Ch 4.7: Gaussian Quadrature

Saturday, December 6, 2025 8:25 PM

Gaussian quadrature is a family of quadrature rules (Gauss-Legendre, Chebyshev-Gauss, Gauss-Laguerre, Gauss-Hermite) that choose n nodes such that the degree of exactness/accuracy/precision is as large as possible ($2n-1$).

Closely related to Gauss-Lobatto and Clenshaw-Curtis

Gauss-Legendre to compute $\int_a^b f(x)dx$

This will be an open formula (nodes do not include endpoints a, b) so (following Burden and Faires) we're going to enumerate the nodes as $\{x_1, x_2, \dots, x_n\}$

⚠ Our previous convention was $n+1$ nodes, $\{x_0, x_1, x_2, \dots, x_n\}$

Just like Newton-Cotes, this will be an interpolatory formula, meaning the high-level idea is
1. polynomial interpolation of f on the nodes
2. integrate the polynomial

The big difference is that now we won't require uniformly spaced nodes, instead we'll pick the nodes to optimize a criterion

we'll use degree of exactness

For Newton-Cotes, recall

⚠ we used to call this "$n=0$" since we were 0-based

Name	# nodes	degree of exactness	error
midpoint rule (open),	1 node ($n=1$),	1 (lucky)	$h^3/3\, f''(\xi)$
trapezoidal rule (closed),	2 nodes ($n=2$)	1	$-h^3/12\, f''(\xi)$
Simpson's rule (closed),	3 nodes ($n=3$)	3 (lucky)	$-h^5/90\, f^{(4)}(\xi)$
Simpson's 3/8 rule (closed),	4 nodes ($n=4$)	3	$-3h^5/80\, f^{(4)}(\xi)$

Saturday, December 6, 2025 8:25 PM

How do we know degree of exactness?

<u>Method 1</u> (general purpose but only a lower bound)

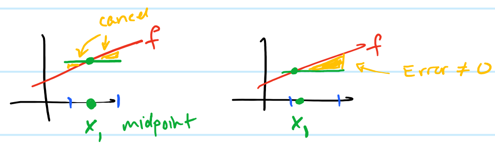
For n nodes, we've seen the $\underbrace{(n-1) \text{ degree interpolating polynomial}}_{p(x)}$ is unique

So if f itself is a $(n-1)$ degree polynomial, since it interpolates itself, it must be p. Hence we integrate it exactly

$\Rightarrow$ For any n nodes, interpolatory quadrature has degree of exactness at least $n-1$

But... doesn't explain why midpoint rule has degree of exactness 1

Here, we got lucky. It was because we used a node exactly in the middle



<u>Method 2</u> (more work to derive)

Derive an error estimate like $h^5 f^{(4)}(\xi)$

hence if $f^{(4)}(x) = 0$ $(\forall x)$ $\Rightarrow$ no error.

If f is a n-degree polynomial, it's $(n+1)$ derivative is 0

ie, $f(x) = ax^2 + bx + c$, $f'''(x) = 0$

So, Newton-Cotes w/ n nodes gives order of exactness $n-1$ (if n even)

or n (if n odd)

Can we do better? We've been thinking of $\{x_1, \ldots, x_n\} \subseteq [a,b]$ as given, and then <u>find weights</u> w_i s.t. our formula is

$$\sum_{i=1}^{n} w_i f(x_i)$$

What if we <u>choose</u> nodes $\{x_i\}$? This gives n more parameters, so might hope we can get order of exactness $\underline{2n-1}$.

In fact, we can!

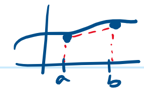
Remember! ☆

<u>Def</u> Gaussian quadrature means picking n nodes such that order of exactness is $\underbrace{2n}-1$

and n weights, so $2n$ variables or "degrees of freedom"

2n constraints/equations

in contrast to trapezoidal rule
which has order-of-exactness 1



Example

$n=2$, find weights and nodes so that $\underbrace{w_1 f(x_1) + w_2 f(x_2)}_{= I_2(f)}$ approximates

$\int_{-1}^{1} f(x)\,dx$ with order of exactness 3 $(= 2n-1)$.

i.e., integrate exactly any $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, i.e., $\int_{-1}^{1} f(x)\,dx = I_2(f)$

since $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

and $I_2(f+g) = I_2(f) + I_2(g)$ for generic f, g

and $\int a \cdot f(x)\,dx = a \cdot \int f(x)\,dx$

$I_2(a \cdot f) = a \cdot I_2(f)$,

i.e., both $\int dx$ and I_2 are linear operators

it suffices to show:

$$\left.\begin{array}{ll} ① & \int_{-1}^{1} x^3\,dx = I_2(x^3) \\ ② & \int_{-1}^{1} x^2\,dx = I_2(x^2) \\ ③ & \int_{-1}^{1} x\,dx = I_2(x) \\ ④ & \int_{-1}^{1} 1\,dx = I_2(1) \end{array}\right\}$$

4 equations, 4 unknowns
x_1, x_2, w_1, w_2

So... compute

① $\int_{-1}^{1} x^3\,dx = 0$ (by odd symmetry) $= w_1 \cdot x_1^3 + w_2 x_2^3$

② $\int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{2}{3}$ $= w_1 x_1^2 + w_2 x_2^2$

③ $\int_{-1}^{1} x\,dx = 0$ (by odd symmetry) $= w_1 x_1 + w_2 x_2$

④ $\int_{-1}^{1} 1\,dx = 2$ $= w_1 + w_2$

These are nonlinear equations.

In general, hard to solve (use ch. 10 techniques)
Here we use tricks, get lucky

you can solve this (to make it more bearable, note we have a symmetry, so expect $w_1 = w_2$ and $x_1 = -x_2$,

so ④ $2 = w_1 + w_2$ and symmetry $(w_1 = w_2)$ $\Rightarrow$ $w_1 = w_2 = 1$

then ③ $\Rightarrow$ $x_1 = -x_2$ (more symmetry) (and ④ is redundant)

so ② $\Rightarrow$ $2/3 = 2 \cdot x_1^2$ $\Rightarrow$ $x_1 = -1/\sqrt{3} = -\sqrt{3}/3$

$x_2 = +1/\sqrt{3} = +\sqrt{3}/3$

Observation:
the Legendre polynomials are $1, x, \underbrace{x^2 - 1/3}_{P_2}, \ldots$

Roots of $P_2(x)$ are $\pm\sqrt{1/3}$. Coincidence?

Saturday, December 6, 2025 8:25 PM

Making it more systematic

Theorem: The nodes of the n-point Gauss-Legendre quadrature rule are given by the roots of the nth Legendre polynomial P_n

Before we prove this, we better say what P_n is!

Notation: let $\mathbb{P}_n = \{$ all polynomials with degree n or less $\}$ and recall from linear algebra that this is a vector space of dimension $n+1$

Def Legendre polynomials $\{P_0, P_1, P_2, \ldots\}$ are polynomials such that

2025, we already saw these in ch. 8

① $\forall\, p \in \mathbb{P}_{n-1} \quad \int_{-1}^{1} p(x)\, P_n(x)\, dx = 0$ "orthogonality"

2a P_n is a polynomial of degree n

2b P_n is monic (= leading coefficient is 1)
(not too important, just a way to ensure uniqueness)

i.e., $\text{Span}(\{P_0, \ldots, P_n\}) = \mathbb{P}_n$

$\Rightarrow \{P_0, P_1, \ldots, P_n\}$ is a basis for $\mathbb{P}_n$
and in fact it's orthogonal by ①
(not ortho<u>normal</u> though)

$\{1, x, x^2, \ldots, x^n\}$ ← "monomial basis" is also a basis for $\mathbb{P}_n$, but not orthogonal and not as nice to work with

If we <u>orthonormalize</u> the monomial basis (eg, Gram-Schmidt) then (up to scaling, 2b) we get the Legendre polynomials. So these are pretty fundamental.

Ex of Legendre polynomials

$P_0(x) = 1$ $\qquad P_3(x) = x^3 - \frac{3}{5}x$

$P_1(x) = x$ $\qquad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

$P_2(x) = x^2 - \frac{1}{3}$

have many known facts, e.g.,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_n(x) = \frac{2n-1}{n} x \cdot P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x) \quad \text{3-term recurrence}$$

The roots of $P_n(x)$ can be efficiently found in $O(n^2)$ time by the 1969 Golub and Welsch algorithm (efficiently finds eigenvalues of a tridiagonal matrix) – before then it was hard to use in practice. (but known since 1814 by Gauss) NOT ON TEST

The weights of the Gauss-Legendre formula are easier: form the interpolating polynomial p (Lagrange interpolating polynomial) and integrate that

⚠ Lagrange ≠ Legendre

Saturday, December 6, 2025 8:25 PM

So..

we'll do $\int_a^b$ later

Def. Gauss–Legendre quadrature for $\int_{-1}^{1} f(x)\,dx$:

Remember

① pick nodes $\{x_1, \ldots, x_n\}$ to be the roots of P_n ← Legendre polynomial

Fact: all roots are simple, real, and lie in $(-1,1)$

② pick weights $w_i = \int_{-1}^{1} \overbrace{\prod_{\substack{j=1 \\ j\neq i}} \frac{x - x_j}{x_i - x_j}}^{L_i(x)} \, dx$

Since our interpolating polynomial is $\sum_{i=1}^{n} f(x_i) L_i(x)$

and define $I_n^{GL}(f) = \sum_{i=1}^{n} w_i f(x_i)$

now back to theorem

Thm if f is a polynomial of degree $2n-1$ or less ($f \in \mathbb{P}_{2n-1}$),

2025, just sketched this

then $\int_{-1}^{1} f(x)\,dx = I_n^{GL}(f)$, i.e., Gauss–Legendre quadrature w/ n nodes has degree of exactness $2n-1$

Proof

① first, suppose $f \in \mathbb{P}_{n-1}$. Then since this is an interpolating quadrature on n nodes, f <u>is</u> its own interpolating polynomial (by uniqueness) hence it is integrated exactly

② now, let f have degree between $[n, 2n-1]$.

see wikipedia "polynomial long division" or "Euclidean division"

Divide f by the Legendre polynomial P_n:

$$P_n(x)\,\overline{)\,f(x)}\quad \text{quotient } Q(x), \ \ldots,\ R(x)$$

$$f(x) = \underbrace{Q(x)}_{\deg \le n-1} \underbrace{P_n(x)}_{\deg = n} + \underbrace{R(x)}_{\substack{\deg \le n \\ \text{remainder}}}$$

where $\text{degree}(R) < \text{degree}(P_n) = n$.

Also, $\text{degree}(f) \in [n, 2n-1]$

$$\text{degree}(f) = \text{degree}(Q \cdot P_n) = \text{degree}(Q) + \text{degree}(P_n)$$
$$= \text{degree}(Q) + n$$

so, $\text{degree}(Q) \in [0, n-1]$

$\Rightarrow \text{degree}(Q) \le n-1$, i.e., $Q \in \mathbb{P}_{n-1}$

Thus

$$\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{1} Q(x) P_n(x)}_{=0 \text{ by orthogonality}} + \underbrace{\int_{-1}^{1} R(x)\,dx}_{\text{equals G-L quadrature by part ①}}$$

$$= 0 + I_n^{GL}(R)$$
$$= I_n^{GL}(f)$$

since $R(x_i) = f(x_i)$. Why?

$$f(x_i) = Q(x_i) \underbrace{P_n(x_i)}_{x_i \text{ is a root of } P_n} + R(x_i) \quad \square$$

p. 6

Saturday, December 6, 2025 8:25 PM

Extending from $\int_{-1}^{1} f(x)\,dx$ to $\int_a^b f(x)\,dx$ ⚠ Important detail!

Just do a change-of-variables

$x = a$ $\quad x = b$

$t = -1$ $\quad t = 1$

Define $t = \dfrac{x - \frac{a+b}{2}}{\frac{1}{2}(b-a)} = \dfrac{2x - a - b}{b-a}$ (center it; normalize it)

So $x = \frac{1}{2}\left[(b-a)t + a + b\right]$, so $dx = \frac{b-a}{2}\,dt$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{(b-a)t + a+b}{2}\right)dt$$

i.e., define $g(t) = f\left(\frac{(b-a)t + a + b}{2}\right)$

so the Gauss-Legendre quadrature estimate of $\int_a^b f(x)\,dx$

is $\frac{b-a}{2} I_n^{GL}(g)$.

More info

- Not only is integration more accurate w/ Gauss-Legendre nodes, but so is interpolation. The Runge phenomenon goes away.

- Nodes are precomputed (see Table 4.2 in book) or see scipy.special.roots_legendre or glint (Driscoll, Braun text)

- A similar method that works almost as well is Clenshaw-Curtis
  see "Is Gaussian Quadrature Better than Clenshaw-Curtis?"
  Lloyd Trefethen, SIAM Review 2008
  (it's faster: $O(n\log n)$ vs $O(n^2)$)

- The Legendre polynomials are orthogonal in the sense

$$\underbrace{\int_{-1}^{1} P_n(x)P_m(x)\,dx}_{\langle P_n, P_m\rangle \text{ an inner-product}} = 0 \text{ if } m \neq n$$

(just as $\vec{x}^T\vec{y}$ is an inner product)

If we make a weighted inner product,

$$\int_{-1}^{1} f(x)g(x)w(x)\,dx = \langle f, g\rangle \quad \text{for } w(x) > 0$$

we get different "orthogonal" polynomials,

ex: $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $(-1,1)$ gives rise to T_n Chebyshev polynomials of the 1st kind

$T_0(x) = 1$ $\qquad T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$

$T_1(x) = x$ $\qquad T_n(x) = \cos(n \cdot \cos^{-1}(x))$ $\quad (*)$

all roots simple, real, lie in $(-1,1)$.

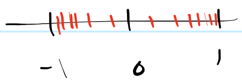
In fact, $(*) \Rightarrow$ roots x_i are $x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ $\quad i = 1,\dots,n$ "Chebyshev nodes of the 1st kind" ex $w(x) = \sqrt{1-x^2}$ on $[-1,1]$, gives rise to U_n Chebyshev polynomials of the 2nd kind

w/ roots

$$x_i = -\cos\left(\frac{i\pi}{n}\right), \quad i = 0, 1, \ldots, n$$

"Chebyshev nodes/points of the 2nd kind"

These are x-projections of equispaced pts on unit circle



and beats Runge phenomenon, so good for interpolation

stronger than C^∞

Thm (9.3.1 Driscoll + Braun) "Spectral Convergence"

Let p_n be the unique degree n polynomial interpolant of f using $n+1$ Chebyshev nodes of the 2nd kind

If f is real analytic* then $\exists\, C > 0$, $\exists\, K > 1$ such that

$$\max_{x\in[-1,1]} |f(x) - p_n(x)| \le C \cdot K^{-n}$$

way better than $n^{-\alpha}$

* real analytic: Taylor series converges everywhere

ex. weight $w(x) = e^{-x}$ (or $e^{-x/2}$) on domain $[0, \infty)$

gives Laguerre polynomials

ex weight $w(x) = e^{-x^2}$ (or $e^{-x^2/2}$) on domain $(-\infty, \infty)$

gives Hermite polynomials

For all these choices of weights, we have a corresponding Gaussian quadrature rule

Name	Interval	Weight
Gauss-Legendre	$[-1,1]$	1
Chebyshev-Gauss	$(-1,1)$	$(1-x^2)^{-1/2}$
	$[-1,1]$	$(1-x^2)^{1/2}$
Gauss-Laguerre	$[0,\infty)$	e^{-x}
Gauss-Hermite	$(-\infty,\infty)$	e^{-x^2}