

Ch 3: Lagrange and Barycentric Interpolation

Friday, October 10, 2025 12:18 PM

One practical method for polynomial interpolation (and can be improved via Barycentric Lagrange interpolation)

Setup

Given $n+1$ distinct nodes $\{x_0, x_1, \dots, x_n\}$ with values $\{y_0, y_1, \dots, y_n\}$,

we want to find the unique n -degree polynomial that interpolates these values. \leftarrow we proved previously

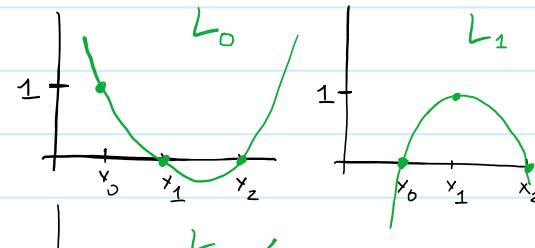
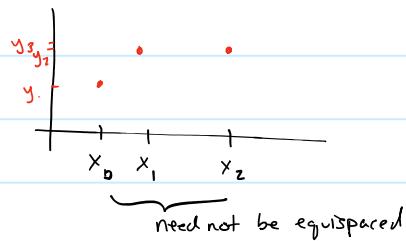
(Note: we could find an interpolating polynomial of degree $>n$, but then (1) it's not unique, (2) more likely to overfit)

GOAL: Avoid using the Vandermonde matrix

One solution: Lagrange polynomials

it's not "the" Lagrange polynomials, since they will change based on the nodes (\neq Lagrange, Laguerre, Legendre, L'Hopital)

Idea: (for illustration, let $n=2$, so looking for a quadratic polynomial to interpolate 3 points)



depends on $\{x_i\}$
but not on $\{y_i\}$

Now, putting it all together is easy:

$$p(x) = y_0 \cdot L_0(x) + y_1 \cdot L_1(x)$$

$$+ y_2 \cdot L_2(x)$$

$$\text{so } p(x_0) = \underbrace{y_0 \cdot L_0(x_0)}_{=1} + \underbrace{y_1 \cdot L_1(x_0)}_{=0} + \underbrace{y_2 \cdot L_2(x_0)}_{=0} = y_0 \quad \checkmark$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

Quadratic

$$L_0(x_0) = 1$$

$$L_0(x_1) = 0$$

$$L_0(x_2) = 0$$

Similarly for



and similarly for x_1, x_2

General Form: (Sometimes we write $L_{n,k}$ to make it clear the dependence on n)

$$\textcircled{*} \quad L_{n,k}(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

Lagrange Polynomial

$$= \prod_{\substack{i=0 \\ i \neq k}}^n \frac{(x - x_i)}{(x_k - x_i)} \quad (\text{so } L_{n,i}(x_k) = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases})$$

and the interpolating polynomial is

$$P(x) = \sum_{k=0}^n y_k \cdot L_{n,k}(x)$$

Lagrange form of the
interpolating polynomial

Comments:

⊕ The slowest part of the algorithm depends only on the nodes $\{x_i\}$, not the function values $\{y_i\}$. Sometimes we need to repeatedly interpolate different sets of $\{y_i\}$ but with the same nodes, so we can amortize the slow part.

- ⊖ Evaluating $p(x)$ takes $O(n^2)$ flops) we often want to evaluate at many points
- ⊖ Adding a new data point (x_{n+1}, y_{n+1}) requires you to start from scratch
- ⊖ Computation can be unstable

ex: we keep adding points until the error in approximation is small

Fix is the

Barycentric version of the Lagrange formula (cf. Berrut and Trefethen, SIAM Review 2004)

p. 3: Barycentric formulation

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Barycentric version of the Lagrange formula (cf. Berrut and Trefethen, SIAM Review 2004)

Define $\ell(x) = (x - x_0)(x - x_1)(x - x_2) \dots (x - x_n)$ (no missing terms)

$$\text{so } L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^n \frac{(x - x_i)}{(x_k - x_i)} = \frac{\prod_{i \neq k} (x - x_i)}{\prod_{i \neq k} (x_k - x_i)} = \frac{\ell(x)/(x - x_k)}{\prod_{i \neq k} (x_k - x_i)} = \ell(x) \cdot \frac{w_k}{(x - x_k)}.$$

So $p(x) = \sum_{k=0}^n y_k L_{n,k}(x) = \ell(x) \cdot \sum_{k=0}^n \frac{w_k}{(x - x_k)} y_k$

"First Form of the Barycentric Formula"

Precompute w_k in $O(n^2)$ flops * (each w_k takes $O(n)$ flops, and do this for $O(n)$ values of k)

* See code on github

Amortize: Spread out the cost over a long period of time (like buying a season pass... it makes sense if you use it a lot)

but now we can evaluate $p(x)$ at many different x , each one costing only $O(n)$. So we've amortized the $O(n^2)$ cost

evaluate $\ell(x)$ once, then $O(n)$ multiplies and sums

Furthermore, we have fast updates:

to add (x_{n+1}, y_{n+1}) , update old w_k ($k=0..n$) by \div by $(x_k - x_{n+1})$ and compute new w_{n+1}

which is $O(n)$. Nice.

Note: an equivalent way to write the Barycentric Formula is

$(p(x) \text{ is continuous})$
use L'Hopital to see

$$p(x) = \begin{cases} \sum_{k=0}^n \frac{w_k}{x - x_k} \cdot y_k & \text{if } x \neq x_k \text{ for } k \in \{0, 1, \dots, n\} \\ y_k & \text{if } x = x_k \end{cases}$$

BARYCENTRIC FORMULA
("Second form" or "true form").
USE THIS ONE

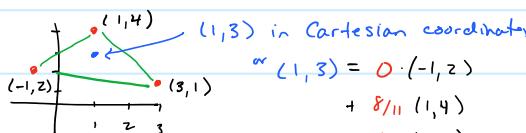
you might be worried that if x is close to some node x_k we get subtractive cancellation, but it turns out in this case it's OK since it happens in numerator and denominator

How to derive?
(if $y_k = 1$, interpolant must be 1 function
(unique!) so
 $I = \ell(x) \sum \frac{w_k}{x - x_k}$
Divide $p(x)$ by 1,
cancel the $\ell(x)$)

(Aside: what does Barycenter mean?)

The Barycenter of two bodies (planets, stars...) is their center of mass, so it's a weighted sum

Barycentric coordinates are coordinates with respect to the vertices of a simplex, ex.



so $(0, 8/11, 3/11)$ in Barycentric coord.

* Chebyshev nodes also simplify

Advantage: we can scale w_k by a constant since it cancels out.
Choose a value to prevent overflow and underflow

p. 4: Special cases and example

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Special Cases

* Chebyshev nodes also simplify,
see Berrut + Trefethen

If nodes are equispaced, $x_{k+1} - x_k = h$

then formulas simplify and are faster,

(skip in lecture)

$$\text{ie., } \underbrace{(x_k - x_0) \cdot (x_k - x_1) \cdots (x_k - x_{k-1})}_{(k)h} \cdot \underbrace{(x_k - x_{k+1}) \cdots (x_k - x_n)}_{-h} = (-1)^{n-k} \cdot h^n \cdot k! \cdot (n-k)! \\ = (-1)^{n-k} h^n \left(\frac{n}{k} / n! \right)^{-1}$$

Please do not memorize!
Not that important

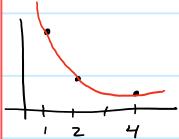
Example

Take

$$x_0 = 1, y_0 = 1 \quad (\text{eg, } y = 1/x)$$

$$x_1 = 2, y_1 = 1/2$$

$x_2 = 4, y_2 = 1/4$. Find quadratic interpolant p and evaluate $p(3)$



Solution: ① One way, compute $L_0(x) = (x-x_1)(x-x_2) \cdot \left(\frac{1}{(x_0-x_1)(x_0-x_2)} \right)$

$$= (x-2)(x-4) \left(\frac{1}{(1-2)(1-4)} \right) = \frac{1}{(-1)(-3)} = \frac{1}{3} = \omega_0$$

$$L_1(x) = (x-x_0)(x-x_2) \cdot \left(\frac{1}{(x_1-x_0)(x_1-x_2)} \right) \\ \omega_1 = \frac{1}{(2-1)(2-4)} = \frac{-1}{2}$$

$$L_2(x) = (x-x_0)(x-x_1) \cdot \left(\frac{1}{(x_2-x_0)(x_2-x_1)} \right) \\ \omega_2 = \frac{1}{(4-1)(4-2)} = \frac{1}{6}$$

then

$$p(x) = 1 \cdot L_0(x) + \frac{1}{2} \cdot L_1(x) + \frac{1}{4} \cdot L_2(x)$$

and plug $x=3$ in (I won't do this now since I'll do it the 2nd way)

② Second way, Barycentric formula. Uses $\omega_0 = 1/3, \omega_1 = -1/2, \omega_2 = 1/6$

$$p(x) = \frac{\sum_{k=0}^n \frac{\omega_k}{x-x_k} \cdot y_k}{\sum_{k=0}^n \frac{\omega_k}{x-x_k}} = \frac{\frac{1}{3} \cdot \frac{1}{(3-1)} \cdot 1 + \frac{-1}{2} \cdot \frac{1}{(3-2)} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{(3-4)} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{(3-1)} + \frac{-1}{2} \cdot \frac{1}{(3-2)} + \frac{1}{6} \cdot \frac{1}{(3-4)}} \quad \text{as before}$$

($x=3$)

$$= \frac{\frac{1}{6} \cdot 1 - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{6} \cdot \frac{1}{4}}{\frac{1}{6} - \frac{1}{2} - \frac{1}{6}} = \frac{\frac{1}{6} - \frac{1}{4} - \frac{1}{24}}{-\frac{1}{2}} = \frac{-\frac{1}{24}}{-\frac{1}{2}}$$

$$= \frac{\frac{4-6-1}{24}}{-\frac{1}{2}} = \frac{+3}{24} \cdot 2 = \boxed{\frac{1}{4}}$$

p. 5: Generalized Rolle's Thm

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Thm 1.7: Rolle's Thm (simplified MVT)

If $f \in C[a,b]$ is differentiable on (a,b) , then if $f(a) = f(b)$ (special case: $f(a) = 0, f(b) = 0$) there exists $\xi \in (a,b)$ s.t. $f'(\xi) = 0$

Thm 1.10: Generalized Rolle's Thm

If $f \in C[a,b]$ is n -times differentiable on (a,b) , then if $f(x) = 0$ for the $n+1$ points $x_0 < x_1 < \dots < x_n$ (all in $[a,b]$) then $\exists \xi \in (x_0, x_n)$ s.t. $f^{(n)}(\xi) = 0$

proof ($n=2$ case only)

via Rolle's, $\exists \xi_1 \in (x_0, x_1)$ s.t. $f(\xi_1) = 0$ since $f(x_0) = f(x_1)$

Similarly, $\exists \xi_2 \in (x_1, x_2)$ s.t. $f(\xi_2) = 0$ $a = x_1, b = x_2$

Now apply Rolle's to f' , using $a = \xi_1, b = \xi_2$

so $\exists \xi \in (\xi_1, \xi_2)$ s.t. $\underbrace{(f')'(\xi)}_{\text{i.e. } f''(\xi)} = 0$

$$\text{i.e. } f''(\xi) = 0.$$

p. 6: Error estimate

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Error

The $\{y_i\}$ can be arbitrary, but sometimes they are generated by a true "underlying" function f . In this case, how close is our interpolant p to f ? Should p be a better approximation as we add nodes? (Yes!) Answered by following theorem:

Thm 3.3 (Burden and Faires)

Let $\{x_0, x_1, \dots, x_n\}$ be distinct points in $[a, b]$ and $f \in C^{n+1}([a, b])$.

Then $\forall x \in [a, b], \exists \xi$ (unknown, depends on x) between x_0, x_1, \dots, x_n (hence $\xi \in (a, b)$) such that if p is the n^{th} degree polynomial interpolant w/ nodes $\{x_i\}_{i=0}^n$ and values $\{y_i = f(x_i)\}_{i=0}^n$, then

$$(*) \quad f(x) = p(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1) \cdots (x-x_n).$$

(i.e., error at $x=x_k$ is 0, and small if x is close to some node.)

Similar to Taylor Series except $(x-x_0) \cdots (x-x_n)$ instead of $(x-x_0)^{n+1}$.

proof Assume $x \neq x_k$ for any k , else it's trivial

$$\text{Fix } x, \text{ define } g(t) = f(t) - p(t) - (f(x) - p(x)) \prod_{i=0}^n \frac{(t-x_i)}{(x-x_i)}$$

then note $f \in C^{n+1}([a, b]) \Rightarrow g \in C^{n+1}([a, b])$

and $g(x_k) = 0$ for $k=0, 1, \dots, n$

$g(x) = 0$ also.

like MVT

So g has $(n+2)$ distinct zeros in $[a, b]$, so by generalized Rolle's Thm

$$\exists \xi \in (a, b) \text{ s.t. } g^{(n+1)}(\xi) = 0$$

$$\begin{aligned} \text{Compute } g^{(n+1)}(\xi) &= f^{(n+1)}(\xi) + p^{(n+1)}(\xi) - \underbrace{(f(x) - p(x))}_{\substack{\text{degree } n \text{ polynomial} \\ \text{so } \frac{d}{dt}^{n+1}() = 0}} \underbrace{\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^n \frac{(t-x_i)}{(x-x_i)}}_{\substack{\text{constants} \\ \text{degree } n+1}} \\ &= f^{(n+1)}(\xi) - (f(x) - p(x)) \cdot \underbrace{\prod_{i=0}^n \frac{(x-x_i)}{(x-x_i)}}_{\substack{\text{so only leading term remains}}} \end{aligned}$$

□

p. 7: Example

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Test yourself :

Let $x_0 = 2, x_1 = 6$

$y_0 = \frac{1}{2}, y_1 = \frac{1}{6}$. Compute $p(4)$ where p is the linear (degree $n=1$) interpolating polynomial.

Note: your answer should be an integer or fraction (simplify fractions),
not a decimal

Answer: $p(x) = -\frac{x}{12} + \frac{2}{3}$
So $p(4) = \frac{1}{3}$

(In)stability of the plain Lagrange interpolation formula

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ASIDE: why is the plain Lagrange method unstable? (for some inputs)

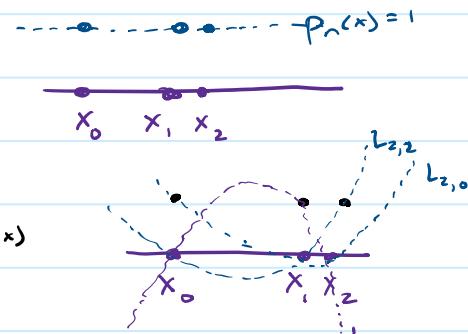
Let $x_0 = 0, x_1 = 1, x_2 = 1 + \varepsilon, y_0 = y_1 = y_2$. So $P_n(x) \equiv 1$.

via Lagrange,

$$L_{2,0}(x) = \frac{(x-1)(x-(1+\varepsilon))}{(-1)(-1-\varepsilon)} = \frac{(x-1)(x-(1+\varepsilon))}{1+\varepsilon}$$

$$L_{2,1}(x) = \frac{(x)(x-(1+\varepsilon))}{(1-\varepsilon)} = -x \cdot \frac{(x-(1+\varepsilon))}{\varepsilon}$$

$$L_{2,2}(x) = \frac{(x)(x-1)}{(\varepsilon)(1+\varepsilon)} \quad P_2(x) = L_{2,0}(x) + L_{2,1}(x) + L_{2,2}(x)$$



look at $x = \frac{1}{2}$

$$\begin{aligned} 1 &= P_2\left(\frac{1}{2}\right) = -\frac{\frac{1}{2} \cdot (-\frac{1}{2}-\varepsilon)}{1+\varepsilon} - \frac{\frac{1}{2}(-\frac{1}{2}-\varepsilon)}{\varepsilon} + \frac{\frac{1}{2}(-\frac{1}{2})}{(1+\varepsilon)\varepsilon} \\ &= \frac{1}{2} \left[\frac{(\frac{1}{2}+\varepsilon)}{1+\varepsilon} + \frac{(\frac{1}{2}+\varepsilon)}{\varepsilon} - \frac{\frac{1}{2}}{(1+\varepsilon)\varepsilon} \right] \end{aligned}$$

as $\varepsilon \rightarrow 0$, these terms are very close in size

$$\frac{\frac{1}{2}+\varepsilon}{\varepsilon} - \frac{\frac{1}{2}}{(\varepsilon+\varepsilon^2)}$$

$\uparrow \varepsilon^2 \approx 0$ when ε is small

$$\underbrace{O(\frac{1}{\varepsilon}) - O(\frac{1}{\varepsilon})}_{\text{large} - \text{large}} = 1.5 \quad (\text{in limit } \varepsilon \rightarrow 0)$$

$$\text{large} - \text{large} = O(1)$$

catastrophic cancellation.

The barycentric formula isn't easy to analyze but is generally more stable.