Green's and Stokes' Theorem
Senior Capstone Project

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1 Introduction

This paper provides an in-depth exploration of Green's Theorem and Stokes' Theorem, two important theorems in vector calculus. These theorems are not mere abstract constructs, but are deeply rooted in the scientific context of their times, reflecting the synergy between theoretical mathematics and its application in understanding the physical world.

George Green's pioneering work, reaching its peak in Green's Theorem, was primarily driven by challenges in understanding electricity and magnetism. His theorem, a cornerstone in vector calculus, provided a novel way to relate line integrals and double integrals within a plane. Green's journey from a self-taught miller's son to a celebrated mathematician is exemplified in his 1828 essay, which laid the groundwork for what would later be known as Green's Theorem. This theorem offered a transformative approach to calculating complex integral expressions, especially useful in electromagnetic theory and potential theory, disciplines where Green made significant contributions [1].

Similarly, Sir George Gabriel Stokes, whose contributions spanned physics and mathematics, brought forth Stokes Theorem. This theorem elegantly bridges the concepts of surface integrals and line integrals in three-dimensional space. Stokes, renowned for his work in fluid dynamics and optics, developed this theorem as a mathematical tool to aid in the understanding of fluid flow and wave phenomena. Stokes's Theorem, in essence, provided a method to simplify the computation of certain surface integrals, which was instrumental in advancing studies in areas like hydrodynamics and the behavior of light waves [2].

First, in this paper, we will provide the mathematical preliminaries needed for the proofs. After that, we will prove Green's Theorem for simple rectangular regions and then for more complicated regions. This will set the stage for Stokes' Theorem, linking surface integrals to line integrals via Green's Theorem. The structure of the proofs is based on the work of Hughes-Hallett et al. [3]. Prior to the discussion of the proofs, we will provide practical examples to emphasize the application of these theorems. Finally, we will wrap up this paper with a conclusion section.

2 Preliminaries

Before stating and proving Green's Theorem and Stokes' Theorem, it is important to revisit some essential concepts in vector calculus. Let us first recall that if $\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$ are two vectors, then their dot (or scalar) product is defined as:

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1}$$

If $\vec{U}(t)$ and $\vec{V}(t)$ are vector functions differentiated with respect to t such that $\vec{U} : \mathbb{R} \to \mathbb{R}^3$ and $\vec{V} : \mathbb{R} \to \mathbb{R}^3$, a straightforward computation reveals:

$$\frac{d}{dt}(\vec{U} \cdot \vec{V}) = \frac{d\vec{U}}{dt} \cdot \vec{V} + \vec{U} \cdot \frac{d\vec{V}}{dt}$$

This formula will be used repeatedly throughout this paper.

We will also rely on a well-known result due to Clairaut's, which states that in a function $f(x,y)$ with continuous second partial derivatives, Clairaut's Theorem ensures the equality of the mixed partials derivatives:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{2}$$

We will frequently encounter situations where a change of variables is necessary for double integrals. This technique is used for transforming the domain of integration from one coordinate system to another, thereby simplifying the computation of the integrals. For a general change of coordinates, where the x, y coordinates are related to s, t coordinates by differentiable functions $x = x(s,t)$ and $y = y(s,t)$, double integrals in the new coordinate system can be expressed as follows:

$$\iint_R f(x,y)\,dx\,dy = \iint_T f(x(s,t), y(s,t)) \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds\,dt \tag{3}$$

where R is a region in the xy plane and T, in the st plane, is the image of R under the change of coordinates. Here, $\left|\frac{\partial(x,y)}{\partial(s,t)}\right|$ is the determinant of the Jacobian matrix[1] of the transformation.

Now we will look into the description of curves. We will need to be familiar with some definitions related to the curves in the plane. We say a function $f : [a,b] \to \mathbb{R}$ is **smooth** if it has derivatives, $f', f'', f^{(3)}, \ldots$ of all orders. Similarly, we say a function $f : [a,b] \to \mathbb{R}$ is **piecewise smooth** if we can split the domain into sub-domains, $[a,b] = [a_0,a_1] \cup [a_1,a_2] \cup \ldots \cup [a_{n-1},a_n]$ such that f is smooth over each sub-domain, where we take the appropriate left or right derivatives at endpoints $a_0, \ldots, a_n$. We say a curve $C \subset \mathbb{R}^2$ is piecewise smooth if it can be parameterized by $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j}$, where $f, g : [a,b] \to \mathbb{R}$ are piecewise smooth functions.

Moreover, we say a curve $C \subset \mathbb{R}^2$ is **simple** if it is parameterized by a 1-to-1 function $\vec{r} : [a,b] \to \mathbb{R}^2$ with the possible exception at endpoints, that is $\vec{r}(a) = \vec{r}(b)$. We say a curve $C \subset \mathbb{R}^2$ is **closed** if it has no endpoints and completely encloses some region $R \subset \mathbb{R}^2$. Lastly, We define a region $R \subset \mathbb{R}^2$ as **open** if for any point $P \in R$ there exists some circle with center P which is covered by R.

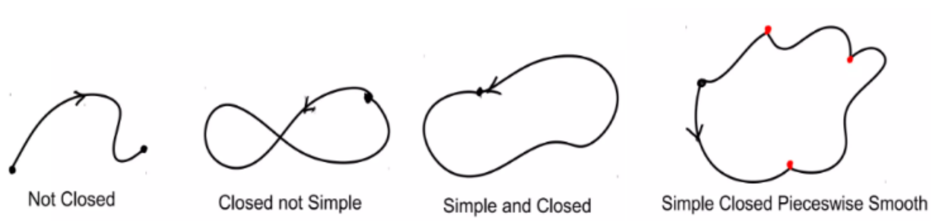

Figure 1: Examples of Curves - Illustrating the concepts of closure, simplicity, and smoothness in curve classification.

[1]For a comprehensive explanation and practice of this concept, refer to [3].

With these preliminaries established, we are now ready to state Green's Theorem.

3 Green's Theorem

Suppose C is a piecewise smooth simple closed curve that is the boundary of a region R in the plane and oriented so that the region is on the left as we move around the curve. Suppose $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$ be defined, where $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field on an open region containing R and C. Then **Green's Theorem** states,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

Before going deep into the proof, we will first illustrate Green's Theorem through some examples that demonstrate how it can simplify computations of line integrals.

Example 1

Consider the following line integral:

$$\oint_C (y^2 dx + x dy)$$

Here, C is the boundary of the rectangle R in the xy-plane with vertices at (0,0), (0,3), (2,0), and (2,3). We will evaluate this integral in two ways, once using Green's Theorem and once without it, to demonstrate how it can be used to speed up a calculation. Let's first evaluate this integral by parameterizing the curve C into 4 straight lines C_1, C_2, C_3, C_4, as shown below.

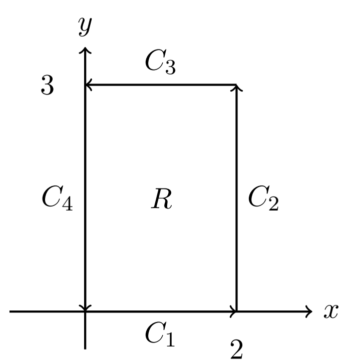

Figure 2: Parameterization of curve C into C_1, C_2, C_3, and C_4.

Note that we orient counterclockwise so that the region R is on the left as we move around the curve (orienting clockwise would introduce a change of sign). Let $\vec{F} = y^2\hat{\imath} + x\hat{\jmath}$. Then,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r} + \oint_{C_4} \vec{F} \cdot d\vec{r}$$

Evaluating each integral:

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \int_0^2 \begin{pmatrix} 0 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx = 0$$

$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \int_0^3 \begin{pmatrix} y^2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dy = 6$$

$$\oint_{C_3} \vec{F} \cdot d\vec{r} = \int_0^2 \begin{pmatrix} 3^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dx = -18$$

$$\oint_{C_4} \vec{F} \cdot d\vec{r} = \int_0^3 \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dy = 0$$

Summing these together gives:

$$\oint_C (y^2 dx + x dy) = 0 + 6 - 18 + 0 = -12$$

This calculation was notably tedious work. Let's now use Green's Theorem to convert the line integral into a double integral over the region R:

$$\oint_C (y^2 dx + x dy) = \int_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y^2) \right) dA = \int_0^3 \int_0^2 (1 - 2y) dx dy = -12$$

We found the same answer but the calculation was much more efficient. Building on this efficiency, let's now explore another practical application of Green's Theorem — calculating the area of planar regions. This approach converts the area calculation into a more manageable line integral, bypassing the need for more complex double integrals.

Example 2

Suppose we wish to find the area of the planar region R with boundary curve C oriented such that R is on the left as we move around C. Since the area of R can be expressed as $\int_R dA$:

$$\begin{aligned} \text{area of } R &= \int_R dA = \int_R \left(\frac{1}{2} + \frac{1}{2} \right) dA \\ &= \int_R \left(\frac{\partial}{\partial x}(\frac{x}{2}) - \frac{\partial}{\partial y}(-\frac{y}{2}) \right) dA \\ &= \frac{1}{2} \oint_C (-y dx + x dy) \end{aligned}$$

In the final line here, we have applied Green's Theorem to the vector field $-y\hat{\imath} + x\hat{\jmath}$ over the region R. We have now obtained a neat formula for the area of R which involves a line integral rather than a double integral. To put this theoretical understanding into practice, let's consider R as the unit circle. We may parameterize C by $\vec{r}(\theta) = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$. Then $dx = \frac{dx}{d\theta} d\theta = -\sin\theta \; d\theta$, and $dy = \frac{dy}{d\theta} d\theta = \cos\theta \; d\theta$. We find that the area of the unit circle is:

$$\frac{1}{2} \oint_C (-y dx + x dy) = \frac{1}{2} \int_0^{2\pi} (\sin^2\theta + \cos^2\theta) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

In this example, we have effectively used Green's Theorem to calculate the area of a unit circle, transforming a potentially complex double integral into a more manageable line integral. This demonstrates Green's Theorem as an effective tool for simplifying calculations in planar regions.

We will now move on to the proof of Green's Theorem. Our ultimate goal is to prove the result for any region that can be partitioned into sub-regions with smooth rectangular parameterizations. We will work towards that result by proving theorems that address increasingly complex regions. Lemma 3.0.1 will demonstrate Green's Theorem for a rectangular region. Theorem 1 will tackle regions resulting from a smooth rectangular parameterization. Finally, Theorem 2 will establish Green's Theorem for regions that can be partitioned into sub-regions with smooth rectangular parameterizations.

Lemma 3.0.1. Green's Theorem over Rectangular Regions

Let $R = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], y \in [c, d]\}$ be a rectangular region with oriented boundary curve C, and let $\vec{F}(x, y) = F_1(x, y)\hat{\imath} + F_2(x, y)\hat{\jmath}$ be a smooth vector field over R. Then Green's Theorem holds over R. That is,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

Proof. Consider a smooth vector field $\vec{F}(x, y) = F_1(x, y)\hat{\imath} + F_2(x, y)\hat{\jmath}$ over a rectangular region R with boundary consisting of four oriented lines C_1, C_2, C_3, and C_4 as illustrated below: The line integral over C can be decomposed into integrals over each side:

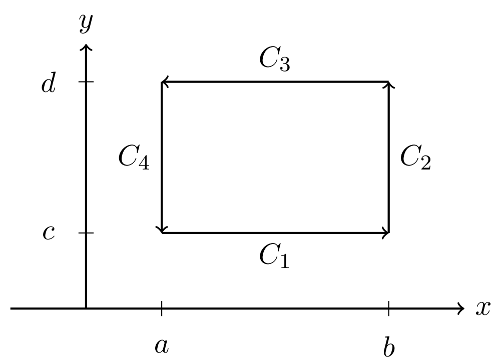

Figure 3: A rectangular region R with boundary C broken into C_1, C_2, C_3, and C_4.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r} + \oint_{C_4} \vec{F} \cdot d\vec{r}$$

Parameterizing each side C_k, we obtain:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_a^b F_1(x, c)\, dx + \int_c^d F_2(b, y)\, dy - \int_a^b F_1(x, d)\, dx - \int_c^d F_2(a, y)\, dy$$

The double integral over R can be expressed as an iterated integral, and applying the Fundamental Theorem of Calculus to the inner integral, yields

$$\begin{aligned}\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA &= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy + \int_a^b \int_c^d -\frac{\partial F_1}{\partial y}\, dy\, dx \\ &= \int_c^d F_2(x,y)\Big|_a^b\, dy + \int_a^b -F_1(x,y)\Big|_c^d\, dx \\ &= \int_c^d (F_2(b,y) - F_2(a,y))\, dy + \int_a^b (-F_1(x,d) + F_1(x,c))\, dx\end{aligned}$$

Since the line and double integrals are equal, Green's Theorem holds over rectangular regions. Now, I will use this proof as a basis to prove Green's Theorem for a transformed rectangle.

Theorem 1. Green's Theorem for Transformed Rectangle

Suppose we have a rectangular region $T = \{(s,t) \in \mathbb{R}^2 : s \in [a,b], t \in [c,d]\}$ in the st-plane and a smooth change of variables from the st-plane to the xy-plane, $x = x(s,t), y = y(s,t)$. Let R in the xy-plane be the image of T under this change of variables. Then $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$

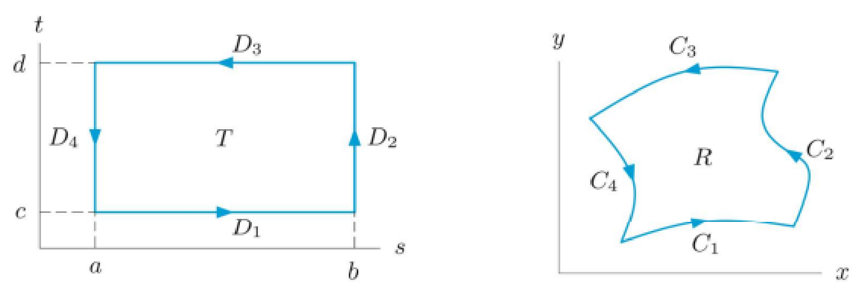

Figure 4: A curved region R in the xy-plane corresponding to a rectangular region T in the st-plane(adapted from [3]).

Proof. Suppose $\vec{F}(x,y) = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$ is a smooth vector field over R. Let $\vec{u} = s(p)\hat{i} + t(p)\hat{j}$, $p_0 \le p \le p_1$ be a parameterization of D, and let $\vec{r} = x(s(p),t(p))\hat{i} + y(s(p),t(p))\hat{j}, p_0 \le p \le p_1$ be a parameterization of C.

Let $\vec{G} = G_1\hat{i} + G_2\hat{j} = (\vec{F} \cdot \frac{\partial \vec{r}}{\partial s})\hat{i} + (\vec{F} \cdot \frac{\partial \vec{r}}{\partial t})\hat{j}$.

Moving forward, our approach involves the usefulness of the following relationship, as we see later in this proof. To establish this key relationship, I have used my original calculations to show the proof of the expression so that everything is clear.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_D \vec{G} \cdot d\vec{u} \tag{4}$$

where C and D are the boundary curves of R and T respectively.

We may express the integral over C like so,

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{p_0}^{p_1} \vec{F}(p) \cdot \left(\frac{dx}{dp}\hat{\imath} + \frac{dy}{dp}\hat{\jmath} \right) dp$$

where $\vec{F}(p)$ is shorthand notation for $\vec{F}\left(x(s(p), t(p)), y(s(p), t(p))\right)$. Applying the multivariate chain rule, the integrand becomes:

$$F_1(p) \left(\frac{\partial x}{\partial s}\frac{ds}{dp} + \frac{\partial x}{\partial t}\frac{dt}{dp} \right) + F_2(p) \left(\frac{\partial y}{\partial s}\frac{ds}{dp} + \frac{\partial y}{\partial t}\frac{dt}{dp} \right)$$

We may express the integral over D so

$$\oint_D \vec{G} \cdot d\vec{u} = \int_{p_0}^{p_1} \left((\vec{F} \cdot \frac{\partial \vec{r}}{\partial s})\hat{\imath} + (\vec{F} \cdot \frac{\partial \vec{r}}{\partial t})\hat{\jmath} \right) \cdot \left(\frac{ds}{dp}\hat{\imath} + \frac{dt}{dp}\hat{\jmath} \right) dp$$

Using formula(1), the integrand becomes,

$$\left(F_1(p)\frac{\partial x}{\partial s} + F_2(p)\frac{\partial y}{\partial s} \right) \frac{ds}{dp} + \left(F_1(p)\frac{\partial x}{\partial t} + F_2(p)\frac{\partial y}{\partial t} \right) \frac{dt}{dp}$$

$$= F_1(p) \left(\frac{\partial x}{\partial s}\frac{ds}{dp} + \frac{\partial x}{\partial t}\frac{dt}{dp} \right) + F_2(p) \left(\frac{\partial y}{\partial s}\frac{ds}{dp} + \frac{\partial y}{\partial t}\frac{dt}{dp} \right)$$

The two integrands are equal, hence establishing the relationship shown in Equation (4).

Next, we will establish this important relationship:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_T \left(\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t} \right) dsdt \tag{5}$$

This equation follows from the change of coordinates formula for double integrals. Going ahead to prove this expression, the integrand in the integral over T can be written as,

$$\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t} = \frac{\partial}{\partial s}(\vec{F} \cdot \frac{\partial \vec{r}}{\partial t}) - \frac{\partial}{\partial t}(\vec{F} \cdot \frac{\partial \vec{r}}{\partial s})$$

$$= \left(\frac{\partial \vec{F}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial t} + \vec{F} \cdot \frac{\partial^2 \vec{r}}{\partial s \partial t} \right) - \left(\frac{\partial \vec{F}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial s} + \vec{F} \cdot \frac{\partial^2 \vec{r}}{\partial t \partial s} \right)$$

Mixed partials cancel by Clairaut's Theorem (see equation (2)). Applying the chain rule:

$$\left(\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t} \right) = \left(\frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial s} \right) \cdot \frac{\partial \vec{r}}{\partial t} - \left(\frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial t} \right) \cdot \frac{\partial \vec{r}}{\partial s}$$

$$
\begin{aligned}
&= \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F_2}{\partial y}\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial t} \\
&- \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial s} - \left(\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F_2}{\partial y}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial s} \\
&= \frac{\partial F_1}{\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} + \frac{\partial F_2}{\partial x}\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial F_1}{\partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial F_2}{\partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right) \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}
\end{aligned}
$$

Using the change of variables for a double integral as mentioned in equation(3), we conclude the relationship shown in equation(5).

Now, using lemma 3.0.1 on equations (4) and (5), we can write, establishing the relationship as stated in equation (5):

$$\int_D \vec{G} \cdot d\vec{u} = \iint_T \left(\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t}\right) dsdt$$

Finally, we can use this result to get,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy,$$

which concludes the proof of Green's Theorem over R. Using this proof as a foundation, we will now establish Green's Theorem for any region that can be partitioned into sub-regions with smooth rectangular parameterizations.

Theorem 2. Pasting regions together

Suppose we have a region $R \subset \mathbb{R}^2$ which can be expressed as a union $R = \bigcup_{i=1}^{n} R_i$, such that $R_i \cap R_j$ has zero areas for all $1 \leq i < j \leq n$ and each R_i can be transformed into a rectangle as described in Lemma 2.0.2. Then $\int_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$

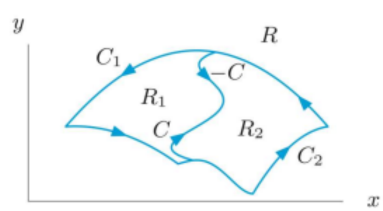

Figure 5: Two regions R_1 and R_2 pasted together to form a region R(adapted from [3]).

Proof. We consider a region R as the union of sub-regions R_1 and R_2. To better understand it, we dissect the boundary of R into two parts: C_1, which overlaps with R_1, and C_2, which overlaps with R_2. Additionally, C represents the boundary segment of R_1 that it has in common with R_2. So, the boundary of R is the sum of C_1 and C_2. The boundary of R_1 is the sum of C_1 and C, while the boundary of R_2 is the sum of C_2 and $-C$, which represents C traversed in the opposite direction. It's imperative to understand that when the curve C is recognized as a segment of the boundary for R_2, its orientation reverses compared to when it's part of the boundary for R_1. Consequently,

$$\begin{aligned}\int_{\text{Boundary of } R_1} \vec{F}\cdot d\vec{r} + \int_{\text{Boundary of } R_2} \vec{F}\cdot d\vec{r} &= \int_{C_1+C} \vec{F}\cdot d\vec{r} + \int_{C_2+(-C)} \vec{F}\cdot d\vec{r} \\ &= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C} \vec{F}\cdot d\vec{r} \\ &= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} \\ &= \int_{\text{Boundary of } R} \vec{F}\cdot d\vec{r}.\end{aligned}$$

So, applying Theorem 2, we get

$$\begin{aligned}\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy &= \iint_{R_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy + \iint_{R_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy \\ &= \int_{\text{Boundary of } R_1} \vec{F}\cdot d\vec{r} + \int_{\text{Boundary of } R_2} \vec{F}\cdot d\vec{r} \\ &= \int_{\text{Boundary of } R} \vec{F}\cdot d\vec{r}\end{aligned}$$

This final result establishes Green's Theorem for the region R.

We've now shown that Green's Theorem is valid for any region that emerges by combining regions with smooth rectangular parameterizations. This conclusion not only solidifies our understanding of Green's Theorem but also sets the stage for our next exploration: Stokes' Theorem.

4 Stokes' Theorem

Similar to our approach with Green's Theorem, we now shift our focus to the formal statement and proof of Stokes' Theorem. We will delve into how Stokes' Theorem can be perceived as a logical extension of the concepts and principles we have just established in Green's Theorem.

Theorem 3. Stokes' Theorem

We will first state the Stokes' Theorem.

Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3, F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ be a continuously differentiable vector field defined on a smooth oriented surface S with boundary curve C. Then Stokes' theorem states:

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \oint_C \vec{F}\cdot d\vec{r},$$

where $\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$ is the curl of $\vec{F}$, $\vec{n}$ is the unit normal vector to S, $\vec{r}$ is the position vector of a point on C, and C is oriented such that $d\vec{r} \times \vec{n}$ points away from the surface.

To prove Stokes' Theorem, we will first provide an example to build intuition by illustrating how the theorem converts surface integrals into simpler line integrals, highlighting its utility.

Example

Let S be the surface of a topless cube with side length 1 as shown in Figure 5. Let C be the boundary curve of the cube's surface at $z = 1$.

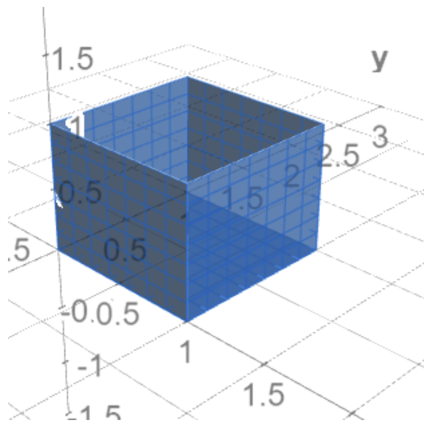

Figure 6: S, a topless cube.

To calculate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ directly, we would need to parametrize C, splitting it up into four oriented lines, and perform a calculation similar to the Green's theorem example 2, which would be tedious. However, we can utilize Stokes' Theorem here.

Note that $\nabla \times \vec{F}$, the curl of $\vec{F}$, is equal to 0. This simplifies calculating the left-hand side of Stokes' Theorem immensely:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int_S (0)\, dS = 0$$

Much like Green's Theorem, Stokes' Theorem significantly simplifies our calculations. To prove it, we'll employ a complex yet fundamental lemma.

Lemma 4.0.1. Suppose $S \subset \mathbb{R}^3$ is a smooth surface where the position vectors of the points on the surface are parameterized $\vec{r} = \vec{r}(s,t)$. Suppose $\vec{F} : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable vector field. Let $\vec{G}_1 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial s}$ and $\vec{G}_2 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial t}$. Then, the following equation holds:

$$(\nabla \times \vec{F}) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) = \frac{\partial \vec{G}_2}{\partial s} - \frac{\partial \vec{G}_1}{\partial t} \tag{6}$$

Proof. For the right hand side:

$$\begin{aligned}
&= \frac{\partial}{\partial s}\left(\vec{F}\cdot\vec{r}_t\right) - \frac{\partial}{\partial t}\left(\vec{F}\cdot\vec{r}_s\right)\\
&= \left(\vec{F}_s\cdot\vec{r}_t + \vec{F}\cdot\vec{r}_{st}\right) - \left(\vec{F}_t\cdot\vec{r}_s + \vec{F}\cdot\vec{r}_{st}\right)\\
&= \vec{F}_s\cdot\vec{r}_t - \vec{F}_t\cdot\vec{r}_s.
\end{aligned}$$

Next, we consider the left-hand side of equation(6)

$$(\nabla\times\vec{F})\cdot\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right) = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\end{pmatrix}\cdot\begin{pmatrix} y_s z_t - z_s y_t\\ z_s x_t - x_s z_t\\ x_s y_t - y_s x_t\end{pmatrix}$$

$$\begin{aligned}
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)(y_s z_t - z_s y_t) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)(z_s x_t - x_s z_t)\\
&+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)(x_s y_t - y_s x_t).
\end{aligned}$$

Distributing within each summand, we get:

$$\begin{aligned}
(\nabla\times\vec{F})\cdot\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right) &= \frac{\partial F_3}{\partial y}y_s z_t - \frac{\partial F_3}{\partial y}z_s y_t - \frac{\partial F_2}{\partial z}y_s z_t + \frac{\partial F_2}{\partial z}z_s y_t\\
&+\frac{\partial F_1}{\partial z}z_s x_t - \frac{\partial F_1}{\partial z}x_s z_t - \frac{\partial F_3}{\partial x}z_s x_t + \frac{\partial F_3}{\partial x}x_s z_t\\
&+\frac{\partial F_2}{\partial x}x_s y_t - \frac{\partial F_2}{\partial x}y_s x_t - \frac{\partial F_1}{\partial y}x_s y_t + \frac{\partial F_1}{\partial y}y_s x_t\\
(\nabla\times\vec{F})\cdot\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right) &= \left(\frac{\partial F_1}{\partial x}x_s\right)x_t + \left(\frac{\partial F_1}{\partial y}y_s\right)x_t + \left(\frac{\partial F_1}{\partial z}z_s\right)x_t\\
&+\left(\frac{\partial F_2}{\partial x}x_s\right)y_t + \left(\frac{\partial F_2}{\partial y}y_s\right)y_t + \left(\frac{\partial F_2}{\partial z}z_s\right)y_t\\
&+\left(\frac{\partial F_3}{\partial x}x_s\right)z_t + \left(\frac{\partial F_3}{\partial y}y_s\right)z_t + \left(\frac{\partial F_3}{\partial z}z_s\right)z_t\\
&-\left(\frac{\partial F_1}{\partial x}x_t\right)x_s - \left(\frac{\partial F_1}{\partial y}y_t\right)x_s - \left(\frac{\partial F_1}{\partial z}z_t\right)x_s\\
&-\left(\frac{\partial F_2}{\partial x}x_t\right)y_s - \left(\frac{\partial F_2}{\partial y}y_t\right)y_s - \left(\frac{\partial F_2}{\partial z}z_t\right)y_s\\
&-\left(\frac{\partial F_3}{\partial x}x_t\right)z_s - \left(\frac{\partial F_3}{\partial y}y_t\right)z_s - \left(\frac{\partial F_3}{\partial z}z_t\right)z_s
\end{aligned}$$

$$= \begin{pmatrix} \frac{\partial \vec{F}_1}{\partial x} x_s + \frac{\partial \vec{F}_1}{\partial y} y_s + \frac{\partial \vec{F}_1}{\partial z} z_s \\ \frac{\partial \vec{F}_2}{\partial x} x_s + \frac{\partial \vec{F}_2}{\partial y} y_s + \frac{\partial \vec{F}_2}{\partial z} z_s \\ \frac{\partial \vec{F}_3}{\partial x} x_s + \frac{\partial \vec{F}_3}{\partial y} y_s + \frac{\partial \vec{F}_3}{\partial z} z_s \end{pmatrix} \cdot \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} - \begin{pmatrix} \frac{\partial \vec{F}_1}{\partial x} x_t + \frac{\partial \vec{F}_1}{\partial y} y_t + \frac{\partial \vec{F}_1}{\partial z} z_t \\ \frac{\partial \vec{F}_2}{\partial x} x_t + \frac{\partial \vec{F}_2}{\partial y} y_t + \frac{\partial \vec{F}_2}{\partial z} z_t \\ \frac{\partial \vec{F}_3}{\partial x} x_t + \frac{\partial \vec{F}_3}{\partial y} y_t + \frac{\partial \vec{F}_3}{\partial z} z_t \end{pmatrix} \cdot \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}$$

$$= \vec{F}_s \cdot \vec{r}_t - \vec{F}_t \cdot \vec{r}_s.$$

Thus, the equality holds, which proves our claim. This confirmation serves as the foundational step towards the subsequent proof of Stokes' Theorem.

Proof. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3, F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ be a continuously differentiable vector field defined on a smooth oriented surface A with boundary curve B. We wish to show that:

$$\int_A (\nabla \times \vec{F}) \cdot d\vec{A} = \int_B \vec{F} \cdot d\vec{r} \tag{7}$$

where $\vec{r}$ is the position vector of a point on B, and B is oriented such that $d\vec{r} \times \vec{n}$ points away from the surface.

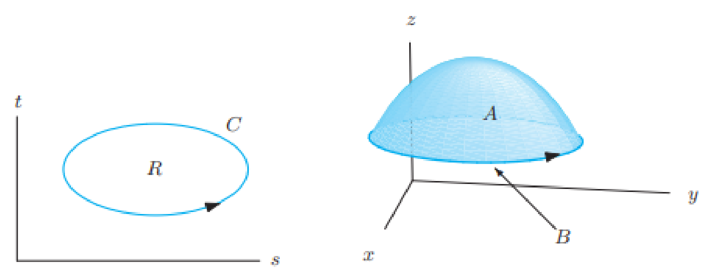

Figure 7: A region R in the st-plane and the corresponding surface A in xyz-space; the curve C corresponds to the boundary B of A(adapted from [3]).

Assume A is smoothly parameterized by $\vec{r} = \vec{r}(s, t)$, corresponding to a region R in the st-plane, that is $A = \{\vec{r}(s, t) : (s, t) \in R\}$. Similarly, B corresponds to the boundary C of R. The goal is to validate Stokes' Theorem for this surface A and the vector field $\vec{F}$ by transcribing the integrals of the theorem in terms of s and t and then applying Green's Theorem in the st-plane.

First, we convert the line integral

$$\int_B \vec{F} \cdot d\vec{r}$$

into a line integral around C. Next, we parameterize $B = \{\vec{r}(p) : p_0 < p < p_1\}$. Then, noting that $\langle s(p), t(p) \rangle$ parameterizes C,

$$\int_B \vec{F} \cdot d\vec{r} = \int_B \left(\vec{F}(\vec{r}(p)) \cdot \frac{d\vec{r}}{dp} \right) dp = \int_B \left(\vec{F}(\vec{r}(p)) \cdot (\frac{\partial \vec{r}}{\partial s} \frac{ds}{dp} + \frac{\partial \vec{r}}{\partial t} \frac{dt}{dp}) \right) dp$$
$$= \int_C \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} ds + \vec{F} \cdot \frac{\partial \vec{r}}{\partial t} dt$$

Here we have used the multivariate chain rule and the distributive property of the dot product which states: $\vec{F} \cdot (\vec{u} + \vec{v}) = \vec{F} \cdot \vec{u} + \vec{F} \cdot \vec{v}$.

So if we define a 2-dimensional vector field $\vec{G} = \langle G_1, G_2 \rangle$ on the st-plane by

$$G_1 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} \quad \text{and} \quad G_2 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial t},$$

then,

$$\int_B \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{u},$$

where u denotes the position vector of a point in the st-plane.

We now look at the surface integral on the left-hand side of Stokes' Theorem in terms of the parameterization $\vec{r} = \vec{r}(s, t)$:

$$\int_A \text{curl } \vec{F} \cdot d\vec{A} = \iint_R \text{curl } \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) dsdt$$

In lemma 4.0.1, we showed that:

$$\nabla \times \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) = \frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t}$$

Hence,

$$\int_A \nabla \times \vec{F} \cdot d\vec{A} = \iint_R \left(\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t} \right) dsdt$$

We have already seen this in equation(4) with its proof:

$$\int_B \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{u}$$

By Green's Theorem, the right-hand sides of the last two equations are equal. Consequently, the equality of the left-hand sides follows. The result is the equation (7), thereby completing our proof of Stokes' Theorem.

5 Conclusion

The evolution of Green's Theorem and Stokes's Theorem from practical tools in physics to foundational elements in mathematical theory is both striking and illuminating. It is worth recognizing the inspiring story of George Green. Rising from a miller's son with limited formal education to a celebrated scholar at Cambridge, Green's journey epitomizes the sheer power of determination and intellectual curiosity in scientific exploration. His work, initially concentrated on electricity and magnetism, had an unforeseen and profound impact on mathematics, a fact that was only fully recognized after Lord Kelvin rediscovered and popularized his work in 1846, five years posthumously [5]. Stories like this remind us of the often unexpected ways practical ideas can evolve into profound theoretical frameworks, significantly enriching our collective understanding of the world. In the grand tapestry of mathematical development, the stories of Green and Stokes remind us that great ideas often begin with addressing practical needs, only to unfold into fundamental theories that profoundly shape our understanding of the world.

We have proved Green's Theorem, a powerful integral theorem relating line integrals and double integrals in the plane, and over progressively more generalized regions in the plane. We have moved on to use Green's Theorem to prove Stokes' Theorem, which relates a surface integral to an integral over the boundary curve in three dimensions.

Assuming Stokes' Theorem holds, we may give an alternative derivation of Green's Theorem. We consider a two-dimensional region R in the plane as a surface embedded into three-dimensional space. We assume the region resides in the xy- plane and the boundary is the simple closed curve C. For a differentiable vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $F(\vec{r}) = \langle P(\vec{r}), Q(\vec{r}), 0\rangle$, where P and Q are scalar functions and $\vec{r} = \langle x, y, z\rangle$ is the position vector in $\mathbb{R}^3$, we apply Stokes' theorem to F over R to get the following equation:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_R \nabla \times \vec{F} \cdot dS$$

where C has positive orientation and dS is oriented as described in Stokes' Theorem.

Let D be the surface R embedded in $\mathbb{R}^2$. Using the natural parameterization for R, $\vec{r}(x, y) = \langle x, y, 0\rangle, \langle x, y\rangle \in D$, we find the the surface element $dS = (\hat{i} \times \hat{j})dxdy = \hat{k}dA$. Note that the orientation of dS is consistent with the direction of $\hat{k}$. We note that the curl of $\vec{F}$ has only one component, which is the z-component. In particular $\nabla \times \vec{F} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\rangle$. So in applying Stokes' Theorem, we precisely obtain Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

This demonstrates that Stokes' Theorem is a powerful generalization of Green's Theorem. Rather remarkably, Stokes' Theorem in three-dimensional space can be generalized further to an arbitrary number of dimensions[4]. The Generalised Stokes' Theorem reads,

$$\int_M d\omega = \int_{\partial M} \omega$$

where M is a compact oriented n-dimensional manifold (the n-dimensional equivalent of a surface) with boundary ∂M and ω is a differential form of degree $n-1$. The details of this theorem are far beyond the scope of this paper. However, we may note that when $n = 1$ the theorem reads,

$$\int_a^b f'(x)dx = f(b) - f(a),$$

when $M = [a, b]$ and $\partial M = \{a, b\}$ is oriented to sum positively $f(b)$ and negatively $f(a)$. This is precisely the familiar Fundamental Theorem of Calculus.

Another special case of the Generalised Stokes' Theorem is the Divergence Theorem. It relates the divergence of a vector field $\vec{F}$, defined $\nabla \cdot \vec{F} = \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{F}}{\partial z}$, in a three-dimensional region, to a surface integral over the boundary of the region.

Theorem 4. Divergence Theorem

Let $R \subset \mathbb{R}^3$ be a three-dimensional region with a piecewise smooth boundary ∂R, and let $\vec{F}$ be a differential vector field on R. Then,

$$\int_R \nabla \cdot \vec{F} dV = \int_{\partial R} \vec{F} \cdot dS$$

where dS is oriented in the direction of the outward pointing normal from R.

We have touched upon its essence. For a thorough and detailed examination of its proofs and multifaceted applications, the reader is directed to the seminal work of Hughes-Hallett et al. [3]. Delving into this comprehensive text not only enriches understanding but also amplifies appreciation for the practical utility and mathematical elegance of integral theorems within the extensive realm of mathematics.

6 Acknowledgement

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References

[1] G. Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Nottingham: T. Wheelhouse, 1828.

[2] G. G. Stokes, *On the Composition and Resolution of Streams of Polarized Light from Different Sources*. Transactions of the Cambridge Philosophical Society, vol. 9, pp. 399-416, 1851.

[3] D. Hughes-Hallett, A. M. Gleason, and W. G. McCallum, *Calculus: Single and Multivariable, 6th Edition*. Wiley, 2012.

[4] S. Waters, *Multivariable Calculus*. Hilary Term 2023. University of Oxford Lecture Notes. [Online]. Available: `https://courses.maths.ox.ac.uk/course/view.php?id=613`.

[5] University of Nottingham, *George Green: Mathematician & Physicist 1793 - 1841 - The Miller of Nottingham*. [Online]. Available: `https://www.nottingham.ac.uk/physics/about/history/george-green.aspx`.