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HW-2

1) Consider the 3-input and 3-output gate given by

$x_1' = x_1$, $x_2' = x_1 \oplus x_2$, $x_3' = x_1 \oplus x_2 \oplus x_3$.

Is this transformation invertible?

Sol:- The truth table is given by:-

x_1	x_2	x_3	x_1'	x_2'	x_3'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	1
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	1	1	0
1	1	0	1	0	0
1	1	1	1	0	1

from the truth table we see that the transformation is invertible. i.e., we have a 1-1 map. The inverse notation is given by

$x_1 = x_1'$, $x_2 = x_1' \oplus x_2'$, $x_3 = x_1' \oplus x_2' \oplus x_3'$.

i.e., we are saying that by doing $x_1' \oplus x_2'$ we get x_2 and by doing the $x_1' \oplus x_2' \oplus x_3'$ we get the x_3. and x_1 is x_1' we can say that the transformation is invertible.

2. Consider a variant of fredkin gate.

$F(a,b,c) = (a, a\cdot b + \bar{a}\cdot c, a\cdot c + \bar{a}\cdot b)$.

(i) Express the NOT(a) gate in terms of fredkin gate.

(ii) Express the AND (a,b) gate in terms of this fredkin gate.

Sol: Given the variant of fredkin gate is given by:

$F(a,b,c) = (a, a\cdot b + \bar{a}\cdot c, a\cdot c + \bar{a}\cdot b)$

fredkin is reversible logic gate that has property of having same no. of inputs as outputs and each input pattern maps to the unique o/p pattern, and the 2 term is the target o/p and first 1,2 o/p's are the control terms.

inputs $\Rightarrow \boxed{(a, 0, 1)}$

(i) $f(a, 0, 1) = (a, a\cdot 0 + \bar{a}\cdot 1, a\cdot 1 + \bar{a}\cdot 0)$

$= (a, \bar{a} + 0, 0 + \bar{a})$

$= (a, \bar{a}, \bar{a}) \Rightarrow \boxed{(a, \bar{a}, a)}$

∴ By the above substitution of values we get the 2 terms as $\boxed{\bar{a}.}$ $\Rightarrow$ NOT(a) = f(a,0,1)

(ii) By substituting (a, b, 0) values in the above we get the AND(a,b)

The Substitution is done as follows:-

$f(a,b,0) = (a,\ a.b + \bar{a}.0,\ a.0 + \bar{a}.b)$

$= (a,\ a.b + 0,\ 0 + \bar{a}.b)$

$= (a.\ a.b,\ \bar{a}.b)$

by the above fredkin gate $a.b = b$ which is the desired value of ours to prove AND(a,b).

3. Is the gate

$(a,b,c) \rightarrow (x,y,z)$,

where $x = a$, $y = a.b \oplus c$, $z = \bar{a}.\bar{c} \oplus \bar{b}$

invertible ?

Sol:- Given above that $x = a$, $y = a.b \oplus c$, $z = \bar{a}.\bar{c} \oplus \bar{b}$ the truth table is given as

a	b	c	x	y	z
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	1
0	1	1	0	1	0
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	1	0
1	1	1	1	0	0

∴ By the above given table we can

Say that the output is one to one

thus the given $\boxed{\text{gate is invertible}}$

4. Show that the magic gate below is an unitary gate

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix}$$

Sol: Given to show that M is the unitary gate if the $MM^{+} = I_4$ then it is unitary.

$$MM^{+} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ -i & 0 & 0 & +i \\ 0 & i & -i & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-i^2+0+0 & 0+0+0+0 & 0+0+0+0 & 1+i^2+0+0 \\ 0+0+0+0 & 0+0-i^2+1 & 0+0-i^2-1 & 0+0+0+0 \\ 0+0+0+0 & 0+0-i^2-1 & 0+0-i^2+1 & 0+0+0+0 \\ 1+i^2+0+0 & 0+0+0+0 & 0+0+0+0 & 1-i^2+0+0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & 0 & 0 & 1-1 \\ 0 & 1+1 & 1-1 & 0 \\ 0 & 1-1 & 1+1 & 0 \\ 1-1 & 0 & 0 & 1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{2}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \boxed{I_4}$$

$$M^\dagger M = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ -i & 0 & 0 & i \\ 0 & i & -i & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{2}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{I_4}$$

$$\boxed{\therefore \quad MM^\dagger = M^\dagger M = \underline{I_4}}$$

5. Show that if we apply quantum fourier transform

$$U_{QFT} = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n} \sum_{k=0}^{2^n-1} e^{-i2\pi kj/2^n} |k\rangle\langle j|$$

to the following state in Hilbert Space $C^8_{(n=3)}$

$$|\psi\rangle = \frac{1}{2} \sum_{j=0}^{7} \cos(2\pi j/8) |j\rangle,$$

the resulting state will given by

$$Q_{FT}|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |7\rangle).$$

Soln

We use the $\{|j\rangle : j=0,1,\ldots 7\}$ as an orthonormal basis in the Hilbert Space $\mathbb{C}^8$, where $|7\rangle = |111\rangle \equiv |1\rangle \otimes |1\rangle \otimes |1\rangle$.

We also use the Eulers identity

$$e^{i\theta} = \cos\theta + i\sin(\theta) \text{ and}$$

$$\sum_{k=0}^{N-1} e^{i2\pi k(n-m)/N} = N\delta_{nm}$$

Thus we have

$$\hat{x}(k) = \sum_{j=0}^{7} e^{-i2\pi kj/8} \cos(2\pi j/8) = \frac{1}{2}\sum_{j=0}^{7}\left(e^{i2\pi(1-k)j/8} + e^{-i2\pi(1+k)j/8}\right)$$

$$= 4(\delta_{k1} + \delta_{k7})$$

and

$$U_{QFT}\frac{1}{2}\sum_{j=0}^{7} \cos(2\pi j/8)|j\rangle = \frac{1}{2\sqrt{8}}\sum_{k=0}^{7} \hat{x}(k)|k\rangle$$

$$\boxed{= \frac{1}{\sqrt{2}}(|1\rangle + |7\rangle).}$$

6 Consider the Hadamard gate U_H and the CNOT gate U_{CNOT}, S.T

$$(I_2 \otimes U_H) U_{CNOT} (I_2 \otimes U_H) |jk\rangle = (-1)^{j \cdot k} |jk\rangle.$$

That is, the above gate implements a phase gate $U_p(\pi)$.

Sol: To S.T given is equal to $(-1)^{j \cdot k} |jk\rangle$ we do the following

$$\Rightarrow (I_2 \otimes U_H) U_{CNOT} (I_2 \otimes U_H) |jk\rangle$$

$$\Rightarrow (I_2 \otimes U_H) U_{CNOT} \left(I_2 \otimes \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{1} |1\rangle\right) |jk\rangle$$

$$\left[\because U_H = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^k |1\rangle)\right]$$

$$\Rightarrow (I_2 \otimes U_H) U_{CNOT} \left(I_2 |j\rangle \otimes \frac{1}{\sqrt{2}} |0\rangle + (-1)^k |1\rangle\right)$$

$$[\because (a \otimes b)(c \otimes d) = ac \otimes bd]$$

$$\Rightarrow (I_2 \otimes U_H) \left(|j\rangle \otimes (|j\rangle \otimes \frac{1}{\sqrt{2}} |0\rangle + (-1)^k |1\rangle\right)$$

$$[\because U_{CNOT} (|j\rangle \otimes |k\rangle = |j\rangle \otimes |j \oplus k\rangle$$

$$\Rightarrow I_2 |j\rangle \otimes U_H \left(|j\rangle \oplus \frac{1}{\sqrt{2}} |0\rangle + (-1)^k |1\rangle\right)$$

$$[\because (a \otimes b)(c \otimes d) = ac \otimes bd)$$

$$\Rightarrow |j\rangle \otimes \frac{1}{\sqrt{2}} |0\rangle + (-1)^j |1\rangle \oplus \frac{1}{\sqrt{2}} |0\rangle + (-1)^k |1\rangle$$

$$\Rightarrow (-1)^{j \cdot k} |j \otimes k\rangle \quad \Rightarrow \boxed{(-1)^{jk} |jk\rangle}$$

(08)

$(I_2 \otimes U_H) U_{CNOT} (I_2 \otimes U_H) |jk\rangle \Rightarrow$

$\Rightarrow (I_2 \otimes U_H) U_{CNOT} \frac{1}{\sqrt{2}} |j\rangle \otimes (|0\rangle + (-1)^k |1\rangle)$

$$\Rightarrow \begin{cases} (I_2 \otimes U_H) \frac{1}{\sqrt{2}} |j\rangle \otimes (|0\rangle + (-1)^k |1\rangle) & j = 0 \\ (I_2 \otimes U_H) \frac{1}{\sqrt{2}} |j\rangle \otimes (|1\rangle + (-1)^k |0\rangle) & j = 1 \end{cases}$$

$\Rightarrow (I_2 \otimes U_H) \frac{1}{\sqrt{2}} |j\rangle \otimes (-1)^{jk} (|0\rangle + (-1)^k |1\rangle)$

$$\boxed{= (-1)^{j \cdot k} |jk\rangle}$$

7. Given an orthonormal basis in $\mathbb{C}^N$,

$|\phi_0\rangle, |\phi_1\rangle, \ldots |\phi_{N-1}\rangle$,

S.T the matrix

$U = \sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|$ is a Unitary matrix

Sol: To S.T that U is unitary we should demonstrate $U^\dagger U = I$

Given that,

$$U = \sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|$$

$$U^\dagger = \left(\sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|\right)^\dagger$$

$$= \sum_{K=0}^{N-2} |\phi_{K+1}\rangle\langle\phi_K| + |\phi_0\rangle\langle\phi_{N-1}|$$

Now $U^{\dagger}U =$

$$\sum_{K=0}^{N-2}\sum_{j=0}^{N-2} |\phi_{K+1}\rangle\langle\phi_K|\phi_j\rangle\langle\phi_{j+1}| + \sum_{K=0}^{N-2} |\phi_{K+1}\rangle\langle\phi_K|\phi_{N-1}\rangle\langle\phi_0|$$

$$+ \sum_{j=0}^{N-2} |\phi_0\rangle\langle\phi_{N-1}|\phi_j\rangle\langle\phi_{j+1}| + |\phi_1$$

Using orthonormal property $\langle\phi_i|\phi_j\rangle = \delta_{ij}$

$$\Rightarrow \sum_{K=0}^{N-2} |\phi_{K+1}\rangle\langle\phi_{K+1}| + \sum_{K=0}^{N-2} |\phi_0\rangle\langle\phi_0| + |\phi_{N-1}\rangle\langle\phi_{N-1}|$$

$$= \sum_{K=0}^{N-2} |\phi_{K+1}\rangle\langle\phi_{K+1}| + (N-1)(|\phi_0\rangle\langle\phi_0| + |\phi_{N-1}\rangle\langle\phi_{N-1}|)$$

$$= I_N$$

Therefore $\boxed{U^{\dagger}U = I_N}$ Confirming that the matrix is Unitary.

8. Consider 8×8 matrix

$$U(\alpha) = e^{i\alpha}/\sqrt{2}\,(I_2 \otimes I_2 \otimes I_2 + i\,\sigma_1 \otimes \sigma_1 \otimes \sigma_1),$$

where σ_1 is the Pauli

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

S.T $U(\alpha)$ is an unitary gate.

Sol:

Given that $U(\alpha) = \frac{e^{i\alpha}}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_1 \otimes \sigma_1)$

& $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\sigma_1^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore \sigma_1^+ = \sigma$

Now Conjugate of $U(\alpha)$ is $U^+(\alpha)$

$$U^+(\alpha) = \frac{e^{-i\alpha}}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 - i\sigma_1 \otimes \sigma_1 \otimes \sigma_1)$$

we need to find $U(\alpha).U^+(\alpha)$

$$\Rightarrow \frac{e^{i\alpha}}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_1 \otimes \sigma_1) . \frac{e^{-i\alpha}}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 - i\sigma_1 \otimes \sigma_1 \otimes \sigma_1)$$

$$\Rightarrow \frac{e^{i\alpha}.e^{-i\alpha}}{2}\left((I_2 \otimes I_2 \otimes I_2)^2 - (i)^2(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)^2\right)$$

$$= \frac{1}{2}\left((I_8)^2 - (-1)(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)^2\right)$$

$$= \frac{1}{2}\left((I_8)^2 + (\sigma_1)^2\right)$$

$$= \frac{1}{2}(I_8 + I_8) \quad [\because I^2 = I \ \& \ \sigma_1^2 = I]$$

$$= \frac{1}{2}(2 I_8)$$

$$\boxed{= I_8}$$

$\therefore \boxed{UU^+ = I}$ we can say it is conjugate.

9. Let σ_3 be the Pauli matrix

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, show that the matrix

$\Pi = \frac{1}{2}(I_2 + \sigma_3) \otimes I_2$ is a projection matrix.

Sol: Given to prove that it is projection matrix.

It should satisfy the following conditions

1) $\Pi^{\dagger} = \Pi$, 2) $\Pi^2 = \Pi$

Given $\Pi = \frac{1}{2}(I_2 + \sigma_3) \otimes I_2$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Condition 1 is satisfied

$$\Pi^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+1+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \Pi$$

$\Rightarrow \Pi^2 = \Pi$

Condition ② is Satisfied

$\therefore$ Given two Conditions is Satisfied then we Can Say that Π is a Projection matrix.

10) The most general state of a Single qubit is described by 3 parameters $\theta, \phi, \sigma \in R$

$$|\psi\rangle = e^{i\sigma}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right).$$

Determine the probability that the State $|\psi\rangle$ is in State

(i) $|0\rangle$, (ii) $|1\rangle$.

Sol. Given in the above data Single qubit is described by real parameters. and given

$$|\psi\rangle = e^{i\sigma}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right).$$

(i) The probability that the state $|\psi\rangle$ is in state $|0\rangle$ is

$$P(|0\rangle) = |\langle 0|\psi\rangle|^2$$

$$= |\langle 0| e^{i\sigma}\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle|^2$$

$$= |e^{i\sigma}\cos(\theta/2)\langle 0|0\rangle + e^{i\phi}\sin(\theta/2)\langle 1|0\rangle|^2$$

$$= |e^{i\sigma}\cos(\theta/2)\langle 0|0\rangle|^2$$

$$= |e^{i\sigma}\cos(\theta/2)|^2$$

$$\boxed{= |\cos(\theta/2)|^2}$$

(ii) The probability that state $|\psi\rangle$ is in state $|1\rangle$ is.

$$P(|1\rangle) = |\langle 1|\psi\rangle|^2$$

$$= |\langle 1| e^{i\sigma}\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle|^2$$

$$= |e^{i\phi}\sin(\theta/2)\langle 1|1\rangle|^2$$

$$= |e^{i\phi}\sin(\theta/2)|^2$$

$$\boxed{= |\sin(\theta/2)|^2}$$

11. Consider the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \text{ \& state } \langle 0| \otimes I_2.$$

prove that the partial measurement result below

$$(\langle 0| \otimes I_2)|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle.$$

Sol: Given to find the partial measurement

$$\left(\langle 0| \otimes I_2\right) |\psi\rangle$$

$$\Rightarrow \langle 0| \otimes I_2 \left(\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle\right)\left(\langle 0| \otimes I_2\right) - \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle\right)\left(\langle 0| \otimes I_2\right)$$

$$\left[\because |01\rangle = |0\rangle \otimes |1\rangle \text{ \& } |10\rangle = |0\rangle \otimes |0\rangle\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \langle 0|0\rangle I_2 |1\rangle - \frac{1}{\sqrt{2}} \langle 0|1\rangle I_2 |0\rangle$$

$$\left[\because \langle a| \otimes b \ |c\rangle \otimes |d\rangle = (a/c) \cdot (b/d)\right]$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cdot (1) \cdot I_2 \cdot |1\rangle - \frac{1}{\sqrt{2}} \cdot (0) \cdot I_2 \cdot |0\rangle$$

$$\left[\because \langle 0|0\rangle = 1 \text{ \& } \langle 0|1\rangle = 0\right]$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cdot I_2 |1\rangle$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{2}} |1\rangle}$$