Homework 3

Problem 1

1a

For a proportionality constant k we can write:

$\pi(x) = k \cdot x^{\alpha-1}(1-x)^{\beta-1}$

$\log \pi(x) = \log k + (\alpha - 1)\log x + (\beta - 1)\log(1-x)$

To use adaptive rejection sampling, we need the first derivative of $\log \pi(x)$ to exist and the second derivative to be non positive.

$\log \pi'(x) = \frac{\alpha-1}{x} - \frac{\beta-1}{1-x}$

$\log \pi''(x) = -\frac{\alpha-1}{x^2} - \frac{\beta-1}{(1-x)^2}$

As we can see, $\log \pi'(x)$ exists for $0 < x < 1$. Let's see which values of α and β make $\log \pi''(x)$ non positive.

$\log \pi''(x) \leq 0 \quad \text{for all } 0 < x < 1$

$-\log \pi''(x) = \frac{\alpha-1}{x^2} + \frac{\beta-1}{(1-x)^2} \geq 0 \quad \text{for all } 0 < x < 1$

Let's study two border cases first.

$$\lim_{x\to 0} -\log \pi''(x) = \lim_{x\to 0} \frac{\alpha-1}{x^2} + \frac{\beta-1}{(1-x)^2} = \lim_{x\to 0} \frac{\alpha-1}{x^2} + \frac{\beta-1}{1^2} = \lim_{x\to 0} \frac{\alpha-1}{x^2} \geq 0$$

Resulting in $\alpha \geq 1$.

$$\lim_{x\to 1} -\log \pi''(x) = \lim_{x\to 1} \frac{\alpha-1}{x^2} + \frac{\beta-1}{(1-x)^2} = \lim_{x\to 1} \frac{\alpha-1}{1^2} + \frac{\beta-1}{(1-x)^2} = \lim_{x\to 1} \frac{\beta-1}{(1-x)^2} \geq 0$$

Resulting in $\beta \geq 1$.

From these border cases we see, that $\alpha \geq 1$ and $\beta \geq 1$. It is also easy to see, that these constraints on α and β make term $-\log \pi''(x) = \frac{\alpha-1}{x^2} + \frac{\beta-1}{(1-x)^2}$ non-negative for all values of $0 < x < 1$.

Therefore, $\log \pi''(x)$ is non positive for $\alpha \geq 1$ and $\beta \geq 1$.

Answer: adaptive rejection sampling can be used for $\alpha \geq 1$ and $\beta \geq 1$.

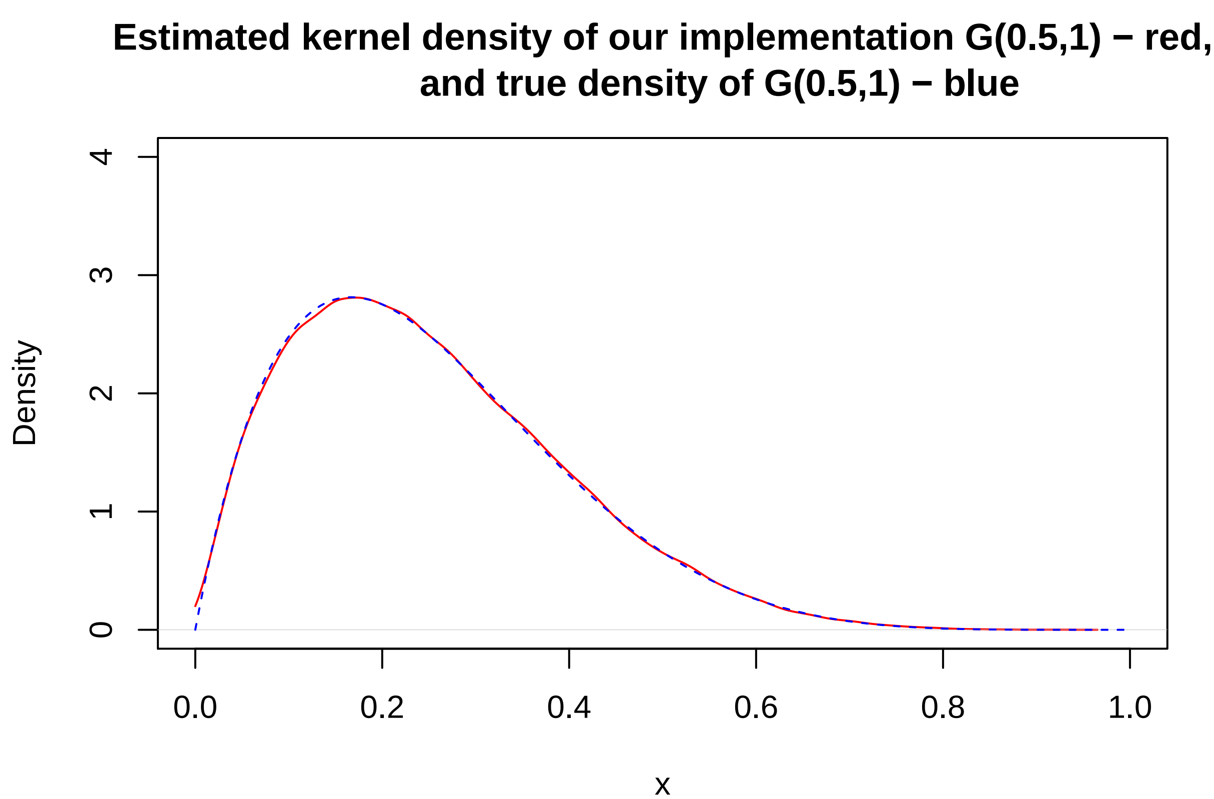
1b

```r
library(ars)
set.seed(42)
```

```r
# log pi(x)
f <- function(x, a, b){
    return((a-1) * log(x) + (b-1) * log(1-x))
}

# log pi'(x)
fprima <- function(x, a, b){
    return((a-1) / (x) - (b-1) / (1-x))
}
```

```r
R = 10^5
a = 2
b = 6

gen_beta <-ars(R,f,fprima, x=c(0.3,0.6),
               m=2, lb=TRUE, xlb=0, ub=TRUE, xub=1,
               a=a ,b=b)
```

```r
x <- seq(0, 1, length=1000)

plot(density(gen_beta, from=0), xlim=c(0,1), ylim=c(0,4), col='red', xlab="x",
     main="Estimated kernel density of our implementation G(0.5,1) - red,
           and true density of G(0.5,1) - blue")
lines(x, dbeta(x, a, b),col='blue', lty=2)
```



Problem 2

Let I denote if individual has HIV and T denoting if the test is positive. Then:

$$\begin{aligned} &P(T=1|I=1)=0.99 \\ &P(T=0|I=0)=0.98 \\ &P(I=1)=0.001 \end{aligned} \tag{1}$$

2.a

$P(I=1|T=1)-?$

$$\begin{aligned} &P(I=1|T=1) \overset{\text{Bayes Theorem}}{=} \tfrac{P(T=1|I=1)\cdot P(I=1)}{P(T=1)} \\ &P(I=0)=1-P(I=1)=0.999 \\ &P(T=1)=P(T=1|I=1)\cdot P(I=1)+P(T=1|I=0)\cdot P(I=0)= \\ &\quad =0.99\cdot 0.001+0.02\cdot 0.999=0.02097 \\ &P(I=1|T=1)=\tfrac{0.99\cdot 0.001}{0.02907}\approx 0.04721=4.721\% \end{aligned} \tag{2}$$

2.b

$percentage(I=0|T=1)-?$

$$percentage(I=0|T=1)=P(I=0|T=1)=1-P(I=1|T=1)\approx\ \approx 0.95279=95.279\% \tag{3}$$

Problem 3

3.a

For i.i.d. data we can write likelihood as follows:

$$\begin{aligned} \ell(\lambda)=f(\mathbf{y}|\lambda) &\overset{i.i.d.}{=} \prod_{i=1}^{n} f\left(y_i|\lambda\right) \\ &=\prod_{i=1}^{n}\frac{\lambda^{y_i}e^{-\lambda}}{y_i!} \\ &\propto \prod_{i=1}^{n}\lambda^{y_i}e^{-\lambda} \\ &=e^{-\lambda n}\cdot\lambda^{\sum_{i=1}^{n}y_i} \end{aligned} \tag{4}$$

Then the maximum likelihood estimate is:

$$\begin{aligned} \hat{\lambda}_{MLE} &= \arg\max_{\lambda}\ell(\lambda)=\arg\max_{\lambda}\log\ell(\lambda) \\ &=\arg\max_{\lambda}\left[-n\cdot\lambda+\log\lambda\cdot\sum_{i=1}^{n}y_i\right] \end{aligned} \tag{5}$$

$$
\begin{aligned}
&\tfrac{d}{d\lambda}\log \ell(\lambda) = -n + \tfrac{1}{\lambda}\textstyle\sum_{i=1}^{n} y_i \overset{!}{=} 0 \\
&\lambda = \tfrac{\sum_{i=1}^{n} y_i}{n} \\
&\hat{\lambda}_{MLE} = \tfrac{\sum_{i=1}^{n} y_i}{n}
\end{aligned} \tag{6}
$$

3.b

$$
\begin{aligned}
p(\lambda) = G(\alpha, \beta) &= \frac{\beta^{\alpha} \cdot \lambda^{\alpha-1} \cdot e^{-\beta\lambda}}{\Gamma(\alpha)} \\
&\propto \lambda^{\alpha-1} e^{-\beta\lambda}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
p(\lambda|\mathbf{y}) = \frac{\ell(\lambda)\cdot p(\lambda)}{f(\mathbf{y})} &= \\
&\propto \ell(\lambda)\cdot p(\lambda) \\
&\propto \lambda^{\alpha-1}\cdot e^{-\beta\lambda}\cdot e^{-n\lambda}\cdot \lambda^{\sum_{i=1}^{n} y_i} \\
&= \lambda^{\left(\alpha-1+\sum_{i=1}^{n} y_i\right)}\cdot e^{-\lambda(\beta+n)} \\
&\propto G(\alpha + \sum_{i=1}^{n} y_i, \beta + n)
\end{aligned} \tag{8}
$$

As we can see, posterior is Gamma-distributed with $\alpha_{post} = \alpha + \sum_{i=1}^{n} y_i$ and $\beta_{post} = \beta + n$.

3.c

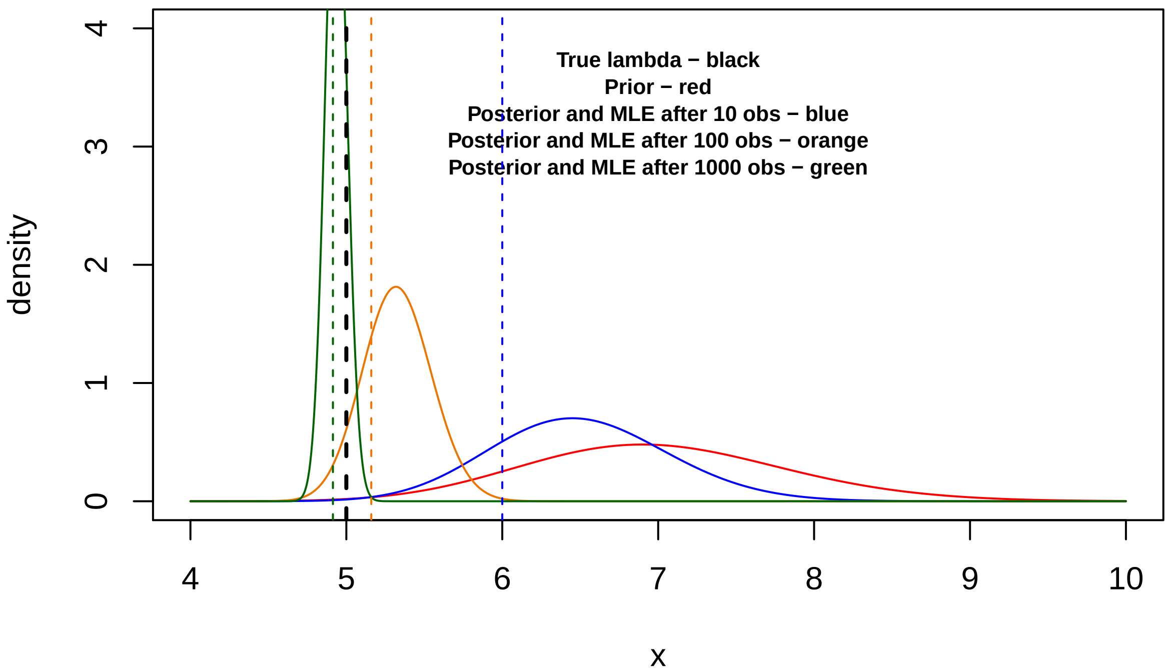
```r
lambda <- 5

n <- 10^3

a_prior <- 70
b_prior <- 10

n_1 <- 10
n_2 <- 100
n_3 <- 1000
```

```r
set.seed(42)
samples <- rpois(n, lambda)
```

```r
ab_posterior <- function(a_prior, b_prior, y){
    a_post = a_prior + sum(y)
    b_post = b_prior + length(y)
    c(a_post, b_post)
}

mle_lambda <- function(y){
    return(mean(y))
}
```

```r
res_1 <- ab_posterior(a_prior, b_prior, samples[1:n_1])
res_2 <- ab_posterior(a_prior, b_prior, samples[1:n_2])
res_3 <- ab_posterior(a_prior, b_prior, samples[1:n_3])
a_post_1 <- res_1[1]
b_post_1 <- res_1[2]
a_post_2 <- res_2[1]
b_post_2 <- res_2[2]
a_post_3 <- res_3[1]
b_post_3 <- res_3[2]
mle_1 <- mle_lambda(samples[1:n_1])
mle_2 <- mle_lambda(samples[1:n_2])
mle_3 <- mle_lambda(samples[1:n_3])
```

```r
x <- seq(4, 10, length=1000)

plot(x, dgamma(x, a_prior, b_prior), ylim=c(0,4), ylab="density",
     lwd=1, col='red', type="l")
title("
True lambda - black
Prior - red
Posterior and MLE after 10 obs - blue
Posterior and MLE after 100 obs - orange
Posterior and MLE after 1000 obs - green
", line = -5, cex.main = 0.7,
)
lines(x, dgamma(x, a_post_1, b_post_1), lwd=1, col='blue')
lines(x, dgamma(x, a_post_2, b_post_2), lwd=1, col='darkorange2')
lines(x, dgamma(x, a_post_3, b_post_3), lwd=1, col='darkgreen')
abline(v=lambda, lty=2, lwd=2, col=c("black"))


abline(v=mle_1, lty=2, lwd=1, col=c("blue"))
abline(v=mle_2, lty=2, lwd=1, col=c("darkorange2"))
abline(v=mle_3, lty=2, lwd=1, col=c("darkgreen"))
```


True lambda − black
Prior − red
Posterior and MLE after 10 obs − blue
Posterior and MLE after 100 obs − orange
Posterior and MLE after 1000 obs − green
density
x
0
1
2
3
4
4
5
6
7
8
9
10