Eigenvalue background (ch 6.4 and ch 9.1-9.2)

Thursday, November 20, 2025 3:57 PM

Determinants (see ch 6.4)

The determinant of a square linear transformation $\vec{x} \mapsto A \cdot \vec{x}$, denoted $\det(A)$ or $|A|$, measures how the transformation changes area/volume, so is used in change-of-variables for integration.

⚠ $|A|$ can be negative, don't let the notation confuse you. It's not absolute value.

$$\boxed{A \text{ is invertible iff } \det(A) \neq 0}$$

Misc. Facts: let $A \in \mathbb{R}^n$ w/ eigenvalues $\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_n}_{\text{duplicates ok}}$

$A = [a_{ij}]$

$$\det(c \cdot A) = c^n \det(A)$$

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

$\det(A+B) \neq \det(A) + \det(B)$ in general

$$\det(A^{-1}) = 1/\det(A)$$

$$\det(A^T) = \det(A)$$

if A is upper or lower triangular (or diagonal)

then $\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \ldots \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

... and $\text{trace}(A) := \sum_{i=1}^{n} a_{ii}$ is equal to $\sum_{i=1}^{n} \lambda_i$

Computing the determinant

Method 1: cofactor expansion

define $M_{ij} = \det(A \text{ with row } i, \text{ col. } j \text{ removed})$ "minor"

then $$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} M_{ij}$$ Recursive!

for any row i you want

and if $A \in \mathbb{R}^{1 \times 1}$, $\det(A) = a_{11}$ [or, use $\det(A) = \det(A^T)$ ...]

Cost: let T = time to compute $\det(A)$ for a $n \times n$ matrix

$$\begin{aligned} T(n) &= n \cdot T(n-1) \\ &= n \cdot (n-1) \cdot T(n-2) \\ &= \ldots \\ &= \text{constant} \cdot n! \end{aligned}$$

So $O(n!)$ time. i.e. NOT RECOMENDED!

Method 2

Gaussian-elimination / LU   Let $A = P^T \cdot L \cdot U$

So $\det(A) = \det(P^T L U)$

$= \underbrace{\det(P)}_{\substack{\text{det is } \pm 1 \\ \text{depending on} \\ \text{exact permutations}}} \cdot \underbrace{\det(L) \cdot \det(U)}_{\substack{\text{triangular, so determinant is} \\ \text{the product of the diagonals}}}$

(every row or column swap flips sign of determinant)

Overall cost is dominated by cost of the LU factorization: $\boxed{O(n^3)}$

MUCH BETTER

Eigenvalues

... of a square matrix A, $n \times n$ (allow $A \in \mathbb{C}^{n \times n}$)

Def $\lambda \in \mathbb{C}$ is an eigenvalue of A if $\exists\ \vec{x} \in \mathbb{C}^n$, $\vec{x} \neq \vec{0}$, s.t.

$A \cdot \vec{x} = \lambda \cdot \vec{x}$ $\vec{x}$ is the eigenvector.

⚠ We exclude $\vec{x} = \vec{0}$ since $A \cdot \vec{0} = \lambda \cdot \vec{0}$ is always true.

we do allow $\lambda = 0$.

There's never a singe unique eigenvector since if $\vec{x}$ is an eigenvector, so is $c \cdot \vec{x}$ for any $c \neq 0$. So instead we speak of an eigenspace of λ: the subspace of $\mathbb{C}^n$ of all eigenvectors for λ (and also include $\vec{0}$).

How to find eigenvalues/vectors?

$\underset{\vec{x} \neq \vec{0}}{A\vec{x} = \lambda \vec{x}} \iff \underset{\vec{x} \neq \vec{0}}{(A - \lambda I) \cdot \vec{x} = 0} \iff A - \lambda I$ is singular (else $\vec{x} = \vec{0}$ is only sol'n)

$\iff \underbrace{\det(A - \lambda I)}_{p_A(\lambda)} = 0$ "characteristic polynomial"

... So, find a root of p_A $\iff \lambda$ is a root of $p_A(\lambda)$

(which is a n-dimensional polynomial, so has exactly n (possibly complex) roots if you count w/ multiplicity)

Thursday, November 20, 2025 4:29 PM

OK, so we compute $p_A(\lambda) = \det(A - \lambda I)$

... how?

(1a) In linear algebra 101, you do this by hand w/ cofactor expansion ... so $O(n!)$. BAD

Also, what to do about λ indeterminate

(1b) Faddeev-Leverrier Expansion

Formula to compute coefficients of p_A,

but involves $\mathrm{tr}(A^k)$ for $k = 1, 2, \ldots, n$

i.e. n matrix-matrix multiplies, each costs $O(n^3)$

so $O(n^4)$ time ... and also not stable either!

(2) or, rather than find the polynomial in closed form, evaluate $\det(A - \lambda I)$, for a given estimate of λ, using LU, and do a generic root-finding method to estimate λ.

This costs $O(n^3)$ every step

and it's not numerically stable.

It would also only give you a single eigenvalue, not all of them.

So hopefully you're convinced that finding eigenvalues isn't trivial.

See related demo Ch9_EigenvaluesTheBadWay.ipynb

More background

Def: multiplicities. Let $A \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$) and $\lambda_k \in \mathbb{C}$ an eigenvalue

The algebraic multiplicity of λ_k is the degree of the root λ_k in p_A,

i.e. alg. mult. is 3 means $p_A(\lambda) = (\lambda - \lambda_k)^3 \cdot q(\lambda)$ and $q(\lambda_k) \neq 0$

ex: $p_A(\lambda) = (\lambda-4)^2(\lambda+5)$

$\lambda=4$ has alg. mult. 2, $\lambda=-5$ has alg. mult. 1

The eigenspace $E_k = \mathrm{Null}(A - \lambda_k I)$

The geometric multiplicity of λ_k is $\dim(E_k)$.

FACTS: $1 \le$ geometric mult. $\le$ algebraic mult. $\le n$

The matrix is diagonalizable iff all geometric mult. = algebraic mult.

Sufficient conditions: $A = A^T$, or, all eigenvalues simple (i.e. alg. mult. $=1$)

Def Similarity If $A, B \in \mathbb{C}^{n\times n}$, they're similar, written $A \sim B$, if $\exists$ nonsingular $P \in \mathbb{C}^{n\times n}$ s.t. $B = P^{-1}AP$

Proposition If $A \sim B$ then

1) $p_A(\lambda) = p_B(\lambda)$
2) $\mathrm{tr}(A) = \mathrm{tr}(B)$, $\det(A) = \det(B)$
3) A and B have the same eigenvalues

Thm Schur Normal Form

If $A \in \mathbb{C}^{n\times n}$, $\exists$ a unitary matrix $U \in \mathbb{C}^{n\times n}$ and upper triangular matrix T

i.e. $U^{-1} = U^*$

$U^* := \bar{U}^T$

s.t. $T = U^* A U$

hence $A = UTU^*$ and $A \sim T$

Thm Spectral Theorem, ver 1

If $A \in \mathbb{C}^{n\times n}$ is Hermitian (i.e. $A = A^*$, or, $A \in \mathbb{R}^{n\times n}$ and $A = A^T$) real symmetric

then $\exists$ diagonal D, unitary U, $\boxed{\text{and } d_{kk} \in \mathbb{R} \text{ for } k=1,\dots,n}$

s.t. $D = U^* A U$ i.e. $A = UDU^*$ i.e. $A \cdot U = D \cdot U$ i.e. $A\vec{u}_k = d_{kk}\vec{u}_k$

i.e. columns of U are eigenvectors, eigenvalues are d_{kk} $\boxed{\text{and eigenvalues are real}}$

version 2

A need not be Hermitian but it's "normal", $AA^* = A^*A$.

Same as version 1 but $d_{kk} \in \mathbb{C}$ is possible

Proof of ver. 1 Use Schur form:

$A = UTU^*$

$A^* = UT^*U$

$\Rightarrow T = T^*$ (triangular)

so T is diagonal

Gerschgorin Circle (or Disc) Theorem 1931

Optional

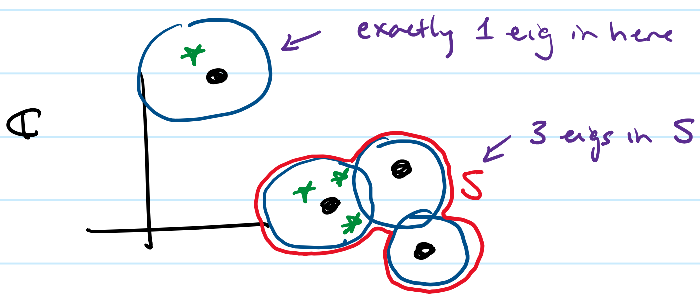
Let $A \in \mathbb{C}^{n\times n}$, $A = [a_{ij}]$

define radii $r_i = \sum_{j\neq i} |a_{ij}|$

and discs $D_i = \{ z \in \mathbb{C} : |z - a_{ii}| \le r_i \}$

Thm

1. All eigenvalues λ of A lie inside $D = \bigcup_{i=1}^{n} D_i$
2. [refined estimate] If m discs form a connected set S, disjoint from other discs, S contains m eigenvalues (counted with multiplicity)
3. $\lambda(A) = \lambda(A^T)$ so you can apply to rows or columns



* = possible eigenvalue.

Corollary: $A = A^*$ and strictly diagonally dominant (i.e., $a_{ii} - r_i > 0$) then it's positive definite.