Homework 5

Problem 1

1.a Gibbs sampler

```
gibbs_sampler <- function(X, Y, alpha_init = rep(0,dim(X)[2]), sigma2_init = 1,
                         delta = 10, lambda = 10, mu = rep(0,dim(X)[2]),
                         Omega = diag(dim(X)[2]), MCsamplesize = 1000){

    # dims of X
  n <- dim(X)[1] # sample size
  p <- dim(X)[2] # number of covariates in linear regression model/alpha-parameters

  # set starting values for parameter chains
    # variables where generated values will be saved
  alpha_chain      <- matrix(NA, MCsamplesize, p)
  sigma2_chain     <- rep   (NA, MCsamplesize)
    # first values for both chains
  alpha_chain[1,]  <- alpha_init  # [0] * p
  sigma2_chain[1]  <- sigma2_init  # [1]

  # create parameter chain
  for (i in 2:(MCsamplesize)){

    alpha_chain[i,]  <- alpha_chain[i-1,]
    sigma2_chain[i]  <- sigma2_chain[i-1]

    for (j in 1:p){

      s_j <- 1 / (Omega[j, j] + sigma2_chain[i]^(-1) * sum(X[, j]^2))
      m_j <- s_j*(mu[j]*Omega[j, j]
                - sum(Omega[j, -j]*(alpha_chain[i, -j] - mu[-j]))
                + sigma2_chain[i]^(-1)*sum(X[, j]*(Y - X[, -j]%*%alpha_chain[i, -j])))
      alpha_chain[i,j] <- rnorm(1, m_j, sqrt(s_j)) # rnorm accepts sd, not var

    }

    sigma2_chain[i] <- 1 / rgamma(n=1, shape=delta + n/2,
                                  rate=lambda + 1/2*sum( (Y - X%*%alpha_chain[i, ])^2 ))
  }

  return(list("alpha" = alpha_chain, "sigma2" = sigma2_chain))

}
```

1.b

```r
set.seed(999)
samplesize <- 200
dim        <- 3
alpha_real <- c(-2, 0, 4)
sigma_real <- 2
X <- matrix(rnorm(samplesize*dim), nrow = samplesize, n = dim)
Y <- as.vector(X%*%alpha_real + rnorm(samplesize, 0, sigma_real))  # sd, not var
```

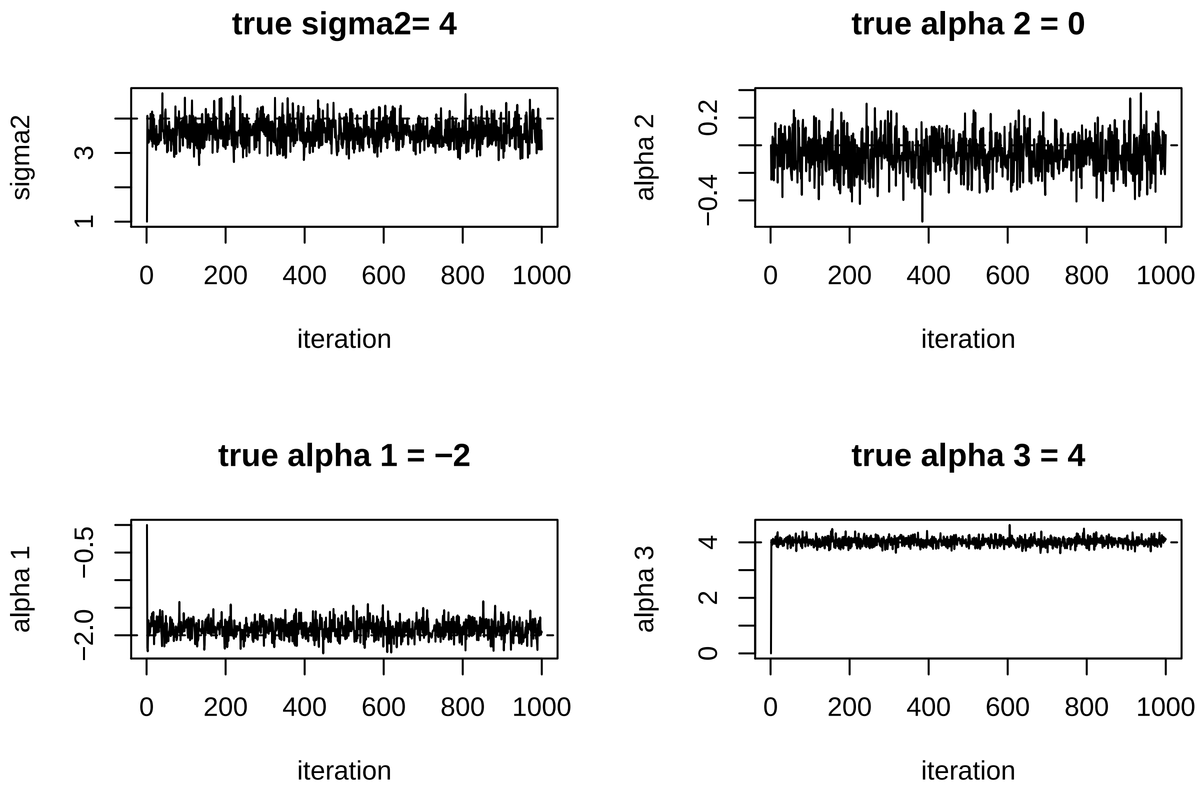
```r
n <- dim(X)[1] # sample size
p <- dim(X)[2] # number of covariates in linear regression model/alpha-parameters

sigma2_real <- sigma_real^2
```

```r
out <- gibbs_sampler(X, Y)
sigma2 <- out$sigma2
alpha <- out$alpha
```

```r
par(mfcol = c(2, 2))
plot(sigma2, ylab = "sigma2", xlab = "iteration", main = paste("true sigma2=",
    round(sigma2_real, digits = 2)), type = "l")
abline(h = sigma2_real, lty = 2)

for (i in 1:p){
    plot(alpha[,i], ylab = paste("alpha",i), xlab = "iteration", main = paste(
        "true alpha", i, "=", round(alpha_real[i], digits = 2)), type = "l")
    abline(h = alpha_real[i], lty = 2)
}
```

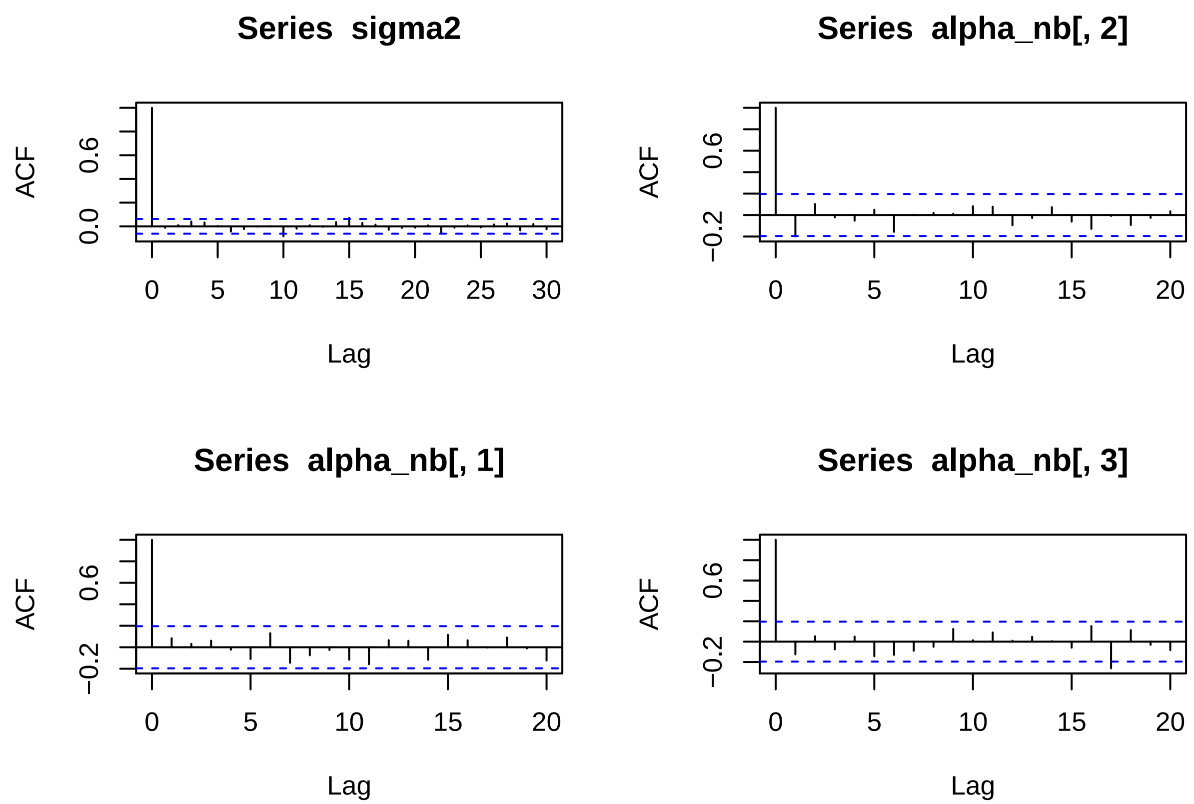


Intrepret traceplots

As we can see, the chains converge quite fast. From the graph it can be seen that the chains converge in 10-20 iterations. Let's choose a burn-in period of 100 to be safe.

```
# no burnin values
sigma2_nb <- sigma2[101:n]
alpha_nb  <- alpha[101:n,]
```

```
par(mfcol = c(2, 2))
acf(sigma2)
acf(alpha_nb[,1])
acf(alpha_nb[,2])
acf(alpha_nb[,3])
```



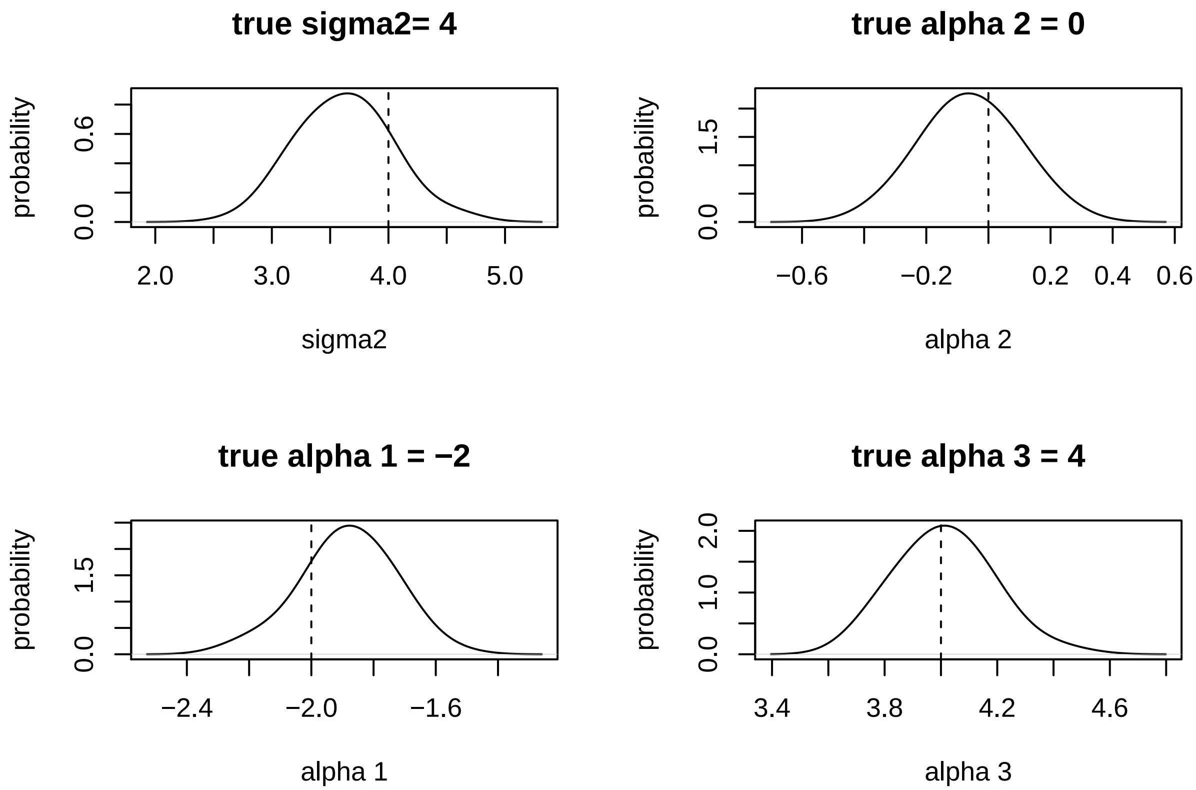
Intrepret ACF plots

We can see, that autocorrelation is low for each of the chains. So we can proceed without throwing away every n'th sample.

```
# density estimates
par(mfcol = c(2, 2))

hw <- 2 * (summary(sigma2_nb)[5] - summary(sigma2_nb)[2])
plot(density(sigma2_nb, width = hw), xlab = "sigma2", ylab = "probability",
    type = "l",main="")
abline(v = sigma2_real, lty = 2)
title(paste("true sigma2=", round(sigma2_real, digits = 2)))

for (i in 1:p){
    hw <- 2 * (summary(alpha_nb[,i])[5] - summary(alpha_nb[,i])[2])
    plot(density(alpha_nb[,i], width = hw), xlab = paste("alpha",i), ylab = "probability",
        type = "l",main="")
    abline(v = alpha_real[i], lty = 2)
    title(paste("true alpha",i, "=", round(alpha_real[i], digits = 2)))
}
```




```r
ord_lr <- lm(Y ~ X - 1)  # -1 to get read of intercept
ord_lr
```

```
##
## Call:
## lm(formula = Y ~ X - 1)
##
## Coefficients:
##      X1       X2       X3
## -1.9238  -0.0587   4.1003
```

```r
summary(ord_lr)
```

```
##
## Call:
## lm(formula = Y ~ X - 1)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.3286 -1.5298 -0.1412  1.1600  5.2405
##
## Coefficients:
##    Estimate Std. Error t value Pr(>|t|)
## X1  -1.9238     0.1510 -12.740   <2e-16 ***
## X2  -0.0587     0.1363  -0.431    0.667
```

```
## X3    4.1003     0.1465  27.980   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.956 on 197 degrees of freedom
## Multiple R-squared:  0.8355, Adjusted R-squared:  0.833
## F-statistic: 333.4 on 3 and 197 DF,  p-value: < 2.2e-16
```

Intrepret `lm` results

Coefficients of the linear regression are `-1.9238  -0.0587   4.1003` and estimated standard error is 1.956, which means that estimated `sigma2=3.83`. These values are closer to the true values, than the maximum of the posterior densities estimated by Gibbs sampler.

Especially bad is the posterior of `sigma2` with peak around 3.5.

1.c

```r
set.seed(42)
samplesize <- 200
dim        <- 3
alpha_real <- c(-2, 0, 4)
sigma_real <- 2
X <- 0.1*matrix(rnorm(samplesize*dim), nrow = samplesize, ncol = dim) +
     matrix(rep(rnorm(samplesize), 3), nrow = samplesize, ncol = dim)
Y <- as.vector(X%*%alpha_real + rnorm(samplesize, 0, sigma_real))
```

```r
alpha_init <- rep(20,3)
```

```r
sigma_init <- 1
tau_init <- 1 / sigma_init^2
```

```r
mah_model <- function(){
    for (i in 1:n){
        Y.mu[i] <- X[i,] %*% alpha
        Y[i] ~ dnorm(Y.mu[i], tau)
    }

    tau ~ dgamma(delta, lambda)
    alpha ~ dmnorm(mu, Omega)

    sigma2 <- 1 / tau
}

mah_data=list(X=X, Y=Y, n=samplesize,
                delta = 10,
                lambda = 10,
                mu = rep(0,dim(X)[2]),
                Omega = diag(dim(X)[2]))
```

```r
mah_mcmc <- R2jags::jags(
    data = mah_data,
    inits=list(list(alpha=alpha_init, tau=tau_init)),
    parameters.to.save = c("alpha","tau", "sigma2"),
    n.iter = samplesize,
    n.chains = 1,
    n.thin = 1,
    n.burnin = 0, # 100 iterations are run for adaption
    model.file = mah_model)
```

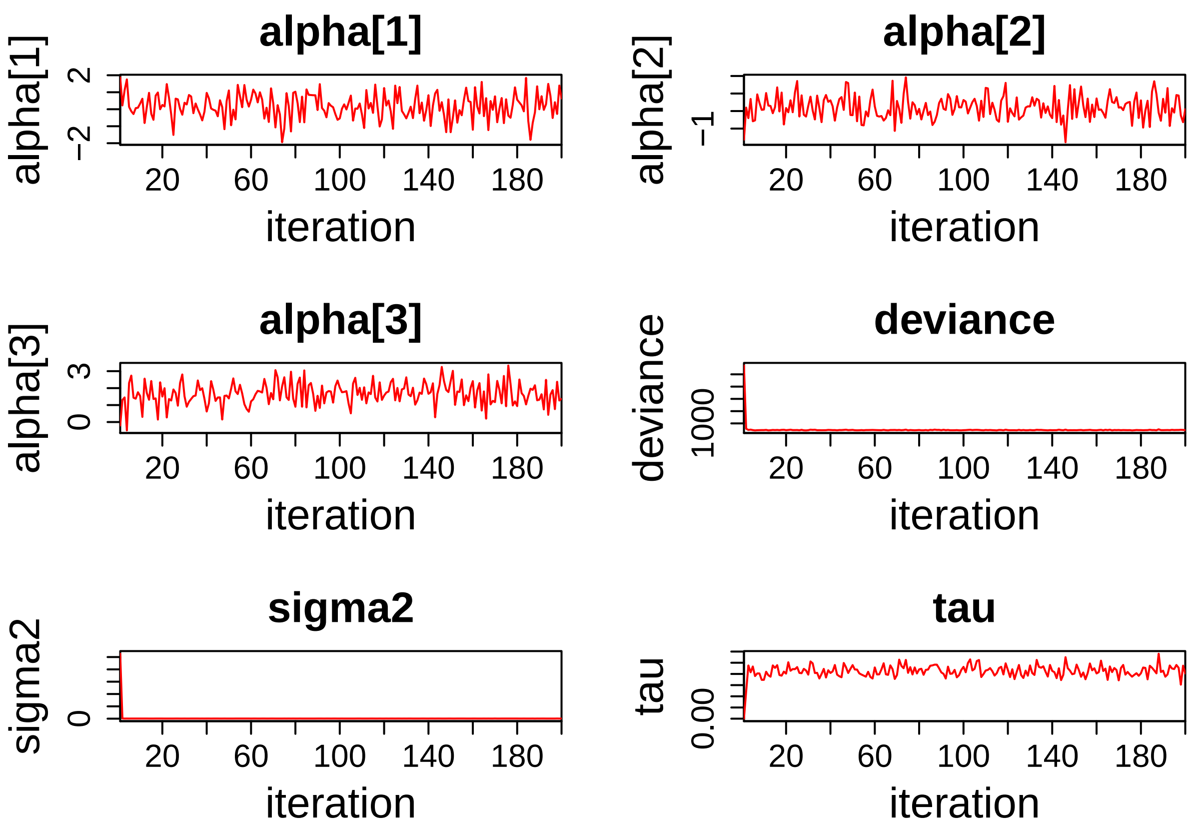
```
## module glm loaded
```

```
## Compiling model graph
##    Resolving undeclared variables
##    Allocating nodes
## Graph information:
##    Observed stochastic nodes: 200
##    Unobserved stochastic nodes: 2
##    Total graph size: 1221
##
## Initializing model
```

```r
structure(mah_mcmc)
```

```
## Inference for Bugs model at "/var/folders/7n/fcgysll141n_czqxfkwtd68r0000gn/T//RtmpV3zXOF/model14dfe5
##  1 chains, each with 200 iterations (first 0 discarded)
##  n.sims = 200 iterations saved
##          mu.vect sd.vect    2.5%     25%     50%     75%   97.5%
## alpha[1]   0.180   0.748  -1.310  -0.301   0.168   0.666   1.481
## alpha[2]   0.212   0.661  -0.853  -0.277   0.193   0.630   1.609
## alpha[3]   1.665   0.668   0.266   1.268   1.659   2.122   2.970
## sigma2    17.903 186.095   3.805   4.387   4.704   5.067   5.772
## tau        0.212   0.028   0.173   0.197   0.213   0.228   0.263
## deviance 896.589  74.490 886.429 889.195 890.731 893.061 898.248
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 2774.4 and DIC = 3671.0
## DIC is an estimate of expected predictive error (lower deviance is better).
```

```r
rafalib::bigpar(3,2)
R2jags::traceplot(mah_mcmc,mfrow=c(3,2))
```




```
mah_mcmc
```

```
## Inference for Bugs model at "/var/folders/7n/fcgysll141n_czqxfkwtd68r0000gn/T//RtmpV3zXOF/model14dfe5
##  1 chains, each with 200 iterations (first 0 discarded)
##  n.sims = 200 iterations saved
##          mu.vect sd.vect    2.5%     25%     50%     75%   97.5%
## alpha[1]   0.180   0.748  -1.310  -0.301   0.168   0.666   1.481
## alpha[2]   0.212   0.661  -0.853  -0.277   0.193   0.630   1.609
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## deviance 896.589  74.490 886.429 889.195 890.731 893.061 898.248
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 2774.4 and DIC = 3671.0
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Problem 2

2.a

Full Conditionals

$$
\begin{aligned}
\pi(\mu_1, \mu_2) &\sim \ell(\mu_1, \mu_2) \cdot p(\mu_1, \mu_2) \\
&= \ell(\mu_1, \mu_2) \cdot 1 \\
&= \ell(\mu_1, \mu_2) \\
&= N_n(y_1, \dots y_n \mid \mu, \Sigma) \\
&\sim \prod_{i=1}^{n} \exp\left\{-\frac{1}{2}(y_i - \mu)^\top \Sigma^{-1}(y_i - \mu)\right\} \\
&\sim \prod_{i=1}^{n} \exp\left\{-\frac{1}{2}\left[y_i^\top \Sigma^{-1} y_i - 2\mu^\top \Sigma^{-1} y_i + \mu^\top \Sigma^{-1}\mu\right]\right\} \\
&\sim \prod_{i=1}^{n} \exp\left\{-\frac{1}{2}\left[-2\mu^\top \Sigma^{-1} y_i + \mu^\top \Sigma^{-1}\mu\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left[-2\mu^\top \Sigma^{-1} y_i + \mu^\top \Sigma^{-1}\mu\right]\right\} \\
&= \exp\left\{-\frac{n}{2}\left[-2\mu^\top \Sigma^{-1} \bar{y} + \mu^\top \Sigma^{-1}\mu\right]\right\} \\
&\sim \exp\left\{-\frac{n}{2}\left[(\mu - \bar{y})^\top \Sigma^{-1}(\mu - \bar{y})\right]\right\} \\
&\sim N_2(\mu \mid \bar{y}, \frac{\Sigma}{n})
\end{aligned}
$$

$$
\begin{gathered}
\pi(\mu_1, \mu_2) \sim N_2(\mu \mid \bar{y}, \tfrac{\Sigma}{n}) \\
\text{where } \bar{y} = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} y_i
\end{gathered}
$$

With formula 1.5:

$$
\begin{aligned}
&\pi(\mu_1 \mid \mu_2) \sim N(\bar{y}_1 + \tfrac{\rho\sigma_1}{\sigma_2}(\mu_2 - \bar{y}_2), \tfrac{\sigma_2^2(1-\rho^2)}{n})) \\
&\pi(\mu_2 \mid \mu_1) \sim N(\bar{y}_2 + \tfrac{\rho\sigma_2}{\sigma_1}(\mu_1 - \bar{y}_1), \tfrac{\sigma_1^2(1-\rho^2)}{n}) \mid \\
&\bar{y}_1 = \textstyle\sum_{i=1}^{n} y_i^{(1)} \\
&\bar{y}_2 = \textstyle\sum_{i=1}^{n} y_i^{(2)}
\end{aligned}
$$

These are the full conditionals

Gibbs sampling algorithm

With $\pi(\mu_1 \mid \mu_2)$ and $\pi(\mu_2 \mid \mu_1)$ as defined above:

1. Set the iteration counter to j=1 and set initial values $\mu^{(0)} = \left(\mu_1^{(0)}, \mu_2^{(0)}\right)'$
2. Obtain a new value $\mu^{(j)} = \left(\mu_1^{(j)}, \mu_2^{(j)}\right)'$ through sucessive generation of values

$$
\begin{aligned}
\mu_1^{(j)} &\sim \pi(\mu_1 \mid \mu_2^{(j-1)}) \\
\mu_2^{(j)} &\sim \pi(\mu_2 \mid \mu_1^{(j)})
\end{aligned}
$$

3. Change counter j to $j+1$ and return to step 2 until convergence is reached.

2.b

```
gibbs_sampler_2 <- function(Y, sigma1, sigma2, rho,
                            R=1000){
  n <- dim(Y)[1]

  # variables where generated values will be saved
  mu_chain      <- matrix(NA, R, 2)
  mu_chain[1,]  <- runif(2,0,10)

  # same for all iterations
  sd1 <- sqrt(sigma2^2*(1-rho^2) / n)
  sd2 <- sqrt(sigma1^2*(1-rho^2) / n)
  y.mean1 <- mean(Y[,1])
  y.mean2 <- mean(Y[,2])

  # create parameter chain
  for (i in 2:(R)){
    mu_chain[i, 1] = rnorm(1,
                           mean=(y.mean1 + rho*sigma1/sigma2 * (mu_chain[i-1,2] - y.mean2)),
                           sd=sd1)
    mu_chain[i, 2] = rnorm(1,
                           mean=(y.mean2 + rho*sigma2/sigma2 * (mu_chain[i,1] - y.mean1)),
                           sd=sd2)
  }

  return(mu_chain)
}
```

```
set.seed(42)
mu_real = c(6,2)
sigma1 = sqrt(2)
sigma2 = sqrt(0.5)
rho = 0.05
n = 100
Sig = matrix(c(sigma1^2, sigma1*sigma2*rho,
               sigma1*sigma2*rho, sigma2^2), nrow = 2, ncol = 2)

Y <- MASS::mvrnorm(n, mu_real, Sig)
```

```
R1 = 100
R2 = 100000
```

```
mu_chain_1 <- gibbs_sampler_2(Y, sigma1, sigma2, rho, R=R1)
mu_chain_2 <- gibbs_sampler_2(Y, sigma1, sigma2, rho, R=R2)
```

```
compute_D <- function(mu, Y, sigma1, sigma2, rho){
    y.mean1 <- mean(Y[,1])
    y.mean2 <- mean(Y[,2])
    result <- 1/(1-rho^2) * (
        (mu[1] - y.mean1)^2 / sigma1^2 +
        (mu[2] - y.mean2)^2 / sigma2^2 +
```

```r
        2*rho*(mu[1] - y.mean1)*(mu[2] - y.mean2) / (sigma1*sigma2)
    )
    return(result)
}

DIC_4.12 <- function(mu_chain, Y, sigma1, sigma2, rho){

    R <- dim(mu_chain)[1]

    mu.mean = c(mean(mu_chain[,1]), mean(mu_chain[,2]))
    D_of_mu.mean <- compute_D(mu.mean, Y, sigma1, sigma2, rho)

    D.mean = 0
    for (i in 1:R){
        D_of_mu.i <- compute_D(mu_chain[i,], Y, sigma1, sigma2, rho)
        D.mean <- D.mean + D_of_mu.i
    }
    D.mean <- 1/R * D.mean

    return(D.mean - D_of_mu.mean)
}

DIC_4.13 <- function(mu_chain, Y, sigma1, sigma2, rho){

    R <- dim(mu_chain)[1]

    D.mean = 0
    for (i in 1:R){
        D_of_mu.i <- compute_D(mu_chain[i,], Y, sigma1, sigma2, rho)
        D.mean <- D.mean + D_of_mu.i
    }
    D.mean <- 1/R * D.mean

    D.var = 0
    for (i in 1:R){
        D_of_mu.i <- compute_D(mu_chain[i,], Y, sigma1, sigma2, rho)
        D.var <- D.var + (D_of_mu.i - D.mean)^2
    }
    D.var <- 1/(2*(R-1)) * D.var

    return(D.var)
}
```

```r
DIC_4.12(mu_chain_1, Y, sigma1, sigma2, rho)
```

```
## [1] 0.3692384
```

```r
DIC_4.12(mu_chain_2, Y, sigma1, sigma2, rho)
```

```
## [1] 0.04280324
```

```r
DIC_4.13(mu_chain_1, Y, sigma1, sigma2, rho)
```

```
## [1] 5.630098
```

```r
DIC_4.13(mu_chain_2, Y, sigma1, sigma2, rho)
```

```
## [1] 0.001665639
```

For both DIC from eq. 4.12 and eq. 4.13, MCMC with more iterations obtained a far lower score, which indicates that longer sampling is beneficial.