# Homework 4

## Problem 1

Let's first deal with the likelihood.

$$
\begin{aligned}
Y &= X\alpha + \varepsilon \\
\varepsilon &= Y - X\alpha \\
\varepsilon_i &= Y_i - X_i\alpha
\end{aligned}
$$

It is given that $\varepsilon$ follows multivariate normal distribution with a diagonal covariance matrix, which means that each $\varepsilon_i$ is independent of others. Therefore:

$$
\begin{aligned}
\ell\left(\sigma^2\right) &= f\left(\varepsilon|\sigma^2\right) = N\left(\varepsilon|0, \sigma^2 I_n\right) \\
&= \prod_{i=1}^{n} N\left(\varepsilon_i|0, \sigma^2\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\varepsilon_i^2}{2\sigma^2}\right] \\
&\propto \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left[-\frac{\varepsilon_i^2}{2\sigma^2}\right] \\
&= \frac{1}{\sigma^n} \cdot \exp\left[-\frac{\sum_{i=1}^{n} \varepsilon_i^2}{2\sigma^2}\right]
\end{aligned}
$$

Likelihood of $\phi$ is equal to the likelihood of $\sigma^2$:

$$
\ell(\phi) = \ell\left(\sigma^2\right) = \phi^{\frac{n}{2}} \exp\left[-\phi \frac{\sum_{i=1}^{n} \varepsilon_i^2}{2}\right]
$$

### 1a

$$
\begin{aligned}
\pi_1\left(\sigma^2\right) &\sim \ell\left(\sigma^2\right) \cdot p(\sigma^2) \\
&= \ell\left(\delta^2\right) \cdot 1 \\
&= l\left(\sigma^2\right) \\
&= \frac{1}{\sigma^n} \cdot \exp\left[-\frac{\sum_{i=1}^{n} \varepsilon_i^2}{2\sigma^2}\right] \\
&\sim \mathrm{IG}\left(\frac{n-1}{2}, \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i^2\right)
\end{aligned}
$$

### 1b

$$\begin{aligned}\pi_2(\phi) &\propto \ell(\phi)\cdot p(\phi)\\ &= \ell(\phi)\cdot 1\\ &= \ell(\phi)\\ &\sim \phi^{\frac{n}{2}}\exp\left[-\phi\frac{\sum_{i=1}^{n}\varepsilon_i^2}{2}\right]\\ &\sim G\left(\frac{n}{2}+1,\frac{\sum\varepsilon_i^2}{2}\right)\end{aligned}$$

# Problem 2

### 2.a

According to Equation 3.7 from the script, stationary distribution is defined as follows:

$$\pi = \pi P$$

Matrix $P$ is known to us. Let's write $\pi$ as:

$$\pi = [xyz]$$

Now let's find $x, y, z$ that satisfy equation 3.7.

- equation 3.7
- multiply left vector with matrix P and write both sides as colomn vectors
- factor $x, y$ and $z$ out of the brackets

$$\begin{gathered}[x \quad y \quad z]P = [x \quad y \quad z]\\ \begin{bmatrix}0,5x+0,3y+0,2z\\0,4x+0,4y+0,3z\\0,1x+0,3y+0,5z\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}\\ \begin{bmatrix}-0,5 & 0,3 & 0,2\\0,4 & -0,6 & 0,3\\0,4 & 0,3 & -0,5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\end{gathered} \tag{1}$$

The probabilities have to sum up to 1, so we also have that $x+y+z=1$. Together we get this system of linear equations.

$$\begin{bmatrix}-0,5 & 0,3 & 0,2\\0,4 & -0,6 & 0,3\\0,4 & 0,3 & -0,5\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{2}$$

Solving this gets us the following result:

$$x = 21/62, y = 32/62, z = 18/62$$

$$\pi = \frac{1}{62}[21 \quad 32 \quad 18]$$

### 2.b

Define transition kernels $p(1,\cdot)$,$p(2,\cdot)$,$p(3,\cdot)$

```
p1 <- c(0.5, 0.4, 0.1)
p2 <- c(0.3, 0.4, 0.3)
p3 <- c(0.2, 0.3, 0.5)
```

```
set.seed(42)
n <- 10^5
```

Sample the $\theta^{(0)}$ state uniformly from {1,2,3}.

```
t <- numeric(n+1)
t[1] <- sample(c(1,2,3), size=1, prob = c(1,1,1))
```

```
for (i in 2:(n+1)){
    if (t[i-1]==1){ p <- p1 }
    else if (t[i-1]==2){ p <- p2 }
    else { p <- p3 }
    t[i] <- sample(c(1,2,3), size=1, prob = p)
}
```

```
rel_freq <- table(t) / n
rel_freq
```

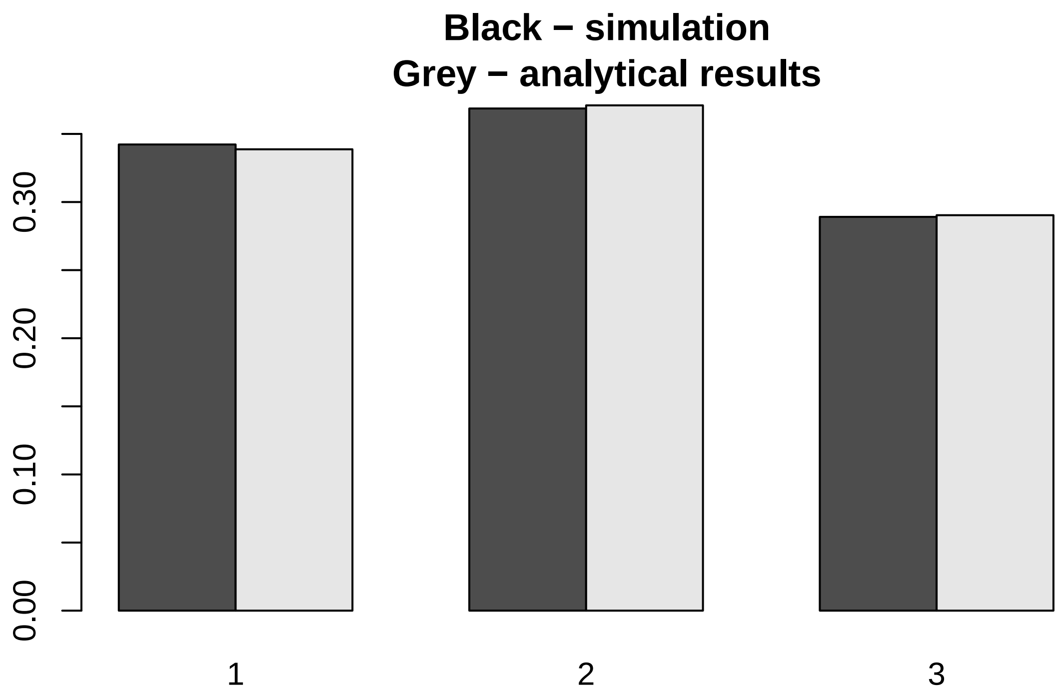
```
## t
##       1       2       3
## 0.34225 0.36867 0.28909
```

Compare this to the analytical results:

```
c(21/62, 23/62, 18/62)
```

```
## [1] 0.3387097 0.3709677 0.2903226
```

```
binded <- rbind(rel_freq, c(21/62, 23/62, 18/62))
barplot(binded, beside=T, main="
    Black - simulation
    Grey - analytical results")
```


Black − simulation
Grey − analytical results
0.00
0.10
0.20
0.30
1
2
3