}1\\-1\end{pmatrix}$

$= \frac{1}{4}(0\ 0)\begin{pmatrix}1\\-1\end{pmatrix}$

$= \frac{1}{4}(0) \Rightarrow \boxed{0}$

$P_{10} = \langle\psi_1|P_0|\psi_1\rangle$

$= (0\ 1)\ \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$= \frac{1}{2}(0+1 \quad 0+1)\begin{pmatrix}0\\1\end{pmatrix}$

$= \frac{1}{2}(1\ 1)\begin{pmatrix}0\\1\end{pmatrix}$

$= \frac{1}{2}(0+1) = \boxed{\frac{1}{2}}$

$P_{01} = \langle\psi_0|P_1|\psi_0\rangle$

$= \frac{1}{\sqrt{2}}(1\ -1)\cdot\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$= \frac{1}{2\times 2}(1+1 \quad -1-1)\begin{pmatrix}1\\-1\end{pmatrix}$

$$= \frac{1}{4}(2 \; -2)\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{4}(2+2)$$

$$= \frac{1}{4}(4) = \boxed{1}$$

$$P_{11} = \langle \psi_1 | P_1 | \psi_1 \rangle$$

$$= (0 \;\; 1)\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{1}{2}(0-1 \;\; 0+1)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{1}{2}(-1 \;\; 1)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{1}{2}(0+1) = \boxed{\frac{1}{2}}$$

$P_{0\cdot} = P_{00} + P_{01} = 0 + 1 = 1$

$P_{1\cdot} = P_{10} + P_{11} = \frac{1}{2} + \frac{1}{2} = 1$

$P_{\cdot 0} = P_{10} + P_{00} = 0 + \frac{1}{2} = \frac{1}{2}$

$P_{\cdot 1} = P_{01} + P_{11} = 1 + \frac{1}{2} = \frac{3}{2}$ [$\because$ values obtained from above]

3. Consider 2×2 unitary matrix

$$U(\theta,\phi) = \begin{pmatrix}\cos(\theta/2) & e^{-i\phi}\sin(\theta/2)\\ -e^{i\phi}\sin(\theta/2) & \cos(\theta/2)\end{pmatrix},$$

Show that the state in $\mathbb{C}^4$,

$$\left(U(\theta_1,\phi_1)\otimes U(\theta_2,\phi_2)\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$ is not entangled.

Sol: To show it is not entangled we need to verify whether it can be written as the product state of two states in $\mathbb{C}^2$.

We denote state $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ as $|0\rangle$ for understanding

Then the state $\mathbb{C}^4$ is written as

$$U(\theta_1, \phi_1) \otimes U(\theta_2, \phi_2)|0\rangle$$

Expanding this out we have

$$U(\theta_1, \phi_1) \otimes U(\theta_2, \phi_2)|0\rangle = \left(U(\theta_1, \phi_1)|0\rangle\right) \otimes \left(U(\theta_2, \phi_2)|0\rangle\right)$$

we need to check if this is factorized into product of two states in $\mathbb{C}^2$,

$$U(\theta_1, \phi_1)|0\rangle = \begin{pmatrix} \cos(\theta_1/2) \\ -e^{i\phi_1}\sin(\theta_1/2) \end{pmatrix}$$

$$U(\theta_2, \phi_2)|0\rangle = \begin{pmatrix} \cos(\theta_2/2) \\ -e^{i\phi_2}\sin(\theta_2/2) \end{pmatrix}$$

Now, let's tensor them together:-

$$\left(U(\theta_1, \phi_1)|0\rangle\right) \otimes \left(U(\theta_2, \phi_2)|0\rangle\right) = \begin{pmatrix} \cos(\theta_1/2)\cos(\theta_2/2) \\ -e^{i\theta_2}\sin(\theta_2/2)\cos(\theta_1/2) \\ e^{-i\phi_1}\sin(\theta_1/2)\cos(\theta_2/2) \\ e^{i(\phi_1+\phi_2)}\sin(\theta_1/2)\sin(\theta_2/2) \end{pmatrix}$$

$$① \longleftarrow$$

we need to compare this with state $U(\theta_1, \phi_1) \otimes U(\theta_2, \phi_2)|0\rangle$, which is

$$\begin{pmatrix} \cos(\theta_1/2)\cos(\theta_2/2) \\ -e^{-i\phi_2}\sin(\theta_2/2)\cos(\theta_1/2) \\ -e^{i\phi_1}\sin(\theta_1/2)\cos(\theta_2/2) \\ \sin(\theta_1/2)\sin(\theta_2/2) \end{pmatrix} \longrightarrow ②$$

comparing ① & ② we need $\phi_1 = 0$ & $\phi_2 = 0$ to have them both equal.

Thus for state in $\mathbb{C}^4$ it is not Entangled.

$$\Rightarrow U(\theta_1,\phi_1)\otimes U(\theta_2,\phi_2)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} =$$
$$\boxed{U(\theta_1,\phi_1)\begin{pmatrix}1\\0\end{pmatrix}\otimes U(\theta_2,\phi_2)\begin{pmatrix}1\\0\end{pmatrix}}$$

4. Let $|0\rangle, |1\rangle$ be an arbitary basis. Can the state in $\mathbb{C}^4$, $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle + \frac{1}{\sqrt{8}}|0\rangle\otimes|1\rangle + \frac{1}{\sqrt{8}}|1\rangle\otimes|0\rangle + \frac{1}{\sqrt{4}}|1\rangle\otimes|1\rangle$. be written as a product state?

Sol: As the given problem we get,

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{8}}|01\rangle + \frac{1}{\sqrt{8}}|10\rangle + \frac{1}{\sqrt{4}}|11\rangle$$

To determine if $|\psi\rangle$ can be written as product state we can express it as tensor product of two states, one for each qubit. lets assume

$$|\psi_1\rangle = c_1|0\rangle + c_2|1\rangle, \quad |\psi_2\rangle = d_1|0\rangle + d_2|1\rangle$$

Then their tensor product $|\psi_1\rangle\otimes|\psi_2\rangle$ should be equal to $|\psi\rangle$:

$$|\psi_1\rangle\otimes|\psi_2\rangle = c_1d_1|00\rangle + c_1d_2|01\rangle + c_2d_1|10\rangle + c_2d_2|11\rangle \quad -(1)$$

Comparing this above equation with eq given in problem we get

$$c_1d_1 = \frac{1}{\sqrt{2}} \;(2), \quad c_1d_2 = \frac{1}{\sqrt{8}} \;(3), \quad c_2d_1 = \frac{1}{\sqrt{8}} \;(4), \quad c_2d_2 = \frac{1}{\sqrt{4}} \;(5)$$

from the ③ & ④ we get $c_1 d_1, c_2 d_2 = \frac{1}{\sqrt{8}}$ — ⑥

from the ② & ⑤ we get $c_1 d_1, c_2 d_2 = \frac{1}{8}$ — ⑦

by the ⑥ & ⑦ we get contradiction

$\therefore$ therefore we have contradiction and $|\psi\rangle$ cannot be as a product state.

5. Consider the state

$$|\psi\rangle = \frac{1}{2}\left(|0\rangle \otimes |0\rangle + e^{i\phi_1}|0\rangle \otimes |1\rangle + e^{i\phi_2}|1\rangle \otimes |0\rangle + e^{i\phi_3}|1\rangle \otimes |1\rangle\right).$$

(i) Let $\phi_3 = \phi_1 + \phi_2$. Is the state $|\psi\rangle$ a product state?

(ii) Let $\phi_3 = \phi_1 + \phi_2 + \pi$. Is the state $|\psi\rangle$ a product state?

Sol: (i) Given the $|\psi\rangle$ can be written as the

$$\frac{1}{2}\left(|00\rangle + e^{i\phi}|01\rangle + e^{i\phi_2}|10\rangle + e^{i\phi_3}|11\rangle\right)$$

Substituting the value $\phi_3 = \phi_1 + \phi_2$ we get

$$= \frac{1}{2}\left(|00\rangle + e^{i\phi}|01\rangle + e^{i\phi_2}|10\rangle + e^{i(\phi_1+\phi_2)}|11\rangle\right)$$

it is written as

$$\boxed{\Rightarrow \frac{1}{2}|0\rangle + e^{i\phi_1}|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi_2}|1\rangle\right)}$$

which is a product state.

(ii) For the above substituting $\phi_3 = \phi_1 + \phi_2 + \pi$ we get

$$|\psi\rangle = \frac{1}{2}\left(|00\rangle + e^{i\phi}|01\rangle + e^{i\phi_2}|10\rangle + e^{i(\phi_1+\phi_2+\pi)}|11\rangle\right.$$

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we do not have a product state for the above equation we have maximally entangled state.

6. Can one find 2×2 matrices S_1 and S_2 such that the product state holds?

$$\left(S_1\begin{pmatrix}1\\0\end{pmatrix}\right)\otimes\left(S_2\begin{pmatrix}1\\0\end{pmatrix}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Sol:- from the above given problem we get

$$\left(S_1\begin{pmatrix}1\\0\end{pmatrix}\right)\otimes\left(S_2\begin{pmatrix}1\\0\end{pmatrix}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Thus by taking example 2×2 matrices of S_1 & S_2 we get,

$$\Rightarrow \begin{pmatrix}S_{11}^{(1)}\\S_{21}^{(1)}\end{pmatrix}\otimes\begin{pmatrix}S_{11}^{(2)}\\S_{21}^{(2)}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}S_{11}^{(1)}\,S_{11}^{(2)}\\S_{11}^{(1)}\,S_{21}^{(2)}\\S_{21}^{(1)}\,S_{11}^{(2)}\\S_{21}^{(1)}\,S_{21}^{(2)}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

from the above we get

$S_{11}^{(1)}S_{11}^{(2)}=\frac{1}{\sqrt{2}}$ — ①, $S_{11}^{(1)}S_{21}^{(2)}=0$ — ②

$S_{21}^{(1)}S_{11}^{(2)}=0$ — ③, $S_{21}^{(1)}S_{21}^{(2)}=\frac{1}{\sqrt{2}}$ — ④

By multiplying (or) by checking the existing possibilities we can say that they are not compatible. Thus no S_1 and S_2 exist such that given condition is satisfied.

7.

Sol:-

We will choose the standard basis in $\mathbb{C}^2$ to calculate the trace. For the density matrix ρ we find

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Therefore

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} +$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix}\right) |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad -(1)$$

Similarly

$$\rho_B = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} - (2)$$

By above (1) = (2)

The eigen values are 0 & 1.

orthonormal eigen vectors are $\frac{1}{\sqrt{2}}(1, 1)^T$ & $\frac{1}{\sqrt{2}}(1, -1)^T$

Thus,

$$-\operatorname{tr}(\rho_A \log_2(\rho_A)) = -\operatorname{tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \log_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right)$$

$$= -\operatorname{tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0\log_2 0 & 0 \\ 0 & 1\log_2 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right)$$

$$= -\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \qquad [\because 0\log_2 = 0,\ 1\log_2 1 = 0]$$

Hence $|\psi\rangle$ is not entangled.

8.

Sol:- Given that

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1 \quad -(1)$$

$$|\phi\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \quad -(2)$$

Inserting (1) into (2) we get

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\right).$$

on the other hand we have

$$\Rightarrow \frac{1}{2\sqrt{2}}\left(|00\rangle + |11\rangle\right) \otimes \left(a|0\rangle + b|1\rangle\right) + \frac{1}{2\sqrt{2}}\left(|00\rangle - |11\rangle\right) \otimes \left(a|0\rangle - b|1\rangle\right)$$

$$+ \frac{1}{2\sqrt{2}}\left(|01\rangle + |10\rangle\right) \otimes \left(a|1\rangle + b|0\rangle\right) + \frac{1}{2\sqrt{2}}\left(|01\rangle - |10\rangle\right) \otimes \left(a|1\rangle - b|0\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(a|000\rangle + a|110\rangle + b|001\rangle + b|111\rangle\right)$$

$$+ \frac{1}{2\sqrt{2}}\left(a|000\rangle - a|110\rangle - b|001\rangle + b|111\rangle\right)$$

$$+ \frac{1}{2\sqrt{2}}\left(a|011\rangle + a|101\rangle + b|010\rangle + b|100\rangle\right)$$

$$+ \frac{1}{2\sqrt{2}}\left(a|011\rangle - a|101\rangle - b|010\rangle + b|100\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\right)$$

$$= \phi$$

$\therefore$ We have shown that the given (388) equation of lecture notes can be written as equation (389).

9. Find the Unitary transform S in Equation (395) of the lecture Notes.

Sol: Given equation (395) is

$$|\psi_3\rangle = S|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0101\rangle - |1010\rangle\right)$$

in the Notes.

Given that $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0101\rangle + |1010\rangle\right)$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|0101\rangle - |1010\rangle\right)$$

We can observe that

$$S \frac{1}{\sqrt{2}}\left(|0101\rangle + |1010\rangle\right) = \frac{1}{\sqrt{2}}\left(|0101\rangle - |1010\rangle\right)$$

from the above

only the sign for the $|1010\rangle$ has changed

$\therefore$ The unitary transform of the solution

S is $I_{16} - 2|1010\rangle\langle 1010|$ i.e., S is the identity except for changing the sign of $|1010\rangle$.