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Point-Form Quantum Field Theory

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Abstract

We outline the general theory of quantizing a relativistic massive scalar field on curved hypersurfaces. The quantization on hyperboloids $X_\mu X^\mu =$ const., which we call Point-form quantization, is then discussed in some detail. The current state of Point-form quantization is reviewed and its present limitations and drawbacks are pointed out. We then propose a generalized Point-form quantization scheme which differs in two issues from the conventional approach, namely in the choice of the time variable and the representation Fock-space. The main feature of Point-form quantization, that interaction terms enter all the components of the 4-momentum operator, however, remains untouched. Unlike the conventional approach, a natural formulation of scattering within the generalized Point-form quantization scheme is possible. As first application we show for our scalar quantum field theory that the S-matrix in leading-order perturbation theory is the same as in covariant perturbation theory.

Contents

Introduction

The objective of this diploma thesis is a comprehensive review of the current status of the 2-dimensional Point-form quantization procedure for the Klein-Gordon field and its further development. In order to gain an overview of the established work, we give a short historical introduction on the development of Point-form quantization.

In the year 1949, an article written by Dirac [1] pointed out the existence of different forms of dynamics for a relativistic system. More precisely, he stated that *one can build up a dynamical theory in terms of dynamical variables that refer to physical conditions on some three-dimensional surface other than an instant.* This famous work gave birth to the parametrized Hamiltonian theory [2] which explicitly takes into account the arbitrariness of choice of such a hypersurface defining an "instant" in time. Since then, very few papers considered quantization methods in curved space-time but the interest peaked sharply in the seventies when Hawking discovered the quantum radiation produced by a black hole. During these years, much work was devoted to the investigation of quantized field theories in curved space-times but an explicit treatment of a quantization on hyperboloids, the Point-form dynamics from Dirac, occurred very rarely. Among these exceptions are papers which deal with the quantization in Euclidean space, instead of Minkowski space. For example, in [3] massless, conformally invariant theories are treated and in [4] this work was extended to the massive case for free scalar and spinor fields in 2-dimensional Euclidean space. Point-form quantization in Minkowski space-time has been performed in [5] for the free Klein-Gordon field and in [6] for the free Klein-Gordon and Dirac fields. The results in the last two papers are discussed and extended in [7] and a little earlier, in [8], the possible non-uniqueness of canonical field quantizations was already observed. Despite the progress made by these researchers, no interacting Point-form quantum field theories have thoroughly been studied yet.

This work does not claim to have established in detail a complete interacting Point-form quantization theory, but rather we provide a consistent review of the already existing results and add some remarks not found in this form elsewhere. In addition, we propose a modified version of the Point-form quantization method used so far and derive first basic results for a quantum field theory with an arbitrary interaction term. The motivation for such a Point-form quantum field theory stems, among other things, from relativistic quantum mechanical systems which derive their interactions and currents from underlying quantum field theories. Being a simple example for quantization on a curved hypersurface, this could of course be considered as interesting enough to be investigated further.

The organization of this thesis is as follows: In Chapter 1 an overview of the causal structure of space-time is given and the idea of Dirac, to use different hypersurfaces for the definition of an instant in time, is presented in the language of modern differential geometry. Section 1.2 is devoted to a general discussion about the problem of rigorously defining the notion of time. Chapter 2 provides the basic results concerning dilatation transformations and the conformal group. Afterwards, the quantization of the real massive scalar field is discussed in Chapter 3. This is done within the context of general relativity, writing down the principal equations for arbitrary spacelike hypersurfaces. In order to provide an example of an application of the results in this chapter, the usual Instant-form quantization is presented in Section 3.4 as a special case. These preparations facilitate the treatment of the Point-form quantization procedure, discussed in Chapter 4. In this chapter, our new ideas are introduced and it is shown, how they fit into the whole theory. Finally, in Chapter 5, our first results for a scattering theory which is compatible with our quantization scheme are presented. To avoid an interruption of the flow of argumentation, most of the lengthier calculations leading to the results in the last two chapters have been moved into two appendices.

Conventions and Notation

In this work, the sign convention for a Lorentzian metric on a manifold is chosen such that the induced metric on any spacelike hypersurfaces is negative definite. This is the opposite convention as usually adopted in differential geometry, where it is more convenient to work with positive definite metrics. In particular, the metric of a 4-dimensional space-time is of signature $(+---)$. The unit system is chosen such, that $\hbar = c = 1$.

The summation convention is used throughout, where Greek letters denote space-time indices which range from 0 to $n-1$ and Latin letters denote purely spatial indices which range from 1 to $n-1$. n always stands for the dimension of the currently used space-time. Exceptions from this rule should be recognizable from the context. Two exceptions worth mentioning are the Greek letters α and β. They will be used as coordinates of a hyperbolic coordinate system and are never to be understood as space-time indices. Since we distinguish between space-time indices and purely spatial indices we cannot use the abstract index notation in order to explicitly denote the type of a given tensor. Nevertheless, we distinguish between covariant and contravariant tensors by placing a bar or tilde sign below and above the tensor, respectively. A bar means, that the tensor lives on a n-dimensional manifold and a tilde means a $(n-1)$-dimensional manifold. We will not need mixed tensors, so no notation is introduced for them. For example, let $(\mathcal{M}, \underline{g})$ be a n-dimensional space-time and $(m, \underset{\sim}{g})$ a $(n-1)$-dimensional Riemannian manifold. Contravariant tensors which are associated with $\mathcal{M}$ are then denoted by $\bar{v}$ and covariant tensors by $\underline{v}$. Those associated with m are then denoted by $\widetilde{v}$ and by $\underset{\sim}{v}$, respectively. Prominent examples are $\bar{\lambda} \in T_X\mathcal{M}$ and $\underline{\lambda} \in T_X^*\mathcal{M}$ which are the tangent and cotangent vectors of $\mathcal{M}$ at the point X, or $\underline{g}$ which is a covariant tensor field of rank two. Note that we can drop this notation when we introduce a coordinate system, since the positions of the indices already allow a distinction between co- and contravariant quantities. X will always stand for a point in $\mathcal{M}$ and x for a point in m.

When we speak of a chart $(\mathcal{U}, \varphi)$ of a manifold $\mathcal{M}$ with $\mathcal{U} \subset \mathcal{M}$ open and $\varphi : \mathcal{U} \to \mathbb{R}^n$ smooth, we usually write it in terms of the coordinate functions $\varphi^\mu : \mathcal{U} \to \mathbb{R}$. The φ^μ are defined in terms of the projection functions $u^\mu : \mathbb{R}^n \to \mathbb{R}$, $u^\mu(y) = y^\mu$, as $\varphi^\mu(X) = u^\mu(\varphi(X))$ with $X \in \mathcal{U}$. Then the point X has the coordinates $(\varphi^0(X), \varphi^1(X), \ldots, \varphi^{(n-1)}(X))$. In the following chapters, the coordinate functions φ^μ are written as X^μ and the coordinates of a point X can then be written as $X^\mu = (X^0(X), X^1(X), \ldots, X^{(n-1)}(X))$. Here, the same symbol has been used to denote the coordinates of a point X with respect to some chart as for the coordinate functions of the chart. The meaning should be clear from the context. The coordinate functions and coordinates of a point associated with a $(n-1)$-dimensional manifold are denoted by x^μ.

Finally, quantities which are operators in Hilbert space are decorated with a $\hat{}$ in order to distinguish them from their classical counterparts.

For convenience, a list of the mathematical symbols used in this work is given below. The order of the symbols in the list corresponds roughly to their order of appearance.

$\mathcal{M}$	n-dimensional space-time manifold	$X \in \mathcal{M}$	point of $\mathcal{M}$
m, Σ	$(n-1)$-dimensional space manifold	$x \in m$	point of m
$T_X\mathcal{M}$	tangent space to $\mathcal{M}$ at X	$T_X^*\mathcal{M}$	cotangent space to $\mathcal{M}$ at X
$\underline{g}$	metric tensor on $\mathcal{M}$	g	the determinant of $\underline{g}$
$\underset{\sim}{g}$	metric tensor on m	g_m	the determinant of $\underset{\sim}{g}$
$\underline{\eta}$	Minkowski metric on $\mathcal{M}$	Σ_τ, σ	hypersurfaces
$D^+(\Sigma)$	future domain of dependence	$D^-(\Sigma)$	past domain of dependence
$D(\Sigma)$	domain of dependence of Σ		
$L_{\bar{\xi}}$	Lie derivative with respect to the vector field $\bar{\xi}$	∇	covariant derivative with respect to a metric
$\mathscr{E}$	space of embeddings	$\mathcal{E}$, $\mathcal{X}$	embeddings
$\mathscr{F}(\mathscr{E})$	differentiable functionals on $\mathscr{E}$	$\mathscr{H}$	hyperspace
$\mathcal{C}$	configuration space	Γ	phase space
S	action functional	$\widetilde{\Gamma}$	constraint surface
P_μ	space-time translation generators	$M_{\mu\nu}$	space-time rotation generators
J_l	space rotation generators	B_l	boost generators
D	dilatation generator	K_μ	generators of special conformal transformations
$\Omega^2(X)$	conformal factor at X		
H	Hamilton operator	$\mathscr{L}$	Lagrange density
ϕ	massive scalar field	$T_{\mu\nu}$	energy-momentum tensor
π	canonical conjugate momentum to ϕ	$\Box$	covariant d'Alembert operator
$\mathring{C}^+$	interior of the future light-cone	J_ν	Bessel function
$H_\nu^{(1)}$	first Hankel function	$H_\nu^{(2)}$	second Hankel function
$\lvert p\rangle$	momentum eigenstate	$\lvert\lambda\rangle 4$	boost eigenstate
$\hat{P}^\mu$	Instant-form momentum operator	$\hat{P}^\mu_{(\alpha)}$	Point-form momentum operator
$\hat{P}^\mu_{(\zeta)}$	generalized Point-form momentum operator	$\hat{H}_{(\zeta)}$	generalized Point-form time evolution operator
T_ζ	time-ordering with respect to ζ	$\hat{S}$	scattering operator

Chapter 1

Time Levels

The evolution of a physical system in time, i. e. its dynamics, is one of the most interesting and important topics in physics. Since the first mathematical description of mechanics by Newton - his dynamical equations for point particles - many other physical theories with dynamical laws were set up. With the introduction of special relativity by Einstein, the former idea of time being some external property of a physical system had been changed.

In classical Newtonian mechanics, the possible world-lines of a particle are not constrained by the speed of light - which is infinite in this theory - and hence the light-cone becomes a plane which represents an instant in time. Thus, the choice of a surface representing an instant in time is unique, and so is time itself. But in special relativity, the finite speed of light introduces many possible definitions of an instant in time. Due to this additional arbitrariness, far-reaching problems arise. They will be summarized in Section 1.2 after some of the mathematical fundaments of space-time and causality are presented in Section 1.1. In Section 1.3 and 1.4 a possible description of physical systems without referring to a special choice of an instant in time is discussed.

1.1 Causal Structure of Space-Time

This section collects the basic and most important definitions and theorems from general relativity referring to the causal structure of space-time. They are included herein to allow easy reference. A more rigorous and complete treatment can be found in the well known books [9] and [10].

Definition 1.1.1. The pair $(\mathcal{M}, \underline{g})$ is called a *n-dimensional space-time*, where $\mathcal{M}$ is a connected n-dimensional smooth manifold and $\underline{g}$ is a Lorentz metric on $\mathcal{M}$. A *Lorentz metric* on a n-dimensional manifold $\mathcal{M}$ is a metric of signature $(1, n-1)$, or sometimes $(n-1, 1)$, depending on the sign convention.

In the language of differential geometry, a space-time is also called a pseudo-Riemannian manifold (the term pseudo is due to the fact, that a Lorentz metric is not positive definite). In this text, only 2 or 4-dimensional space-times are considered when doing explicit calculations. The used signatures for the Lorentz metrics are $(+-)$ or $(+---)$, respectively.

Definition 1.1.2. *Minkowski* space-time is the space-time $(\mathbb{R}^n, \underline{\eta})$ of n-dimensional special relativity with $\underline{\eta}$ being the usual flat Lorentz metric.

Definition 1.1.3. Let $\mathcal{M}$ be a n-dimensional manifold. An embedded submanifold of dimension $(n-1)$ is called a *hypersurface*.

Hypersurfaces are sometimes classified as spacelike, timelike or null (lightlike). Roughly speaking this depends on the normal vectors of the hypersurface being timelike, spacelike or null (lightlike), respectively. For a rigorous definition see the references given above.

Definition 1.1.4. A *smooth curve*, C, on a manifold $\mathcal{M}$ is a C^∞ map of $\mathbb{R}$ (or an interval of $\mathbb{R}$) into $\mathcal{M}$, $C : \mathbb{R} \to \mathcal{M}$. By a *(past) future inextendible curve* we mean a smooth curve which has no (starting-) end-point (the domain of C is not bounded below or above, respectively).

Curves, not extendible into the past, future or both (which are just called inextendible), are basic concepts necessary for the more important definitions and theorems which will follow.

Definition 1.1.5. Let Σ be a spacelike hypersurface. The *future (past) domain of dependence* of Σ, denoted by $D^+(\Sigma)$ $(D^-(\Sigma))$, is defined as the set of all points $q \in \mathcal{M}$ such that each past-directed (future-directed) inextendible non-spacelike curve through q intersects Σ. The *domain of dependence* $D(\Sigma)$ is defined as $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$. If $D(\Sigma) = \mathcal{M}$, i.e. if every inextendible non-spacelike curve in $\mathcal{M}$ intersects Σ, then Σ is said to be a *Cauchy surface*. In other words, a Cauchy surface is a spacelike hypersurface which is intersected by every non-spacelike curve exactly once.

A simple example for a space-time which admits Cauchy surfaces is Minkowski space-time with the surfaces $\Sigma_t : t = \text{const}$. On the other hand, the surfaces $\Sigma_\tau : \eta_{\mu\nu} X^\mu X^\nu = \tau = \text{const.}$ with $\tau > 0$ are not Cauchy (see Figure 1.1).

Definition 1.1.6. A space-time which possesses a Cauchy surface Σ is called *globally hyperbolic.*

In the literature the definitions of global hyperbolicity sometimes differ from the one given above. Nevertheless, it can be shown that they are all equivalent.

Theorem 1.1.7. *If $(\mathcal{M}, \underline{g})$ is globally hyperbolic with a Cauchy surface Σ, $\mathcal{M}$ is homeomorphic to $\mathbb{R} \times \Sigma$. Furthermore, a global time function τ can be chosen, such that each surface of constant τ is a Cauchy surface. Thus, $\mathcal{M}$ can be foliated by this one-parameter family of smooth Cauchy surfaces.*

This theorem is of great importance when considering the time-evolution of solutions of wave equations in a general space-time. If the space-time in question is globally hyperbolic, a well defined, deterministic classical evolution from initial conditions given on some Cauchy surface is guaranteed. The possibility to choose among an infinite number of different foliations of a space-time and hence different time parameters, leads to a variety of nontrivial consequences. Some of them will be discussed in the following section.

1.1.1 Killing Fields

Killing vector fields are an important notion in differential geometry, that allows a precise mathematical formulation of the term symmetry and symmetric space. Here we will focus mainly on the application of the theory of Killing fields to causality. Because Killing vector fields correspond to symmetry transformations, the existence of a global timelike Killing field guarantees the time translational invariance of the physical system, thus establishing the link to causality and time developments.

Definition 1.1.8. Let $(\mathcal{M}, \underline{g})$ be a space-time and $\phi_t : \mathcal{M} \to \mathcal{M}$ a local one-parameter group of isometries, i.e. $\phi_t^* \underline{g} = \underline{g}$. The vector field $\bar{\xi}$ which generates ϕ_t is called a *Killing vector field.*

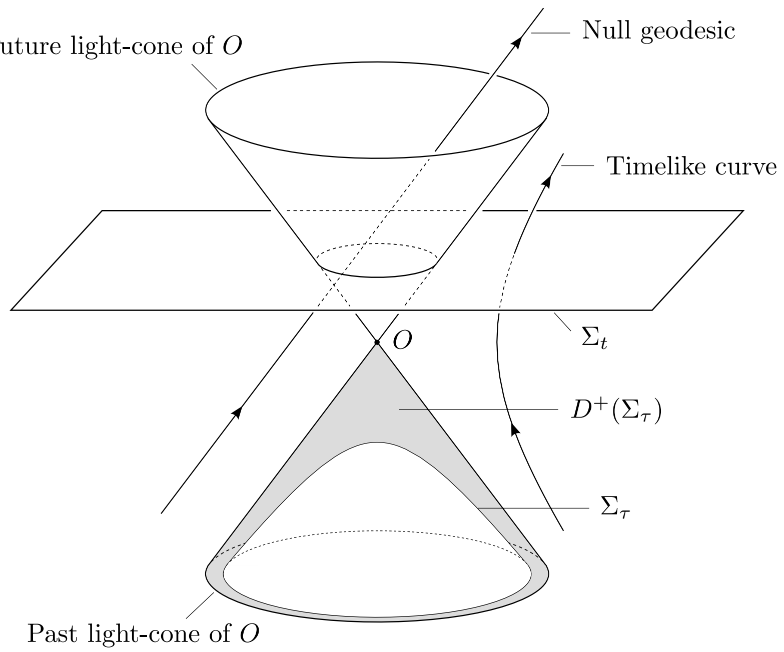


Figure 1.1: Hypersurfaces in Minkowski space-time. The surfaces Σ_t are Cauchy surfaces but the surfaces $\Sigma_\tau : \eta_{\mu\nu}X^\mu X^\nu = \tau = \text{const.}$ with $\tau > 0$ are not. One can find null geodesics which go through points outside of the past light-cone that do not intersect the spacelike surface Σ_τ with $-\tau, X^0 < 0$. The future domain of dependence of Σ_τ, $D^+(\Sigma_\tau)$, is bounded by what is called a future Cauchy horizon. In this example, the future Cauchy horizon of Σ_τ is the surface of the past light-cone.

The condition that the metric be invariant under a diffeomorphism is the same condition as normally used for the definition of Poincaré transformations in Minkowski space-time. Thus, Poincaré transformations are merely flows generated by Killing fields. Because the metric is invariant under the flow of a Killing vector field, it follows from the definition of the Lie derivative that $L_{\bar{\xi}}\underline{g} = 0$. This is equivalent to Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,, \tag{1.1}$$

where ∇_μ is the covariant derivative associated with the metric $\underline{g}$.

A general, curved n-dimensional space-time may possess no symmetry at all and therefore may not admit Killing fields. It can be shown, that there can be at most $n(n+1)/2$ independent Killing fields and thus symmetries. A simple example of a maximally symmetric space-time is that of special relativity, as already mentioned. It admits 10 independent Killing vector fields, namely (in cartesian coordinates) ∂_μ and $X_\mu\partial_\nu - X_\nu\partial_\mu$. They generate 4-dimensional translations and rotations and their corresponding conserved quantities are known as the Poincaré generators. The global timelike Killing vector field $\partial/\partial t$ generates a flow of isometries, which is orthogonal to the hypersurfaces $\Sigma_t : t = \text{const.}$, being the usual Cauchy surfaces. The resulting invariance of the system under time translations is expressed in the time-independence of the time evolution operator of the system.[1] In curved space-time,

[1] The choice of the "right" time evolution operator is not trivial. See the following sections and especially Section 1.4 for a discussion of time evolution.

the Poincaré group is in general no longer a global symmetry group.

There exists a generalization of the notion of Killing vector fields, which is quite useful within the context of conformal transformations. We will give the basic definitions here and refer to Chapter 2 for a short introduction to the conformal group and especially to the dilatation transformation.

Definition 1.1.9. A *conformal isometry*, ϕ, on a (pseudo-) Riemannian manifold $(\mathcal{M}, \underline{g})$ is a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ for which a function Ω exists such that $\phi^* \underline{g} = \Omega^2 \underline{g}$. Further, the generator $\bar{\psi}$ of a local one-parameter group of conformal isometries is called a *conformal Killing vector field*.

The Lie derivative of $\underline{g}$ along the conformal Killing field $\bar{\psi}$ must be proportional to $\underline{g}$, $L_{\bar{\psi}} \underline{g} = c\underline{g}$. Thus, $\bar{\psi}$ satisfies

$$\nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu = c g_{\mu\nu} \,. \tag{1.2}$$

The function c can be evaluated by multiplying the previous equation with $g^{\sigma\mu}$ and taking the trace, which leads to

$$\nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu = \frac{2}{n} (\nabla^\sigma \psi_\sigma) g_{\mu\nu} \tag{1.3}$$

with $n = \dim \mathcal{M}$. This is the, so-called, conformal Killing equation. Although conformal Killing vector fields can not be associated with symmetries related to the Poincaré group, they play an important role in its generalization - the conformal group. Many models which exhibit conformal symmetries are known and especially in two dimensions these symmetries very often allow an exact solution of the problem.

1.2 The Problem of Time

As already shortly mentioned at the beginning of this chapter, the definition of time and of an instant in time becomes non-trivial already within special relativity. Due to Theorem 1.1.7, the space-time in question can be foliated in various ways and thus allows for many definitions of the time parameter. This was first investigated by Dirac in the context of classical relativistic mechanics of point particles [1]. In fact, generalizations of his theory are used in quantization procedures for a much broader class of parametrized systems (see Section 1.4 for details and references). In relativistic quantum theories additional problems arise which stem from the special role of "time" in quantum mechanics.

The following remarks about the conceptual problems of a definition of time have their origin in discussions about quantum gravity, but are equally true for any parametrized system. Only the parts relevant for the other chapters in this text are presented - for a more complete exposition, see [11, 12, 13].

- *The global time problem.* It is possible that a simple constrained dynamical system does not admit a global time function, as defined in Theorem 1.1.7. Hence, the theory is only valid for a subset of space-time and the arising global issues do not seem to be analyzed in detail yet.

- *The multiple-choice problem.* Being able to choose among many different definitions of a time variable, the question of which being the "right" or most appropriate one arises. In special cases, the symmetries of a system can single out one possibility (see Section 1.1.1). In quantum mechanics, the Schrödinger equation based on one particular choice of time may give a different quantum theory than the Schrödinger equation based on another choice of time. This was discovered very early, for example in [3] and [5].

Although theories describing the same system may be unitary inequivalent in the usual sense of quantum mechanics, they may generate unitary equivalent representations of the Poincaré group (this is explicitly shown in [14] for the Instant- and Front-form, which will be presented in Section 1.5).

Subsequently, a short overview of the problem of time in quantum mechanics will be given. These issues are well known and are accomodated within the framework of quantum physics by explicitly giving time a special role as an external (or background) parameter of the system. Trying to quantize a relativistic system, where time is an internal property, brings these issues to our attention again.

- Time is not a physical observable in the normal sense since it can not be represented by a self-adjoint operator. As mentioned above, it is treated as a background parameter which is used to mark the evolution of the system (as in classical mechanics). This usually applies to non-relativistic quantum theory, to relativistic particle dynamics and quantum field theory (although a generalized notion of time obtained by replacing the plane by an arbitrary spacelike hypersurface was realized very early). Thus, the uncertainty relation between time and energy is quite different from those between other conjugate pairs of operators.
- In the Copenhagen interpretation, measurements made at a particular point in time are a fundamental part. Observables are interpreted as objects whose values can be measured at a fixed time.
- Any scalar product on a Hilbert space of states is required to be conserved under the time evolution, governed by the Schrödinger equation. Moreover, complete sets of observables are required to commute at a fixed value of time.
- Concerning the problem of the representation of time by an operator in quantum mechanics, it was shown in [15] that no physical clock can provide a precise measure of it. This interesting result can also be derived very quickly by imposing a slightly stronger requirement on a hypothetical time operator $\hat{T}$, namely that it should be a "Hamiltonian time observable" in the sense that
$$[\hat{T}, \hat{H}] = i\hbar\,. \tag{1.4}$$
This implies that $U(t)|T\rangle = |T+t\rangle$ where $\hat{T}|T\rangle = T|T\rangle$, which is precisely the type of behaviour that is required for a perfect clock. However, self-adjoint operators satisfying representations of (1.4) are known to have spectra equal to the whole real line $\mathbb{R}$. Therefore, condition (1.4) is incompatible with the requirement that $\hat{H}$ be a positive operator, i. e. that the system has positive energy.

As illustrated above, there are many conceptual and technical problems with the notion of time, in classical and in quantum theories alike. In this context, it is worth mentioning another approach, which completely overcomes some of these difficulties. This is achieved by constructing systems with no explicit reference to a time variable at all. Interesting original papers are [16, 17] and an overview and comparison between different approaches is given in [11, 13]. Finally, a summary of the discussion above for different physical theories is given, but it is far from being complete.

- *Newtonian mechanics.* Neither the global time problem nor the multiple-choice problem exists for parametrized Newtonian dynamics [11]. This is due to the fact that the velocity of particles is not limited and the absolute Newtonian time always increases along any dynamical trajectory.

- *Relativistic mechanics.* Both the global time and the multiple-choice problems occur. Because the world-lines of particles are required to lie in the light-cone, multiple choices of a spacelike surface representing an instant in time are possible. For certain systems, it may not be possible to define one time function for the whole space-time.
- *Quantum mechanics.* Standard quantum mechanics uses an externally prescribed time variable and thus the global time and multiple-choice problem are irrelevant. Of course, the problems of measurements and commutators at a fixed time are omnipresent.
- *Relativistic quantum mechanics.* This theory evidently inherits both sets of problems from relativistic mechanics and quantum mechanics. Usually, the time variable of the theory is chosen to be the standard Minkowski time $X^0 = t$ from special relativity.
- *Quantum field theories.* Taking relativistic quantum mechanics as a starting point and constructing an infinite-dimensional system, quantum field theory obviously inherits all the problems about time from its underlying theories.

1.3 Hyperspace

By now we have seen that the problem of choosing a foliation of space-time for a relativistic system is a very fundamental one. If there is no timelike Killing vector field, how can one find the "right" hypersurface representing an instant in time? Even a more interesting question could be asked, namely why should it be necessary to fix the foliation, since the properties of a physical system should not depend on a particular splitting of space-time into space and time? A coordinate-dependent formalism of the dynamical evolution of fields in a given space-time with a fixed foliation has been developed by Dirac and by Arnowitt, Deser and Misner (the ADM formalism). The hypersurfaces of the foliation are characterized as the coordinate hypersurfaces $\tau = X^0 =$ const. in a coordinate system X^μ. In this approach, the change of coordinates $X^\mu \to X'^\mu(X^\nu)$ induces a change of the foliation. Nevertheless, the way how the foliation is chosen does not reflect the full arbitrariness because it is either done by "coordinate conditions" or by hand (Dirac).

A more geometric, global approach was developed by Kuchař in [18, 19, 20] where field dynamics is described to take place in hyperspace, which is the manifold of all spacelike hypersurfaces drawn in a given space-time. Some of the basic facts about hyperspace will be presented below - they will prove to be very helpful in understanding the role of different foliations.

In Definition 1.1.3 we already introduced the notion of a hypersurface. Let us denote the embedding of m (space) to $\mathcal{M}$ (space-time) by

$$\mathcal{E} : m \ni x \mapsto X \in \mathcal{M}\,, \qquad X = \mathcal{E}(x)\,. \tag{1.5}$$

The image of m under the embedding $\mathcal{E}$ is a hypersurface in $\mathcal{M}$. A hypersurface $\mathcal{E}(m)$ is only determined up to a $(n-1)$-dimensional diffeomorphism[2] φ, thus one must speak of a hypersurface h as an equivalence class of embeddings:

$$h \equiv \{\mathcal{E} = \mathring{\mathcal{E}} \circ \varphi \mid \varphi \in \mathrm{Diff}(m)\}\,. \tag{1.6}$$

The push-forward of the tangent space of m into the tangent space of $\mathcal{M}$ induced by the embedding $\mathcal{E}$ is

$$\mathcal{E}_* = \frac{\partial \mathcal{E}(x)}{\partial x} \equiv \underset{\sim}{\bar{\mathcal{E}}}\,, \qquad \underset{\sim}{\bar{\mathcal{E}}} : T_x m \ni \tilde{\lambda} \mapsto \bar{\lambda} \in T_{\mathcal{E}(x)}\mathcal{M}\,, \tag{1.7}$$

[2] This is generally true for every manifold, since they are actually defined as equivalence classes with respect to diffeomorphisms. This is why general relativity is often called a diffeomorphism-invariant theory.

which is a vector in $T_X\mathcal{M}$ and a co-vector in $T_x m$. If we introduce the unit normal vector field $\bar{n}$ with $\bar{n}(x) \in T_{\mathcal{E}(x)}\mathcal{M}$ which is orthogonal to all tangent vectors in $\mathcal{E}_*(T_x m)$ and normalized to 1 with respect to the metric $\underline{g}$ (future pointing), the vectors $\bar{n}(x)$ and $\bar{\underset{\sim}{\mathcal{E}}}$ form a basis in $T_X(M)$. Every tangent vector in $\mathcal{M}$ can thus be decomposed with respect to this basis as

$$\bar{\lambda} = \lambda^{\perp}\bar{n} + \langle \bar{\underset{\sim}{\mathcal{E}}}, \widetilde{\lambda}^{\parallel} \rangle\,, \tag{1.8}$$

where $\langle \cdot, \cdot \rangle$ is the space scalar product with respect to the induced metric on the hypersurface.

If $(\mathcal{M}, \underline{g})$ is a 4-dimensional space-time with the pseudo-Riemannian metric $\underline{g}$ having the signature $(+ - --)$, an embedding $\mathcal{E}$ induces a metric $\underset{\sim}{g}$ on m,

$$\underset{\sim}{g} = \mathcal{E}^* \underline{g}\,. \tag{1.9}$$

The induced metric is interpreted as the intrinsic metric of the hypersurface. If this spatial metric is non-degenerate and negative definite, the hypersurface is said to be spacelike.[3] The components of the induced metric can be calculated via the definition of the pull-back,

$$(\mathcal{E}^* g)_{ab}(x) = g_{\mu\nu}(\mathcal{E}(x))\,\partial_a \mathcal{E}^{\mu}(x)\,\partial_b \mathcal{E}^{\nu}(x), \quad \text{with} \quad x \in m\,, \tag{1.10}$$

where $\mathcal{E}^{\mu}(x)$ are the components of the image of $\mathcal{E}(x)$ under a chart on $\mathcal{M}$.

Definition 1.3.1. The *space of embeddings* is an infinite dimensional manifold, formed by all embeddings $\mathcal{E}$ which lead to a spacelike hypersurface in a space-time $(\mathcal{M}, \underline{g})$. It is denoted by $\mathscr{E}$,

$$\mathscr{E} = \{\mathcal{E} \mid \underline{g}(\bar{n}, \bar{n}) = 1\} \equiv \mathrm{Emb}_{\underline{g}}(m, \mathcal{M})\,, \tag{1.11}$$

with $\mathrm{Emb}_{\underline{g}}(m, \mathcal{M}) \subset \mathrm{Emb}(m, \mathcal{M})$ being the subset of all these spacelike embeddings.

Using the equivalence of embeddings under diffeomorphisms, one can define a manifold formed by all hypersurfaces (1.6).

Definition 1.3.2. The infinite-dimensional manifold

$$\mathscr{H} \equiv \mathscr{E}/\mathrm{Diff}(m) \tag{1.12}$$

is called *hyperspace*. It is the collection of all hypersurfaces.

The manifolds $\mathscr{E}$ and $\mathscr{H}$ have a rich geometrical structure, but we will need only very little of it. A useful concept is the one-parameter family of embeddings,

$$\mathcal{E}_t : t \in \mathbb{R}, \quad m \ni x \mapsto X = \mathcal{E}_t(x) \in \mathcal{M}\,, \tag{1.13}$$

which is a curve in $\mathscr{E}$ and represents a continuous deformation of a hypersurface in space-time. The tangent vectors

$$\mathcal{T} = \left.\frac{d\mathcal{E}_t}{dt}\right|_{t=t_0} \tag{1.14}$$

to all such curves passing through the same point $\mathcal{E} \equiv \mathcal{E}(t_0)$ fill the tangent space $T_{\mathcal{E}}\mathscr{E}$. Using the algebraic approach to tangent vectors, elements of $T_{\mathcal{E}}\mathscr{E}$ can be viewed as linear differential operators on the ring $\mathscr{F}(\mathscr{E})$ of differentiable functionals of the embedding $\mathcal{E}$. By

[3] If we had chosen the opposite sign convention such that the metric $\underline{g}$ would have signature $(- + ++)$, the spatial metric $\underset{\sim}{g}$ would be required to be positive definite and non-degenerate for the hypersurface to be spacelike.

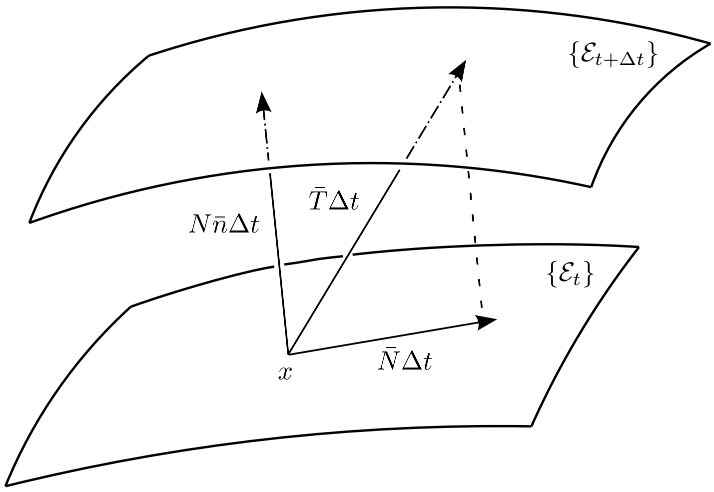


Figure 1.2: Decomposition of the deformation vector $\bar{T}$. The scalar space function $N(x)$ is called lapse function and the space vector $\tilde{N}(x)$ shift vector.

choosing variational derivatives as a coordinate basis in $T_{\mathcal{E}}\mathscr{E}$, the component expression of $\mathcal{T}$ can be viewed as a space-time vector field $\bar{T}(x)$ (see [18]),

$$\bar{T}(x) \equiv \left.\frac{\partial \mathcal{E}_t(x)}{\partial t}\right|_{t=t_0} \quad \text{with} \quad \bar{T}(x) : m \ni x \mapsto \bar{T}(x) \in T_{X=\mathcal{E}(x)}(\mathcal{M}) \,. \tag{1.15}$$

The vector field $\bar{T}$ is known as the deformation vector of the foliation and can be interpreted as representing the flow of time. We decompose the deformation vector into components perpendicular and parallel to a hypersurface according to equation (1.8),

$$\bar{T} = N\bar{n} + \langle \bar{\underset{\sim}{\mathcal{E}}}, \tilde{N} \rangle \,. \tag{1.16}$$

The components N and $\tilde{N}$ are called the lapse function and shift vector (see Figure 1.2), respectively. In local coordinates, they can be calculated by

$$N = T^\alpha n_\alpha \qquad \text{and} \qquad N^a = -T^\alpha \partial_\alpha \mathcal{E}^a \,. \tag{1.17}$$

1.4 Parametrized Systems

The idea of treating time and space equally is ultimatively expressed in general relativity, where each coordinate system is put on the same footing. Thus, the field equations of general relativity are generally covariant and the feasibility to write down the dynamical equations of any physical theory in a covariant way is often regarded to be an essential feature of this theory. Surprisingly, Hamiltonian mechanics, which is very often the starting point for a relativistic theory (classical or quantum), breaks the space-time covariance at the very beginning by the definition of the Lagrangian, from which the corresponding Hamiltonian is derived. However, in [2] Dirac reformulated the Hamiltonian formalism as a parametrized theory, where a particular time does not play a special role. In a more modern language, the idea is to treat embeddings of Cauchy hypersurfaces in the space-time as an additional degree of freedom of the system. Together with the conjugate momenta of these embeddings they define an extended phase space, on which constraints have to be imposed in order to retrieve the original dynamics. This leads to an elegant generalization of standard Hamiltonian mechanics, known as presymplectic mechanics and the resulting systems can be classified as first-class parametrized systems. The resulting freedom of time choice is similar to a choice of gauge, although gauge systems differ in certain aspects from parametrized ones. We will briefly review the most important concepts of this generalization for finite-dimensional systems, based on [2] and [21]. Field theories with a fixed background space-time can also be made generally covariant by the same procedure as was shown in [20] for example, but a complete, satisfactory quantization formalism for systems with an infinite number of degrees of freedom (quantum field theories) on a fixed background is not established yet.[4] However, see Chapter 3 for some remarks about the quantization of the parametrized scalar field.

We start with a n-dimensional manifold $\mathcal{C}$ as the configuration space with some coordinates $\{q^\mu\}$. The cotangent bundle $\Gamma = T^*\mathcal{C}$ forms the phase space of the system, with coordinates $\{q^\mu, p_\mu\}$. The action of a constrained system can then be written as

$$S = \int (p_\mu \dot{q}^\mu - \sum_{i=1}^{m} N_i C_i)\, d\tau\,. \tag{1.18}$$

The dot denotes the derivative with respect to τ, the N_i are Lagrange multipliers and the C_i are the so-called *constraint functions*. The idea is to treat the Lagrange multipliers as additional independent variables. Then varying the action with respect to the N_is leads to the following system of equations:

$$C_i = 0 \quad \forall i\,. \tag{1.19}$$

These equations are the constraints and they define a subset $\widetilde{\Gamma} \subset \Gamma$ which is called *constraint surface*. In general $\widetilde{\Gamma}$ is not a surface in the strict sense, but by a suitable construction it can be turned into a proper $(2n-m)$-dimensional surface.

Now, considering the time evolution, Dirac introduced spacelike hypersurfaces, the "time surfaces" in space-time and looked at the evolution from one hypersurface to another. In parametrized theories, this idea is extended to the phase space where a *transversal surface* is the equivalent to a hypersurface. It is defined such that it is intersected by an arbitrary orbit in $\widetilde{\Gamma}$ in at most one point. Further, Dirac extended the notion of a "Hamiltonian" operator to be any observable which, via Poisson brackets, moves the time surface non-trivially. The parametrized version of a Hamiltonian is a perennial[5] which moves a transversal surface $\Gamma_0 \subset \widetilde{\Gamma}$ to another transversal surface Γ_t. The 1-dimensional family $\{\Gamma_t\}$, which is generated by a particular perennial, sweeps a $(2n-2m+1)$-dimensional surface within the constraint surface $\widetilde{\Gamma}$. The surfaces Γ_t are called *time levels.*

In this context, the role of a Hamiltonian is merely to generate time levels. It does not need to be the total energy of the system, it does not need to be positive, or define a ground state.

[4] Not speaking of a quantized field theory where the metric itself is regarded as a dynamical variable, which would thus lead to a theory of quantum gravity.

[5] A perennial can be thought of as an operator, but not every operator is a perennial. See [22, 21] for details.

1.5 Dirac's Forms of Relativistic Dynamics

In his early paper about relativistic dynamics on different hypersurfaces [1], Dirac discussed the arbitrariness of time choice, but picked out three hypersurfaces by hand - his "three forms of relativistic dynamics". This is done by considering the Poincaré group $\mathcal{G}$ which is a symmetry group of any (special) relativistic system, because it leaves the constraint surface $\widetilde{\Gamma}$ invariant. With q^μ labeling a point in space-time and p_μ being its conjugate momentum, the infinitesimal generators of the Poincaré group can be written as:

$$\begin{aligned} \text{space-time translations} \qquad & P_\mu = p_\mu \\ \text{space rotations} \qquad & J_l = \epsilon_{lmn} q^m p_n \\ \text{boosts} \qquad & B_l = q^0 p_l + q^l p_0 \end{aligned}$$

The conserved quantities associated with the Poincaré transformations then generate the symmetry transformations via Poisson brackets. Note that in the expressions above, the summation convention applies for any pair of indices, not just co- and contravariant ones. If one chooses a transversal surface $S \subset \widetilde{\Gamma}$ and applies elements of $\mathcal{G}$ to it, a family $\mathcal{F}$ of transversal surfaces in $\widetilde{\Gamma}$ is obtained. $\mathcal{F}$ forms a manifold diffeomorphic to $\mathcal{G}/\mathcal{G}_S$, where $\mathcal{G}_S$ is the stability group of S. The generators of the stability group are called *kinematical* operators, whereas the others are called *dynamical* operators or *Hamiltonians* (as was already shortly discussed at the end of Section 1.4). Dirac chose three surfaces such that $\mathcal{G}_S$ is among the largest subgroups of $\mathcal{G}$.[6] This leads to the three most popular forms of relativistic dynamics.

	Hypersurfaces Σ_τ	Stability group $\mathcal{G}_\Sigma$
Instant-form	$\tau = X^0$	P_l, J_l
Front-form	$\tau = X^0 - X^3$	B_3, J_3, 6 others
Point-form	$\tau = X^\mu X_\mu, \quad \tau > 0$	B_l, J_l

Table 1.1: Dirac's three forms of dynamics. The stability groups are the three largest subgroups of the Poincaré group. The stability group of the Front-form is not fully given in order to avoid the introduction of unnecessary notation (the 6 other kinematical operators are merely linear combinations of Poincaré generators).

The hypersurfaces in Table 1.1 are plotted in Table 1.2 for a 3-dimensional Minkowski space-time.

Although there are five subgroups of the Poincaré group, these are the most widely used surfaces because of their larger stability group and therefore higher symmetry. In Section 1.1.1 we have seen that the concept of symmetry is closely related to Killing fields and this relation is emphasized in Figure 1.3.

The *Instant-form* is the most popular form of dynamics. Its hypersurfaces are planes, transversal to the Minkowskian time coordinate and thus the generators of their stability group are P_l and J_l. Although it is not the largest subgroup of the Poincaré group, the dynamical evolution of a system from one plane to another becomes very often particularly simple. It is the standard form of dynamics used in most textbooks for quantizing relativistic particle theories or field theories. Nevertheless, other forms may be better suited for

[6] In Dirac's original paper, the Poincaré group $\mathcal{G}$ acts on Minkowski space-time instead of the constraint surface $\widetilde{\Gamma}$, as in the modern parametrized approach. Therefore, the orbits generated by the action of subgroups of the Poincaré group on points in Minkowski space-time are hypersurfaces instead of transversal surfaces.

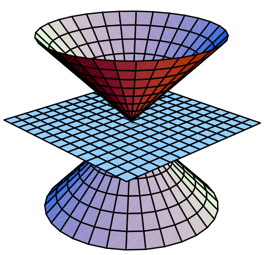
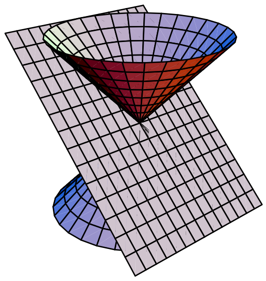
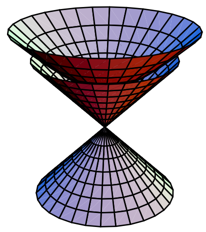
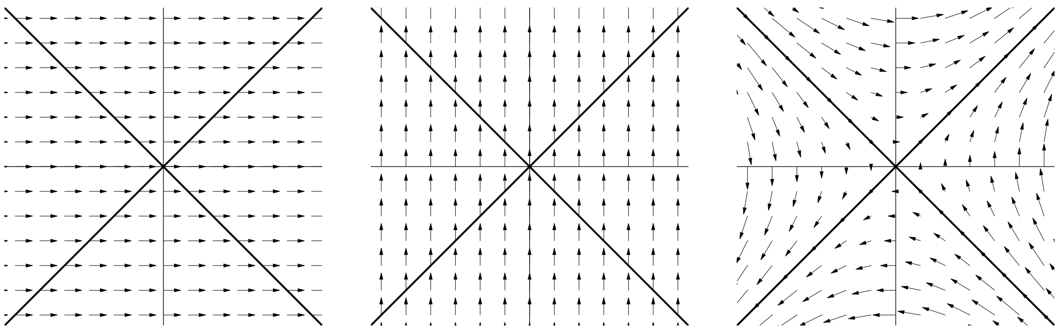
Instant-form	Front-form	Point-form
		
$X^0 = \tau$	$X^0 - X^3 = \tau$	$X^\mu X_\mu = \tau, \quad \tau > 0$

Table 1.2: Hypersurfaces of Dirac's three forms of dynamics defining an instant in time.



(a) Space-translation Killing field (b) Time-translation Killing field (c) Boost Killing field

Figure 1.3: The space-time translation and boost Killing fields for a 2-dimensional Minkowski space-time. The orbits of (a) and (c) correspond to the Instant-form and Point-form hypersurfaces, respectively.

the description of the dynamics of special systems. The hypersurfaces associated with the *Front-form*, planes tangent to the light-cone, are invariant under the largest subgroup of the Poincaré group and the evolution of systems from one such light front to another received much attention during the last years. Front-form quantization schemes for field theories have also been studied extensively. The *Point-form* is the least explored one of Dirac's forms of dynamics. This form uses hyperboloids in space-time to define an instant in time. Point-form quantization has only been studied for very simple cases, like free or massless theories, but no comprehensive understanding has been achieved as yet.

Chapter 2

Dilatation Transformations

In this chapter, dilatation transformations - which belong to the conformal group - will be studied. This is somewhat out of the main line, concerning the rest of this work but the results here will prove to be useful when we consider the usual Point-form quantization method in Chapter 4. In the past, the transformations which were most often considered by physicists were those belonging to the Poincaré group as the flows generated by the Poincaré generators are isometries of the Minkowski metric of special relativity. In other words, physical theories are normally required to be invariant under these flows (the flows are supposed to correspond to symmetry transformations of the system). But for some theories, there exists a broader class of transformations under which they are invariant, i.e. the conformal transformations. The first theory that was discovered to exhibit conformal invariance was Maxwell's source-free electrodynamics. Since then many theories where found to be conformally invariant. We will review the conformal group in the next section and, subsequently, the effects of broken scale invariance to the Lie algebra of the conformal generators will be discussed.

2.1 The Conformal Group

By definition, the conformal transformations are coordinate transformations within a n-dimensional space-time $(\mathcal{M}, \underline{g})$ which change the metric only by a scale factor, $\underline{g}'(X) = \Omega^2(X)\underline{g}(X)$. They form a group of which the Poincaré group clearly is a subgroup with $\Omega = 1$. Conformal transformations preserve angles and from $ds^2 = g_{\mu\nu}dX^\mu dX^\nu = 0$ it is obvious that the light-cone is preserved. In order to find an explicit form for the conformal transformations, we start with an infinitesimal coordinate transformation $X'^\mu = X^\mu + \epsilon^\mu(X)$. The functional change of the metric, at first order in ϵ, is therefore

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \,. \tag{2.1}$$

For an infinitesimal transformation, we can write the conformal factor as $\Omega^2(X) = 1 + \kappa(X)$ with κ being infinitesimally small. Thus, the metrical change $\delta g_{\mu\nu}$ is also $\kappa g_{\mu\nu}$. With (2.1) we get the condition

$$-\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = \kappa g_{\mu\nu} \,. \tag{2.2}$$

Applying $g^{\mu\nu}$ to both sides of the previous equation, the factor κ can be determined as $\kappa = -\frac{2}{n}\partial_\mu \epsilon^\mu$. This finally leads to

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{n} g_{\mu\nu} \partial_\rho \epsilon^\rho \tag{2.3}$$

which is precisely equation (1.3) for conformal Killing vector fields. Solving this equation for the ϵ^μ leads to the infinitesimal form of the conformal transformations. Their finite counterparts form the conformal group and are given by [23]

$$\begin{aligned} &\text{translations} && X'^\mu = X^\mu + A^\mu && \text{(2.4a)}\\ &\text{rotations} && X'^\mu = \Lambda^\mu_\nu X^\nu && \text{(2.4b)}\\ &\text{dilatations} && X'^\mu = \varrho X^\mu \qquad (\varrho > 0) && \text{(2.4c)}\\ &\text{special conformal transformations (SCT)} && X'^\mu = \frac{X^\mu - B^\mu X^2}{1 - 2B_\mu X^\mu + B^2 X^2} && \text{(2.4d)} \end{aligned}$$

with A^μ, B^μ being constant four-vectors, $\Lambda_{\mu\nu}$ being the well-known matrix-representation of the restricted Lorentz group $SO^+(1, n-1)$ in n-dimensional space-time and ϱ being a positive dilatation parameter. The first three transformations are relatively simple, whereas the special conformal transformations change the length scale point by point, thus generating position-dependent dilatations. As can be seen, the Poincaré transformations form a $\frac{1}{2}n(n-1)$-dimensional subgroup of the conformal group, which is itself $\frac{1}{2}(n+1)(n+2)$-dimensional (it is isomorphic to $SO^+(1, n+1)$). The infinitesimal generators of the conformal group are:

$$\begin{aligned} &\text{translations} && P_\mu = -i\partial_\mu \\ &\text{rotations} && M_{\mu\nu} = i(X_\mu \partial_\nu - X_\nu \partial_\mu) \\ &\text{dilatations} && D = -iX^\mu \partial_\mu \\ &\text{SCT} && K_\mu = -i(2X_\mu X^\nu \partial_\nu - X^2 \partial_\mu) \end{aligned} \tag{2.5}$$

They obey the following commutation relations, which are known as the conformal algebra:

$$[D, P_\mu] = iP_\mu \tag{2.6a}$$
$$[D, K_\mu] = -iK_\mu \tag{2.6b}$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu}) \tag{2.6c}$$
$$[K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \tag{2.6d}$$
$$[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \tag{2.6e}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}) \tag{2.6f}$$

All other commutators vanish. The realations have been stated for Minkowski space-time and for simplicity, we will restrict ourselves to this case (a flat metric) for the rest of this discussion.

Since our focus in this chapter is on dilatation transformations, we will now investigate the effect of this transformation on a massive scalar field ϕ (which can be classical or quantized). The following treatment can be found in most textbooks about conformal field theory and interesting original papers are [24, 25, 26] and [27]. To explicitly determine the transformation properties of ϕ under dilatations, note that the field ϕ is supposed to change under dilatations as

$$\phi'(X') = \varrho^{-d}\phi(X) \qquad \text{with} \qquad X'^\mu = \varrho X^\mu\,, \tag{2.7}$$

where ϱ is the dilatation factor and d is called the scaling dimension of ϕ. The variation of X and ϕ under an infinitesimal transformation with $\varrho = 1 + \epsilon$, ϵ being infinitesimally small, is

$$\delta_D X = \epsilon X^\mu \qquad \text{and} \qquad \delta_D \phi(X) = -\epsilon d \phi(X)\,. \tag{2.8}$$

We are now interested in the total variation of ϕ, $\bar{\delta}_D \phi(X) = \phi'(X) - \phi(X)$, and thus the commutation relation between D and ϕ. It is sufficient to consider the total variation of

$\phi(X)$ at the origin of X, because the translation operator can then be used to specify the transformation for an arbitrary X. Since the coordinate transformation (2.4c) leaves the origin invariant, the transformation of the field at the origin should be expressible in terms of the field at the origin:

$$\bar{\delta}_D\phi(0) = i\epsilon[D,\phi(0)] = -\epsilon d\phi(0)\,. \tag{2.9}$$

The right hand side follows from (2.8) and also the fact that $\bar{\delta}_D\phi(0) = \delta_D\phi(0)$.[1] Before applying the translation operator we need the following identity,

$$e^{iX^\mu P_\mu} D e^{-iX^\mu P_\mu} = D + X^\mu \partial_\mu\,, \tag{2.10}$$

where the Hausdorff formula and (2.6a) have been used. This finally leads to the transformation rule

$$\bar{\delta}_D\phi(X) = i\epsilon[D,\phi(x)] = -\epsilon(X^\mu\partial_\mu + d)\phi(X)\,. \tag{2.11}$$

Another derivation of this formula is gained by using the following equation, which relates the local to the total variation:

$$\bar{\delta}\phi(X) = \delta\phi(X) - \delta X^\mu \partial_\mu \phi(X)\,. \tag{2.12}$$

With (2.8) this immediately leads to the result given above.

In order to see if the dilatation transformation is a symmetry of a given system, one has to verify that the Lagrangian density changes only by a total divergence under infinitesimal dilatations. This condition guarantees the invariance of the action $\int \mathscr{L} d^n X$. Thus, in general, one has to show that

$$\bar{\delta}\mathscr{L} = \frac{\partial \mathscr{L}}{\partial\phi}\bar{\delta}\phi + \pi^\mu \partial_\mu \bar{\delta}\phi = \partial_\mu \Lambda^\mu \tag{2.13}$$

holds, with

$$\pi^\mu \equiv \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\,. \tag{2.14}$$

With the help of the Euler-Lagrange equation, one gets $\bar{\delta}\mathscr{L} = \partial_\mu(\pi^\mu\bar{\delta}\phi)$ which leads to the condition

$$\partial_\mu(\pi^\mu\bar{\delta}\phi - \Lambda^\mu) = 0\,. \tag{2.15}$$

In the case of a dilatation transformation, the total variation of the Lagrangian reads

$$\begin{aligned} \bar{\delta}_D\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi}\bar{\delta}_D\phi + \pi^\mu\partial_\mu\bar{\delta}_D\phi \\ &= -\epsilon\frac{\partial\mathscr{L}}{\partial\phi}(X^\mu\partial_\mu + d)\phi - \epsilon\pi^\mu\partial_\mu[(X^\nu\partial_\nu + d)\phi]\,. \end{aligned} \tag{2.16}$$

If we assume translation invariance of the Lagrangian, the equation above can be simplified by using

$$\partial_\mu\mathscr{L} = \pi^\nu\partial_\nu\partial_\mu\phi + \frac{\partial\mathscr{L}}{\partial\phi}\partial_\mu\phi\,, \tag{2.17}$$

which is a constraint on $\mathscr{L}$ resulting from the invariance under translations. Equation (2.16) now simplifies to

$$\bar{\delta}_D\mathscr{L} = -\epsilon\partial_\mu(X^\mu\mathscr{L}) - \epsilon\left[\frac{\partial\mathscr{L}}{\partial\phi}d\phi + \pi^\nu(1+d)\partial_\nu\phi - n\mathscr{L}\right]\,. \tag{2.18}$$

[1]Note that this is only the case for scalar fields. In the general approach of a multi-component field, d has to be replaced by a matrix which is afterwards seen to be a multiple of the identity.

In order for the dilatation transformation to be a symmetry, the total variation of the Lagrangian must equal a total divergence, which gives

$$\pi^\mu(1+d)\partial_\mu\phi + \frac{\partial\mathscr{L}}{\partial\phi}d\phi - n\mathscr{L} = 0\,. \tag{2.19}$$

If the scale dimension d is chosen such that $d = n/2-1$, massless free scalar field theories exhibit dilatation invariance, as can be seen from the equation above. We can now proceed to define the dilatation current j_D^μ with the help of (2.15) and with $\Lambda^\mu = X^\mu\mathscr{L}$ from (2.18):

$$j_D^\mu = X_\nu T^{\mu\nu} + \pi^\mu d\phi\,, \tag{2.20}$$

where $T^{\mu\nu}$ is the canonical energy-momentum tensor

$$T^{\mu\nu} = \pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathscr{L}\,. \tag{2.21}$$

With this current, the generator of dilatation transformations is

$$D(\sigma) = \int\limits_\sigma j_D^\mu\, d\sigma_\mu \tag{2.22}$$

with σ being a spacelike hypersurface.

Finally, it is clear from (2.15) that a theory is dilatation invariant if the dilatation current vanishes,

$$\partial_\mu j_D^\mu = T_\mu^\mu + d\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\,\phi\right) = 0\,. \tag{2.23}$$

In 2-dimensional theories, this condition expresses the fact, that the energy-momentum tensor must be traceless for the theory to be dilatation invariant. Many treatments of conformal transformations now introduce the symmetric Belinfante tensor and add conserved terms such that the current can be written as $j_D^\mu = X_\nu\Theta^{\mu\nu}$ with $\Theta^{\mu\nu}$ being the modified energy-momentum tensor. The details can be looked up in any standard textbook, but they will not be needed for our further developments.

2.2 Broken Scale Invariance

Many theories are not scale invariant, i.e. the current (2.20) is not conserved. Nevertheless, the dilatation charge (2.22) can still be defined, but then the commutation relations (2.6a), (2.6b) and $[D, M_{\mu\nu}] = 0$ are no longer satisfied. We will derive the improper commutation relations for the non-conserved dilatation charge $D(\sigma)$ with P_μ explicitly. To calculate $[D(\sigma), P_\mu]$ we make use of the Heisenberg equation of motion and the identity

$$\int\limits_\sigma \partial_\nu F(X)\, d\sigma^\mu - \int\limits_\sigma \partial_\mu F(X)\, d\sigma^\nu = 0\,, \tag{2.24}$$

which is valid for $x^2F(X) \to 0$ as $|x| \to 0$ (see [25] for a proof). By noting that (2.22) explicitly depends on X, the Heisenberg equation gives

$$\begin{aligned}[D(\sigma), P_\mu] &= i\int\limits_\sigma T_\mu^\lambda\, d\sigma_\lambda - i\int\limits_\sigma \partial_\mu j_D^\lambda\, d\sigma_\lambda \\ &= iP_\mu - i\int\limits_\sigma \partial_\lambda j_D^\lambda\, d\sigma_\mu\,.\end{aligned} \tag{2.25}$$

In a similar way the following commutation relation of $M_{\mu\nu}$ with the non-conserved dilatation charge is gained,

$$[D(\sigma), M_{\mu\nu}] = i \int_\sigma X_\nu \partial_\lambda j_D^\lambda \, d\sigma_\mu - i \int_\sigma X_\mu \partial_\lambda j_D^\lambda \, d\sigma_\nu \, . \tag{2.26}$$

Note that in a scale invariant theory where $\partial_\mu j_D^\mu = 0$, the commutators (2.6a) and (2.6b) are restored.

Chapter 3

Quantization of a Scalar Field

In Chapter 1, we have reviewed the ideas about different forms of dynamics in relativistic theories. The generalization of these ideas to parametrized, generally covariant quantum field theories on a fixed background are often supposed to be analogous to the finite-dimensional case. Nevertheless, explicit treatments exist only for very simple, non-interacting models. A quantization of the free, massive scalar field within this framework has been done in [28]. The authors started with a Cauchy surface Σ and its embeddings $\mathcal{X} : \Sigma \to \mathcal{M}$ into a curved space-time $(\mathcal{M}, \underline{g})$. Then, the extended phase space Γ is constructed by considering all spacelike embeddings $\mathcal{X}$ and their conjugate momenta $\mathcal{P}$ as additional variables. This leads to a parametrized theory, which gives the original dynamics by imposing proper constraints.

In this work, we will not adopt this formalism, but we will rather investigate the canonical quantization procedure for a massive scalar field in 2-dimensional Minkowski space-time on different hypersurfaces, thus explicitly breaking general covariance. This is done by introducing a new coordinate system on $\mathcal{M}$, adopted to the chosen embedding of our 1-dimensional manifold, representing space. Then the canonical commutation relations on a fixed hypersurface are imposed. This procedure eventually implies solving the field equations in Minkowski space-time using curvilinear coordinates. Since one chart does not necessarily cover the whole of $\mathcal{M}$, it may not be possible to carry out the quantization for the whole space-time at once and one has to split it (this will be the case in Chapter 4 where we will discuss standard Point-form dynamics). Within the framework of general relativity, every coordinate system may be used to label points in space-time and therefore one can adopt curvilinear coordinates along with the new associated metric. This idea is taken up again in Chapter 4.

First, we will briefly elaborate on the general idea of time developments and in Section 3.2 the Klein-Gordon equation and the canonical quantization procedure in curved space-time are reviewed. Afterwards the general theory will be applied in some detail to the Instant-form (which reduces to the standard quantization in Minkowski space-time) and to the Point-form (in Chapter 4). The latter gives rise to some non-trivial questions about the necessity of quantizing the whole space-time and the interpretation of the theory in terms of particles.

3.1 General Time Development

We have already discussed the necessity of a generalization of the concept of time in Chapter 1 (in particular, see the discussion in 1.2). According to Section 1.4, the evolution of a physical system (the fields in question) from one hypersurface to another is governed by what is called the Hamiltonian of this system. In the general approach, all operators which move hypersurfaces in space-time are called Hamiltonians. In the following, to avoid confusion with this terminology, we define precisely what is meant by the words Hamiltonian and time evolution (development) operator within this work.

The definitions actually apply to any space-time $(\mathcal{M}, \underline{g})$ which allows a foliation but will make more sense if it possesses some symmetries, as will be seen. Space will be represented by a manifold Σ and, by abuse of language, its image under an embedding $\mathcal{E} : \Sigma \to \mathcal{M}$ into space-time will also be denoted by Σ. Usually Σ is a Cauchy surface.

Definition 3.1.1. The *Hamiltonians* of a system in a space-time with a given foliation Σ_τ are those operators which generate displacements of a hypersurface Σ_{τ_0}.

Introducing a coordinate system in which the displacement of hypersurfaces is along the coordinate θ, we can describe the flow by a vector field $\bar{\xi}(X)$. This vector field, together with the condition $\xi^\mu \nabla_\mu \theta = 1$, can then be used to formally define operators by

$$\hat{\Xi}(\tau) = \int\limits_{\Sigma_\tau} \hat{T}_{\mu\nu} \xi^\mu d\Sigma_\tau^\nu \,. \tag{3.1}$$

These operators then generate displacements of a given hypersurface Σ_τ along the orbits of $\bar{\xi}$. Furthermore, the commutation relations among the generators corresponding to different vector fields are isomorphic to the Lie algebra of these vector fields [29]. Note that the generator $\hat{\Xi}$ will not be called a Hamiltonian if the hypersurfaces Σ_τ are invariant under the flow of $\bar{\xi}$ for all τ.[1] This definition is the geometric version of the definition used in [3]. If $\bar{\xi}$ is a Killing vector field, then $\hat{\Xi}$ is conserved and therefore independent of τ.

Definition 3.1.2. The *time evolution operator H* generates the time evolution of the physical system in question. Let the orbits of a timelike vector field $\bar{\tau}$ represent the flow of time. Then H is governed by (3.1)

$$\hat{H}(\tau) = \int\limits_{\Sigma_\tau} \hat{T}_{\mu\nu} \tau^\mu d\Sigma_\tau^\nu \,. \tag{3.2}$$

In this definition for a time evolution operator we did not require $\bar{\tau}$ to be a Killing vector. Therefore, the energy of the system is not conserved in the case of a non-Killing vector field. Note that in the usual case where $\bar{\tau}$ is a Killing vector the time evolution operator is the familiar conserved Hamiltonian operator of the system.

3.2 The Klein-Gordon Equation

This section presents the Klein-Gordon equation in the context of curved space-times. Although we will always stay in flat Minkowski space-time, the general definitions are also quite useful when working with curvilinear coordinates. The results stated here will be of great use in Section 3.4 and 4, where we will discuss the Klein-Gordon field in Minkowski space-time with different coordinate systems.

The generalization of the Klein-Gordon Lagrangian density to n-dimensional curved space-time $(\mathcal{M}, \underline{g})$ is given by [30]

$$\mathscr{L}(X) = \frac{1}{2}[-g(X)]^{\frac{1}{2}} \left\{ g^{\mu\nu}(X) \nabla_\mu \phi(X) \nabla_\nu \phi(X) - [m^2 + \xi R(X)] \phi^2(X) \right\} \,, \tag{3.3}$$

[1] When we speak of a deformation of a hypersurface generated by an operator $\hat{\Xi}$, we actually mean the transformation of a physical system given on a hypersurface Σ_τ to another hypersurface $\Sigma_{\tau'}$ generated via the commutators.

where $\phi(X)$ is a real scalar field, m the mass of the field quanta, $g(X)$ the determinant of the metric tensor $g_{\mu\nu}(X)$ at the space-time point X, ∇_μ the covariant derivative, ξ a dimensionless constant and $R(X)$ the curvature scalar of the manifold.

The constant ξ is discussed in several texts about the Klein-Gordon equation in curved space-time. The main reasons for the term $\xi R(X)$ are that (i) for massless fields and a particular value for ξ the action and equation of motion are conformally invariant and (ii) in interacting field theories in curved space-time the renormalization of the field involves a counter-term proportional to $R\phi^2$. In n-dimensional space-time the values for ξ in question turn out to be $\xi = 0$ and $\xi = (n-2)/(4(n-1))$. The scalar field is then called minimally coupled or conformally coupled, respectively. Note that for $n = 2$ the field is always minimally coupled and thus the term ξR will always be zero in 2-dimensional space-times.

The action associated with this Lagrangian is

$$S = \int_{\mathcal{M}} \mathscr{L}(X) d^n X \tag{3.4}$$

and the scalar field equation resulting from the least action principle is

$$[\Box + m^2 + \xi R(X)]\phi(X) = 0\,, \tag{3.5}$$

where $\Box$ is the covariant d'Alembert operator with

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = (-g)^{-\frac{1}{2}}\partial_\mu[(-g)^{\frac{1}{2}}\, g^{\mu\nu}\partial_\nu\phi] \tag{3.6}$$

and $g = \det(g_{\mu\nu})$. If one wants to canonically quantize equation (3.5), the first step involves finding a complete set of classical solutions. Depending on the metric, this often reduces the problem to looking for eigenfunctions of a second order elliptic linear differential operator on a manifold.[2] In quantum mechanics these solutions are given a particle interpretation and the Hilbert space of states of the quantum theory could be constructed out of the vector space of solutions of equation (3.5). If f and h are two such solutions, their inner product could be defined by

$$(f,h) = i\int_{\Sigma} (f^*\nabla_\mu h - h\nabla_\mu f^*)\, d\Sigma^\mu\,, \tag{3.7}$$

where Σ is a Cauchy surface and $d\Sigma^\mu = n^\mu d\Sigma$ is the future directed volume element of the Cauchy surface with the unit vector n^μ orthogonal to Σ. That this definition does not lead to a positive definite inner product can already be seen in the case of Minkowski space-time, where the solutions are plane waves. But if we take the subspace of positive frequency functions with finite norm (induced by the inner product above), we can define the one-particle Hilbert space $\mathcal{H}$ as the Hilbert space completion of this subspace with the inner product (3.7).[3] Note that the negative frequency solutions can be obtained from the complex conjugate space, $\mathcal{H}^*$ which is naturally isomorphic to the topological dual of $\mathcal{H}$. The restriction to positive frequency functions implies the existence of a coordinate system with a global timelike coordinate, which plays the role of "time". Therefore, only globally hyperbolic space-times are considered in practice. That the definition of positive frequency solutions is in general quite troublesome will be seen in Section 4.3. For a more complete discussion about the construction of a one-particle Hilbert space in stationary space-times, see for example [32].

[2] See for example [31] for a treatment of eigenfunction expansion and elliptic differential operators.

[3] Using the Minkowski space-time example again, the plane wave solutions have obviously no finite norm. Therefore one has to construct a rigged Hilbert space, instead of using the described construction.

A very important property of (3.7) is that it is independent of the chosen hypersurface Σ. To show this, we need Gauss' theorem (see [9])

$$\int\limits_{\partial\mathcal{U}} \zeta^\mu d\sigma_\mu = \int\limits_{\mathcal{U}} \nabla_\mu \zeta^\mu dv \,, \tag{3.8}$$

where $\mathcal{U}$ is a compact n-dimensional submanifold of $\mathcal{M}$, $\bar{\zeta}$ is a vector field and dv is a volume measure on $\mathcal{U}$. Now, let Σ_1 and Σ_2 be two different spacelike hypersurfaces with $\Sigma_1 \cap \Sigma_2 = \emptyset$. If Σ_1 or Σ_2 is not compact, assume f or h has compact support, respectively. Denote by $\mathcal{U}$ the compact submanifold bounded by Σ_1, Σ_2 (or their compact subset on which f and h live) and suitable timelike hypersurfaces where $f = h = 0$. Then we can write

$$\begin{aligned}(f,h)_{\Sigma_2} - (f,h)_{\Sigma_1} &= i \int\limits_{\partial\mathcal{U}} (f^* \nabla_\mu h - h \nabla_\mu f^*) \, d\sigma^\mu \\ &= i \int\limits_{\mathcal{U}} \nabla^\mu (f^* \nabla_\mu h - h \nabla_\mu f^*) \, dv \,. \end{aligned} \tag{3.9}$$

With the help of equation (3.5), the integrand in the last integral becomes

$$\nabla^\mu (f^* \nabla_\mu h - h \nabla_\mu f^*) = f^* \nabla^\mu \nabla_\mu h - h \nabla^\mu \nabla_\mu f^* = 0 \,. \tag{3.10}$$

This proves that (3.7) is independent of the chosen hypersurface.

If the question about the choice of a suitable time-coordinate has been settled, maybe by identifying the positive frequency solutions, the canonical momentum π can be defined. Let $X^0 = \tau$ be the chosen time-coordinate and $\bar{\xi}$ the vector field which generates the associated "time" orbits and satisfies $\xi^\mu \nabla_\mu \tau = 1$. By introducing local coordinates $X^\mu = (\tau, x)$ with $x \in \Sigma_\tau$, the canonical conjugate to ϕ on Σ_τ is [32]

$$\pi = \frac{\delta \mathscr{L}}{\delta(\partial_0 \phi)} = \sqrt{-g_{\Sigma_\tau}} \, n^\mu \nabla_\mu \phi \,, \tag{3.11}$$

which is a scalar field density of weight one. n^μ is the usual unit normal to Σ_τ and g_{Σ_τ} is the induced metric on Σ_τ. Now we can second-quantize the field ϕ and its canonical conjugate momentum and impose the equal time, i.e. fixed hypersurface, canonical commutation relations:

$$[\hat{\phi}(\tau, x), \hat{\pi}(\tau, y)] = i\delta^{n-1}(x - y) \,, \tag{3.12a}$$

$$\left[\hat{\phi}(\tau, x), \hat{\phi}(\tau, y)\right] = [\hat{\pi}(\tau, x), \hat{\pi}(\tau, y)] = 0 \,, \tag{3.12b}$$

with x and y being the $(n-1)$-dimensional spatial coordinates, i. e. $x, y \in \Sigma_\tau$.

Finally, we note that the stress-energy-momentum tensor in curved space-time can be obtained by variation of the action (3.4) with respect to the metric [30]:

$$T^{\text{grav}}_{\mu\nu}(X) = 2[-g(X)]^{-\frac{1}{2}} \frac{\delta S}{\delta g^{\mu\nu}(X)} \,, \tag{3.13}$$

where the factor $[-g(x)]^{-\frac{1}{2}}$ has been added to yield a tensor, instead of a tensor density. The factor 2 was chosen such, that $T^{\text{grav}}_{\mu\nu}$ yields the Einstein equations. Further, this factor ensures that the gravitational stress tensor for a minimally coupled scalar field is equivalent to the canonical stress tensor,

$$T^{\text{grav}}_{\mu\nu} = T^{\text{can}}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - m^2 \phi^2 \right) \,. \tag{3.14}$$

In order to get a symmetric tensor, we must symmetrize the expression above, which can be done by using anti-commutators. This leads to the following form, to which we will refer as the stress tensor for a operator-valued scalar field:

$$\hat{T}_{\mu\nu} = \frac{1}{2}\{\nabla_\mu\hat{\phi}, \nabla_\nu\hat{\phi}\} - \frac{1}{4}g_{\mu\nu}\left(g^{\rho\sigma}\{\nabla_\rho\hat{\phi}, \nabla_\sigma\hat{\phi}\} - m^2\{\hat{\phi}, \hat{\phi}\}\right) . \tag{3.15}$$

With this expression and equation (3.1), operators which generate transformations of a given hypersurface can be computed. For example, the operators $\hat{P}^\sigma$ which generate translations along the coordinate lines are

$$\hat{P}^\sigma = \int_{\Sigma_\tau} \hat{T}^\sigma_\nu \, d\Sigma^\nu_\tau \,, \tag{3.16}$$

where the σ vector fields $g^{\mu\sigma}$ where used. In the case of a flat space-time with a cartesian coordinate system, this is of course the energy-momentum four-vector, but in general it is neither conserved nor is it correlated with the translation generators of the Poincaré group.

3.2.1 The Initial Value Formulation

In classical physics, solving the dynamical equations for a given system usually involves a set of initial conditions and possibly constraints. If the initial conditions are specified and the system is allowed to evolve freely, its state at all times is completely determined. A theory which allows appropriate initial data such that the dynamical evolution of the system is uniquely determined is said to possess an initial value formulation. However, for a physical theory to be viable, the property of having an initial value formulation is not sufficient. One can think of two other conditions to be satisfied. First, "small changes" in the initial data should produce only correspondingly "small changes" in the solution (over any fixed compact region of space-time). Second, if one changes the initial data inside a region of the initial data surface, this should not produce any changes outside of the causal future of this region. If these two properties are also satisfied by the physical theory, the initial value formulation is said to be well posed.

A theorem which states the well-posedness of the initial value formulation for a system of partial differential equations with arbitrary order was found by Cauchy and Kowalewski for analytic initial data. Unfortunately, this is not sufficient for the Klein-Gordon theory, because the Cauchy-Kowalewski theorem cannot deal with the causal propagation of the field. To see this, note that an analytic function is uniquely determined by its values and all its derivatives at one point. Thus, if we alter the initial data in any open region of the initial data surface Σ_0, the initial data must be altered over the entire surface Σ_0. In order to be able to investigate the causal behaviour of the field, it is therefore necessary to consider non-analytic initial data. The following theorem renders the Klein-Gordon theory in curved space-time into a theory with a well posed initial value formulation. It is taken from [10] and simplified from a system of n linear equations to one linear equation.

Theorem 3.2.1. *Let $(\mathcal{M}, \underline{g})$ be a globally hyperbolic space-time (or a globally hyperbolic region of an arbitrary space-time) with an arbitrary smooth Lorentz metric $\underline{g}$. Let ∇_μ be the covariant derivative associated with $\underline{g}$ and Σ a smooth, spacelike Cauchy surface. Consider the linear equation for the function ϕ of the form*

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + A^\mu\nabla_\mu\phi + B\phi + C = 0 \tag{3.17}$$

with A^μ being an arbitrary smooth vector field and B and C arbitrary smooth functions on $\mathcal{M}$. Then (3.17) has a well posed initial value formulation on Σ. More precisely, given *arbitrary smooth initial data* $(\phi, n^\mu \nabla_\mu \phi)$ *on* Σ *there exists a unique solution of equation (3.17) throughout* $\mathcal{M}$*. Furthermore, the solutions depend continuously on the initial data and a variation of the initial data outside of a closed subset,* $\mathcal{S}$*, of* Σ *does not affect the solution in the domain of dependence of* $\mathcal{S}$*,* $D(\mathcal{S})$*.*

This version of the initial value theorem will suffice for all our further discussions, although most of the differentiability assumptions can be weakened.

3.3 Bogolubov Transformations

In this section we will investigate how different Fock space representations are related to each other. As was pointed out in the previous section, in general curved space-times the Poincaré group may no longer play the role of a symmetry group. Through the choice of the natural coordinate system $X^\mu = (t, x^1, x^2, x^3)$ in Minkowski space-time, the solutions for the Klein-Gordon equation are eigenfunctions of the Killing field $\partial/\partial t$ and can be decomposed with respect to this vector field in positive and negative frequency functions. But in the general case, there will be no timelike Killing vector which identifies a preferred set of positive and negative frequency modes. This gives rise to the possibility of many different Fock space representations. Normally, these are obtained by solving equation (3.5) on different hypersurfaces, if the problem of identifying the positive and negative frequency solutions can be solved. But there may be other technical issues, for example the domain of validity of the different sets of solutions may be different. In this section, these problems will be ignored.

Let us therefore consider two different decompositions of our field $\phi(X)$, each with a different complete set of orthonormal solutions (modes),

$$\phi(X) = \sum_i \left[\phi_i(X) a_i + \phi_i^*(X) a_i^\dagger\right], \tag{3.18a}$$

$$\phi(X) = \sum_i \left[\tilde{\phi}_i(X) \tilde{a}_i + \tilde{\phi}_i^*(X) \tilde{a}_i^\dagger\right]. \tag{3.18b}$$

Here the index i stands symbolically for the quantity which is necessary to label the modes. It may be continuous and the sums may be replaced by integrations over appropriate measure spaces. The modes are orthonormal with respect to the inner product (3.7),

$$(\phi_i, \phi_j) = \delta_{ij} \qquad (\phi_i^*, \phi_j^*) = -\delta_{ij} \qquad (\phi_i, \phi_j^*) = 0 \tag{3.19a}$$

$$(\tilde{\phi}_i, \tilde{\phi}_j) = \delta_{ij} \qquad (\tilde{\phi}_i^*, \tilde{\phi}_j^*) = -\delta_{ij} \qquad (\tilde{\phi}_i, \tilde{\phi}_j^*) = 0\,. \tag{3.19b}$$

Also, the annihilation and creation operators satisfy the usual commutation relations,

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad \text{and} \qquad [\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij}\,. \tag{3.20}$$

Having two different pairs of annihilation and creation operators, we will also have two, generally distinct, definitions of the vacuum state:

$$a_i|0\rangle = 0 \qquad \text{and} \qquad \tilde{a}_i|\tilde{0}\rangle = 0\,. \tag{3.21}$$

Since both sets are complete, we express the $\tilde{\phi}_i$ modes in terms of the others:

$$\tilde{\phi}_i = \sum_k (\alpha_{ik}\phi_k + \beta_{ik}\phi_k^*)\,. \tag{3.22}$$

Using the relations (3.19), we see that the coefficients α_{ik} and β_{ik} are subject to the following constraints:

$$\sum_k (\alpha^*_{ik}\alpha_{jk} - \beta^*_{ik}\beta_{jk}) = \delta_{ij} \, , \tag{3.23a}$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0 \, . \tag{3.23b}$$

The inverse relation to (3.22) reads

$$\phi_i = \sum_k (\alpha^*_{ki}\tilde{\phi}_k - \beta_{ki}\tilde{\phi}^*_k) \, , \tag{3.24}$$

which can be checked by reinserting it into (3.22) and using relations (3.19).

The expressions (3.22) and (3.24), which relate one set of modes to another, are known as Bogolubov transformations and the matrices α_{ij} and β_{ij} are called Bogolubov coefficients. We can also relate the creation and annihilation operators from the two different representations to each other, by using $a_i = (\phi, \phi_i)$ and $\tilde{a}_i = (\tilde{\phi}, \tilde{\phi}_i)$ and equating the two expansions (3.18):

$$a_i = \sum_k (\alpha_{ki}\tilde{a}_k + \beta^*_{ki}\tilde{a}^\dagger_k) \, , \tag{3.25}$$

$$\tilde{a}_i = \sum_k (\alpha^*_{ik}a_k - \beta^*_{ik}a^\dagger_k) \, . \tag{3.26}$$

Note also that using the expressions (3.19), the Bogolubov coefficients can be evaluated as

$$\alpha_{ij} = (\tilde{\phi}_i, \phi_j)^* \qquad \text{and} \qquad \beta_{ij} = -(\tilde{\phi}_i, \phi^*_j)^* \, . \tag{3.27}$$

It is now interesting to look at the expectation value of the number operator of one representation with respect to the vacuum of the other. With $N_i = a^\dagger_i a_i$, the number of ϕ_i-mode particles in the state $|\tilde{0}\rangle$ is

$$\langle\tilde{0}|N_i|\tilde{0}\rangle = \sum_k |\beta_{ki}|^2 \, . \tag{3.28}$$

The crucial question now is, if this expectation value is finite. The answer may not easily be given, since the calculation possibly involves complicated integrals, whose convergence properties have to be determined. Note, that if the $\tilde{\phi}_i$ - modes are a linear combination of ϕ_j alone, the coefficients $\beta_{ij} = 0$. Thus, the two Fock space representations share a common vacuum state, $a_i|0\rangle = 0$ as well as $\tilde{a}_j|0\rangle = 0$ and the particle interpretation of the theory is the same for both representations. Otherwise, the two sets of solutions (bases) which may viewed as the solutions on different hypersurfaces and therefore different coordinate systems, define different vacua. It is this arbitrariness of choice of a hypersurface (and therefore possibly inequivalent Fock space representations), which renders the notion of particles observer dependent. The vacuum for a Minkowski observer may not be a vacuum for another, instead particle production may occur,[4] according to (3.28). However, if for one observer, the particle density relative to another vacuum becomes infinite, the physical interpretation of the theory has to be carefully reconsidered.

[4]This is essentially the idea which leads to the well known Hawking and Unruh effects.

3.4 Instant-Form Quantization

We will now apply the general theory developed so far to canonically quantize the Klein-Gordon field in 2-dimensional Minkowski space-time $(\mathbb{R}^2, \underset{\sim}{\eta})$ on the surfaces $\Sigma_t : t = \text{const.}$, where t is the time coordinate of the global cartesian coordinate chart $X^\mu = (t, x)$. As usual, Σ is used to denote the manifold representing space and $\Sigma_t = \mathcal{E}_t(\Sigma)$ denotes the image of the manifold Σ under the natural embedding $\mathcal{E}_t : \Sigma = \mathbb{R} \ni x \mapsto X = (t, x) \in \mathbb{R}^2$. The quantization of classical fields on the hypersurfaces Σ_t is then called Instant-form quantization. This is just the usual canonical quantization procedure found in textbooks about quantum field theory which focus mainly on Minkowski space-time. For the sake of simplicity, space-time is taken to be 2-dimensional and the discussion will be limited to the very basics of quantum field theory.

So the setting is 2-dimensional Minkowski space-time $(\mathbb{R}^2, \underset{\sim}{\eta})$ with a global cartesian chart $X^\mu = (t, x)$ where the one-parameter family of hypersurfaces

$$\Sigma_t : t = \text{const.} \tag{3.29}$$

forms a foliation of $\mathbb{R}^2$. Since the domain of dependence for each hypersurface is the whole manifold, $D(\Sigma_\tau) = \mathbb{R}^2$, they are Cauchy surfaces and $(\mathbb{R}^2, \underset{\sim}{\eta})$ is globally hyperbolic. See Figure 3.1 for an overview.

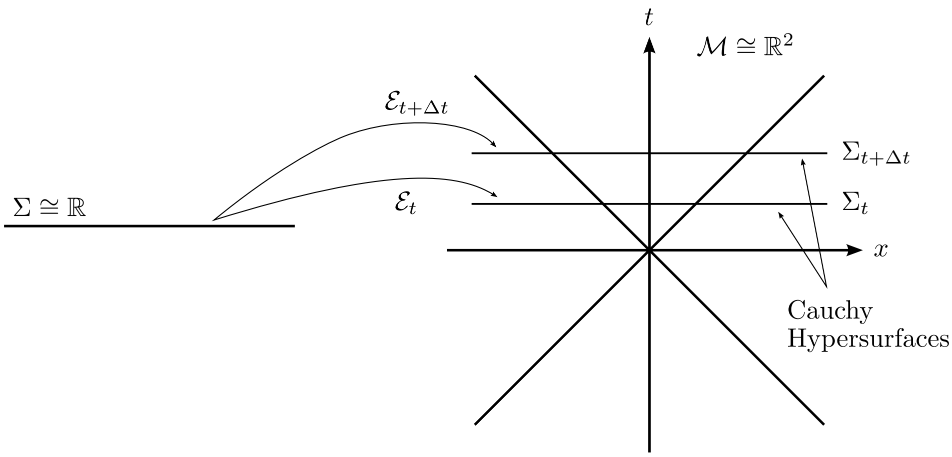


Figure 3.1: Instant-form quantization method. The scalar field ϕ is quantized on the equal-time hypersurfaces Σ_t which are planes in Minkowski space-time.

Due to Theorem 1.1.7, we can choose a global time function, which will be the projection $\tau(X) = X^0(X)$ where $X \in \mathbb{R}^2$. Additionally, the global timelike vector field ∂_t is a Killing field which can easily be seen by using the defining equation (1.1) - this ensures the invariance of the Klein-Gordon theory under time translations in Minkowski space-time and hence we have energy conservation. Using the chosen cartesian coordinate system, the components of the metric tensor are $\eta_{00} = 1$, $\eta_{11} = -1$ and zero for the off-diagonal terms and the covariant derivative reduces to the ordinary partial derivatives, $\nabla_\mu = \partial_\mu$. The general field equation (3.5) can now be written as

$$(\partial^\mu \partial_\mu + m^2)\phi(t, x) = 0\,. \tag{3.30}$$

Because of Theorem 3.2.1, equation (3.30) has a well posed initial value formulation on Σ_t

for all $t \in \mathbb{R}$. A complete set of solutions of (3.30) is

$$\phi_p^{(+)}(t,x) = (2\pi)^{-\frac{1}{2}} e^{-i\omega t + ipx}\,, \tag{3.31a}$$
$$\phi_p^{(-)}(t,x) = [\phi_p^{(+)}(t,x)]^*\,, \tag{3.31b}$$

where $\omega(p) = (p^2+m^2)^{\frac{1}{2}}$. The modes (3.31a) are said to be positive frequency and (3.31b) negative frequency solutions. They are normalized such that using the Instant-form version of the inner product (3.7),

$$(f,h)_{\mathrm{IF}} = i\int_{\mathbb{R}} (f^*\partial_t h - h\partial_t f^*)\,dx\,, \tag{3.32}$$

gives:

$$(\phi_p^{(\pm)}, \phi_{p'}^{(\pm)})_{\mathrm{IF}} = \pm 2\omega(p)\,\delta(p-p')\,, \tag{3.33a}$$
$$(\phi_p^{(\pm)}, \phi_{p'}^{(\mp)})_{\mathrm{IF}} = 0\,. \tag{3.33b}$$

Our general solution, $\phi(t,x)$, can now be expanded in terms of the solutions (3.31) with coefficients $a(p)$ and $a^\dagger(p)$,[5]

$$\phi(t,x) = \int_{\mathbb{R}} [\phi_p^{(+)}(t,x)a(p) + \phi_p^{(-)}(t,x)a^\dagger(p)]\frac{dp}{2\omega(p)}\,. \tag{3.34}$$

Note that the integral measure $dp/(2\omega(p))$ is chosen such that it is Lorentz invariant as is the normalization condition (3.33). The canonical conjugate momentum is derived from (3.11), being

$$\pi(t,x) = \partial_t\phi(t,x)\,. \tag{3.35}$$

To obtain a quantum field theory we promote the coefficients $a(p)$ and $a^\dagger(p)$ to operators, $\hat{a}(p)$ and $\hat{a}^\dagger(p)$, acting on the symmetrized Fock space of the one-particle Hilbert space constructed out of the positive frequency solutions (3.31a) (see Section 3.2). Then the canonical commutation relations (3.12) are imposed. If one uses the field expansion (3.34) and works out the commutation relations, the equivalent relations for the creation and annihilation operators,

$$[\hat{a}(p), \hat{a}^\dagger(p')] = 2\omega(p)\,\delta(p-p')\,, \tag{3.36a}$$
$$[\hat{a}(p), \hat{a}(p')] = [\hat{a}^\dagger(p), \hat{a}^\dagger(p')] = 0\,, \tag{3.36b}$$

are found. The Instant-form vacuum is now defined to be the state which is annihilated by all $\hat{a}(p)$ operators (p ranging over the spectrum of the momentum operator $\hat{P}^1$):

$$\hat{a}(p)|0\rangle_{\mathrm{IF}} = 0 \qquad \text{for all} \quad p \in \sigma(\hat{P}^1) = \mathbb{R}\,. \tag{3.37}$$

We can now proceed to work out the conserved operators corresponding to the symmetries in Minkowski space-time, known as the Poincaré generators. In Section 1.1.1 we already stated the 10 linearly independent Killing fields in 4-dimensional Minkowski space-time as an example. In Table 3.1 their 2-dimensional counterparts and their associated flows are listed. The corresponding generators can be computed with (3.1) and they are conserved,

[5]Since the Klein-Gordon field is a real scalar field, the notation for the coefficient functions is already chosen such, that after the quantization the field operator will be hermitian.

	Killing Vector $\bar{\xi}$	Flow $\Phi^{\bar{\xi}}_{\epsilon} : \mathbb{R}^2 \to \mathbb{R}^2$
Time Translation	∂_t	$(t, x) \mapsto (t + \epsilon, x)$
Space Translation	∂_x	$(t, x) \mapsto (t, x + \epsilon)$
Boost	$t\partial_x + x\partial_t$	$(t, x) \mapsto (t \cosh \epsilon + x \sinh \epsilon, t \sinh \epsilon + x \cosh \epsilon)$

Table 3.1: The 3 linearly independent, complete Killing vector fields in 2-dimensional Minkowski space-time with coordinates $X^\mu = (t, x)^\top$.

because the vector fields are Killing vectors. We can calculate the translation operators $\hat{P}^\sigma$ simultaneously if we replace ξ^μ by $\eta^{\sigma\mu}$ in (3.1) and use $d\Sigma^\nu = n^\nu dx$ with $n^\nu = (1, 0)^\top$:

$$\hat{P}^\sigma = \int_{\mathbb{R}} \hat{T}^{\sigma 0}\, dx\,. \tag{3.38}$$

Here, $\hat{P}^0 \equiv \hat{H}$ is also the conserved time evolution operator in the sense of Definition 3.1.2 because ∂_t is a timelike Killing vector field - this is *the* Hamiltonian in Instant-form quantum field theory. Using (3.15), the expression for the Hamiltonian operator becomes

$$\hat{H} = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_t \hat{\phi})^2 + (\partial_x \hat{\phi})^2 + m^2 \hat{\phi}^2 \right] dx\,. \tag{3.39}$$

Note that this is equivalent to the spatial integral over the canonical Hamiltonian density, $\mathscr{H} = \pi\dot{\phi} - \mathscr{L}$, which is often used as the definition for the Hamiltonian (time evolution operator). Plugging in the field expansion (3.34) in (3.38) , the familiar form in terms of the creation and annihilation operators is found:

$$\hat{P}^\mu = \frac{1}{2} \int_{\mathbb{R}} p^\mu [a(p) a^\dagger(p) + a^\dagger(p) a(p)] \,\frac{dp}{2\omega(p)}\,. \tag{3.40}$$

Doing the same for the boost vector field $\xi^\mu = (x, t)^\top$, the following expression for the generator of 2-dimensional boosts $\hat{B} \equiv \hat{M}^{01}$ is found:

$$\hat{B} = \frac{i}{2} \int_{\mathbb{R}} \left[(\partial_p \hat{a}^\dagger(p)) \hat{a}(p) - (\partial_p \hat{a}(p)) \hat{a}^\dagger(p) \right] \frac{dp}{2}\,. \tag{3.41}$$

We will now look at some of the properties of the Klein-Gordon field and its associated Poincaré generators with respect to dilatation transformations. The general theory has been developed in Chapter 2 and we can use equation (2.20) to calculate the dilatation current for our scalar field with scale dimension $d = 0$ (for the 2-dimensional case):

$$\hat{\jmath}^\mu_D = X_\nu \hat{T}^{\mu\nu} = \frac{1}{2} X_\nu \left[\partial^\mu \hat{\phi}\, \partial^\nu \hat{\phi} + \partial^\nu \hat{\phi}\, \partial^\mu \hat{\phi} - \eta^{\mu\nu} (\partial_\rho \hat{\phi}\, \partial^\rho \hat{\phi} - m^2 \hat{\phi}^2) \right], \tag{3.42}$$

where (3.15) has been used. By calculating the divergence of this equation one sees that the massive scalar field is not invariant under dilatation transformations,

$$\partial_\mu \hat{\jmath}^\mu_D = m^2 \hat{\phi}^2\,. \tag{3.43}$$

Therefore, the commutation relations of the dilatation generator $\hat{D}$ and the translation generators are given by (2.25), which are in our case

$$[\hat{D}, \hat{P}^0] = i\hat{P}^0 - im^2 \int\limits_{\mathbb{R}} \hat{\phi}^2 dx \tag{3.44}$$

and

$$[\hat{D}, \hat{P}^1] = i\hat{P}^1 \,. \tag{3.45}$$

The first relation can also easily be derived by a direct computation of the commutator of the dilatation generator and the Hamiltonian. The dilatation generator is again derived via equation (3.1) with the vector field $\xi^\mu = X^\mu$. Note that this is not a Killing field and therefore the generator $\hat{D}$ is not conserved.

Chapter 4

Point-Form Quantization

In Section 3.4 we have discussed the common procedure how to quantize the Klein-Gordon field in Minkowski space-time starting from classical Instant-form dynamics. Now we will focus on quantizing the scalar field on the hypersurfaces

$$\Sigma_\tau : \eta_{\mu\nu} X^\mu X^\nu = \tau^2 = \text{const.}\,, \tag{4.1}$$

again in 2-dimensional Minkowski space-time $(\mathbb{R}^2, \underline{\eta})$. This is called the Point-form because the origin is invariant under the action of the stability group of the hyperboloids (4.1) (see Section 1.5). To emphasize the main feature of a Point-form quantum field theory, we repeat that the advantage of using hyperboloids as quantization surfaces lies in the resulting interaction-independence of the boost and rotation generators of the Poincaré group. The drawback, of course, is that the space translation generators become interaction-dependent, but having simple boost and rotation generators can be quite useful in many applications. Besides, Point-form quantum field theory is not very far developed yet (an overview has been given in the introduction) and it is interesting by itself as a simple example for quantization on curved hypersurfaces.

We will make use of two different coordinate charts on $\mathbb{R}^2$ where one will be the global cartesian chart $X^\mu = (t, x)$ which is – of course – just the identity map on $\mathbb{R}^2$ and Y^μ, which will be introduced below. Further, we will always restrict ourselves to the forward light-cone, namely $\tau, t > 0$. The embeddings of the manifold Σ representing space into space-time can then be uniquely written as $\mathcal{E}_\tau : \Sigma = \mathbb{R} \ni x \mapsto X = (\sqrt{\tau^2 + x^2}, x) \in \mathbb{R}^2$. The images of Σ, $\Sigma_\tau = \mathcal{E}_\tau(\Sigma)$, are then hyperboloids in $\mathbb{R}^2$. Obviously, they do not form a foliation of $\mathbb{R}^2$, neither are they Cauchy surfaces (see Figure 1.1 for an example of a timelike curve outside the light-cone which does not intersect any of the hyperboloids (4.1)). Nevertheless, they are spacelike and the domain of dependence for each hypersurface Σ_τ coincides with the chronological future of the origin (the set of points inside the future light-cone), which will be denoted by $\mathring{C}^+$, hence $D(\Sigma_\tau) = \mathring{C}^+$.[1] Endowed with the induced metric, $(\mathring{C}^+, \underline{\eta})$ is a proper pseudo-Riemannian 2-dimensional submanifold of $(\mathbb{R}^2, \underline{\eta})$. Now the hypersurfaces Σ_τ are Cauchy surfaces with respect to this space-time (which can just be viewed as the restriction of Minkowski space-time to the interior of the forward light-cone) and it is therefore globally hyperbolic. See Figure 4.1 for an illustration of these ideas. The second chart, $Y^\mu = (\alpha, \beta)$, which we will frequently make use of, is related to the cartesian chart in the following way:

$$t = e^\alpha \cosh\beta\,, \qquad x = e^\alpha \sinh\beta\,, \qquad \text{where} \qquad \alpha, \beta \in \mathbb{R}\,, \tag{4.2}$$

[1] There are other definitions for the domain of dependence, for example in [33], where $D(\Sigma_\tau)$ would be the future light-cone C^+ itself. Physically, this is of no importance.

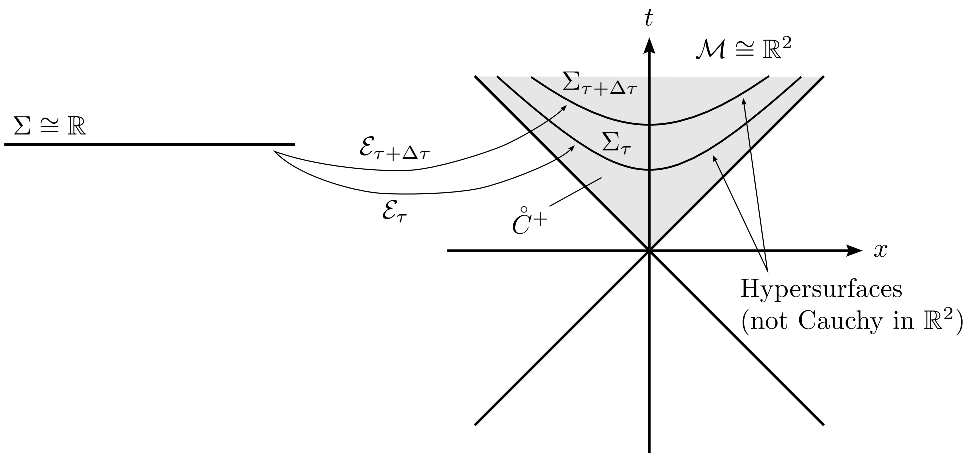


Figure 4.1: Point-form quantization method. The scalar field ϕ is quantized on the equal-time hypersurfaces Σ_τ which are hyperboloids in Minkowski space-time.

and $e^\alpha = \tau$. This chart only covers the region inside the future light-cone, hence it is a global chart for $(\mathring{C}^+, \underline{\eta})$. Nevertheless, this is quite a crucial point, since global issues must now be addressed when dealing with non-local techniques in only a patch of Minkowski space-time. Note that we can represent the overlap function for the two coordinate charts in the following way:

$$X^\mu(X) = e^\alpha \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} Y^\mu(X) \qquad \text{for all} \quad X \in \mathring{C}^+ . \tag{4.3}$$

As in Instant-form dynamics, we can now apply Theorem 1.1.7 to $(\mathring{C}^+, \underline{\eta})$ and choose our global time function to be $\tau(X) = Y^0(X)$ with $X \in \mathring{C}^+$, because the coordinate α is constant on each hypersurface Σ_τ. The Killing fields in Table 3.1 are equally important in Point-form and their discussion is spread among the following sections. In Section 4.4 an explicit discussion about the choice of a timelike vector field which generates the flow of time can be found.

Maybe some words about coordinate charts and reference systems in general relativity are appropriate here. When we work with a given space-time $(\mathcal{M}, \underline{g})$, no atlas on $\mathcal{M}$ is preferred and the choice of an atlas must not change the statements of a physical theory. General relativity tells us that the (generally covariant) physical laws stay the same if we make an arbitrary change of the reference system (by coordinate transformations) and transform the metric accordingly. Therefore, it is important not to mix up the concepts of a change of the atlas (or a local change of charts) – being just a (local) reparametrization – and the change of the reference system. In order to clarify the notion of a reference system (sometimes also called an observer's system), one can adopt the idea of identifying it with an irrotational fluid (see for example [34]). The matter points of this fluid can than be labeled by different coordinate charts. Such a fluid also represents a special way of foliating space-time into space and time and therefore introducing a natural notion of time T. This natural time is a quantity which is constant on hypersurfaces orthogonal to the world-lines of free particles and a coordinate chart with $X^0 = T$ is called an adapted chart. In Minkowski space-time, the adapted chart is of course the usual cartesian coordinate system.

Having clarified these points, we start by explicitly discussing the metric and Killing fields with respect to the charts X^μ and Y^μ introduced above. Afterwards, the canonical quantization procedure for Point-form dynamics is sketched and then solutions for the Klein-Gordon equation in hyperbolic coordinates will be given. Because the standard procedure is subject to some oddities, especially concerning the time development, we will introduce another coordinate chart and foliation in Section 4.4 and finally present Fock space representations of the Point-form Poincaré generators in Section 4.5.

4.1 Metric and Symmetries

Now we are taking a closer look on the hyperbolic chart Y^μ and the reference system associated with it. If we change to the hyperbolic reference system via the inverse relations of (4.2), the principles of general relativity tell us how to transform the Minkowski metric in order to ensure the validity of the framework presented in Section 3.2. The space-time $(\mathbb{R}^2, \underline{g})$, where $\underline{g}$ is given by

$$g_{\mu\nu}(\alpha,\beta) = \eta_{\rho\sigma}\frac{\partial X^\rho}{\partial Y^\mu}\frac{\partial X^\sigma}{\partial Y^\nu} = e^{2\alpha}\eta_{\mu\nu}\,, \tag{4.4}$$

is then the space-time of an observer described by the curvilinear coordinates Y^μ within $(\mathbb{R}^2, \underline{\eta})$. It is known as the Milne universe and treated as an example in [30, 35, 36]. As we will see in the following sections, the Milne universe is one of the rare exceptions for which it is possible to solve the conformally non-trivial field equation exactly. Note that although the Milne universe has a time-dependent metric, it is merely flat Minkowski space-time with an unconvential coordinatization – hence it is clearly not permeated by any additional gravitational field.

Because the Milne universe is flat, it shares the same symmetries with the Minkowski universe. The Killing vectors could easily be obtained from Table 3.1 and the relation (4.3), but for illustration, we will calculate all Killing fields $\bar{\xi}$ for the metric (4.4) by directly solving Killings equation (1.1). We start by writing down Killings equation in terms of the components of the metric tensor:

$$\frac{\partial\, g_{\mu\nu}}{\partial Y^\rho}\xi^\rho + g_{\rho\nu}\frac{\partial\,\xi^\rho}{\partial Y^\mu} + g_{\mu\rho}\frac{\partial\,\xi^\rho}{\partial Y^\nu} = 0\,. \tag{4.5}$$

It can be seen that this is equivalent to (1.1) by using $\nabla_\mu\xi_\nu = \partial_\mu\xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho$ and the definition of the Christoffel symbols. From the equation above the following set of partial differential equations for the components of the Killing vector fields are obtained:

$$\xi^0 + \partial_\alpha\xi^0 = 0\,, \qquad \partial_\beta\xi^0 - \partial_\alpha\xi^1 = 0\,, \qquad \xi^0 + \partial_\beta\xi^1 = 0\,. \tag{4.6}$$

Three linearly independent solutions of (4.6) are given by

$$\xi^\mu_{(1)} = e^{-\alpha}\begin{pmatrix}\cosh\beta\\ -\sinh\beta\end{pmatrix}\,, \qquad \xi^\mu_{(2)} = e^{-\alpha}\begin{pmatrix}-\sinh\beta\\ \cosh\beta\end{pmatrix}\,, \qquad \xi^\mu_{(3)} = \begin{pmatrix}0\\ 1\end{pmatrix}\,, \tag{4.7}$$

which can now be checked to lead to the known Minkowski representation via (4.3). There are some interesting points to note: First, the flow generated by $\bar{\xi}_{(1)}$ could be used to describe the flow of time since the vector field is globally timelike. Additionally, because it is a Killing vector, the generator constructed via equation (3.1) would be conserved. On the other hand, the generated transformations of the hypersurfaces $\Sigma_\alpha : \alpha = \text{const.}$ (they are shifted along the Minkowski X^0 coordinate) are not contained in the set $\{\Sigma_\alpha\}_{\alpha\in\mathbb{R}}$ and are therefore not compatible with the given foliation. We will come back to this point later in Section 4.4. Secondly, $\bar{\xi}_{(2)}$ and $\bar{\xi}_{(3)}$ are globally spacelike and $\bar{\xi}_{(3)}$, which represents boosts in Minkowski space-time, acquires a very simple form,

$$\bar{\xi}_{(3)} = \frac{\partial}{\partial \beta}\,, \tag{4.8}$$

as it was intended by the construction of Y^μ. It represents the invariance of the spatial section under translations. In Minkowski coordinates this is expressed by the invariance of the hyperboloids under Lorentz boosts. Thus we have found the generator of Lorentz boosts in hyperbolic coordinates.

Another interesting quantity is the spatial metric, i. e. the induced metric on a hypersurface Σ_α. To calculate the induced metric we use the natural embeddings of the space manifold Σ into the manifold $\mathbb{R}^2$ given by

$$\mathcal{E}_\alpha : \Sigma \ni \beta \mapsto (\alpha, \beta) \in \mathbb{R}^2. \tag{4.9}$$

Then the induced metric $\underset{\sim}{g}_{\Sigma_\alpha}$ is the pull-back of $\underset{\sim}{g}$, $\underset{\sim}{g}_{\Sigma_\alpha} = \mathcal{E}^*_\alpha \underset{\sim}{g}$. Since, in the present case, the hypersurfaces are 1-dimensional, the induced metric is a scalar and therefore identical to its determinant. With equation (1.10) and our coordinate system $Y^\mu = (\alpha, \beta)$, we get

$$(g_{\Sigma_\alpha})_{11} = e^{2\alpha} \eta_{\mu\nu} \partial_1 Y^\mu \partial_1 Y^\nu = -e^{2\alpha}. \tag{4.10}$$

This expression for the spatial metric reveals the fact that, if one evolves from one hypersurface to another, two initially neighbouring points get separated further and further.

Finally, we note that the dilatation transformation generated by the vector field $\bar{\xi} = X^\mu \partial_\mu$ is a conformal isometry in the sense of Definition 1.1.9, where the conformal factor $\Omega^2 = e^{2\alpha}$. Thus, $X^\mu \partial_\mu$ is a conformal Killing vector field.

4.2 Canonical Quantization

To canonically quantize the Klein-Gordon field on the hypersurfaces Σ_α, we start with the Lagrangian (3.3) in hyperbolic coordinates,

$$\mathscr{L} = \frac{1}{2} \left[(\partial_\alpha \phi)^2 - (\partial_\beta \phi)^2 - m^2 e^{2\alpha} \phi^2 \right] . \tag{4.11}$$

From this we can get, via the least action principle, the field equation

$$(\partial_\alpha^2 - \partial_\beta^2 + m^2 e^{2\alpha}) \phi(\alpha, \beta) = 0\,, \tag{4.12}$$

which is the hyperbolic version of (3.5). It is of similar form as (3.30) with a time-dependent mass. Solutions of this equation will be given in the next section. The canonical conjugate π to ϕ is given by (3.11), where g_{Σ_τ} equals (4.10) and $n^\mu = X^\mu/\tau$, leading to

$$\pi(\alpha, \beta) = \partial_\alpha \phi(\alpha, \beta)\,. \tag{4.13}$$

If we choose a basis of solutions for (4.12), we can build up the general solution ϕ as a superposition of these solutions and promote the coefficient functions to operators. The usual construction then requires an identification of positive frequency solutions with which a one-particle Hilbert space can be constructed. We shall suppose that this is possible and can then impose the canonical commutation relations on the operator-valued fields $\hat{\phi}$ and $\hat{\pi}$ for equal time arguments α:

$$[\hat{\phi}(\alpha,\beta),\hat{\pi}(\alpha,\beta')] = i\delta(\beta-\beta')\,, \tag{4.14a}$$

$$[\hat{\phi}(\alpha,\beta),\hat{\phi}(\alpha,\beta')] = [\hat{\pi}(\alpha,\beta),\hat{\pi}(\alpha,\beta')] = 0\,. \tag{4.14b}$$

The Point-form analogue to the Hamiltonian (3.39), which moves one hyperboloid Σ_α to another hyperboloid $\Sigma_{\alpha'}$, is given by equation (3.1) with the vector field $\xi^\mu = (1,0)^\top$ (using the chart Y^μ, of course). A short calculation gives

$$\hat{D}(\alpha) \equiv \hat{H}(\alpha) = \frac{1}{2}\int\limits_{\mathbb{R}} \Big[(\partial_\alpha\hat{\phi})^2 + (\partial_\beta\hat{\phi})^2 + m^2 e^{2\alpha}\hat{\phi}^2\Big]\,d\beta\,, \tag{4.15}$$

see Section B.1.1 for the details. Above, we have written $\hat{D}$ for the Hamiltonian because the vector field which generates the transformations is the same as for the dilatation generator in (2.5). Hence the Hamiltonian generating the time evolution in α is the dilatation operator. Note that it does not belong to the Poincaré group and that it is not conserved, because $\xi^\mu = (1,0)^\top$ cannot be written as a linear combination of the Killing vectors (4.7). It rather belongs to the category of conformal Killing vectors. Therefore, an analogous definition for a conserved energy associated with this operator is quite problematic. The explicit α-dependence of $\hat{D}$ is also a direct consequence of this issue. Nevertheless, the fact that $\hat{D}$ is not conserved does not alter its property of generating the equations of motion

$$i[\hat{D}(\alpha),\hat{\phi}(\alpha,\beta)] = \frac{\partial\hat{\phi}}{\partial\alpha} \tag{4.16}$$

and

$$i[\hat{D}(\alpha),\hat{\pi}(\alpha,\beta)] = \frac{\partial^2\hat{\phi}}{\partial\alpha^2} = \frac{\partial^2\hat{\phi}}{\partial\beta^2} - m^2 e^{2\alpha}\hat{\phi}\,. \tag{4.17}$$

With the help of equation (3.1) we can formally write down the Point-form Poincaré generators. The operators $\hat{P}^\sigma$, generating translations along the X^σ coordinates, can be treated simultaneously if we replace the vector field ξ^μ by $\eta^{\mu\sigma}$. For the surface element, we have $d\Sigma^\nu = (dx,dt)^\top = X^\nu d\beta$ which leads to the expression

$$\hat{P}^\mu = \int\limits_{\mathbb{R}} X_\nu \hat{T}^{\mu\nu} d\beta = \int\limits_{\mathbb{R}} [e^\alpha \cosh\beta\,\hat{T}^{\mu 0} - e^\alpha \sinh\beta\,\hat{T}^{\mu 1}]\,d\beta\,. \tag{4.18}$$

If we add an additional integration by writing $\delta(e^\alpha - e^{\alpha'})\,d\alpha' d\beta$ instead of $d\beta$ alone and use the identity $\delta(e^\alpha - e^{\alpha'}) = 2e^{2\alpha}\delta(e^{2\alpha} - e^{2\alpha'})$, the following integral is obtained:

$$\hat{P}^\mu = 2\int\limits_{\mathbb{R}^2} X_\nu \hat{T}^{\mu\nu}\,\delta(X^\sigma X_\sigma - \tau^2)\Theta(X^0)\,d^2X\,, \tag{4.19}$$

where we have used (4.2) and $\tau = e^{\alpha'}$. The commutation relation between the Point-form Hamiltonian $\hat{D}$ and the translation generators $\hat{P}^\mu$ can now be written down with the help of equation (2.25) and (3.43), where we have calculated the divergence of the dilatation current for the Klein-Gordon field. Hence, the commutator is given by

$$[\hat{D},\hat{P}^\mu] = i\hat{P}^\mu - im^2\int\limits_{\mathbb{R}} X^\mu\hat{\phi}^2 d\beta\,. \tag{4.20}$$

The boost operator $\hat{B} \equiv \hat{M}^{01}$ is obtained in a similar way as the momentum operators, by using the Killing vector field $\xi^\mu = (X^1, X^0)^\top$.[2] This leads to:

$$\hat{B} = 2 \int\limits_{\mathbb{R}^2} X_\nu [X^0 \hat{T}^{1\nu} - X^1 \hat{T}^{0\nu}]\, \delta(X^\sigma X_\sigma - \tau^2) \Theta(X^0)\, d^2X \,. \tag{4.21}$$

These expressions will be our starting point for the calculation of a representation of these operators with respect to a set of normal modes of the Klein-Gordon equation. Normal modes will be investigated in the following section and in Section 4.5 we will discuss different representations.

4.3 Normal Modes

In this section, we look for solutions of the Klein-Gordon equation (3.5) in the curvilinear coordinates $Y^\mu = (\alpha, \beta)$ in order to be able to expand the field $\phi(\alpha, \beta)$ in terms of these hyperbolic normal modes. We will mainly follow [5] with some additions now and then. The Klein-Gordon equation with respect to the chart Y^μ has already been stated in the previous section, being

$$(\partial_\alpha^2 - \partial_\beta^2 + m^2 e^{2\alpha}) \chi_\lambda(\alpha, \beta) = 0 \,, \tag{4.22}$$

where we have introduced the functions χ_λ, being the different modes (solutions) labeled by λ. Because of the form of (4.22), we can make an ansatz by separating the variables α and β and looking for Lorentz-boost eigenfunctions as solutions, meaning functions $\chi_\lambda(\alpha, \beta)$ which satisfy

$$- i\partial_\beta \chi_\lambda(\alpha, \beta) = \lambda \chi_\lambda(\alpha, \beta) \,. \tag{4.23}$$

This property is naturally required from the invariance of the hyperboloids under boosts. Note that this is analogous to the Instant-form procedure, where the quantization hypersurfaces are invariant under spatial translations. Thus, our solutions are of the form

$$\chi_\lambda(\alpha, \beta) = e^{i\lambda\beta} \chi_\lambda(\alpha) \,, \tag{4.24}$$

where $\chi_\lambda(\alpha)$ satisfies

$$\left(\frac{d^2}{d\alpha^2} + m^2 e^{2\alpha} + \lambda^2 \right) \chi_\lambda(\alpha) = 0 \,. \tag{4.25}$$

This is Bessel's differential equation (see Section B.1.2) which is known to be solved by the complete sets of modes $\{J_{i\lambda}(me^\alpha), J_{-i\lambda}(me^\alpha)\}_{\lambda\in\mathbb{R}}$ and $\{H^{(1)}_{i\lambda}(me^\alpha), H^{(2)}_{i\lambda}(me^\alpha)\}_{\lambda\in\mathbb{R}}$, where $J_{i\lambda}$ is the Bessel function with imaginary order and $H^{(1)}_{i\lambda}$ and $H^{(2)}_{i\lambda}$ are the first and second Hankel functions with imaginary order [37]. Of course, one can find other complete sets of modes but in the following we will only need these two.

We concentrate now on the Bessel functions because they will lead to a different vacuum definition for the Milne universe $(\mathring{C}^+, g)$, as will be seen (compared to the vacuum in Minkowski space-time). In order to be able to proceed with our quantization scheme, we need to identify the positive and negative frequency solutions with respect to α. As discussed in Section 3.2 this is necessary for the construction of a one-particle Hilbert space with a positive-definite scalar product given by (3.7). To accomplish this task we look at the

[2] The simultaneous treatment of the space-time rotation generators $M_{\rho\sigma}$ in a higher-dimensional space-time is achieved by replacing ξ^μ for $X_\rho g^\mu_\sigma - X_\sigma g^\mu_\rho$.

massless limit $m \to 0$ (which renders the Klein-Gordon equation conformally invariant) or, equivalently, $\alpha \to -\infty$ of our preliminary solution

$$\chi_\lambda(\alpha,\beta) = \mathcal{N} e^{i\lambda\beta} J_{\pm i\lambda}(me^\alpha) \xrightarrow[\text{or } \alpha\to-\infty]{m\to 0} \mathcal{N} e^{i\lambda\beta} e^{\pm i\lambda(\alpha+\ln m)} \left[2^{\pm i\lambda}\Gamma(1\pm i\lambda)\right]^{-1}, \tag{4.26}$$

where $\mathcal{N}$ is a normalization constant. Hence we see that $J_{-i\lambda}(me^\alpha \to 0)$ behaves like a positive frequency solution with respect to the conformal Killing vector ∂_α. This justifies the following definition of positive and negative frequency solutions, respectively:

$$\chi_\lambda^{(+)}(\alpha,\beta) = -i\mathcal{N} e^{i\lambda\beta} J_{-i|\lambda|}(me^\alpha) \tag{4.27a}$$

and

$$\chi_\lambda^{(-)}(\alpha,\beta) = [\chi_\lambda^{(+)}(\alpha,\beta)]^*, \tag{4.27b}$$

where we have extracted an $-i$ of the normalization constant in order to be able to write the negative frequency solution as a complex conjugate of the positive frequency solution and still being a Lorentz-boost eigenfunction (the minus sign is for cosmetic reasons considering the equations below). From now on, we will speak of t-positive frequency and α-positive frequency functions, where we mean of course functions which are positive frequency solutions with respect to Instant-form time $X^0 = t$ or Point-form time $Y^0 = \alpha$, respectively. For a thorough discussion about the positive frequency property of solutions of the Klein-Gordon equation in Bianchi type I universes, see [38].

Coming back to our solutions (4.27), the normalization constant $\mathcal{N}$ can be determined via the hyperbolic version of the scalar product (3.7), which is

$$(f,h)_{\mathrm{PF}} = i \int_{\mathbb{R}} [f^*(\alpha,\beta)\partial_\alpha h(\alpha,\beta) - h(\alpha,\beta)\partial_\alpha f^*(\alpha,\beta)]\, d\beta, \tag{4.28}$$

and requiring the solutions to be normalized with respect to this scalar product:

$$(\chi_\lambda^{(\pm)}, \chi_{\lambda'}^{(\pm)})_{\mathrm{PF}} = \pm\delta(\lambda-\lambda'), \tag{4.29}$$

$$(\chi_\lambda^{(\pm)}, \chi_{\lambda'}^{(\mp)})_{\mathrm{PF}} = 0. \tag{4.30}$$

In Section B.1.2 the normalization constant is calculated, giving $\mathcal{N} = \frac{1}{2}\sinh(\pi|\lambda|)^{-1/2}$ and the relations above are verified. This leads to a complete set of normalized modes for (4.22):

$$\chi_\lambda^{(+)}(\alpha,\beta) = -\frac{i}{2}\sinh(\pi|\lambda|)^{-1/2} e^{i\lambda\beta} J_{-i|\lambda|}(me^\alpha), \tag{4.31a}$$

$$\chi_\lambda^{(-)}(\alpha,\beta) = [\chi_\lambda^{(+)}(\alpha,\beta)]^*, \tag{4.31b}$$

which are Lorentz-boost eigenfunctions, i. e. satisfying (4.23).

We are now in the position to construct the general solution $\phi(\alpha,\beta)$ as a superposition of the modes $\chi_\lambda^{(\pm)}(\alpha,\beta)$ with the coefficients $b(\lambda)$ and $b^\dagger(\lambda)$:

$$\phi(\alpha,\beta) = \int_{\mathbb{R}} d\lambda \left[\chi_\lambda^{(+)}(\alpha,\beta) b(\lambda) + \chi_\lambda^{(-)}(\alpha,\beta) b^\dagger(\lambda)\right]. \tag{4.32}$$

Quantizing the field ϕ and imposing the commutation relations (4.14) we get, by using the field expansion above, the equivalent commutation relations for the operator-valued coefficients $\hat{b}$ and $\hat{b}^\dagger$:

$$[\hat{b}(\lambda), \hat{b}^\dagger(\lambda')] = \delta(\lambda - \lambda')\,, \tag{4.33a}$$

$$[\hat{b}(\lambda), \hat{b}(\lambda')] = [\hat{b}^\dagger(\lambda), \hat{b}^\dagger(\lambda')] = 0\,. \tag{4.33b}$$

The definition of the Point-form vacuum is analogous to (3.37), which is now the state that is annihilated by all $\hat{b}(\lambda)$ operators,

$$\hat{b}(\lambda)|0\rangle_{\mathrm{PF}} = 0 \qquad \text{for all} \quad \lambda \in \sigma(\hat{D}) = \mathbb{R}\,. \tag{4.34}$$

Since the dilatation generator $\hat{D}(\alpha)$ depends explicitly on α, the interpretation of the vacuum $|0\rangle_{\mathrm{PF}}$ as the state of lowest eigenvalue λ is not so straightforward as it is in Instant-form. Nevertheless, in Section B.2 the expression (4.15) for $\hat{D}(\alpha)$ is calculated in the limit $\alpha \to -\infty$ being

$$\hat{D}(-\infty) = \frac{1}{4} \int_{\mathbb{R}} |\lambda| \left[\hat{b}(\lambda)\hat{b}^\dagger(\lambda) + \hat{b}^\dagger(\lambda)\hat{b}(\lambda)\right] d\lambda\,. \tag{4.35}$$

In this limit, the state $|0\rangle_{\mathrm{PF}}$ can be viewed as the state of lowest eigenvalue. In Section 4.5.1 we will see how this limit is connected with the Point-form boost operator.

4.3.1 Mode Transformations

An interesting question now is how the Point-form (Milne) vacuum $|0\rangle_{\mathrm{PF}}$ is related to the Instant-form (Minkowski) vacuum $|0\rangle_{\mathrm{IF}}$ or rather how the particle interpretations of the two quantization methods are connected. To investigate these questions we seek the Bogolubov transformations connecting the modes $\chi_\lambda^{(\pm)}$ and $\phi_p^{(\pm)}$ or, which is equivalent, the operators $\{\hat{b}(\lambda), \hat{b}^\dagger(\lambda)\}$ and $\{\hat{a}(p), \hat{a}^\dagger(p)\}$. We start by constructing t-positive frequency solutions $|\lambda\rangle$ of the Klein-Gordon equation which are also Lorentz-boost eigenfunctions in the sense of (4.23):

$$|\lambda\rangle = \int_{\mathbb{R}} |p\rangle\langle p|\lambda\rangle \frac{dp}{2\omega(p)}\,. \tag{4.36}$$

The coefficients $\langle p|\lambda\rangle$ are calculated in Section B.3.1 to be

$$\langle p|\lambda\rangle = \frac{1}{\sqrt{\pi}} \left(\frac{p + \omega_p}{m}\right)^{i\lambda}\,. \tag{4.37}$$

They are normalized such that $\langle\lambda|\lambda'\rangle = \delta(\lambda - \lambda')$ is satisfied (which is also shown in Section B.3.1). The corresponding wave-function $\phi_\lambda^{(+)}(t,x)$ is calculated in Section B.3.2, being

$$\phi_\lambda^{(+)}(t,x) = \langle 0|\hat{\phi}(t,x)|\lambda\rangle \tag{4.38a}$$

$$= \frac{i e^{i\lambda\beta}}{\sqrt{8}\sinh(\pi\lambda)} \left[e^{-\frac{\pi}{2}\lambda} J_{i\lambda}(m e^\alpha) - e^{\frac{\pi}{2}\lambda} J_{-i\lambda}(m e^\alpha)\right] \tag{4.38b}$$

$$= -\frac{i e^{i\lambda\beta}}{\sqrt{8}} e^{\frac{\pi}{2}\lambda} H_{i\lambda}^{(2)}(m e^\alpha)\,, \tag{4.38c}$$

with the domain of $\phi_\lambda^{(+)}$ being restricted to the forward light-cone, i. e. $\phi_\lambda^{(+)}|_{\mathring{C}^+}(t,x)$. In (4.38c) we find again the second Hankel functions, which have been initially claimed to be equally well solutions of the Klein-Gordon equation (4.22). Now we see that they actually are superpositions of t-positive frequency solutions and if we had chosen them at the beginning, no different Fock space or vacuum would have been found. This can also be shown by using an integral representation for the second Hankel function [39],

$$H^{(2)}_{i\lambda}(me^{\alpha}) = \frac{i}{\pi} e^{-\frac{\pi}{2}\lambda} \int_{\mathbb{R}} e^{-ime^{\alpha}\cosh\gamma'} e^{-i\lambda\gamma'} d\gamma' \,. \tag{4.39}$$

Setting $\gamma = \gamma' - \beta$ gives

$$H^{(2)}_{i\lambda}(me^{\alpha}) = \frac{i}{\pi} e^{-(\frac{\pi}{2}\lambda + i\lambda\beta)} \int_{\mathbb{R}} e^{-imt\cosh\gamma - imx\sinh\gamma} e^{-i\lambda\gamma} d\gamma \,. \tag{4.40}$$

With $p(\gamma) = -m\sinh\gamma$ and $\omega(\gamma) = \sqrt{p^2 + m^2}$ the wave-function $\phi^{(+)}_{\lambda}(t,x)$, expressed through the integral representation of the Hankel function is

$$\phi^{(+)}_{\lambda}(t,x) = \frac{1}{\sqrt{8}\pi} \int_{\mathbb{R}} e^{-i\omega(\gamma)t + ip(\gamma)x} e^{-i\lambda\gamma} d\gamma \,, \tag{4.41}$$

which clearly is a superposition of t-positive frequency functions, as it was constructed. This leads to a somewhat more natural set of complete modes $\phi^{(\pm)}_{\lambda}$ for the equation (4.22) which involves the first and second Hankel functions:

$$\phi^{(+)}_{\lambda}(\alpha,\beta) = -\frac{ie^{i\lambda\beta}}{\sqrt{8}} e^{\frac{\pi}{2}\lambda} H^{(2)}_{i\lambda}(me^{\alpha}) \,, \tag{4.42a}$$

$$\phi^{(-)}_{\lambda}(\alpha,\beta) = \left[\phi^{(+)}_{\lambda}(\alpha,\beta)\right]^* = \frac{ie^{-i\lambda\beta}}{\sqrt{8}} e^{\frac{\pi}{2}\lambda} H^{(1)}_{-i\lambda}(me^{\alpha}) \,. \tag{4.42b}$$

The positive-negative frequency decomposition is now with respect to Minkowski time. In [7], a thorough quantization using these modes is carried out. It can be seen that no differences occur in comparison with the Instant-form, which is essentially due to equation (4.41).

Interestingly, the solutions which require Bessel functions lead to a different Fock space and we proceed with the investigation of their relationship with the plane wave Instant-form modes. The transformation between the Bessel and Hankel functions could be read off from equations (4.38b) and (4.38c) immediately, although some care must be taken considering the sign of λ. Following the line of [5], if we take a closer look at (4.38b) we see that this expression can be simply written in terms of the $\chi^{(\pm)}_{\lambda}(\alpha,\beta)$:

$$\phi^{(+)}_{\lambda}(t,x) = [2\sinh(\pi|\lambda|)]^{-\frac{1}{2}} \left[e^{\frac{\pi}{2}|\lambda|}\chi^{(+)}_{\lambda}(\alpha,\beta) + e^{-\frac{\pi}{2}|\lambda|}\chi^{(-)}_{-\lambda}(\alpha,\beta)\right] . \tag{4.43}$$

Introducing a new set of creation and annihilation operators $\hat{a}(\lambda)$ and $\hat{a}^{\dagger}(\lambda)$, defined such that

$$\hat{a}^{\dagger}(\lambda) = \int_{\mathbb{R}} \hat{a}^{\dagger}(p)\langle p|\lambda\rangle \frac{dp}{2\omega(p)} \qquad \text{so that} \qquad |\lambda\rangle = \hat{a}^{\dagger}(\lambda)|0\rangle_{\text{IF}} \tag{4.44}$$

holds, we can write down another expansion of the field $\hat{\phi}(t,x)$, using the modes (4.43):

$$\hat{\phi}(t,x) = \int_{\mathbb{R}} \left[\phi_\lambda^{(+)}\hat{a}(\lambda) + \phi_\lambda^{(-)}\hat{a}^\dagger(\lambda)\right] d\lambda \tag{4.45}$$

$$\begin{aligned} = \int_{\mathbb{R}} [2\sinh(\pi|\lambda|)]^{-\frac{1}{2}} \Big\{ \chi_\lambda^{(+)}(\alpha,\beta) \left[e^{\frac{\pi}{2}|\lambda|}\hat{a}(\lambda) + e^{-\frac{\pi}{2}|\lambda|}\hat{a}^\dagger(-\lambda)\right] \\ + \chi_\lambda^{(-)}(\alpha,\beta)\left[e^{\frac{\pi}{2}|\lambda|}\hat{a}^\dagger(\lambda) + e^{-\frac{\pi}{2}|\lambda|}\hat{a}(-\lambda)\right]\Big\} . \end{aligned} \tag{4.46}$$

Comparing this with the quantized version of (4.32) gives us the transformation between the $\hat{a}(\lambda), \hat{a}^\dagger(\lambda)$ and $\hat{b}(\lambda), \hat{b}^\dagger(\lambda)$ which is

$$\hat{b}(\lambda) = [2\sinh(\pi|\lambda|)]^{-\frac{1}{2}} \left[e^{\frac{\pi}{2}|\lambda|}\hat{a}(\lambda) + e^{-\frac{\pi}{2}|\lambda|}\hat{a}^\dagger(-\lambda)\right] . \tag{4.47}$$

This is a diagonal Bogolubov transformation and with (3.25) we identify the Bogolubov coefficients in $\hat{b}(\lambda) = \alpha_\lambda \hat{a}(\lambda) + \beta_{-\lambda}\hat{a}^\dagger(-\lambda)$ as

$$\alpha_\lambda = \left[\frac{e^{\pi|\lambda|}}{2\sinh(\pi|\lambda|)}\right]^{\frac{1}{2}} \qquad \text{and} \qquad \beta_\lambda = \beta_{-\lambda} = \left[\frac{e^{-\pi|\lambda|}}{2\sinh(\pi|\lambda|)}\right]^{\frac{1}{2}} . \tag{4.48}$$

With (3.26) we can immediately write the inverse relation as $\hat{a}(\lambda) = \alpha_\lambda \hat{b}(\lambda) - \beta_\lambda \hat{b}^\dagger(-\lambda)$. Note that we can also give the transformations between the two sets of modes with the help of (3.22) and (3.24) being

$$\chi_\lambda^{(+)} = \alpha_\lambda \phi_\lambda^{(+)} - \beta_\lambda \phi_{-\lambda}^{(-)} \qquad \text{and} \qquad \phi_\lambda^{(+)} = \alpha_\lambda \chi_\lambda^{(+)} + \beta_\lambda \chi_{-\lambda}^{(-)} . \tag{4.49}$$

These relations can easily be checked by direct calculation. Having established the relationship between the Instant-form and Point-form modes, we can now change from plane waves to the Bessel modes by Fourier-transforming them via (4.41) and using the Bogolubov transformation (4.49). The same procedure would apply for the transition from the $\hat{a}(p), \hat{a}^\dagger(p)$ operators to the $\hat{b}(\lambda), \hat{b}^\dagger(\lambda)$ operators.

Another important point revealed by the equations (4.49) is that mode mixing occurs. A particle in the Milne universe will look like a superposition of a particle and an anti-particle in the Minkowski universe. The particle creation rate (3.28) in this case is $e^{-\pi|\lambda|}/(2\sinh(\pi|\lambda|) = (e^{2\pi|\lambda|}-1)^{-1}$ which is the Planck spectrum for radiation at temperature $T = (2\pi k_B)^{-1}$. This is a first glimpse at the Hawking effect, but in our special case it is just some kind of artificial by-product and of no physical relevance. This is due to the fact that the interpretations of the quantizations in terms of particles and anti-particles rest only on the choice of different global charts on the flat space-time $(\mathring{C}^+, \underline{\eta})$ and the choice of suitable normal modes. Further discussions about the particle interpretation of quantum field theories on curved hypersurfaces can be found in the textbook [30] and in a few papers, for example [8].

4.3.2 Extension Outside the Light-Cone

So far, our investigation of the Point-form quantization method has been restricted to the forward light-cone only. Original studies of the region outside the light-cone, parametrized similarly as in (4.2) – known as the Rindler space – can be found in [40]. Quantization in Rindler space is often treated as an example, see for example [30, 35, 36] and [8].

The basic idea is to interchange the variables t and x in (4.2) and to quantize on the spacelike hypersurfaces where β = const. The solutions of the Klein-Gordon equation are then found to be the modified Bessel functions of the third kind - but we will not go into the details here. In [5] the extension to Rindler space is discussed and in [7] a thorough discussion about the time evolution in full space-time is given. The ideas are sketched in Figure 4.2.

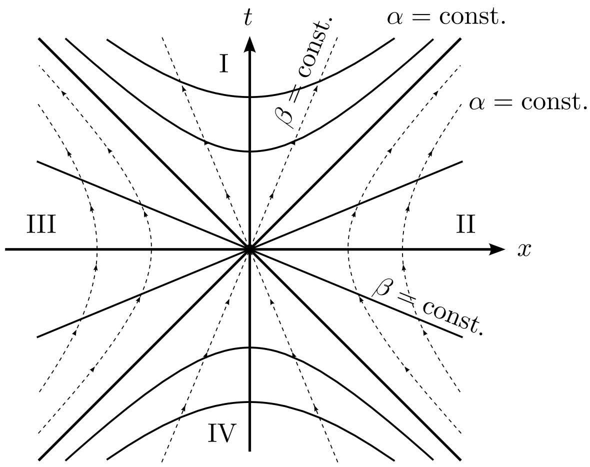


Figure 4.2: Point-form time evolution in full space-time. Regions I and IV are parametrized such that $t = \pm e^\alpha \cosh\beta$ and $x = \pm e^\alpha \sinh\beta$, respectively.[3] In the regions II and III $t = \pm e^\alpha \sinh\beta$ and $x = \pm e^\alpha \cosh\beta$ are used for the parametrization, respectively. The solid lines represent spacelike hypersurfaces (except for the axes and the light-cone, of course) and the dashed lines illustrate the time flow in the different regions. The time-ordering is along α and β in the regions I and II and along $-\beta$ and $-\alpha$ in the regions III and IV, respectively.

Carefully linking the time evolutions of the different parts at the horizons, one acquires a functional picture of the evolution from one point in space-time to another. In [7] it is shown that the propagator

$$_{\mathrm{IF}}\langle 0|\mathcal{D}\hat{\phi}(X)\hat{\phi}(X')|0\rangle_{\mathrm{IF}} = \Delta(X,X') \tag{4.50}$$

is defined for all space-time points X if one uses a field expansion with respect to the modes (4.42) and the time evolution sketched in Figure 4.2 (where $\mathcal{D}$ denotes the corresponding time ordering). Furthermore, it is shown that the propagator is the same as in Instant-form. Nevertheless, defining a Point-form scattering theory seems a little awkward within this approach and finding a representation for the Point-form Poincaré operators is still a problematic task, as will be discussed in Section 4.5.1.

4.4 Time Development Revisited

All the difficulties concerning the Point-form quantization discussed so far actually stem from the foliation induced by the parametrization (4.2), or rather by the chart Y^μ defined only for a region of Minkowski space-time. Thus we propose a slightly more general parametrization given by the following relations:

$$t = e^\alpha \cosh\beta + \zeta a^0 \qquad \text{and} \qquad x = e^\alpha \sinh\beta + \zeta a^1\,, \qquad \text{where} \qquad \zeta,\beta,\alpha \in \mathbb{R}\,. \tag{4.51}$$

[3]We used the same parametrization as in [7], concerning the minus sign of the $\sinh\beta$ term. Hence, in regions III and IV β increases along the direction of the integral curves of the boost vector field, as does α along the direction of the dilatation vector field in the regions I and IV.

In addition we require that the vector $\bar{a}$ is timelike and without loss of generality normalized to unity, i. e. $\eta_{\mu\nu}a^\mu a^\nu = 1$. Similar to Section 4.3.2, time evolution could be considered with respect to α and by setting $\zeta = 0$ (or $\bar{a} = 0$), the usual parametrization is recovered. Contrary, we can now regard α as a fixed parameter and consider time evolution in ζ. This parametrization is quite useful as we see by looking at the expansion of the field operator $\hat{\phi}$ in terms of plane waves (3.31) like in Instant-form:

$$\hat{\phi}(\zeta, \beta; \alpha, \bar{a}) = \int\limits_{\mathbb{R}} \left[\phi_p^{(+)}(\zeta, \beta; \alpha, \bar{a})\hat{a}(p) + \phi_p^{(-)}(\zeta, \beta; \alpha, \bar{a})\hat{a}^\dagger(p)\right] \frac{dp}{2\omega(p)}\,, \tag{4.52}$$

with

$$\phi_p^{(+)}(\zeta, \beta; \alpha, \bar{a}) = \frac{1}{\sqrt{2\pi}} e^{-i\zeta a^\mu p_\mu} e^{-ie^\alpha(\omega(p)\cosh\beta - p\sinh\beta)}\,, \tag{4.53a}$$

$$\phi_p^{(-)}(\zeta, \beta; \alpha, \bar{a}) = \left[\phi_p^{(+)}(\zeta, \beta; \alpha, \bar{a})\right]^*\,. \tag{4.53b}$$

This motivates the choice of ζ for the time variable since the ζ- and β-dependence separates in the modes above, whereas the α- and β-dependence does not. Thus, one can again easily establish the Heisenberg and Interaction pictures with respect to this parametrization. Furthermore, the relations (4.51) are defined everywhere in $\mathbb{R}^2$. We shall call the dynamics associated with the parametrization (4.51) *generalized Point-form (GPF) dynamics.*

Hence, we are led to the following embeddings of the space manifold Σ into our Minkowski space-time $(\mathbb{R}^2, \underline{\eta})$ with its natural chart X^μ,

$$\mathcal{E}_\zeta^{(\alpha,\bar{a})} : \Sigma = \mathbb{R} \ni x \mapsto (\zeta a^0 + \sqrt{e^{2\alpha} + (x - \zeta a^1)^2}, x) \in \mathbb{R}^2\,. \tag{4.54}$$

The hypersurfaces $\Sigma_\zeta^{(\alpha,\bar{a})} = \mathcal{E}_\zeta^{(\alpha,\bar{a})}(\Sigma)$ and their evolution along ζ are shown in Figure 4.3. The collection of all $\Sigma_\zeta^{(\alpha,\bar{a})}$ with $\zeta \in \mathbb{R}$ forms a foliation of $\mathbb{R}^2$ but they are, of course, not Cauchy.[4] To ensure the well-posedness of the initial value formulation of the Klein-Gordon equation with initial data on a hypersurface $\Sigma_\zeta^{(\alpha,\bar{a})}$, we make again use of Theorem 3.2.1. But now, since $\Sigma_\zeta^{(\alpha,\bar{a})}$ is not Cauchy, we must be more careful. We note that the domain of dependence for $\Sigma_\zeta^{(\alpha,\bar{a})}$ can be identified with the interior of a future light-cone, $\mathring{C}_\zeta^+$, whose surface is the past Cauchy horizon of $\Sigma_\zeta^{(\alpha,\bar{a})}$. Thus, if we have some initial data on a hypersurface $\Sigma_{\zeta_0}^{(\alpha,\bar{a})}$, we can apply Theorem 3.2.1 to the globally hyperbolic space-time $(\mathring{C}_{\zeta_0}^+, \underline{\eta})$ with Cauchy surface $\Sigma_{\zeta_0}^{(\alpha,\bar{a})}$.

Following the line of the preceding discussions about different foliations, we denote the global charts belonging to the bijections (4.51) by $Z_{(\alpha,\bar{a})}^\mu = (\zeta, \beta)$ where α is just the parameter characterizing the hyperboloid. The global time function for $(\mathbb{R}^2, \underline{\eta})$ can then be given by $\tau(X) = Z_{(\alpha,\bar{a})}^0(X)$ for all $X \in \mathbb{R}^2$. The overlap function between the two charts X^μ and $Z_{(\alpha,\bar{a})}^\mu$ is given by

$$X^\mu(X) = \begin{pmatrix} X^0(\bar{a}) & e^\alpha \sinh\beta \\ X^1(\bar{a}) & e^\alpha \cosh\beta \end{pmatrix} Z_{(\alpha,\bar{a})}^\mu(X) \qquad \text{for all} \quad X \in \mathbb{R}^2\,. \tag{4.55}$$

[4]Note that the converse of Theorem 1.1.7 is not true. The existence of a foliation for a space-time does not imply that it is globally hyperbolic, i. e. that the leaves of the foliation are Cauchy surfaces. On the other hand, one can of course find foliations of a globally hyperbolic space-time where the leaves are not Cauchy, as in the construction above.

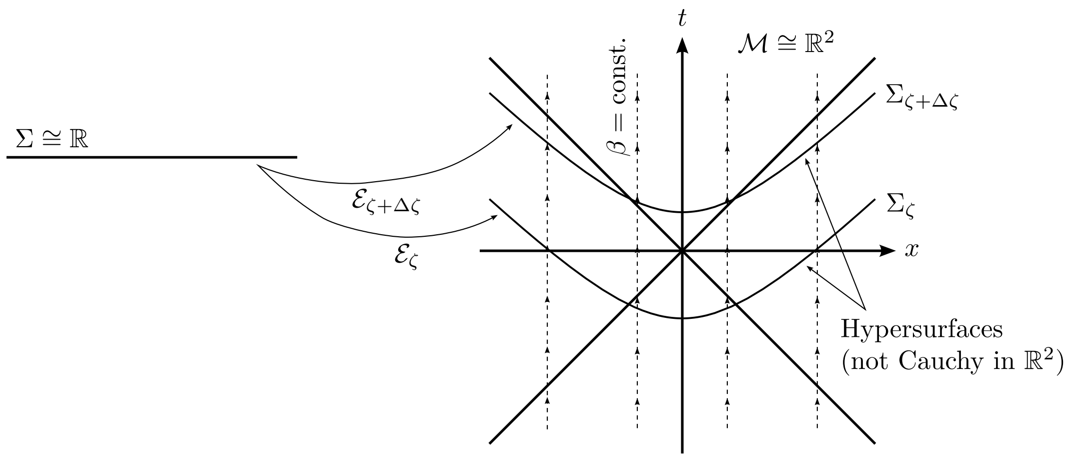


Figure 4.3: Generalized Point-form quantization method. Minkowski space-time is foliated by the equal hyperboloids Σ_ζ embedded via $\mathcal{E}_\zeta$ (α fixed and $X^\mu(\bar{a}) = (1,0)^\top$ in this illustration). The evolution from one hyperboloid to another is accomplished by translation along the vector $\bar{a}$. The flow of time, represented by the dashed lines, is in the present case equivalent to the time-flow in Instant-form.

For completeness, we state the new metric tensor, belonging to the reference system with coordinates $Z^\mu_{(\alpha,\bar{a})}$,

$$\begin{aligned} g_{\mu\nu}(\zeta,\beta;\alpha,\bar{a}) &= \eta_{\rho\sigma}\frac{\partial X^\rho}{\partial Z^\mu_{(\alpha,\bar{a})}}\frac{\partial X^\sigma}{\partial Z^\nu_{(\alpha,\bar{a})}} \\ &= \begin{bmatrix} (X^0(\bar{a}))^2 + (X^1(\bar{a}))^2 & e^\alpha(X^0(\bar{a})\sinh\beta + X^1(\bar{a})\cosh\beta) \\ e^\alpha(X^0(\bar{a})\sinh\beta - X^1(\bar{a})\cosh\beta) & -e^{2\alpha} \end{bmatrix}_{\mu\nu}. \end{aligned} \tag{4.56}$$

The induced metric on the hypersurfaces $\Sigma^{(\alpha,\bar{a})}_\zeta$ is again, as in (4.10), $g_{\Sigma^{(\alpha,\bar{a})}_\zeta}(\zeta,\beta) = -e^{2\alpha}$ but it is now coordinate-independent (since α is kept constant).

A nice feature of the time evolution along ζ is that the associated vector field $\bar{\xi} = a^\mu\partial_\mu$ is a globally timelike Killing vector field, which is clear from our initial condition $\eta_{\mu\nu}a^\mu a^\nu = 1$. Thus, the operator which generates the evolution in time will be conserved. If we chose $X^\mu(\bar{a}) = (1,0)^\top$, the time flow would be equivalent to Instant-form, and $X^\mu \circ \bar{\xi}(X) = Z^\mu_{(\alpha,\bar{a})} \circ \bar{\xi}(X)$ for all $X \in \mathbb{R}^2$. The boost Killing field looks rather unpleasant with respect to the chart $Z^\mu_{(\alpha,\bar{a})}$ when compared to (4.8), namely

$$\xi^\mu_{(3)} = \frac{1}{e^\alpha(X^0(\bar{a})\cosh\beta - X^1(\bar{a})\sinh\beta)}\begin{pmatrix} \zeta e^\alpha(X^1(\bar{a})\cosh\beta - X^0(\bar{a})\sinh\beta) \\ \zeta + e^\alpha(X^0(\bar{a})\cosh\beta - X^1(\bar{a})\sinh\beta) \end{pmatrix}. \tag{4.57}$$

For $\zeta = 0$, on the other hand, it is again very simple and reflects the boost invariance of the hypersurface $\Sigma^{(\alpha,\bar{a})}_{\zeta=0}$. This could be considered as the main disadvantage of our new parametrization but we will see in Section 4.5.3 that the introduced ζ-dependence of the Poincaré generators is rather simple.

We could now try to solve equation (3.5) with respect to $Z^\mu_{(\alpha,\bar{a})}$, but we already know that the modes (4.53) form a complete set of solutions and, due to the invariance of the scalar product (3.7) under a change of hypersurfaces, they are also orthonormal on the hyperboloids $\Sigma_\zeta^{(\alpha,\bar{a})}$. Therefore, we actually have the identities

$$(\phi_p^{(\pm)}, \phi_{p'}^{(\pm)})_{\mathrm{GPF}} = (\phi_p^{(\pm)}, \phi_{p'}^{(\pm)})_{\mathrm{PF}} = (\phi_p^{(\pm)}, \phi_{p'}^{(\pm)})_{\mathrm{IF}} = \pm 2\omega(p)\delta(p-p')\,, \tag{4.58}$$

$$(\phi_p^{(\pm)}, \phi_{p'}^{(\mp)})_{\mathrm{GPF}} = (\phi_p^{(\pm)}, \phi_{p'}^{(\mp)})_{\mathrm{PF}} = (\phi_p^{(\pm)}, \phi_{p'}^{(\mp)})_{\mathrm{IF}} = 0\,. \tag{4.59}$$

This leads to the idea of calculating the Point-form and generalized Point-form Poincaré generators by using a field expansion with the usual t-positive and t-negative frequency functions $\phi_p^{(\pm)}$. In Section 4.5 the results of this calculation will be presented.

4.5 Representations of the PF Poincaré Generators

If one wants to apply a specific Poincaré generator to a given state in order to carry out the associated (symmetry) transformation, a Fock space representation of this operator proves to be very useful. So far, we have not given any representation of one of the Poincaré generators, only the dilatation operator $\hat{D}(\alpha)$ has been stated in (4.35) in terms of the $\hat{b}(\lambda)$ and $\hat{b}^\dagger(\lambda)$ operators for the limit $\alpha \to -\infty$. Following the terminology in [41], we will call the normal modes $\chi_\lambda^{(\pm)}$ belonging to the Fock space generated by $\hat{b}(\lambda)$ a Lorentz basis (the Poincaré group is reduced with respect to the Lorentz subgroup). The normal modes $\phi_p^{(\pm)}$ belonging to the Fock space generated by $\hat{a}(p)$ are called a Wigner basis (the Poincaré group is reduced with respect to the translation subgroup).

In Section 4.5.1 we discuss the problems related to the representation in the Lorentz basis, in Section 4.5.2 we present our results for the representation of the PF Poincaré generators in the Wigner basis and in Section 4.5.3 the relation to the representation of the GPF Poincaré generators is discussed. To clarify the notation, we generally denote by $\hat{P}^\mu$ the operators which generate translations along the X^μ coordinates and by $\hat{B} \equiv \hat{M}^{10}$ the operator which generates boosts (in 2-dimensional Minkowski space-time). To distinguish between the Point-form and our generalized Point-form version of these operators, we will use different subscripts which will indicate the time-coordinate, i. e. $\hat{P}^\mu_{(\alpha)}$ and $\hat{P}_{(\zeta)}$ for the PF and GPF version respectively. We also set $\hat{P}^\mu \equiv \hat{P}^\mu_{(t)}$, so the Instant-form versions are only provided with a subscript if we need to explicitly point out a particular relationship.

4.5.1 Lorentz Representation

We will now discuss the representation of the Poincaré generators in 2-dimensional Minkowski space-time in the Lorentz basis. The attempt to calculate a Lorentz representation for the translation generators $\hat{P}^\mu$ is foredoomed by the results in [41]. The essential result concerns the matrix elements of $\hat{P}^\mu$ which cannot be evaluated in the Lorentz basis. This makes it impossible to compute momentum expectation values of boost eigenstates in the Lorentz basis. A short discussion about these problems can also be found in [3].

Nevertheless, we expect the boost operator $\hat{B}_{(\alpha)}$ to be of an especially simple form because of the boost invariance of the hypersurfaces Σ_α. The representation is calculated in Section B.4 and is:

$$\hat{B}_{(\alpha)} = \frac{1}{2}\int\limits_{\mathbb{R}} \lambda[\hat{b}^\dagger(\lambda)\hat{b}(\lambda) + \hat{b}(\lambda)\hat{b}^\dagger(\lambda)]\, d\lambda\,. \tag{4.60}$$

Thus, the boost operator is diagonal and in addition, it is interaction-independent. Moreover, it is closely related to the dilatation operator (4.35) in the limit $\alpha \to -\infty$. In this limit, the hypersurface $\Sigma_{\alpha\to-\infty}$ equals the surface of the future light-cone and the two operators both generate translations along it. See Figure 4.4 for a pictorial explanation.

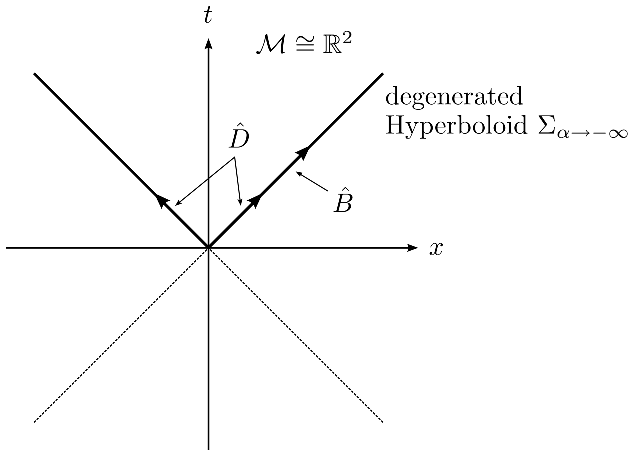


Figure 4.4: Relation between the dilatation and boost operators in the limit $\alpha \to -\infty$. The dilatation operator stretches the degenerated hyperboloid, leaving the origin fixed. The boost operator shifts points along the same direction in the region $t, x > 0$ and along the opposite direction in the region $t, -x > 0$.

4.5.2 Wigner Representation

Although the boost operator naturally exhibits a very simple form in the Lorentz basis, not being able to give a representation of the momentum operators $\hat{P}^\mu$ is somewhat uncomfortable. But since we remarked at the beginning of Section 4.5 that the plane wave solutions $\phi_p^{(\pm)}$ form a complete orthonormal set on any hypersurface, we can use them as a basis for a representation of the Poincaré generators. The explicit calculation is presented in Section B.5 where the representations in 2-dimensional Minkowski space-time are seen to be:

$$\hat{P}^\mu_{(\alpha)} = \frac{1}{2} \int\limits_{\mathbb{R}} p^\mu \left[\hat{a}(p)\hat{a}^\dagger(p) + \hat{a}^\dagger \hat{a}(p)\right] \frac{dp}{2\omega(p)}\,, \tag{4.61}$$

$$\hat{B}_{(\alpha)} = \frac{i}{2} \int\limits_{\mathbb{R}} \left[(\partial_p \hat{a}^\dagger(p))\hat{a}(p) - (\partial_p \hat{a}(p))\,\hat{a}^\dagger(p)\right] \frac{dp}{2}\,. \tag{4.62}$$

Indeed, these expressions are the same as (3.40) and (3.41) in Instant-form dynamics, at least for the free field theory. Expressed in formulae, we have found the following relations between the Instant-form and Point-form representations:[5]

$$\hat{P}^\mu \equiv \hat{P}^\mu_{(t)} = \hat{P}^\mu_{(\alpha)} \qquad \text{and} \qquad \hat{B} \equiv \hat{B}_{(t)} = \hat{B}_{(\alpha)}\,. \tag{4.63}$$

[5]Unfortunately, we have only recently seen that equation (3.1) is in fact independent of the spacelike hypersurface Σ, if the field $\hat{\phi}$ falls off sufficiently rapidly at infinity. This was remarked in [42] and can be proven with the same technique as in Section 3.2, where we have shown the invariance of the scalar product under a change of spacelike hypersurfaces. Therefore, after using Gauss' law and $\nabla^\nu(\hat{T}_{\mu\nu}\xi^\mu) = 0$, it follows immediately that the Instant-form representations are the same for every form of dynamics using spacelike hypersurfaces.

4.5.3 Wigner Representation of the GPF Poincaré Generators

In order to calculate the Wigner representation for the generalized Point-form Poincaré generators it is fortunately not necessary to repeat the equivalent steps in the Appendix B.5. We can rather use the results in the preceding section to obtain a representation for the operators $\hat{P}^{\mu}_{(\zeta)}$ and $\hat{B}_{(\zeta)}$.

For the translation generators $\hat{P}^{\sigma}_{(\zeta)}$, we rely again on equation (3.1). By replacing ξ^{μ} by $\eta^{\mu\sigma}$ we can treat them simultaneously and with $d\Sigma^{\nu} = (dx, dt)^{\top} = (X^{\nu} - \zeta a^{\nu})\, d\beta$ we get

$$\hat{P}^{\mu}_{(\zeta)} = \int_{\mathbb{R}} \hat{T}^{\mu\nu}(X_{\nu} - \zeta a_{\nu})\, d\beta = \int_{\mathbb{R}} [e^{\alpha} \cosh\beta\, \hat{T}^{\mu 0} - e^{\alpha} \sinh\beta\, \hat{T}^{\mu 1}]\, d\beta\,. \tag{4.64}$$

But this is the same expression as the Point-form version (4.18) for a fixed α.[6] Thus we have easily found that

$$\hat{P}^{\mu}_{(\zeta)} = \hat{P}^{\mu}_{(\alpha)} = \hat{P}^{\mu}\,. \tag{4.65}$$

This result could also be obtained in a different way, using the Instant-form translation operators (3.40), which have already been shown in Section 4.5.2 to be equivalent to the Point-form operators $\hat{P}^{\mu}_{(\alpha)}$. With these operators we can therefore shift the Point-form operators $\hat{P}^{\mu}_{(\alpha)}$ along the ζ-coordinate in order to get from the $\Sigma^{(\alpha,\bar{a})}_{\zeta=0} = \Sigma_{\alpha}$ hypersurface to the hypersurface $\Sigma^{(\alpha,\bar{a})}_{\zeta}$. With the help of the Baker-Campell-Hausdorff formula, this procedure gives:

$$\hat{P}^{\mu}_{(\zeta)} = e^{i\zeta a_{\nu}\hat{P}^{\nu}_{(\alpha)}}\hat{P}^{\mu}_{(\alpha)} e^{-i\zeta a_{\nu}\hat{P}^{\nu}_{(\alpha)}} = \hat{P}^{\mu}_{(\alpha)}\,, \tag{4.66}$$

where we have used $[\hat{P}^{\mu}, \hat{P}^{\nu}] = 0$. Thus, we have found again that the relation (4.65) holds.

The computation of $\hat{B}_{(\zeta)}$ is done analogously. With the vector field $\xi^{\mu} = (X^1, X^0)$ we obtain from equation (3.1) the expression

$$\hat{B}_{(\zeta)} = \int_{\Sigma^{(\alpha,\bar{a})}_{\zeta}} [X^0 \hat{T}^{1\nu} - X^1 \hat{T}^{0\nu}]\, d\Sigma_{\nu} \tag{4.67}$$

$$= \int_{\Sigma^{(\alpha,\bar{a})}_{\zeta}} [(X^0 - \zeta a^0)\hat{T}^{1\nu} - (X^1 - \zeta a^1)\hat{T}^{0\nu}]\, d\Sigma_{\nu} + \zeta \int_{\Sigma^{(\alpha,\bar{a})}_{\zeta}} [a^0 \hat{T}^{1\nu} - a^1 \hat{T}^{0\nu}]\, d\Sigma_{\nu}\,. \tag{4.68}$$

As above for the momentum operators, the first integral in (4.68) can now be seen to be equivalent to the Point-form boost operator $\hat{B}_{(\alpha)}$ for a fixed α and the second integral is precisely the sum of two momentum operators. Since in Section 4.5 we have seen that $\hat{B}_{(\alpha)}$ does not depend on α, we can write

$$\hat{B}_{(\zeta)} = \hat{B}_{(\alpha)} + \zeta \left[a^0 \hat{P}^1_{(\alpha)} - a^1 \hat{P}^0_{(\alpha)}\right]\,. \tag{4.69}$$

Again, the same result is produced with the shift method using the momentum operators $\hat{P}^{\mu}_{(\alpha)}$. In order to see this, we write

$$\begin{aligned}\hat{B}_{(\zeta)} &= e^{-i\zeta a_{\nu}\hat{P}^{\nu}_{(\alpha)}}\hat{B}_{(\alpha)} e^{i\zeta a_{\nu}\hat{P}^{\nu}_{(\alpha)}} \\ &= \hat{B}_{(\alpha)} - i\zeta a_{\nu}[\hat{P}^{\nu}_{(\alpha)}, \hat{B}_{(\alpha)}] - \frac{\zeta^2 a_{\sigma} a_{\nu}}{2}[\hat{P}^{\sigma}_{(\alpha)}, [\hat{P}^{\nu}_{(\alpha)}, \hat{B}_{(\alpha)}]] + \ldots \end{aligned} \tag{4.70}$$

[6] Note that the operators $\hat{T}^{\mu\nu}$ are of course not parametrization-dependent and thus $\hat{T}^{\mu\nu}(Z^{\mu}_{(\alpha,\bar{a})}(X)) = \hat{T}^{\mu\nu}(Y^{\mu}(X))$ for all $X \in \mathbb{R}^2$.

The commutation relation $[\hat{P}^\nu, \hat{B}] = i(\eta^{\nu 0}\hat{P}^1 - \eta^{\nu 1}\hat{P}^0)$ obtained from (2.6e) with $\hat{B} = \hat{M}^{01}$ then leads to the same result as above. Therefore, the generalized Point-form boost operator is just the usual boost operator plus additional translations. The main difference of $\hat{B}_{(\zeta)}$ compared to $\hat{B}_{(\alpha)} = \hat{B}$ is, that it is now ζ-dependent. In the special case $\zeta = 0$ the normal Point-form case is restored, due to the invariance of $\Sigma^{(\alpha,\bar{a})}_{\zeta=0}$ under boosts. For the general case though, a state which is defined on a hypersurface $\Sigma^{(\alpha,\bar{a})}_{\zeta\neq 0}$ will be moved to another hypersurface, if a boost is applied. This is illustrated in Figure 4.5.

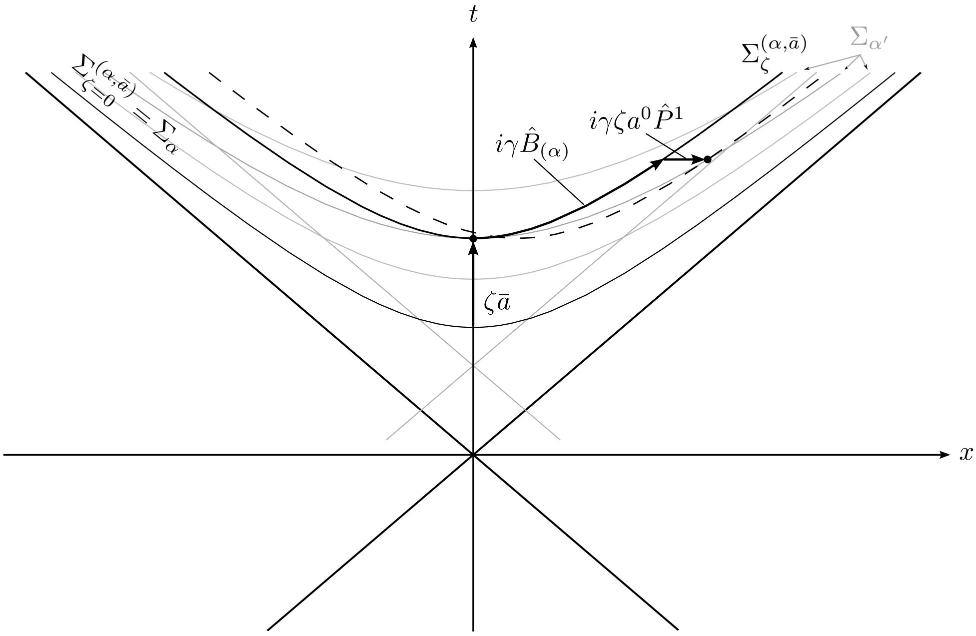


Figure 4.5: Action of the generalized boost generator. We used $a^\mu = (a^0, 0)^\top$ in the figure and the action of an infinitesimal boost is shown for the point $(\zeta, 0)^\top$. $\hat{B}_{(\alpha)}$ moves the point along the hyperboloid $\Sigma^{(\alpha,\bar{a})}_{\zeta}$ and $\hat{P}^1$ translates it parallel to the x-axis (γ is the boost parameter). The grey hyperboloids $\Sigma_{\alpha'}$ are the usual Point-form hyperboloids and the dashed hyperboloid represents the boosted hyperboloid $\Sigma^{(\alpha,\bar{a})}_{\zeta}$.

Chapter 5

Scattering Theory

Very few papers exist, which discuss a possible approach to Point-form scattering theory and in none of them, the theory is worked out in detail. In [5] a Point-form S-Matrix is formally written down in an analogous way to Instant-form, but the matrix can, by construction, only be defined in the forward light-cone. It is argued, that the restriction to the forward light-cone is not as problematic as it seems, because one could use a light-cone whose origin coincides with the big-bang. Nevertheless, no explicit calculations are presented. In [41] the time development picture for full space-time, as discussed in Section 4.3.2, is carefully developed in order to provide a firm basis for a possible Point-form S-Matrix, defined everywhere in space-time. It has also been shown that this picture can not be applied to the massless field but again, no explicit calculations are presented.

The big advantage of our generalized Point-form parametrization, introduced in Section 4.4, is evidently the simple time evolution, following the pattern of Instant-form dynamics. Therefore we naturally have an evolution from one hypersurface $\Sigma_{\zeta}^{(\alpha,\bar{a})}$ to another $\Sigma_{\zeta'}^{(\alpha,\bar{a})}$ defined for all of space-time. The only major flaw is the use of hypersurfaces which are not Cauchy in Minkowski space-time, as was already discussed. In order to avoid an ill-posed initial value formulation stemming from the use of initial data given on non-Cauchy hypersurfaces (see Theorem 3.2.1), we make use of the proposition above. That is, we lay the origin of the light-cone in the past of any relevant events (see the following section). Though it seems that not much has been gained with this construction when compared to the normal Point-form scattering approach, we still have a very simple, conserved time evolution operator (compared to the non-conserved dilatation operator $\hat{D}(\alpha)$ in Point-form).

Having briefly reviewed the work about Point-form scattering theory we know of, we will present the very basics of the generalized Point-form approach to scattering theory in the following section.

5.1 First Order Perturbation Theory

All discussions preceding the current chapter did not explicitly involve interactions and, in fact, most of the framework in Chapter 3 is valid only for the free Klein-Gordon field. Now, we are going to add a (polynomial) interaction term $V(\hat{\phi})$ to the Lagrangian (3.3) in Minkowski space-time,

$$\mathscr{L} = \frac{1}{2}\left[\partial_\mu\hat{\phi}\,\partial^\mu\hat{\phi} - m^2\hat{\phi}^2\right] - V(\hat{\phi})\,. \tag{5.1}$$

This leads to the following symmetrized energy-momentum tensor, analogous to the free version (3.15),

$$\hat{T}_{\mu\nu} = \frac{1}{2}\left[\partial_\mu\hat{\phi}\,\partial_\nu\hat{\phi} + \partial_\nu\hat{\phi}\,\partial_\mu\hat{\phi} - \eta_{\mu\nu}\left(\partial_\rho\hat{\phi}\,\partial^\rho\hat{\phi} - m^2\hat{\phi}^2\right)\right] + \eta_{\mu\nu}V(\hat{\phi})\,. \tag{5.2}$$

Stepping through the same procedure as in Section 4.2 for the generalized Point-form, we write down the interaction-dependent time evolution operator $\hat{H}_{(\zeta)}$, which is actually just a linear combination of the $\hat{P}^\mu_{(\zeta)}$'s:[1]

$$\hat{H}_{(\zeta)} = \int\limits_{\Sigma_\zeta^{(\alpha,\bar{a})}} \hat{T}^{\mu\nu}X_\mu(\bar{a})\,d\Sigma_\nu = \hat{H}^{\text{free}}_{(\zeta)} + \underbrace{e^\alpha\int\limits_{\mathbb{R}}\left(X^0(\bar{a})\cosh\beta - X^1(\bar{a})\sinh\beta\right):V(\hat{\phi}):\,d\beta}_{:=\hat{H}^{\text{int}}_{(\zeta)}}\,. \tag{5.3}$$

The term $\hat{H}^{\text{int}}_{(\zeta)}$ is actually ζ-dependent, although we have not explicitly expressed it in the notation. Furthermore, we have introduced normal ordering in the expression for $\hat{H}_{(\zeta)}$. Switching to the interaction picture, we can now formally define the $\hat{S}$-operator as in Instant-form by

$$\begin{aligned}\hat{S} &= T_\zeta\exp\left(-i\int\limits_{\mathbb{R}}\hat{H}^{\text{int}}_{(\zeta)}\,d\zeta\right)\\ &= T_\zeta\exp\left(-ie^\alpha\int\limits_{\mathbb{R}^2}\left(X^0(\bar{a})\cosh\beta - X^1(\bar{a})\sinh\beta\right):V(\hat{\phi}):\,d\zeta d\beta\right)\,,\end{aligned} \tag{5.4}$$

where T_ζ denotes time-ordering with respect to ζ. If we perform an integral transformation induced by changing from the variables $Z^\mu_{(\alpha,\bar{a})}$ to X^μ using $dtdx = e^\alpha(X^0(\bar{a})\cosh\beta - X^1(\bar{a})\sinh\beta)\,d\zeta d\beta$, the quite simple result

$$\hat{S} = T_\zeta\exp\left(-i\int\limits_{\mathbb{R}^2}:V(\hat{\phi}):\,dtdx\right) \tag{5.5}$$

is obtained. This has striking similarities with the Instant-form version and in fact, it is instantly seen that up to first order in perturbation theory, the matrix elements are the same. For higher orders, we must be more careful, since the time-ordering in (5.5) is still with respect to ζ and the corresponding foliation of Minkowski space-time. Figure 5.1 finally illustrates the scattering process in the generalized Point-form.

[1] Note, that although we have shown in Section 4.5.3 that $\hat{P}^\mu_{(\zeta)} = \hat{P}^\mu$ for free scalar field theories, we cannot a priori state the same for interacting theories.

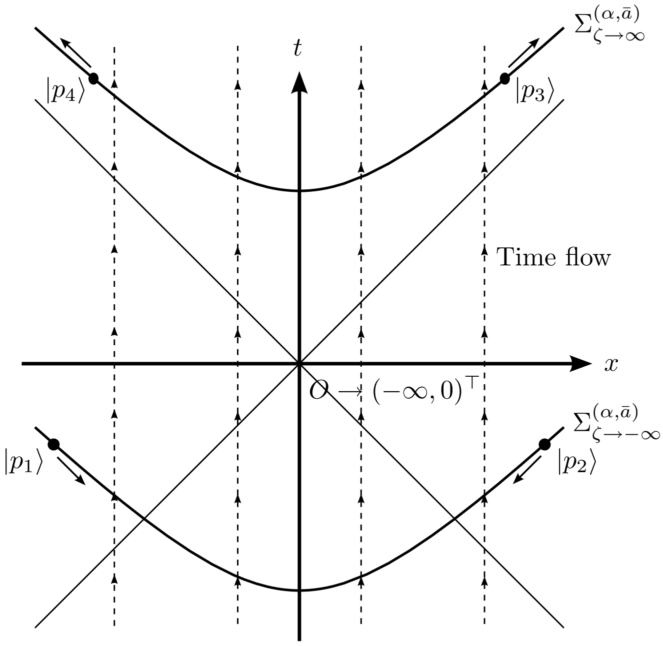


Figure 5.1: Generalized Point-form scattering process. In the figure, $X^\mu(\bar{a}) = (1,0)^\top$. The origin of the light-cone is considered to lie at $-\infty$ in order to obtain a well posed initial value formulation on the hypersurface $\Sigma^{(\alpha,\bar{a})}_{\zeta\to-\infty}$. The illustrated process is a generic two-particle scattering process.

Conclusions

Although the free scalar quantum field theory has been formulated as a parametrized system in [28], the introduction of a certain foliation of space-time is still necessary for a more detailed investigation, especially for interacting systems.

Hence, quantization of field theories on curved hypersurfaces is a vast area of active research and the use of hyperboloids as quantization surfaces is, within this general context, just a simple special case. The advantages of this specialization are obvious: The hyperboloids are highly symmetric and the field equation for the Klein-Gordon field can be solved analytically. Therefore, the Point-form quantization method serves as an important special case in order to study the properties of quantum field theories in curved space-time. Although the employed hypersurfaces are relatively simple, a comprehensive understanding of a massive, interacting scalar Point-form quantum field theory has not been achieved as yet. This might be due to the fact, that the natural time evolution from one hyperboloid to another is not easily extended to full space-time, which complicates the definition of a suitable scattering theory. In this thesis we made some contributions to this and related problems.

We pointed out how to explicitly transform plane wave momentum eigenmodes to the Lorentz boost eigenmodes. This might be useful for translating states defined in one quantization scheme into appropriate states for the other scheme. Furthermore, we mentioned the inherent impossibility of a representation of all Point-form Poincaré generators in the Lorentz basis. Thus, we proposed to use the Wigner basis for a representation and we found, that all free Point-form Poincaré generators can be represented with respect to this basis. As it turned out, the expressions for the free generators exactly match those of the usual representation of the free generators in Instant-form. This shows that the Minkowski translation operators in Point-form are also diagonalized by the usual momentum creation and annihilation operators.

With applications to scattering theory in mind, we then proposed a generalized version of the Point-form parametrization, which leads to a natural time evolution similar as in Instant-form. This resulted in a conserved time evolution operator $\hat{H}_{(\zeta)}$ and we saw how to relate the generalized Point-form Poincaré generators to the ones used in the usual Point-form quantization approach, which relativized the disadvantage of losing the boost invariance for our hypersurfaces $\Sigma_{\zeta}^{(\alpha,\bar{a})}$. Subsequently, we applied our generalized Point-form approach to scattering theory and saw that the $\hat{S}$-operator looks formally the same as in Instant-form, except for the time-ordering. This led us to the result, that the elements of the generalized Point-form $\hat{S}$-matrix to leading order perturbation theory are the same as in Instant-form.

Much work has still to be done for a deeper understanding of the quantization procedure on hyperboloids. Obviously, a generalization of our results to 4-dimensional Minkowski space-time would be the next important step. This should be straightforward and some of the calculations in the appendix were already written down with this generalization in mind.

Furthermore the generalized Point-form approach could equally well be applied to the (massive) Dirac field and to massless gauge-boson fields, which would lead to quantization of these fields on the hyperboloid. As a next step one could think of using these results to construct effective interactions and conserved currents which can be used in Point-form quantum mechanical models with a restricted number of particles. Considering all these open tasks it is clear that we are still far away from a comprehensive understanding of Point-form quantum field theory and its possible applications and a lot of work still has to be done.

Appendix A

Formulae Related to Bessel Functions

A.1 Basic Relations

In this section we present basic relations between Bessel and Hankel functions for easy reference. We start with the formulae, which relate the Bessel to the first and second Hankel functions:

$$H_\nu^{(1)}(z) = \frac{1}{i\sin(\nu\pi)}\left[J_{-\nu}(z) - J_\nu(z)\,e^{-i\nu\pi}\right], \tag{A.1}$$

$$H_\nu^{(2)}(z) = \frac{1}{i\sin(\nu\pi)}\left[J_\nu(z)\,e^{i\nu\pi} - J_{-\nu}(z)\right]. \tag{A.2}$$

The complex conjugates of these functions, denoted in this section by a bar, are given by:

$$\overline{J_\nu(z)} = J_{\overline{\nu}}(\overline{z}), \qquad \overline{H_\nu^{(1)}(z)} = H_{\overline{\nu}}^{(2)}(\overline{z}) \qquad \text{and} \qquad \overline{H_\nu^{(2)}(z)} = H_{\overline{\nu}}^{(1)}(\overline{z}). \tag{A.3}$$

The Wronskian of two functions, defined as $W(w_1, w_2) = w_1 w_2' - w_1' w_2$, can be calculated for the Bessel and Hankel functions to be:

$$W(J_\nu, J_{-\nu}) = -\frac{2\sin(\nu\pi)}{\pi z} \qquad \text{and} \qquad W(H_\nu^{(1)}, H_\nu^{(2)}) = -\frac{4i}{\pi z}. \tag{A.4}$$

Denote by $C_\nu(z)$ any of the functions $J_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ then the recurrence relations

$$C_{\nu-1}(z) - C_{\nu+1}(z) = 2C_\nu'(z) \qquad \text{and} \qquad C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z}C_\nu(z) \tag{A.5}$$

hold, where the prime denotes the derivation with respect to z.

A.2 An Integral Representation

We frequently need an integral formula involving exponentiations of hyperbolic functions. Such a formula is found to be related to the generalization of Bessel's integrals, found in [43]:

$$\begin{aligned}\pi\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}\nu} J_\nu\left[(a^2-b^2)^{1/2}\right] &= \int_0^\pi e^{b\cos t}\cos(a\sin t-\nu t)\,dt \\ &\quad -\sin(\nu\pi)\int_0^\infty e^{-a\sinh t-b\cosh t-\nu t}\,dt\,,\end{aligned} \tag{A.6}$$

where the second integral converges if $\Re(a+b)>0$. Rearranging the terms gives

$$\begin{aligned}\int_0^\infty e^{-a\sinh t-b\cosh t-\nu t}\,dt &= \frac{1}{\sin(\nu\pi)}\int_0^\pi e^{b\cos t}\cos(a\sin t-\nu t)\,dt \\ &\quad -\frac{\pi}{\sin(\nu\pi)}\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}\nu} J_\nu\left[(a^2-b^2)^{1/2}\right]\,.\end{aligned} \tag{A.7}$$

To obtain a similar formula for an integration over the whole real line, note that

$$\begin{aligned}\int_{-\infty}^\infty e^{-a\sinh t-b\cosh t-\nu t}dt &= \int_0^\infty e^{-a\sinh t-b\cosh t-\nu t}dt+\int_0^\infty e^{a\sinh t-b\cosh t+\nu t}dt \\ &= \frac{\pi}{\sin(\nu\pi)}\left(\frac{a-b}{a+b}\right)^{-\frac{1}{2}\nu}\left(J_{-\nu}\left[(a^2-b^2)^{1/2}\right]-J_\nu\left[(a^2-b^2)^{1/2}\right]\right)\end{aligned} \tag{A.8}$$

which is convergent only if $\Re(b)-|\Re(a)|>0$.

Appendix B

Point-Form Calculations

B.1 The Klein-Gordon Equation

B.1.1 Dilatation Generator

To calculate the dilatation generator – the time evolution operator in Point-form – in terms of the Klein-Gordon field $\hat{\phi}$, we start with the expression (3.1) for the generator of transformations along the flow of a vector field $\bar{\xi}$. For $\hat{T}_{\mu\nu}$ we use (3.15) which leads to

$$\hat{D} = \frac{1}{2} \int\limits_{\Sigma_\tau} \left[\partial_\mu \hat{\phi} \partial_\nu \hat{\phi} + \partial_\nu \hat{\phi} \partial_\mu \hat{\phi} - g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \hat{\phi} \partial_\sigma \hat{\phi} - m^2 \hat{\phi}^2) \right] \delta^\mu d\Sigma_\tau^\nu \,, \tag{B.1}$$

where $\bar{\delta}$ is now the vector field generating dilatation transformations. The calculation is straightforward in the reference frame of the hyperbolic coordinates $Y^\mu = (\alpha, \beta)$ where $\delta^\mu = (1,0)^\top$. With $g_{\mu\nu} = e^{2\alpha} \eta_{\mu\nu}$ from (4.4) and $d\Sigma_\tau^\nu = (1,0)^\top d\beta$ we get

$$\hat{D}(\alpha) = \frac{1}{2} \int\limits_{\mathbb{R}} \left[(\partial_\alpha \hat{\phi})^2 + (\partial_\beta \hat{\phi})^2 + e^{2\alpha} m^2 \hat{\phi}^2 \right] d\beta \,. \tag{B.2}$$

B.1.2 Solutions of the Hyperbolic Klein-Gordon Equation

To verify that (4.25) is actually Bessel's differential equation, we write $z = me^\alpha$ and with $\frac{d}{d\alpha} = z \frac{d}{dz}$ equation (4.25) becomes

$$z \frac{d}{dz} \left(z \frac{dw}{dz} \right) + \left(z^2 - (i\lambda)^2 \right) w = 0 \tag{B.3}$$

with w denoting any function solving this equation.

We now show that the solutions (4.27) satisfy (4.29) with the appropriate normalization. Since the scalar product (3.7) was defined with the linear dependence of functions in mind, i.e. it is based on the Wronskian of two functions, it is not surprising that (4.29) is satisfied. Starting from our solutions $\chi_\lambda^{(\pm)} = \mathcal{N} e^{\pm i\lambda\beta} J_{\mp i|\lambda|}(z)$ we first show that positive and negative solutions are always orthogonal:

$$(\chi_\lambda^{(\pm)}, \chi_{\lambda'}^{(\mp)})_{\mathrm{PF}} = i \int\limits_{\mathbb{R}} \left[\chi_\lambda^{(\mp)}(\alpha,\beta)\,\partial_\alpha \chi_{\lambda'}^{(\mp)}(\alpha,\beta) - \chi_{\lambda'}^{(\mp)}(\alpha,\beta)\,\partial_\alpha \chi_\lambda^{(\mp)}(\alpha,\beta)\right] d\beta \tag{B.4}$$

$$= i\,|\mathcal{N}|^2 m e^\alpha \int\limits_{\mathbb{R}} e^{\mp i\beta(\lambda+\lambda')} \left[J_{\pm i|\lambda|} J'_{\pm i|\lambda'|} - J_{\pm i|\lambda'|} J'_{\pm i|\lambda|}\right] d\beta \tag{B.5}$$

$$= i\,|\mathcal{N}|^2 m e^\alpha 2\pi\,\delta(\lambda+\lambda') \left[J_{\pm i|\lambda|} J'_{\pm i|\lambda|} - J_{\pm i|\lambda|} J'_{\pm i|\lambda|}\right] = 0\,. \tag{B.6}$$

The scalar product of two positive or two negative solutions is computed similarly:

$$(\chi_\lambda^{(\pm)}, \chi_{\lambda'}^{(\pm)})_\alpha = i\,|\mathcal{N}|^2 m e^\alpha \int\limits_{\mathbb{R}} e^{\mp i\beta(\lambda-\lambda')} \left[J_{\pm i|\lambda|} J'_{\mp i|\lambda'|} - J_{\mp i|\lambda'|} J'_{\pm i|\lambda|}\right] d\beta \tag{B.7}$$

$$= i\,|\mathcal{N}|^2 m e^\alpha 2\pi\,\delta(\lambda-\lambda') \left[J_{\pm i|\lambda|} J'_{\mp i|\lambda|} - J_{\mp i|\lambda|} J'_{\pm i|\lambda|}\right]\,. \tag{B.8}$$

The expression in the square brackets is just the Wronskian of $J_{i|\lambda|}$ and $J_{-i|\lambda|}$ and can be evaluated to [37]

$$W(J_\nu, J_{-\nu}) = J_\nu(z) J'_{-\nu}(z) - J_{-\nu}(z) J'_\nu(z) = -2(\pi z)^{-1}\sin(\nu\pi)\,. \tag{B.9}$$

Using this relation we get

$$(\chi_\lambda^{(\pm)}, \chi_{\lambda'}^{(\pm)})_{\mathrm{PF}} = \pm|\mathcal{N}|^2 4\sinh(\pi|\lambda|)\,\delta(\lambda-\lambda')\,. \tag{B.10}$$

With the normalization constant $\mathcal{N}$ being

$$\mathcal{N} = \frac{1}{2}\sinh(\pi|\lambda|)^{-1/2} \tag{B.11}$$

we have successfully shown the orthonormality relation (4.29).

B.2 The Conformal Limit of $\hat{D}$

We wish to calculate the expression for the dilatation generator (4.15),

$$\hat{D}(\alpha) = \frac{1}{2}\int\limits_{\mathbb{R}} \left[(\partial_\alpha\hat{\phi})^2 + (\partial_\beta\hat{\phi})^2 + m^2 e^{2\alpha}\hat{\phi}^2\right] d\beta\,, \tag{B.12}$$

using the field expansion (4.32) for the quantized field $\hat{\phi}$,

$$\hat{\phi}(\alpha,\beta) = \int\limits_{\mathbb{R}} \left[\chi_\lambda^{(+)}(\alpha,\beta)\hat{b}(\lambda) + \chi_\lambda^{(-)}(\alpha,\beta)\hat{b}^\dagger(\lambda)\right] d\lambda\,, \tag{B.13}$$

in the conformal limit. The conformal limit $m \to 0$ is equivalent to the limit $\alpha \to -\infty$ and therefore the two can be equally used in the interpretation of the results. The mass term in (B.12) vanishes in this limit and the derivative terms can be evaluated by using the explicit form of the modes $\chi_\lambda^{(\pm)}(\alpha,\beta)$ given in (4.31),

$$\chi_\lambda^{(+)}(\alpha,\beta) = -\frac{i}{2}\sinh(\pi|\lambda|)^{-1/2} e^{i\lambda\beta} J_{-i|\lambda|}(me^\alpha)\,, \tag{B.14a}$$

$$\chi_\lambda^{(-)}(\alpha,\beta) = [\chi_\lambda^{(+)}(\alpha,\beta)]^*\,. \tag{B.14b}$$

From this we see that the $(\partial_\alpha\hat{\phi})^2$ term vanishes too because it is essentially of the form $m^2 e^{2\alpha} J'_a(z) J'_b(z)$. Thus, using the shorter notation $\hat{b}_\lambda \equiv \hat{b}(\lambda)$ we are left with

$$\hat{D}(-\infty) = \lim_{z\to 0} \frac{\pi}{4} \int_{\mathbb{R}} \lambda^2 \sinh(\pi|\lambda|)^{-1} \Big[J_{i|\lambda|}(z) J_{-i|\lambda|}(z) \left(\hat{b}_\lambda \hat{b}^\dagger_\lambda + \hat{b}^\dagger_\lambda \hat{b}_\lambda\right) - J^2_{-i|\lambda|}(z) \hat{b}_\lambda \hat{b}_{-\lambda} - J^2_{i|\lambda|}(z) \hat{b}^\dagger_\lambda \hat{b}^\dagger_{-\lambda} \Big] \, d\lambda\,. \tag{B.15}$$

The asymptotic behaviour of the Bessel function is easily computed by using its definition via the series of Gamma functions,

$$\begin{aligned} J_\nu(z) &= \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{2m+\nu} [m!\,\Gamma(m+\nu+1)]^{-1} \\ &= \left(\frac{z}{2}\right)^{\nu} \Gamma(1+\nu)^{-1} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{2m+\nu} [m!\,\Gamma(m+\nu+1)]^{-1} \end{aligned} \tag{B.16}$$

where we have in the last line just pulled out the first term of the sum. In the limit $z \to 0$ only this first term survives which leads to

$$\begin{aligned} J_{i|\lambda|} J_{-i|\lambda|} &= [\Gamma(1+i|\lambda|)\Gamma(1-i|\lambda|)]^{-1} = [i|\lambda|\Gamma(i|\lambda|)\Gamma(1-i|\lambda|)]^{-1} \\ &= \frac{\sin(i\pi|\lambda|)}{i\pi|\lambda|} = \frac{\sinh(\pi|\lambda|)}{\pi|\lambda|}\,, \end{aligned} \tag{B.17}$$

where we have used Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$ and the identity $\Gamma(1+z) = z\Gamma(z)$ [44]. Neglecting the small oscillating terms $\hat{b}_\lambda \hat{b}_{-\lambda}$ and $\hat{b}^\dagger_\lambda \hat{b}^\dagger_{-\lambda}$ finally reveals the diagonal form of the dilatation generator in the conformal limit:

$$\hat{D}(-\infty) = \frac{1}{4} \int_{\mathbb{R}} |\lambda| \left[\hat{b}_\lambda \hat{b}^\dagger_\lambda + \hat{b}^\dagger_\lambda \hat{b}_\lambda\right] d\lambda\,. \tag{B.18}$$

B.3 Mode Transformations

B.3.1 The Coefficients $\langle p|\lambda\rangle$

To compute the coefficients $\langle p|\lambda\rangle$ we will use the Point-form scalar product (4.28) for which we need the functions $\langle \alpha,\beta|p\rangle$ and $\langle \alpha,\beta|\lambda\rangle$. The first one is easily derived by recalling that

$$\langle t,x|p\rangle = \phi^{(+)}_p(t,x) \propto e^{i(px-\omega_p t)} \tag{B.19}$$

and using the equations (4.2) which relate the Minkowski coordinates to the hyperbolic ones. The second one is the usual positive frequency solution of the hyperbolic Klein-Gordon equation, which we also write down in its unnormalized form since we are going to normalize the coefficients $\langle p|\lambda\rangle$ later on:

$$\langle \alpha,\beta|\lambda\rangle = \chi^{(+)}_\lambda(\alpha,\beta) \propto e^{i\lambda\beta} J_{-i|\lambda|}(me^\alpha)\,. \tag{B.20}$$

We can now evaluate $\langle p|\lambda\rangle$ with $\mathcal{N}$ being the normalization constant and $f\overleftrightarrow{\partial}_\alpha h$ being an abbreviation for $f\partial_\alpha h - h\partial_\alpha f$. Hence,

$$\langle p|\lambda\rangle = \mathcal{N}' \left(\phi_p^{(+)}(\alpha,\beta), \chi_\lambda^{(+)}(\alpha,\beta)\right)_{\mathrm{PF}} = \mathcal{N} \int_{\mathbb{R}} \phi_p^{(-)}(\alpha,\beta)\overleftrightarrow{\partial}_\alpha \chi_\lambda^{(+)}(\alpha,\beta)\, d\beta \tag{B.21}$$

$$= \mathcal{N} \int_{\mathbb{R}} e^{-i(pe^\alpha \sinh\beta - \omega_p e^\alpha \cosh\beta) + i\lambda\beta} \overleftrightarrow{\partial}_\alpha J_{i|\lambda|}(me^\alpha)\, d\beta \tag{B.22}$$

$$\begin{aligned} &= \mathcal{N} m e^\alpha J'_{i|\lambda|} \int_{\mathbb{R}} e^{-i(pe^\alpha \sinh\beta - \omega_p e^\alpha \cosh\beta) + i\lambda\beta}\, d\beta \\ &\quad - \mathcal{N} J_{i|\lambda|} \frac{d}{d\alpha} \int_{\mathbb{R}} e^{-i(pe^\alpha \sinh\beta - \omega_p e^\alpha \cosh\beta) + i\lambda\beta}\, d\beta\,. \end{aligned} \tag{B.23}$$

To evaluate the remaining integral in the two terms we use formula (A.8). With $a = ipe^\alpha$, $b = -i\omega_p e^\alpha + \epsilon$ and $\nu = -i\lambda$ in (A.8) we get

$$\int_{\mathbb{R}} e^{-i(pe^\alpha \sinh\beta - \omega_p e^\alpha \cosh\beta) + i\lambda\beta}\, d\beta = \frac{i\pi}{\sinh(\pi\lambda)} \left(\frac{p+\omega_p}{p-\omega_p}\right)^{\frac{i}{2}\lambda} \left[J_{i\lambda}(me^\alpha) - J_{-i\lambda}(me^\alpha)\right], \tag{B.24}$$

after taking the limit $\epsilon \to 0$. Noting that differentiating the integral above gives the same integral times me^α with the Bessel functions replaced by their derivatives we finally get

$$\langle p|\lambda\rangle = \mathcal{N} \frac{i\pi m e^\alpha}{\sinh(\pi\lambda)} \left(\frac{p+\omega_p}{p-\omega_p}\right)^{\frac{i}{2}\lambda} \left(J'_{i|\lambda|}J_{i\lambda} - J'_{i|\lambda|}J_{-i\lambda} - J_{i|\lambda|}J'_{i\lambda} + J_{i|\lambda|}J'_{-i\lambda}\right). \tag{B.25}$$

With (B.9) the bracket with the Bessel functions can be written as

$$(\dots) = -\frac{2i}{\pi m e^\alpha} \sinh(\pi\lambda) \tag{B.26}$$

and the coefficients become

$$\langle p|\lambda\rangle = 2\mathcal{N} \left(\frac{p+\omega_p}{p-\omega_p}\right)^{\frac{i}{2}\lambda} = 2\mathcal{N} \left(\frac{p+\omega_p}{m}\right)^{i\lambda} e^{\frac{\pi}{2}\lambda}\,. \tag{B.27}$$

Now we normalize our coefficients in order to satisfy $\langle\lambda|\lambda'\rangle = \delta(\lambda-\lambda')$:

$$\langle\lambda|\lambda'\rangle = \int_{\mathbb{R}} \langle\lambda|p\rangle\langle p|\lambda'\rangle \frac{dp}{2\omega_p} \tag{B.28}$$

$$= 2|\mathcal{N}|^2 m^{i(\lambda-\lambda')} e^{\frac{\pi}{2}(\lambda+\lambda')} \int_{\mathbb{R}} (p+\omega_p)^{-i(\lambda-\lambda')} \frac{dp}{\omega_p} \tag{B.29}$$

$$= |\mathcal{N}|^2 4\pi e^{\pi\lambda} \delta(\lambda-\lambda')\,. \tag{B.30}$$

Thus $\mathcal{N} = (2\sqrt{\pi}e^{(\pi\lambda)/2})^{-1}$ and the final result is

$$\langle p|\lambda\rangle = \frac{1}{\sqrt{\pi}} \left(\frac{p+\omega_p}{m}\right)^{i\lambda}. \tag{B.31}$$

B.3.2 The Function $\phi_\lambda^{(+)}(t,x)$

The Minkowski positive frequency wave-function $\phi_\lambda^{(+)}(t,x)$, which is also a Lorentz-boost eigenfunction, can be explicitly calculated in the following way:

$$\phi_\lambda^{(+)}(t,x) = \langle 0|\hat{\phi}(t,x)|\lambda\rangle = \int_{\mathbb{R}} \langle 0|\hat{\phi}(t,x)|p\rangle\langle p|\lambda\rangle \frac{dp}{2\omega(p)} \tag{B.32}$$

$$= \frac{m^{-i\lambda}}{\sqrt{8}\pi} \int_{\mathbb{R}} e^{i(pe^\alpha \sinh\beta - \omega_p e^\alpha \cosh\beta)} (p+\omega_p)^{i\lambda} \frac{dp}{\omega_p} \,. \tag{B.33}$$

With $\omega_p = m\cosh\gamma$ and $p = m\sinh\gamma$ we can rewrite the expression above, such that

$$\phi_\lambda^{(+)}(t,x) = \frac{1}{\sqrt{8}\pi} \int_{\mathbb{R}} e^{ime^\alpha \sinh\beta\sinh\gamma - ime^\alpha\cosh\beta\cosh\gamma + i\lambda\gamma} d\gamma \,. \tag{B.34}$$

Using the integral-formula (A.8) with $a = -ime^\alpha \sinh\beta$, $b = ime^\alpha\cosh\beta$ and $\nu = -i\lambda$ we get

$$\phi_\lambda^{(+)}(t,x) = \frac{i}{\sqrt{8}\sinh(\pi\lambda)} \left(\frac{\sinh\beta+\cosh\beta}{\sinh\beta-\cosh\beta}\right)^{\frac{i}{2}\lambda} \left[J_{i\lambda}(me^\alpha) - J_{-i\lambda}(me^\alpha)\right] \tag{B.35}$$

$$= \frac{ie^{i\lambda\beta}}{\sqrt{8}\sinh(\pi\lambda)} \left[e^{-\frac{\pi}{2}\lambda} J_{i\lambda}(me^\alpha) - e^{\frac{\pi}{2}\lambda} J_{-i\lambda}(me^\alpha)\right] \tag{B.36}$$

$$= -\frac{ie^{i\lambda\beta}}{\sqrt{8}} e^{\frac{\pi}{2}\lambda} H_{i\lambda}^{(2)}(me^\alpha) \,. \tag{B.37}$$

B.4 Lorentz Representation of the Boost Generator

In this section a representation of the Point-form boost generator in terms of the $\hat{b}_\lambda$ and $\hat{b}_\lambda^\dagger$ is calculated. We use equation (3.1) with the vector field $\xi^\mu = (0,1)^\top$ in the coordinates (α,β). With $d\Sigma^\nu = (1,0)^\top d\beta$ the operator is

$$\hat{B} = -\int_{\mathbb{R}} \hat{T}_{10}\, d\beta = -\frac{1}{2}\int_{\mathbb{R}} \left[\partial_\alpha\hat{\phi}\,\partial_\beta\hat{\phi} + \partial_\beta\hat{\phi}\,\partial_\alpha\hat{\phi}\right] d\beta \,. \tag{B.38}$$

With the field expansion (4.32), the $\hat{b}_\lambda\hat{b}_{\lambda'}^\dagger$ contribution is

$$-\frac{ime^\alpha}{8} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{\lambda J_{-i|\lambda|}J'_{i|\lambda'|} - \lambda' J'_{-i|\lambda|}J_{i|\lambda'|}}{[\sinh(\pi|\lambda|)\sinh(\pi|\lambda'|)]^{\frac{1}{2}}} e^{i\beta(\lambda-\lambda')} \hat{b}_\lambda\hat{b}_{\lambda'}^\dagger \, d\lambda d\lambda' d\beta =$$

$$-\frac{im\pi e^\alpha}{4} \int_{\mathbb{R}} \frac{\lambda(J_{-i|\lambda|}J'_{i|\lambda|} - J'_{-i|\lambda|}J_{i|\lambda|})}{\sinh(\pi|\lambda|)} \hat{b}_\lambda\hat{b}_\lambda^\dagger \, d\lambda = \frac{1}{2}\int_{\mathbb{R}} \lambda\hat{b}_\lambda\hat{b}_\lambda^\dagger \, d\lambda \,. \tag{B.39}$$

The analogous result holds for the $\hat{b}^\dagger_\lambda \hat{b}_{\lambda'}$ contribution. The $\hat{b}_\lambda \hat{b}_{\lambda'}$ contribution is computed in a similar way:

$$-\frac{ime^\alpha}{8}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^2}\frac{\lambda' J'_{-i|\lambda|}J_{-i|\lambda'|}+\lambda J_{-i|\lambda|}J'_{-i|\lambda'|}}{[\sinh(\pi|\lambda|)\sinh(\pi|\lambda'|)]^{\frac{1}{2}}}\,e^{i\beta(\lambda+\lambda')}\hat{b}_\lambda\hat{b}_{\lambda'}\,d\lambda d\lambda' d\beta =$$

$$-\frac{im\pi e^\alpha}{4}\int\limits_{\mathbb{R}}\frac{\lambda(J_{-i|\lambda|}J'_{-i|\lambda|}-J'_{-i|\lambda|}J_{-i|\lambda|})}{\sinh(\pi|\lambda|)}\,\hat{b}_\lambda\hat{b}_{-\lambda}\,d\lambda = 0\,. \quad \text{(B.40)}$$

The $\hat{b}^\dagger_\lambda \hat{b}^\dagger_{\lambda'}$ contribution equally vanishes leading to the following expression for the Point-form boost operator:

$$\hat{B} = \frac{1}{2}\int\limits_{\mathbb{R}}\lambda[\hat{b}^\dagger(\lambda)\hat{b}(\lambda)+\hat{b}(\lambda)\hat{b}^\dagger(\lambda)]d\lambda\,. \quad \text{(B.41)}$$

B.5 Wigner Representation of the Poincaré Generators

In order to calculate the Poincaré generators in the Wigner basis we make use of a trick pointed out by F. Coester. Since in the plane wave basis the integrals over space involve exponential functions with argument $ix^\mu p_\mu$, the idea is to boost the expression into a frame where $\tilde{p}_\mu = (0, \tilde{\mathbf{p}})$ with $\tilde{p} = \Lambda p$. Then the spatial integrations can eventually be carried out. In the whole section we deviate from our usual notation by denoting the space part of a vector also by a bold letter whenever there is no summation over its components.

To illustrate the main technique we show that plane waves are orthogonal on hyperboloids. We start by writing down the scalar product (3.7) over a hyperboloid in Minkowski coordinates:

$$(f,h) = 2i\int\limits_{\mathbb{R}^n}\delta(x^\mu x_\mu - \tau^2)\Theta(x^0)x^\rho(f^*\partial_\rho h - h\partial_\rho f^*)d^n x\,, \quad \text{(B.42)}$$

where $d^n x$ actually stands for $dx^0 dx^1 \cdots d^{n-1}x$. With the plane waves

$$f(x) = (2\pi)^{-(n-1)/2}e^{ix^\mu p_\mu} \qquad \text{and} \qquad h(x) = (2\pi)^{-(n-1)/2}e^{ix^\mu p'_\mu}$$

the scalar product reads

$$W(p,p') = (f,h) = -2(2\pi)^{-(n-1)}\int\limits_{\mathbb{R}^n}\delta(x^\mu x_\mu - \tau^2)\Theta(x^0)x^\rho(p+p')_\rho e^{-ix^\mu(p-p')_\mu}\,d^n x\,. \quad \text{(B.43)}$$

We now define the new variables $P = (p+p')$ and $Q = (p-p')$ which satisfy $P^\mu Q_\mu = 0$. In doing so, we get

$$W(p,p') = -2(2\pi)^{-(n-1)}\int\limits_{\mathbb{R}^n}\delta(x^\mu x_\mu - \tau^2)\Theta(x^0)x^\rho P_\rho e^{-ix^\mu Q_\mu}\,d^n x\,. \quad \text{(B.44)}$$

To be able to evaluate the integration, we perform a boost Λ which transforms Q such that it only has spatial components

$$\Lambda^\mu_\nu Q^\nu = \tilde{Q}^\mu = (0,\tilde{\mathbf{Q}})^\top \qquad \text{and} \qquad \Lambda^\mu_\nu P^\nu = \tilde{P}^\mu = (\tilde{P}^0, 0)^\top\,, \quad \text{(B.45)}$$

where the transformation for P follows from the fact that $P^\mu Q_\mu = 0$. Because (B.44) is Lorentz invariant, i.e. $W(\Lambda p, \Lambda p') = W(p,p')$ and with $\delta(x^\mu x_\mu - \tau^2)\Theta(x^0) = \delta(x^0 - \sqrt{\mathbf{x}^2+\tau^2})/2x^0$ the scalar product becomes

$$\begin{aligned} W(p,p') &= -(2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} \tilde{P}^0 e^{-i\tilde{x}^k \tilde{Q}_k} d\tilde{x}^1 \cdots d\tilde{x}^{n-1} \\ &= -\tilde{P}^0 \delta^{n-1}(\tilde{\mathbf{Q}}) \,. \end{aligned} \tag{B.46}$$

By writing down the explicit form of the boost Λ and using the relation $P^\mu Q_\mu = 0$, the boosted components $\tilde{P}^0$ and $\tilde{\mathbf{Q}}$ can be expressed through the original components. This finally gives

$$W(p,p') = -2p^0 \delta^{n-1}(\mathbf{p} - \mathbf{p}') \tag{B.47}$$

which is in accordance with the Instant-form scalar product (3.33). Of course, this is just a special case of the fact that, if solutions of the Klein-Gordon equation are orthogonal on one spacelike hypersurface, they are orthogonal on any spacelike hypersurface, as was shown in Section 3.2.

In the following derivation of the Wigner representation of the Poincaré generators for a 2-dimensional Minkowski space-time, we will need two different boosts. $\Lambda(\gamma_P)$, which boosts P such that is has only spatial coordinates, and $\Lambda(\gamma_Q)$ which boosts P into a frame where it has only a time coordinate. Table B.1 summarizes the required expressions.

	$P = p + p'$	$Q = p - p'$
$\Lambda(\gamma_P)$	$\tilde{P}^\mu = (0, \tilde{P}^1)^\top$	$\tilde{Q}^\mu = (\tilde{Q}^0, 0)^\top$
	$P^\mu = (-\sinh\gamma_P \tilde{P}^1, \cosh\gamma_P \tilde{P}^1)^\top$	$Q^\mu = (\cosh\gamma_P \tilde{Q}^0, -\sinh\gamma_P \tilde{Q}^0)^\top$
	$x^\mu = (\cosh\gamma_P \tilde{x}^0 - \sinh\gamma_P \tilde{x}^1, -\sinh\gamma_P \tilde{x}^0 + \cosh\gamma_P \tilde{x}^1)^\top$	
$\Lambda(\gamma_Q)$	$\tilde{P}^\mu = (\tilde{P}^0, 0)^\top$	$\tilde{Q}^\mu = (0, \tilde{Q}^1)^\top$
	$P^\mu = (\cosh\gamma_Q \tilde{P}^0, -\sinh\gamma_Q \tilde{P}^0)^\top$	$Q^\mu = (-\sinh\gamma_Q \tilde{Q}^1, \cosh\gamma_Q \tilde{Q}^1)^\top$
	$x^\mu = (\cosh\gamma_Q \tilde{x}^0 - \sinh\gamma_Q \tilde{x}^1, -\sinh\gamma_Q \tilde{x}^0 + \cosh\gamma_Q \tilde{x}^1)^\top$	

Table B.1: Relations between P, Q and their boosted values $\tilde{P}$, $\tilde{Q}$ for the two special boosts $\Lambda(\gamma_P)$ and $\Lambda(\gamma_Q)$ in 2-dimensional Minkowski space-time.

Another abbreviation in terms of notation must be mentioned. When carrying out boosts, we never explicitly write down the transformed versions of the creation and annihilation operators. Thus, after some boost Λ the creation operator $\hat{a}_{\mathbf{p}}$, for example, should strictly be written as $\hat{a}_{\Lambda^k_\nu p^\nu}$, or even as $\hat{a}_{\frac{1}{2}\Lambda^k_\nu (P+Q)^\nu}$. Because of this clumsy notation we just write $\hat{a}_p$ everywhere and the meaning should be clear from the context. In addition we denote in all following formulae the coordinates of a space-time point by x^μ, instead of X^μ.

B.5.1 Translation Generators

The Point-form translation generators $\hat{P}^\sigma$ in Minkowski coordinates X^μ are simultaneously obtained from (3.1) by replacing ξ^μ by $\eta^{\mu\sigma}$ and integrating over the hyperboloid Σ_α. The details have been presented in Section 4.2 and we will use the following generalization of (4.19):

$$\hat{P}^\mu = 2 \int\limits_{\mathbb{R}^n} \delta(x^\mu x_\mu - \tau^2)\Theta(x^0) x_\nu \hat{T}^{\mu\nu} d^n x \,. \tag{B.48}$$

With (3.15) in a Minkowski space-time $(\mathbb{R}^n, \underline{\eta})$,

$$\hat{T}^{\mu\nu} = \frac{1}{2}(\partial^\mu\hat{\phi}\,\partial^\nu\hat{\phi} + \partial^\nu\hat{\phi}\,\partial^\mu\hat{\phi}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\hat{\phi}\,\partial^\rho\hat{\phi} - m^2\hat{\phi}^2)\,, \tag{B.49}$$

the translation generator is

$$\hat{P}^\mu = \int\limits_{\mathbb{R}^n} \delta(x^\mu x_\mu - \tau^2)\Theta(x^0)\left[\partial^\mu\hat{\phi}\,x_\rho\partial^\rho\hat{\phi} + x_\rho\partial^\rho\hat{\phi}\,\partial^\mu\hat{\phi} - x^\mu(\partial_\rho\hat{\phi}\,\partial^\rho\hat{\phi} - m^2\hat{\phi}^2)\right] d^n x\,. \tag{B.50}$$

The expansion (3.34) of the field $\hat{\phi}$,

$$\hat{\phi}(x) = (2\pi)^{-(n-1)/2}\int\limits_{\mathbb{R}^{n-1}} \left(e^{-ix^\mu p_\mu}\hat{a}_{\mathbf{p}} + e^{ix^\mu p_\mu}\hat{a}^\dagger_{\mathbf{p}}\right)\frac{d\mathbf{p}}{2p^0} \tag{B.51}$$

is used to calculate the Fock space representation of the translation generators in the Wigner basis. We first investigate the $\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}$ contribution:

$$(2\pi)^{-(n-1)}\int\limits_{\mathbb{R}^n}\delta(x^\mu x_\mu - \tau^2)\Theta(x^0)\,d^n x\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}\frac{d\mathbf{p}\,d\mathbf{p}'}{4p^0p'^0}\times$$
$$\times\left[p^\mu x_\rho p'^\rho + p'^\mu x_\rho p^\rho - x^\mu(p_\rho p'^\rho - m^2)\right]e^{ix^\sigma(p-p')_\sigma}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}\,. \tag{B.52}$$

Now substituting p and p' for P and Q, the term in the square brackets becomes

$$[\ldots] = \frac{1}{4}\left[2x_\rho P^\rho P^\mu - 2x_\rho Q^\rho Q^\mu - x^\mu(P_\rho P^\rho - Q_\rho Q^\rho - 4m^2)\right]\,. \tag{B.53}$$

In order to evaluate the space integration in (B.52) we apply the boost $\Lambda(\gamma_Q)$. Note that the integral measures $\delta(x^\mu x_\mu - \tau^2)\Theta(x^0)d^n x$ and $\frac{d\mathbf{p}\,d\mathbf{p}'}{p^0p'^0}$ are Lorentz invariant and that

$$\frac{d\mathbf{p}\,d\mathbf{p}'}{4p^0p'^0} = 2^{-n}\frac{d\mathbf{P}\,d\mathbf{Q}}{(P^0)^2 - (Q^0)^2}\xrightarrow{\Lambda(\gamma_Q)} 2^{-n}\frac{d\tilde{\mathbf{P}}\,d\tilde{\mathbf{Q}}}{(\tilde{P}^0)^2 - (\tilde{Q}^0)^2}\,. \tag{B.54}$$

Since the expression (B.53) is Lorentz-covariant, the $\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}$ contribution of the boosted translation generators $[\Lambda(\gamma_Q)]^\mu_{\ \nu}\hat{P}^\nu$ is simply

$$\frac{2^{-n-2}}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^n}\delta(\tilde{x}^\mu\tilde{x}_\mu - \tau^2)\Theta(\tilde{x}^0)\,d^n\tilde{x}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}\frac{d\tilde{\mathbf{P}}\,d\tilde{\mathbf{Q}}}{(\tilde{P}^0)^2 - (\tilde{Q}^0)^2}\times$$
$$\times\left[2\tilde{x}_\rho\tilde{P}^\rho\tilde{P}^\mu - 2\tilde{x}_\rho\tilde{Q}^\rho\tilde{Q}^\mu - x^\mu(\tilde{P}_\rho\tilde{P}^\rho - \tilde{Q}_\rho\tilde{Q}^\rho - 4m^2)\right]e^{i\tilde{x}^\sigma\tilde{Q}_\sigma}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}\,. \tag{B.55}$$

Looking at the $\mu = 0$ component of the expression above and using the special form of $\tilde{P}^\mu$ and $\tilde{Q}^\mu$ leads to

$$2^{-n-3}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}\left[2(\tilde{P}^0)^2 - (\tilde{P}_\rho\tilde{P}^\rho - \tilde{Q}_\rho\tilde{Q}^\rho - 4m^2)\right]\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}\delta^{n-1}(\tilde{\mathbf{Q}})\frac{d\tilde{\mathbf{P}}\,d\tilde{\mathbf{Q}}}{(\tilde{P}^0)^2 - (\tilde{Q}^0)^2}\,. \tag{B.56}$$

The integral over $\tilde{\mathbf{Q}}$ of $(\tilde{P}_\rho\tilde{P}^\rho - \tilde{Q}_\rho\tilde{Q}^\rho - 4m^2)\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}\delta^{n-1}(\tilde{\mathbf{Q}})$ is easily seen to be zero when it is evaluated in the original frame. Therefore, the result for the $\mu = 0$ component in the boosted frame is

$$\Lambda(\gamma_Q)^0_\nu \hat{P}^\nu = 2^{-n-2} \int\limits_{(\mathbb{R}^{n-1})^2} (\tilde{P}^0)^2 \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}'} \delta^{n-1}(\tilde{\mathbf{Q}}) \frac{d\tilde{\mathbf{P}} d\tilde{\mathbf{Q}}}{(\tilde{P}^0)^2 - (\tilde{Q}^0)^2} + \text{other comb. of } \hat{a}_{\mathbf{p}} \text{ and } \hat{a}^\dagger_{\mathbf{p}}. \tag{B.57}$$

We will now look at the spatial components of (B.55) and show that they vanish. For $\mu = k$ we have

$$\frac{2^{-n-3}}{(2\pi)^{n-1}} \int\limits_{(\mathbb{R}^{n-1})^3} \Bigg[\underbrace{\frac{2\tilde{\mathbf{x}} \cdot \tilde{\mathbf{Q}}\, \tilde{Q}^k}{\sqrt{\tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} + \tau^2}}}_{A(\tilde{\mathbf{x}})} - \underbrace{\frac{\tilde{x}^k}{\sqrt{\tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} + \tau^2}} (\tilde{P}_\rho \tilde{P}^\rho - \tilde{Q}_\rho \tilde{Q}^\rho - 4m^2)}_{B(\tilde{\mathbf{x}})} \Bigg] e^{i\tilde{\mathbf{x}} \cdot \tilde{\mathbf{Q}}} d\tilde{\mathbf{x}}\, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}'} \frac{d\tilde{\mathbf{P}}\, d\tilde{\mathbf{Q}}}{(\tilde{P}^0)^2}\,.$$

The partial integration of $A(\tilde{\mathbf{x}}) e^{i\tilde{\mathbf{x}} \cdot \tilde{\mathbf{Q}}}$ over $\tilde{\mathbf{x}}$ gives $\delta(\tilde{\mathbf{Q}}) Q^k \times$ (terms with $\tilde{\mathbf{x}}$) which vanishes when carrying out the $\tilde{\mathbf{Q}}$ integration. Similarly, a partial integration of $B(\tilde{\mathbf{x}}) e^{i\tilde{\mathbf{x}} \cdot \tilde{\mathbf{Q}}}$ over $\tilde{\mathbf{x}}$ vanishes because the term $(\tilde{P}_\rho \tilde{P}^\rho - \tilde{Q}_\rho \tilde{Q}^\rho - 4m^2)\delta(\tilde{\mathbf{Q}})$ equals zero when carrying out the $\tilde{\mathbf{Q}}$ integration. Thus, the spatial components are zero and, together with (B.57), this result can be written down as

$$\Lambda(\gamma_Q)^\mu_\nu \hat{P}^\nu = 2^{-n-2} \int\limits_{(\mathbb{R}^{n-1})^2} \tilde{P}^\mu \tilde{P}^0 \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}'} \delta^{n-1}(\tilde{\mathbf{Q}}) \frac{d\tilde{\mathbf{P}} d\tilde{\mathbf{Q}}}{(\tilde{P}^0)^2 - (\tilde{Q}^0)^2} + \text{other comb. of } \hat{a}_{\mathbf{p}} \text{ and } \hat{a}^\dagger_{\mathbf{p}}.$$

Boosting back into the original frame with $\Lambda^{-1}(\gamma_Q)$ and switching back to the p and p' variables finally leads to the $\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}$ contribution of the representation of $\hat{P}^\mu$ in the Wigner basis:

$$\frac{1}{2} \int\limits_{\mathbb{R}^{n-1}} p^\mu \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \frac{d\mathbf{p}}{2p^0}\,. \tag{B.58}$$

In a very similar way, the $\hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{p}'}$ and $\hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}'}$ contributions are seen to vanish, which finally leads to the following expression for representation of the Point-form translation generators in the Wigner basis:

$$\hat{P}^\mu = \frac{1}{2} \int\limits_{\mathbb{R}^{n-1}} p^\mu \left[\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \hat{a}_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}}\right] \frac{d\mathbf{p}}{2p^0}\,. \tag{B.59}$$

B.5.2 Boost Generator

The Point-form boost generator $\hat{B}$ in Minkowski coordinates X^μ is obtained from (3.1) by setting $\xi^\mu = (X^1, X^0)^\top$ and integrating over the hyperboloid Σ_α. The details have been presented in Section 4.2 and we will directly use (4.21):

$$\hat{B}^k = 2 \int\limits_{\mathbb{R}^n} \delta(x^\mu x_\mu - \tau^2)\Theta(x^0) x_\rho [x^0 \hat{T}^{\rho k} - x^k \hat{T}^{\rho 0}] d^n x\,. \tag{B.60}$$

Adding an arbitrary interaction term $V(\hat{\phi})$ to (B.49),

$$\hat{T}^{\mu\nu} = \frac{1}{2}(\partial^\mu \hat{\phi}\, \partial^\nu \hat{\phi} + \partial^\nu \hat{\phi}\, \partial^\mu \hat{\phi}) - \frac{1}{2}\eta^{\mu\nu} \left[\partial_\rho \hat{\phi}\, \partial^\rho \hat{\phi} - m^2 \hat{\phi}^2 - V(\hat{\phi})\right] \tag{B.61}$$

gives

$$\hat{B}^k = \int\limits_{\mathbb{R}^n} \delta(x^\mu x_\mu - \tau^2)\Theta(x^0) \left[x^\rho \partial_\rho \hat{\phi}(x^0 \partial^k \hat{\phi} - x^k \partial^0 \hat{\phi}) + (x^0 \partial^k \hat{\phi} - x^k \partial^0 \hat{\phi}) x^\rho \partial_\rho \hat{\phi}\right] d^n x\,. \tag{B.62}$$

Here it can explicitly be seen that interaction terms do not enter in the expressions for the boost generators in Point-form. Again, we use the expansion (B.51) of the field $\hat{\phi}$ to calculate the Fock space representation of the boost generator in the Wigner basis. We will first investigate the $\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}'}$ contribution:

$$(2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^n} \delta(x^\mu x_\mu - \tau^2)\Theta(x^0)\times$$
$$\times \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \left[x^\rho p_\rho (x^k p'^0 - x^0 p'^k) + (x^k p^0 - x^0 p^k) x^\rho p'_\rho\right] e^{-ix^\sigma (p+p')_\sigma} \hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}'} \frac{d\mathbf{p}\, d\mathbf{p}'}{4p^0 p'^0} d^n x\,. \tag{B.63}$$

Substituting now P and Q for p and p', the term in the square brackets becomes

$$[\dots] = \frac{1}{2}\left[x^0(x^\rho Q_\rho Q^k - x^\rho P_\rho P^k) + x^k(x^\rho P_\rho P^0 - x^\rho Q_\rho Q^0)\right]\,. \tag{B.64}$$

To evaluate the integral (B.63) we perform the boost $\Lambda(\gamma_P)$. With (B.64) transforming under the 2-dimensional version of the Lorentz boost $\Lambda(\gamma_P)$ as

$$[\dots] \xrightarrow{\Lambda(\gamma_P)} \frac{1}{2}\tilde{x}^0\tilde{x}^1\left[(\tilde{P}^1)^2 - (\tilde{Q}^0)^2\right] \tag{B.65}$$

and with (B.54) we can express the boosted $\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}'}$ contribution (B.63) in a 2-dimensional Minkowski space-time, i.e. $n = 2$ and $k = 1$, in the following way

$$\frac{1}{16\pi}\int\limits_{\mathbb{R}^2} \delta(\tilde{x}^0 - \sqrt{(\tilde{x}^1)^2 + \tau^2}) \int\limits_{\mathbb{R}^2} \frac{(\tilde{Q}^0)^2 - (\tilde{P}^1)^2}{(\tilde{Q}^0)^2} \tilde{x}^1 e^{i\tilde{x}^1\tilde{P}^1} \hat{a}_{p^1}\hat{a}_{p'^1}\, d\tilde{P}^1 d\tilde{Q}^1 d\tilde{x}^0 d\tilde{x}^1 =$$
$$-\frac{i}{8}\int\limits_{\mathbb{R}^2} \frac{(\tilde{Q}^0)^2 - (\tilde{P}^1)^2}{(\tilde{Q}^0)^2} \hat{a}_{p^1}\hat{a}_{p'^1}\partial_{\tilde{P}^1}\delta(\tilde{P}^1)\, d\tilde{P}^1 d\tilde{Q}^1\,. \tag{B.66}$$

Integrating by parts and carrying out the $d\tilde{P}^1$ integration afterwards leads to

$$\frac{i}{8}\int\limits_{\mathbb{R}} \frac{\partial}{\partial\tilde{P}^1}\bigg|_{\tilde{P}^1=0} \left[\hat{a}_{p^1}\hat{a}_{p'^1}\right] d\tilde{Q}^1\,. \tag{B.67}$$

Boosting back into the original frame with $\Lambda^{-1}(\gamma_P)$ then gives

$$-\frac{i}{8}\int\limits_{\mathbb{R}} \frac{\partial}{\partial P^1}\bigg|_{P^1=0} \left[\hat{a}_{\frac{1}{2}(P^1+Q^1)}\hat{a}_{\frac{1}{2}(P^1-Q^1)}\right] dQ^1 = 0\,. \tag{B.68}$$

The same result can be obtained for the $\hat{a}^\dagger_{p^1}\hat{a}^\dagger_{p'^1}$ contribution.

For the $\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}$ contribution we start again from (B.62) and get

$$(2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^n} \delta(x^\mu x_\mu - \tau^2)\Theta(x^0)\times$$
$$\times \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \left[x^\rho p_\rho (x^0 p'^k - x^k p'^0) + (x^0 p^k - x^k p^0) x^\rho p'_\rho\right] e^{ix^\sigma (p-p')_\sigma} \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'} \frac{d\mathbf{p}\, d\mathbf{p}'}{4p^0 p'^0} d^n x\,. \tag{B.69}$$

Substituting again p and p' for P and Q the term in the square brackets becomes

$$[\dots] = \frac{1}{2}\left[x^0(x^\rho P_\rho P^k - x^\rho Q_\rho Q^k) + x^k(x^\rho Q_\rho Q^0 - x^\rho P_\rho P^0)\right] . \tag{B.70}$$

Now we perform the boost $\Lambda(\gamma_Q)$ which transforms (B.70) as

$$[\dots] \xrightarrow{\Lambda(\gamma_Q)} \frac{1}{2}\tilde{x}^0\tilde{x}^1\left[(\tilde{Q}^1)^2 - (\tilde{P}^0)^2\right] . \tag{B.71}$$

With (B.54) the $\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}'}$ contribution (B.69) for $n = 2$ and $k = 1$ is

$$\frac{1}{16\pi}\int\limits_{\mathbb{R}^2}\delta(\tilde{x}^0 - \sqrt{(\tilde{x}^1)^2+\tau^2})\int\limits_{\mathbb{R}^2}\frac{(\tilde{Q}^1)^2-(\tilde{P}^0)^2}{(\tilde{P}^0)^2}\tilde{x}^1 e^{-i\tilde{x}^1\tilde{Q}^1}\,\hat{a}^\dagger_{p^1}\hat{a}_{p'^1}\,d\tilde{P}^1 d\tilde{Q}^1 d\tilde{x}^0 d\tilde{x}^1 =$$
$$\frac{i}{8}\int\limits_{\mathbb{R}^2}\frac{(\tilde{Q}^1)^2-(\tilde{P}^0)^2}{(\tilde{P}^0)^2}\hat{a}^\dagger_{p^1}\hat{a}_{p'^1}\partial_{\tilde{Q}^1}\delta(\tilde{Q}^1)\,d\tilde{P}^1 d\tilde{Q}^1 . \tag{B.72}$$

Integrating by parts and carrying out the $d\tilde{Q}^1$ integration leads to

$$-\frac{i}{8}\int\limits_{\mathbb{R}}\left.\frac{\partial}{\partial\tilde{Q}^1}\right|_{\tilde{Q}^1=0}\left[\hat{a}^\dagger_{p^1}\hat{a}_{p'^1}\right]d\tilde{P}^1 . \tag{B.73}$$

If we boost back into the original frame with $\Lambda^{-1}(\gamma_Q)$, we get

$$-\frac{i}{8}\int\limits_{\mathbb{R}}\left.\frac{\partial}{\partial Q^1}\right|_{Q^1=0}\left[\hat{a}^\dagger_{\frac{1}{2}(P^1+Q^1)}\hat{a}_{\frac{1}{2}(P^1-Q^1)}\right]dP^1 = \frac{i}{8}\int\limits_{\mathbb{R}}\left[\hat{a}^\dagger_{p^1}(\partial_{p^1}\hat{a}_{p^1}) - (\partial_{p^1}\hat{a}^\dagger_{p^1})\hat{a}_{p^1}\right]dp^1 . \tag{B.74}$$

Together with the $\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}'}$ contribution, the representation of the Point-form boost generator in the Wigner basis finally is:

$$\hat{B} = \frac{i}{2}\int\limits_{\mathbb{R}}\left[(\partial_{p^1}\hat{a}^\dagger_{p^1})\hat{a}_{p^1} - \left(\partial_{p^1}\hat{a}_{p^1}\right)\hat{a}^\dagger_{p^1}\right]\frac{dp^1}{2} . \tag{B.75}$$

Bibliography

[1] P. A. M. Dirac. Forms of relativistic dynamics. *Reviews of Modern Physics*, 21(3):392–399, 1949.

[2] P. A. M. Dirac. *Lectures on Quantum Mechanics*. Number 2 in Belfer Graduate School of Science. Yeshiva University, New York, 1964.

[3] S. Fubini, A. J. Hanson, and R. Jackiw. New approach to field theory. *Physical Review D*, 7(6):1732–1760, 1973.

[4] A. di Sessa. Quantization of a field with mass in two-dimensional euclidean space. *Physical Review D*, 9(10), 1974.

[5] Sommerfield. Quantization on spacetime hyperboloids. *Annals of Physics*, 84:285–302, 1974.

[6] D. Gromes, H. J. Rothe, and B. Stech. Field quantization on the surface $x^2 =$ constant. *Nuclear Physics B*, 75:313–332, 1974.

[7] A. diSessa. Quantization on hyperboloids and full space-time field expansion. *Journal of Mathematical Physics*, 15(11):1892–1900, 1974.

[8] S. A. Fulling. Nonuniqueness of canonical field quantization in Riemannian space-time. *Physical Review D*, 7(10):2850–2862, 1973.

[9] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, 1973.

[10] R. M. Wald. *General Relativity*. The University of Chicago Press, 1984.

[11] K. V. Kuchař. Time and interpretations of quantum gravity. In G. Kunstatter, D. Vincent, and J. Williamas, editors, *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, pages 211–314. World Scientific, Singapore, 1992.

[12] C. J. Isham. Conceptual and geometrical problems in quantum gravity. In H. Mitter and H. Gausterer, editors, *Recent Aspects of Quantum Fields*, volume 396 of *Lecture Notes in Physics*, pages 123–229. Springer-Verlag, 1991.

[13] C. J. Isham. Canonical quantum gravity and the problem of time. In L. A. Ibort and M. A. Rodríguez, editors, *Integrable Systems, Quantum Groups, and Quantum Field Theories*, volume 409 of *Mathematical and Physical Sciences*, pages 157–287. Kluwer Academic Publishers, 1992.

[14] P. Hájiček. Quantization of systems with constraints. pages 1–41. WE-Heraeus Seminar. Presented at 117th WE-Heraeus Seminar: The Canonical Formalism in Classical and Quantum General Relativity, Bad Honnef, Germany, 13-17 Sep 1993.

[15] W. G. Unruh and R. M. Wald. Time and the interpretation of canonical quantum gravity. *Physical Review D*, 40(8):2598–2614, 1989.

[16] C. Rovelli. Time in quantum gravity: An hypothesis. *Physical Review D*, 43(2):442–456, 1991.

[17] C. Rovelli. Is there incompatibility between the ways time is treated in general relativity and in standard quantum mechanics? In A. Ashtekar and J. Stachel, editors, *Conceptual Problems of Quantum Gravity*, pages 126–140. Birkhäuser, Boston, 1991.

[18] K. Kuchař. Geometry of hyperspace. I. *Journal of Mathematical Physics*, 17(5):777–791, 1976.

[19] K. Kuchař. Kinematics of tensor fields in hyperspace. II. *Journal of Mathematical Physics*, 17(5):792–800, 1976.

[20] K. Kuchař. Dynamics of tensor fields in hyperspace. III. *Journal of Mathematical Physics*, 17(5):801–820, 1976.

[21] P. Hájíček. Group quantization of parametrized systems. I. Time levels. *Journal of Mathematical Physics*, 36:4612–4638, 1995.

[22] K. V. Kuchař. Canonical quantum gravity. In C. Kozameh, editor, *Proceedings of the 13th International Conference on General Relativity and Gravitation*. IOP Publishing, Bristol, 1993.

[23] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. Springer, 1997.

[24] S. Coleman and R. Jackiw. Why dilatation generators do not generate dilatations. *Annals of Physics*, 67:552–598, 1971.

[25] Y. Takahashi. Some covariant identities and the generators of the Poincaré group. *Proceedings of the Royal Irish Academy*, 71:1–11, 1971.

[26] Y. Takahashi. Note on the scale transformation. *Physical Review D*, 3(2):622–625, 1971.

[27] J. Katz. On broken conformal invariance. *Il Nuovo Cimento*, 3A(2):263–270, 1971.

[28] P. Hájíček and C. J. Isham. Perennials and the group-theoretical quantization of a parametrized scalar field on a curved background. *Journal of Mathematical Physics*, 37(7):3522–3538, 1996.

[29] G. W. Gibbons. Quantum field theory in curved spacetime. In S. W. Hawking and W. Israel, editors, *General Relativity, An Einstein Centenary Survey*. Cambridge Press, 1979.

[30] N. D. Birrell and P. C. W. Davies. *Quantum fields in curved space*. Cambridge University Press, 1984.

[31] S. A. Fulling. *Aspects of Quantum Field Theory in Curved Space-Time*. Cambridge University Press, 1989.

[32] Robert M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. The University of Chicago Press, 1994.

[33] R. P. Geroch. The domain of dependence. *Journal of Mathematical Physics*, 11:437–449, 1970.

[34] M. Castagnino and R. Ferraro. Observer-dependent quantum vacua in curved space. *Physical Review D*, 34(2):497–503, 1986.

[35] T. S. Bunch, S. M. Christensen, and S. A. Fulling. Massive quantum field theory in two-dimensional Roberston-Walker space-time. *Physical Review D*, 18(12):4435–4459, 1978.

[36] P. C. W. Davies and S. A. Fulling. Quantum vacuum energy in two dimenstional space-times. *Proceedings of the Royal Society of London*, A354:59–77, 1977.

[37] A. Erdélyi, editor. *Higher Transcendental Functions*, volume II of *Bateman Manuscript Project*. McGraw-Hill, 1953.

[38] S. A. Fulling. Remarks on positive frequency and hamiltonians in expanding universes. *General Relativity and Gravitation*, 10(10):807–824, 1979.

[39] S. A. Fulling, L. Parker, and B. L. Hu. Conformal energy-momentum tensor in curved spacetime: Adiabtic regularization and renormalization. *Physical Review D*, 10(12):3905–3924, 1974.

[40] W. Rindler. Kruskal space and the uniformly accelerated frame. *American Journal of Physics*, 34:1174–1178, 1966.

[41] W. W. MacDowell and R. Roskies. Reduction of the Poincaré group with respect to the Lorentz group. *Journal of Mathematical Physics*, 13(10):1585–1592, 1972.

[42] G. W. Gibbons. Vacuum polarization and the spontaneous loss of charge by black holes. *Commun. math. Phys.*, 44:245–264, 1975.

[43] C. G. Lambe. A generalization of Bessel's integrals. *Journal of the London Mathematical Society*, 6:257–259, 1931.

[44] A. Erdélyi, editor. *Higher Transcendental Functions*, volume I of *Bateman Manuscript Project*. McGraw-Hill, 1953.

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