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Surveillance of the mean behavior of multivariate time series

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In this paper several cumulative sum (CUSUM) charts for the mean of a multivariate time series are introduced. We extend the control schemes for independent multivariate observations of Crosier [*Technometrics* (1988) Vol. 30, pp. 187–194], Pignatiello and Runger [*Journal of Quality Technology* (1990) Vol. 22, pp. 173–186], and Ngai and Zhang [*Statistica Sinica* (2001) Vol. 11, pp. 747–766] to multivariate time series by taking into account the probability structure of the underlying stochastic process. We consider modified charts and residual schemes as well. It is analyzed under which conditions these charts are directionally invariant. In an extensive Monte Carlo study these charts are compared with the CUSUM scheme of Theodossiu [*Journal of the American Statistical Association* (1993) Vol. 88, pp. 441–448], the multivariate exponentially weighted moving-average (EWMA) chart of Kramer and Schmid [*Sequential Analysis* (1997) Vol. 16, pp. 131–154], and the control procedures of Bodnar and Schmid [*Frontiers of Statistical Process Control* (2006) Physica, Heidelberg]. As a measure of the performance, the maximum expected delay is used.

Keywords and Phrases: statistical process control, multivariate time series, CUSUM charts.

1 Introduction

Juran (1997) reports that Walter Shewhart invented the control chart on May 16, 1924. Since the pioneering contributions of Shewhart in the 1920s and 1930s of the last century (e.g. Shewhart, 1926, 1931), control charts have become an important and extremely useful tool in statistical process control (SPC). Many new control schemes have been proposed in the meantime (e.g. Page, 1954; Roberts, 1959). In a great number of papers these schemes have been analyzed in detail.

While it is common in industry to monitor individual process characteristics separately, multivariate methods use the relationship between the components of a multivariate stochastic process to generate more powerful control algorithms. The

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aim is to detect changes in the process, which can hardly be detected by a univariate attempt. A survey is given in WIERDA (1994).

HOTELLING (1947) adapted the Shewhart control chart to monitor several quality characteristics by considering the Mahalanobis distance of the observed data and the target mean of the process. The extension of the cumulative sum (CUSUM) chart of PAGE (1954) to multivariate data is not obvious. In the univariate case a CUSUM chart can be derived via the sequential probability ratio test of Wald. In the multivariate case this approach leads to a scheme which depends on the size and the direction of the expected shift. This is a very unpleasant property. For that reason several authors proposed schemes which exclusively depend on the magnitude of the expected shift (e.g. CROSIER, 1988; PIGNATIELLO and RUNGER, 1990; HAWKINS, 1991, 1993; NGAI and ZHANG, 2001). An extension of the exponentially weighted moving-average (EWMA) chart of ROBERTS (1959) to multivariate data was given by LOWRY *et al.* (1992). All these charts deal with independent samples.

By the end of the 1980s, a new orientation started. ALWAN and ROBERTS (1988) and ALWAN (1989) proposed control charts for time series. ALWAN (1989) analyzed 235 datasets which were frequently used as examples of SPC. He showed that in about 85% the control limits were misplaced. In nearly half of these cases neglected correlation was responsible. Although some authors had already treated or at least mentioned that problem (e.g. GOLDSMITH and WHITFIELD, 1961; HARRISON and DAVIES, 1964), their papers had not been widely noticed in the literature.

The paper of ALWAN and ROBERTS (1988) followed a large number of contributions dealing with control charts for time series. Nowadays we can distinguish between two different types of control schemes for time series: (i) 'Residual charts' based on the application of control charts for independent and identically distributed (i.i.d.) process to the residuals (e.g. ALWAN and ROBERTS, 1988; HARRIS and ROSS, 1991; WARDELL, MOSKOWITZ and PLANTE, 1994a,b; LU and REYNOLDS, 1999). (ii) 'Modified control charts' are based on the consideration of the original process, however, the control design is calculated with respect to the time-series structure (e.g. VASILOPOULOS and STAMBOULIS, 1978; SCHMID, 1995, 1997a,b).

There are only a few papers dealing with the surveillance of the mean behavior of time series. THEODOSSIOU (1993) derived a multivariate CUSUM control chart via the sequential probability ratio. KRAMER and SCHMID (1997) generalized the control scheme of LOWRY *et al.* (1992) to time series. BODNAR and SCHMID (2006) extended some CUSUM control charts for independent samples to Gaussian processes by standardizing the observations with respect to the time series.

For a long time the methods of SPC have been mainly applied in engineering in order to monitor the parameters of a quality characteristic. Because of the extension of control charts to more general processes, SPC has become quite attractive for many other disciplines. Interesting new applications are, for example in medicine (e.g. FRISEN, 1992; SONESSON and BOCK, 2003; LAWSON and KLEINMAN, 2005) and finance (e.g. SCHIPPER and SCHMID, 2001; ANDERSSON, BOCK and FRISEN, 2004;

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SCHMID and TZOTCHEV, 2004; BODNAR, 2007). In that context, the data mainly have a memory and are modeled by time series.

The remainder of the paper is organized as follows. In section 2 models for the in-control (no change) and the out-of-control (change) situation are introduced. The target process is always assumed to be a multivariate stationary Gaussian process. Furthermore we focus on the surveillance of the mean behavior of the time series.

In section 3 several new multivariate CUSUM control charts for time series are proposed. These charts are extensions of the CUSUM schemes for independent multivariate observations of CROSIER (1988), PIGNATIELLO and RUNGER (1990), and NGAI and ZHANG (2001). Therefore, these control schemes are called modified charts. Our presentation is chosen as follows. First, we briefly review these charts for i.i.d. variables. After that it is shown how they can be extended to time series. We present the procedure of BODNAR and SCHMID (2006) and a new approach which takes into account the probability distribution of the time series. One important property of these CUSUM charts for i.i.d. variables is that they are directionally invariant. It is shown that this property is in general no longer fulfilled for time series. However, we derive a sufficient condition under which the introduced CUSUM charts for time series and those discussed in BODNAR and SCHMID (2006) are directionally invariant (cf. Theorem 1). In section 4 we consider CUSUM residual charts. It is again examined under which conditions these charts are directionally invariant.

Within an extensive simulation study all proposed charts are compared with each other in section 5. We include the CUSUM chart of THEODOSSIU (1993), the multivariate EWMA scheme of KRAMER and SCHMID (1997), and the schemes of BODNAR and SCHMID (2006) in our study as well. As a measure of performance the maximum expected delay (MED) is used. The expected delay is equal to the average number of observations of a control chart from the change point until the time point of the signal. The MED is obtained by maximizing the expected delay over all possible positions of the change point. The control limits of all charts are determined such that the in-control average run length (ARL), i.e. the average number of observations taken until a signal is given, is equal to a prespecified value. Using these limits the MEDs are compared with each other. The in-control process is taken as a vector autoregressive process of order 1 and dimensions 2 and 10.

Our study shows that no chart is superior to all others. Because of the large number of parameters of the in-control process and the fact that the charts depend on several smoothing parameters it is difficult to make general recommendations. Based on our results we recommend a practitioner to apply the modified charts if the process is positively correlated and residual schemes if it is negatively correlated. In case of a positive correlation the best results for moderate and large shifts are obtained for the modified versions of the charts of CROSIER (1988) and NGAI and ZHANG (2001) based on the varying Mahalanobis distance. For small shifts both modified approaches obtained by extending the chart of PIGNATIELLO and RUNGER (1990) overperform. It has to be emphasized that the implementation of the residual charts is much easier. Their decisive advantage is that the control limits do not

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depend on the parameter matrices of the in-control process. The paper ends with an appendix where all proofs are given.

2 Model

In the following we distinguish between the target process and the observed (actual) process. In engineering the target process is equal to the process that fulfills the quality requirements. Its parameters are frequently known values (target values). In most of the applications in economics and natural sciences the target process is determined by fitting a model, e.g. a time series, to a set of previous (historical) observations of the process.

We denote the target process by $\{\mathbf{Y}_t\}$. Its components are denoted by $\mathbf{Y}_{t1},\ldots,\mathbf{Y}_{tp}$. It is assumed that $\{\mathbf{Y}_t\}$ is a p-dimensional stationary Gaussian process with mean $\boldsymbol{\mu}_0=(\mu_{01},\ldots,\mu_{0p})'$ and cross-covariance matrix

$$\begin{aligned}\boldsymbol{\Gamma}(h)&=E((\mathbf{Y}_{t+h}-\boldsymbol{\mu}_0)(\mathbf{Y}_t-\boldsymbol{\mu}_0)')\\&=[E((Y_{ti}-\mu_{0i})(Y_{t-h,j}-\mu_{0j}))]_{i,j=1}^{p}=[\gamma_{ij}(h)]_{i,j=1}^{p}\end{aligned}$$

at lag h.

In the rest of the paper we assume that all parameters of the process $\{\mathbf{Y}_t\}$, i.e. $\boldsymbol{\mu}_0$ and $\boldsymbol{\Gamma}(h)$ for $h\geq 0$, are known. Although it is very restrice to assume that the parameters of the in-control process are known it is made in nearly all papers on SPC. Recently, some papers deal with the influence of parameter estimation on the performance of a control chart. Kramer and Schmid (2000) consider a Shewhart chart for time series. It is assumed that the parameters are estimated by an independent prerun. The prerun is modeled by an autoregressive (AR) process of order 1. As a performance measure the conditional ARL, given the estimators, is calculated. The authors discuss several estimators for the AR parameters. Recommendations about the choice of the sample size are given. Albers and Kallenberg (2004) focus on an independent random sample. They use the unconditional distribution of the run length as a measure of the performance. It is proposed to correct the control limits, taking into account the influence of the parameter estimator. Such investigations have not been made for a multivariate time series up to now. This is a very important point for future research.

We deal with the problem whether sequentially taken observations can be considered as a realization of a given target process or not. A change should be detected as soon as possible after its occurrence. This is a typical question treated within SPC.

We denote the present observations by $\mathbf{x}_1,\mathbf{x}_2,\ldots$. They are assumed to be a realization of the observed process $\{\mathbf{X}_t\}$. Of course changes may influence the target process in many different ways. Our aim is the surveillance of the mean behavior of the observed process. The relationship between the observed process and the target process is described by a change-point model. This means that

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$$\mathbf{X}_t = \mathbf{Y}_t + \mathbf{a}\mathbf{1}_{\{q, q+1, \ldots\}}(t), \tag{1}$$

where $\mathbf{a} \in \mathbb{R}^p$, $q \in \mathbb{N}$, and $t \in \mathbb{Z}$. $\mathbf{1}_A(t)$ denotes the indicator function of the set A at point t; $\mathbf{a}$ and q are unknown quantities; and $\mathbf{a}$ describes the size and the direction of the change. If $\mathbf{a} \neq \mathbf{0}$ a change point at time q is present and the observed process is said to be out of control. Note that in the out-of-control case the observed process is not stationary. From the other side no change means that the target and observed processes coincide. In this case the observed process is said to be in control. The in-control mean $\boldsymbol{\mu}_0$ is called the target value. Note that in our model it holds that $\mathbf{X}_t = \mathbf{Y}_t$ for $t \leq 0$, i.e. both processes are the same up to time point 0.

Since the pioneering contributions of SHEWHART (1926, 1931), control charts have become the most important statistical technique for detecting changes within a process. For determining the control design only the target process is of relevance. It is necessary to know its complete probability structure. The knowledge of the relationship between the observed and the target process is important for the construction of the chart. It is essential to know the type of change (e.g. mean shift, variance change, drift, outlier) against which the process has to be protected. Here this is done by applying model (1).

We will make use of the notation $E_{\mathbf{a},q}$ for the expectation taken with respect to model (1). E_0 means that no shift is present. By analogy the notations $\mathrm{cov}_{\mathbf{a},q}$, cov_0, etc. are used.

3 Modified CUSUM control charts

In the univariate case CUSUM charts have been shown to be successful. Because they are derived using the sequential probability ratio test (SPRT) of WALD (1947) they are based on a fundamental statistical principle. HEALY (1987) applied the SPRT to an independent sample of multivariate normal variables. He obtained a CUSUM chart which depends on the direction of the change. The scheme is very sensitive to deviations from the true direction of the change. For that reason further CUSUM-type control charts have been proposed for independent multivariate samples by, e.g. CROSIER (1988), PIGNATIELLO and RUNGER (1990), and NGAI and ZHANG (2001).

Only a few papers deal with CUSUM charts for multivariate time series. THEODOSSIOU (1993) derived a chart by using the SPRT approach. His chart suffers from the same disadvantages as the scheme of HEALY (1987). BODNAR and SCHMID (2006) extended some CUSUM charts for independent multivariate data to time series. The main idea of their approach is to standardize $\mathbf{X}_t$ by $\boldsymbol{\Gamma}(0)^{-1/2}(\mathbf{X}_t - \boldsymbol{\mu}_0)$ instead of $\boldsymbol{\Sigma}^{-1/2}(\mathbf{X}_t - \boldsymbol{\mu}_0)$. Here $\boldsymbol{\Sigma}$ denotes the covariance matrix of $\mathbf{X}_t$ assuming an independent sample. Next we want to introduce further schemes for time series. For the calculation of the control design the time-series structure is taken into account.

All charts of this section are the so-called modified control charts (cf. KNOTH and SCHMID, 2004). This means that similar control statistics are used as in the i.i.d.

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case, however, the control design is calculated with respect to the distribution of the underlying stochastic process. Contrary to residual charts (cf. section 4) they are based on original observations.

In the following we briefly review several CUSUM charts for i.i.d. data. A detailed description of these schemes can be found in the original literature. After that it is described how these charts can be extended to time series. Despite the approach of Bodnar and Schmid (2006) a new procedure is introduced.

We make use of the notation $\|\mathbf{x}\|_C = \sqrt{\mathbf{x}'\mathbf{C}^{-1}\mathbf{x}}$ where $\mathbf{C}$ is assumed to be positive-definite. $\|\mathbf{x}\|_C$ denotes a norm.

3.1 Modified MC1

Pignatiello and Runger (1990) introduced two multivariate CUSUM charts, MC1 and MC2.

Let

$$\mathbf{S}_{m,l} = \sum_{i=m+1}^{l} (\mathbf{X}_i - \boldsymbol{\mu}_0) \quad \text{for } l, m \geq 0.$$

The control chart is based on the MC1 statistic which is defined as

$$\text{MC1}_t = \max\{\|\mathbf{S}_{t-n_t,t}\|_{\boldsymbol{\Sigma}} - kn_t, 0\}, \quad t \geq 1. \tag{2}$$

n_t is equal to the number of observations taken until the last reconstruction of the CUSUM chart, i.e.

$$n_t = \begin{cases} n_{t-1} + 1 & \text{if } \text{MC1}_{t-1} > 0 \\ 1 & \text{if } \text{MC1}_{t-1} = 0 \end{cases} \tag{3}$$

for $t \geq 1$ with $\text{MC1}_0 = 0$. If MC1_t exceeds an upper control limit then it is concluded that the process is out of control. For independent samples the ARL of this chart is directionally invariant, i.e. it depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only through the noncentrality parameter $\lambda = \|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|^2_{\boldsymbol{\Sigma}}$. This is an advantage with respect to Healy's scheme.

The motivation of the control statistic is as follows. $\sum_{i=t-n_t+1}^{t} \mathbf{X}_i / n_t$ can be considered as an estimator of the mean $\boldsymbol{\mu}$ using the data after the last reconstruction. The distance between $\sum_{i=t-n_t+1}^{t} \mathbf{X}_i / n_t$ and the mean $\boldsymbol{\mu}_0$ is measured by the Mahalanobis distance. We get

$$\left(\left(\frac{\sum_{i=t-n_t+1}^{t} \mathbf{X}_i}{n_t} - \boldsymbol{\mu}_0 \right)' \boldsymbol{\Sigma}^{-1} \left(\frac{\sum_{i=t-n_t+1}^{t} \mathbf{X}_i}{n_t} - \boldsymbol{\mu}_0 \right) \right)^{1/2} = \frac{\|S_{t-n_t,t}\|_{\boldsymbol{\Sigma}}}{n_t}.$$

If this quantity is large enough the statistic indicates a possible change. A disadvantage of this scheme is that the derivation of the scheme does not provide any conclusion on the choice of k. Pignatiello and Runger (1990) recommended the use of $k = \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0\|_{\boldsymbol{\Sigma}}/2$ where $\boldsymbol{\mu}_1$ is a specified off-target state.

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BODNAR and SCHMID (2006) extended this scheme to time series by taking into account the covariance matrix of the time series $\mathbf{\Gamma}(0)$. Consequently, $\mathbf{\Sigma}$ is replaced by $\mathbf{\Gamma}(0)$ in (2) and (3). We denote the corresponding control chart as $\mathrm{MC1}_{\mathbf{\Gamma}}$.

Here we introduce a new approach. We make use of the covariance matrix of the mean process. Let $\{\mathbf{Y}_t\}$ be a p-dimensional stationary process. Then it holds that for all $m, l \in \mathbb{N}$

$$\mathrm{cov}_0(\mathbf{S}_{m,l}) = (l-m)\mathbf{\Gamma}(0) + \sum_{h=1}^{l-m-1}(l-m-h)(\mathbf{\Gamma}(h)+\mathbf{\Gamma}(h)') = (l-m)\mathbf{\Delta}_{l-m}. \quad (4)$$

While in BODNAR and SCHMID (2006) the norm of the vector $\mathbf{S}_{t-n_t,t}$ is calculated with respect to the matrix $\mathbf{\Gamma}(0)$ here we consider the norm with respect to the matrix $\mathbf{\Delta}_{n_t}$ at time t. Hence the covariance matrix used for the determination of the norm is time-varying. Then the control chart for time series gives a signal as soon as for $t \geq 1$

$$\mathrm{MC1}_{\Delta,t} = \max\left\{\|\mathbf{S}_{t-n_{\Delta,t},t}\|_{\Delta_{n_{\Delta,t}}} - kn_{\Delta,t}, 0\right\} > h_1, \quad (5)$$

where h_1 is a given constant; $n_{\Delta,t}$ is defined by analogy to (3) but with respect to $\mathrm{MC1}_{\Delta,t-1}$; and $n_{\Delta,t}\mathbf{\Delta}_{n_{\Delta,t}}$ is the covariance matrix of $\mathbf{S}_{t-n_{\Delta,t},t}$. The constant h_1 is chosen to achieve a given in-control average run length ξ.

3.2 Modified MC2

PIGNATIELLO and RUNGER (1990) introduced a second control chart. It is based on the application of the one-sided CUSUM recursion to $D_t^2 = (\mathbf{X}_t - \boldsymbol{\mu}_0)'\mathbf{\Sigma}^{-1}(\mathbf{X}_t - \boldsymbol{\mu}_0)$. Replacing again $\mathbf{\Sigma}$ by $\mathbf{\Gamma}(0)$ we get the control statistic

$$\mathrm{MC2}_{\Gamma,t} = \max\left\{0, \mathrm{MC2}_{\Gamma,t-1} + D_{\Gamma,t}^2 - p - k\right\}, \quad t \geq 1 \quad (6)$$

with $\mathrm{MC2}_{\Gamma,0} = 0$ and $D_{\Gamma,t}^2 = (\mathbf{X}_t - \boldsymbol{\mu}_0)'\mathbf{\Gamma}(0)^{-1}(\mathbf{X}_t - \boldsymbol{\mu}_0)$. The chart indicates a change at time t if the control statistic is sufficiently large.

We observe that $E(D_{\Gamma,t}^2) = p + \|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|_{\Gamma(0)}^2$ and thus for a specified out-of-control change $\boldsymbol{\mu}_1$ it is recommended to choose $k = \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0\|_{\Gamma(0)}^2/2$.

3.3 Modified vector-valued CUSUM

Using the shrinking method, CROSIER (1986) introduced a CUSUM chart for univariate data. CROSIER (1988) generalized the univariate proposal to the multivariate case by replacing the scalar quantities in the univariate CUSUM recursion by vectors in the multivariate case.

The direct extension of the CUSUM chart of PAGE (1954) to the multidimensional case leads to the recursion $\mathbf{S}_t = \max\{\mathbf{0}, \mathbf{S}_{t-1} + (\mathbf{X}_t - \boldsymbol{\mu}_0) - \mathbf{k}\}$. Now it is unclear how the maximum over vectors has to be understood and how $\mathbf{k}$ has to be chosen. To overcome these problems CROSIER (1988) chose the following procedure. First, $\mathbf{k}$ must have the same direction as $\mathbf{S}_{t-1} + (\mathbf{X}_t - \boldsymbol{\mu}_0)$. Its length is equal to $k = \sqrt{\mathbf{k}'\mathbf{\Sigma}^{-1}\mathbf{k}}$.

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Second, the vector $\mathbf{S}_{t-1}+(\mathbf{X}_t-\boldsymbol{\mu}_0)$ is shrunk toward $\mathbf{0}$ by the vector $k(\mathbf{S}_{t-1}+(\mathbf{X}_t-\boldsymbol{\mu}_0))/C_t$, where C_t is the length of $\mathbf{S}_{t-1}+(\mathbf{X}_t-\boldsymbol{\mu}_0)$, i.e.

$$C_t=\|\mathbf{S}_{t-1}+\mathbf{X}_t-\boldsymbol{\mu}_0\|_{\boldsymbol{\Sigma}}. \tag{7}$$

$k>0$ stands for a reference value. Hence the multivariate CUSUM (MCUSUM) chart of Crosier (1988) makes use of the statistic

$$\mathbf{S}_t=\begin{cases}\mathbf{0} & \text{if } C_t\le k\\ (\mathbf{S}_{t-1}+\mathbf{X}_t-\boldsymbol{\mu}_0)(1-\frac{k}{C_t}) & \text{if } C_t>k\end{cases} \tag{8}$$

for $t\ge 1$ with $\mathbf{S}_0=\mathbf{0}$. The chart gives an out-of-control signal as soon as the length of the vector $\mathbf{S}_t$, i.e.

$$\text{MCUSUM}_t=(\mathbf{S}_t'\boldsymbol{\Sigma}^{-1}\mathbf{S}_t)^{1/2}=\max\{0,C_t-k\}$$

exceeds a preselected value h_2.

Crosier (1988) proved that the distribution of the run length depends on the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ only through the noncentrality parameter $\lambda=\|\boldsymbol{\mu}-\boldsymbol{\mu}_0\|^2_{\boldsymbol{\Sigma}}$, i.e. it is directionally invariant. He recommended choosing the reference value as $k=\|\boldsymbol{\mu}_1-\boldsymbol{\mu}_0\|_{\boldsymbol{\Sigma}}/2$ for the specified off-target value $\boldsymbol{\mu}_1$.

In order to extend the scheme to multivariate stationary processes Bodnar and Schmid (2006) calculated the norm in (8) and (7) with respect to $\boldsymbol{\Gamma}(0)$ instead of $\boldsymbol{\Sigma}$. The resulting control chart is denoted as $\text{MCUSUM}_{\boldsymbol{\Gamma}}$.

Following the idea presented in the last section we want to take into account the covariance structure of the mean process. Hence we make use of the norm taken with respect to $\boldsymbol{\Delta}_t$ because

$$\operatorname{cov}\left(\sum_{i=1}^{t}(\mathbf{X}_i-\boldsymbol{\mu}_0)\right)=t\boldsymbol{\Delta}_t.$$

Consequently we consider

$$\mathbf{S}_{\Delta,t}=\begin{cases}\mathbf{0} & \text{if } C_{\Delta,t}\le k\\ (\mathbf{S}_{\Delta,t-1}+\mathbf{X}_t-\boldsymbol{\mu}_0)(1-\frac{k}{C_{\Delta,t}}) & \text{if } C_{\Delta,t}>k\end{cases} \tag{9}$$

for $t\ge 1$ with $\mathbf{S}_{\Delta,0}=\mathbf{0}$ and

$$C_{\Delta,t}=\|\mathbf{S}_{\Delta,t-1}+(\mathbf{X}_t-\boldsymbol{\mu}_0)\|_{\boldsymbol{\Delta}_t}.$$

The control scheme signals as soon as

$$\text{MCUSUM}_{\Delta,t}=(\mathbf{S}_{\Delta,t}'\boldsymbol{\Delta}_t^{-1}\mathbf{S}_{\Delta,t})^{1/2} \tag{10}$$

is sufficiently large. The control limit is again determined in such a way that the in-control ARL of the chart is equal to a prespecified value.

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3.4 Modified projected pursuit CUSUM

Another type of CUSUM chart is based on the idea of projection pursuit. This approach was proposed by NGAI and ZHANG (2001) for independent vectors. Here it will be denoted as PPCUSUM.

First, it is assumed that the the direction of the change $\mathbf{a}_0$ with $\|\mathbf{a}_0\|_2 = 1$ (Euclidean norm) is known. NGAI and ZHANG (2001) consider the normalized projected univariate variables $\mathbf{a}_0'\boldsymbol{\Sigma}^{-1/2}(\mathbf{X}_t - \boldsymbol{\mu}_0)$. They apply the CUSUM recursion to these variables leading to

$$C_0^{\mathbf{a}_0} = 0, \quad C_t^{\mathbf{a}_0} = \max\{0, C_{t-1}^{\mathbf{a}_0} + \mathbf{a}_0'\boldsymbol{\Sigma}^{-1/2}(\mathbf{X}_t - \boldsymbol{\mu}_0) - k\}, \quad t \geq 1. \tag{11}$$

Several authors proved that the CUSUM chart has some optimality properties (e.g. POLLAK, 1985; MOUSTAKIDES, 1986). For a known direction the projected variables possess these desirable properties as well. In practice, however, the direction is unknown in most cases and therefore the statistic cannot be directly applied.

NGAI and ZHANG (2001) proposed to calculate $\max_{\|\mathbf{a}\|_2=1} C_t^{\mathbf{a}}$. They proved that $\max_{\|\mathbf{a}\|_2=1} C_t^{\mathbf{a}} = \mathrm{PPCUSUM}_{\Sigma,t}$ with

$$\mathrm{PPCUSUM}_{\Sigma,t} = \max\{0, \|\mathbf{S}_{t-1,t}\|_{\Sigma} - k, \|\mathbf{S}_{t-2,t}\|_{\Sigma} - 2k, \ldots, \|\mathbf{S}_{0,t}\|_{\Sigma} - tk\} \tag{12}$$

for $t \geq 1$ with $\mathbf{S}_{t-v,t}$ as in section 3.1. The control scheme gives an alarm as soon as $\mathrm{PPCUSUM}_{\Sigma,t} > h_3$.

If the chart gives a signal at time t_0 then it follows that there is $t_1 < t_0$ such that

$$\sqrt{\mathbf{S}_{t_1,t_0}'\boldsymbol{\Sigma}^{-1}\mathbf{S}_{t_1,t_0}} - (t_0 - t_1)k = \max_{\|\mathbf{a}\|_2=1} C_{t_0}^{\mathbf{a}} > h_3.$$

Then the direction of the shift is estimated by

$$\hat{\mathbf{a}}_0 = \frac{\boldsymbol{\Sigma}^{-1/2}\mathbf{S}_{t_1,t_0}}{\sqrt{\mathbf{S}_{t_1,t_0}'\boldsymbol{\Sigma}^{-1}\mathbf{S}_{t_1,t_0}}}.$$

There are several ways to extend the PPCUSUM chart to time series. Because the process should be normalized with respect to the covariance matrix of time series the first idea is to normalize the data by $\boldsymbol{\Gamma}(0)^{-1/2}(\mathbf{X}_t - \boldsymbol{\mu}_0)$. Then $\boldsymbol{\Sigma}$ has to be replaced by $\boldsymbol{\Gamma}(0)$ in (11) and thus the norm in (12) is taken with respect to $\boldsymbol{\Gamma}(0)$ (see BODNAR and SCHMID, 2006). We denote the control statistic by $\mathrm{PPCUSUM}_{\Gamma,t}$.

Another approach is based on the observation that the covariance matrix of $\mathbf{S}_{t-v,t}$ is equal to $v\boldsymbol{\Delta}_v$. This fact gives rise to the idea to calculate the norm of $\mathbf{S}_{t-v,t}$ with respect to the matrix $\boldsymbol{\Delta}_v$. Hence the second control chart for time series gives a signal as soon as

$$\mathrm{PPCUSUM}_{\Delta,t} = \max\{0, \|\mathbf{S}_{t-1,t}\|_{\Delta_1} - k, \ldots, \|\mathbf{S}_{0,t}\|_{\Delta_t} - tk\} \tag{13}$$

is larger than the control limit which is again chosen as described above.

If the sequence of the random vectors $\{\mathbf{Y}_t\}$ is i.i.d. then both modified control schemes are equal to the PPCUSUM chart proposed by NGAI and ZHANG (2001).

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3.5 Invariance properties of the modified MC1, PPCUSUM, MCUSUM Charts

For independent variables the MC1, MC2, modified CUSUM, and PPCUSUM charts have the property that the distributions of their run lengths depend on the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only over the quantity $\lambda = \|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|_{\boldsymbol{\Sigma}}^2$. They do not directly depend on the direction of the change but only on the distance of the change and the target value. Consequently, the distribution of a chart has the same values for all directions $\boldsymbol{\mu}$ for which $\|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|_{\boldsymbol{\Sigma}}$ is the same. This property facilitates the analysis of the charts dramatically. However, the charts have a second interesting property which is of value for the determination of the control limits. It holds that the distributions of the run lengths in the in-control state do not depend on $\boldsymbol{\Sigma}$.

Next we analyze whether the control charts for time series possess similar properties. It turns out to be rather difficult to make any statement about the distribution of a run length for dependent data. Moreover, the invariance properties are not fulfilled for all processes.

In Theorem 1 we prove for a general family of Gaussian processes having a special cross-covariance structure that the charts are directionally invariant. We show that this class includes all VARMA(1, 1) processes whose coefficient matrices are a multiple of the identity matrix.

THEOREM 1. *Let $\{\mathbf{Y}_t\}$ be a p-variate stationary Gaussian process with mean $\boldsymbol{\mu}_0$ and cross-covariance matrix $\boldsymbol{\Gamma}(h)$. Assume that for all $h \geq 0$ there is $d_h \in \mathbb{R}$ such that $\boldsymbol{\Gamma}(h) = d_h \boldsymbol{\Gamma}(0)$. Then the distributions of the run lengths of all control charts considered in section 3 (i.e. the MC1, the MCUSUM, and the PPCUSUM charts based on $\|\cdot\|_{\boldsymbol{\Gamma}(0)}$ or $\|\cdot\|_{\boldsymbol{\Delta}_t}$) depend on $\mathbf{a}$ only through the parameter $\lambda_{\boldsymbol{\Gamma}(0)} = \mathbf{a}'\boldsymbol{\Gamma}(0)^{-1}\mathbf{a}$, i.e. if N denotes the run length of such a chart then for all $\mathbf{a}_1$, $\mathbf{a}_2$ with $\mathbf{a}_1'\boldsymbol{\Gamma}(0)^{-1}\mathbf{a}_1 = \mathbf{a}_2'\boldsymbol{\Gamma}(0)^{-1}\mathbf{a}_2$ it holds that $P_{\mathbf{a}_1, q}(N > i) = P_{\mathbf{a}_2, q}(N > i)$ for all q, i.*

The proof of Theorem 1 is given in the Appendix.

EXAMPLE. *Let $\{\mathbf{Y}_t\}$ be a p-variate stationary* VARMA(1, 1) *process, i.e.*

$$\mathbf{Y}_t - \boldsymbol{\mu}_0 = \boldsymbol{\Phi}(\mathbf{Y}_{t-1} - \boldsymbol{\mu}_0) + \boldsymbol{\varepsilon}_t - \boldsymbol{\Theta}\boldsymbol{\varepsilon}_{t-1}.$$

Let $\{\boldsymbol{\varepsilon}_t\}$ be independent and normally distributed with zero-mean and positive-definite covariance matrix $\boldsymbol{\Sigma}$. Suppose that

$$\boldsymbol{\Phi} = \phi\mathbf{I}, \quad \boldsymbol{\Theta} = \theta\mathbf{I} \quad \text{with } |\phi| < 1, \ \theta \in \mathbb{R}. \tag{14}$$

Then it holds that $\boldsymbol{\Gamma}(h) = d_h\boldsymbol{\Gamma}(0)$ for $h \geq 0$. Note that the assumption (14) *is equivalent to*

$$\boldsymbol{\Phi}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\boldsymbol{\Phi}', \quad \boldsymbol{\Theta}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\boldsymbol{\Theta}' \quad \text{for all } \boldsymbol{\Sigma}. \tag{15}$$

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This result is proved in the Appendix. If (14) *holds then we have*

$$\mathbf{\Gamma}(0) = \frac{1+\theta^2-2\theta\phi}{1-\phi^2}\mathbf{\Sigma} \quad \textit{and} \quad \lambda_{\Gamma(0)} = \frac{1-\phi^2}{1+\theta^2-2\theta\phi}\mathbf{a}'\mathbf{\Sigma}^{-1}\mathbf{a}.$$

It is important to note that the charts are no longer directionally invariant if the condition (14) is not fulfilled. In section 5 counter-examples are given.

Kramer and Schmid (1997) proved for a VAR(1) process satisfying $\mathbf{\Phi\Sigma} = \mathbf{\Sigma\Phi}'$ for all $\mathbf{\Sigma}$ that the in-control ARL of the modified EWMA chart does not depend on $\mathbf{\Sigma}$. Following the arguments of their proof it can be shown that the CUSUM charts considered in Theorem 1 have the same property. A detailed proof can be found in Bodnar (2005).

4 Multivariate CUSUM residual charts

Residual control charts make use of a transformation of the observations (e.g. Alwan and Roberts, 1988; Wardell *et al.*, 1994a,b). The aim of this approach is to simplify the correlation structure of the original process. There are various proposals about the choice of the transformation (e.g. Montgomery and Mastrangelo, 1991; Kramer and Schmid, 1997; Jiang, Tsui and Woodall 2000). Under weak assumptions on the target process it can be found that the residuals are independent. For that reason the determination of the control design turns out to be much easier than for modified charts. However, it is important to note that instead of the original observations the residual process is monitored. This is one reason why some practitioners favor modified schemes.

4.1 Residual charts

Here we determine the residuals in the same way as proposed by Kramer and Schmid (1997). In order to predict $\mathbf{X}_t$ based on the known history $\mathbf{X}_1, \ldots, \mathbf{X}_{t-1}$ the best linear predictor is taken. This is done under the assumption that the process is in control. We get $\hat{\mathbf{X}}_t = f(\mathbf{X}_1, \ldots, \mathbf{X}_{t-1})$, where $f(\mathbf{y}_1, \ldots, \mathbf{y}_{t-1})$ is the best linear predictor of $\mathbf{Y}_t$ given that $\mathbf{Y}_1 = \mathbf{y}_1, \ldots, \mathbf{Y}_{t-1} = \mathbf{y}_{t-1}$.

Brockwell and Davis (1991, Ch. 11) describe how the best linear predictor can be calculated recursively. For a stationary VARMA(1, 1) process with mean $\boldsymbol{\mu}_0$ the best linear predictor is given by

$$\hat{\mathbf{X}}_t = \boldsymbol{\mu}_0 + \mathbf{\Phi}(\mathbf{X}_{t-1} - \boldsymbol{\mu}_0) - \mathbf{\Theta}_t(\mathbf{X}_{t-1} - \hat{\mathbf{X}}_{t-1}) \quad \text{for } t \geq 2$$

with $\hat{\mathbf{X}}_1 = \boldsymbol{\mu}_0$ and the matrix $\mathbf{\Theta}_t = \mathbf{\Theta\Sigma V}_{t-1}^{-1}$ where

$$\mathbf{V}_t = \begin{cases} \mathbf{\Gamma}(0) & \text{for } t = 1 \\ \mathbf{\Sigma} + \mathbf{\Theta\Sigma\Theta}' - \mathbf{\Theta}_t \mathbf{V}_{t-1} \mathbf{\Theta}_t' & \text{for } t \geq 2 \end{cases}.$$

If the process $\{\mathbf{Y}_t\}$ is invertible then it follows that $\mathbf{\Theta}_t \to \mathbf{\Theta}$ and $\mathbf{V}_t \to \mathbf{\Sigma}$ with $t \to \infty$.

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The residual schemes of this section are based on the normalized residuals $\boldsymbol{\eta}_t = \mathbf{V}_t^{-1/2}(\mathbf{X}_t - \hat{\mathbf{X}}_t)$ for $t \geq 1$ with $\mathbf{V}_t = \mathrm{var}_0(\mathbf{X}_t - \hat{\mathbf{X}}_t)$. Note that $E_0(\boldsymbol{\eta}_t) = \mathbf{0}$, $\mathrm{cov}_0(\boldsymbol{\eta}_t) = \mathbf{I}$ and $E_0(\boldsymbol{\eta}_t \boldsymbol{\eta}_s') = \mathbf{0}$ for $t \neq s$. For a Gaussian process $\{\mathbf{X}_t\}$ it follows that in the in-control state the variables $\{\boldsymbol{\eta}_t\}$ are independent and normally distributed. Consequently it is possible to apply all the well-known control charts for independent observations to the residuals $\{\boldsymbol{\eta}_t\}$. It is important to note that in the out-of-control case the residual charts show a different behavior than the charts for an independent sample. The reason for this property lies in the fact that in case of a shift in the mean the normalized residuals are no longer identically distributed but still normally distributed. This unpleasant property is based on a starting value problem (see, Kramer and Schmid, 1997). The whole history of the process is not known but only a finite number of observations. This fact is tacitly ignored by many authors. However, in the out-of-control situation the normalized residuals are still independent.

In many studies of univariate charts the limit of $\boldsymbol{\Theta}_t$ and $\mathbf{V}_t$ is used instead of the exact quantities. Using the asymptotic values the corresponding residuals are not independent. This is a further drawback of this procedure.

Now the control statistics of section 3 are determined for the residuals. Because in the in-control case the normalized residuals are i.i.d. with variance $\mathbf{I}$, the quantities are recalculated with respect to the norm $\|\cdot\|_{\mathbf{I}}$. This is equal to the Euclidean norm for which we use the symbol $\|\cdot\|_2$. Thus for $\mathbf{x} \in \mathbb{R}^p$ we define

$$\|\mathbf{x}\|_2 = (\sum_{i=1}^{p} x_i^2)^{1/2}.$$

The MC1 chart for the residuals is based on the control statistic

$$\mathrm{MC1}_{r,t} = \max\{\|\mathbf{S}_{r,t-n_{r,t},t}\|_2 - k n_{r,t}, 0\} \quad \text{with } \mathbf{S}_{r,t-v,t} = \sum_{i=t-v+1}^{t} \boldsymbol{\eta}_i. \tag{16}$$

$n_{r,t}$ is defined by the analogy to (3).

The MC2 scheme for residuals makes use of $D_{r,t}^2 = \boldsymbol{\eta}_t' \boldsymbol{\eta}_t$. The control statistic is given by

$$\mathrm{MC2}_{r,t} = \max\{0, \mathrm{MC2}_{r,t-1} + D_{r,t}^2 - p - k\}, \quad t \geq 1 \tag{17}$$

with $\mathrm{MC2}_{r,0} = 0$.

In order to extend the MCUSUM chart of section 3.3 we calculate $C_{r,t} = (\mathbf{S}_{r,t-1} + \boldsymbol{\eta}_t)'(\mathbf{S}_{r,t-1} + \boldsymbol{\eta}_t)$ with $\mathbf{S}_{r,t}$ as in (8). Then,

$$\mathrm{MCUSUM}_{r,t} = \|\mathbf{S}_{r,t}\|_2. \tag{18}$$

The PPCUSUM control statistic for residuals is defined as

$$\mathrm{PPCUSUM}_{r,t} = \max\{0, \|\mathbf{S}_{r,t-1,t}\|_2 - k, \ldots, \|\mathbf{S}_{r,0,t}\|_2 - tk\}. \tag{19}$$

A large value of the control statistic is always a hint that something has changed. If the control statistic is larger than a given constant then the charts give a signal. For all charts the control limits are determined such that the in-control ARL of

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each chart is equal to a fixed value ξ. Because no explicit formulas for the ARLs are available we determine approximate solutions by simulations.

Residual charts are based on independent variables. In the in-control case the residuals are identically distributed as well. Because these charts have the same in-control ARL as the corresponding scheme for an i.i.d. sequence with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}$ it follows that their in-control ARL does not depend on the parameters of the process, e.g. for a VARMA(1,1) process on $\mathbf{\Phi}$, $\mathbf{\Theta}$, and $\mathbf{\Sigma}$. This fact facilitates the determination of the control limits a lot because they do not depend on the process parameters. This is an advantage of the residual charts when compared with the modified charts.

If a chart signals a change then the analyst must decide what to do and how to interpret the alarm. Frequently this is difficult for residual charts because the residual process is monitored and thus the chart signals a deviation in the residual process and not directly in the observed process.

4.2 Invariance properties of the residual charts

As seen in section 3.2, modified control charts for time series are in general not directionally invariant. This is a drawback of these schemes. It makes the analysis of these charts more difficult. Next we deal with the question whether residual charts are directionally invariant.

Let us consider a VAR(1) process with mean $\mathbf{0}$. Then the normalized residuals are

$$\boldsymbol{\eta}_t = \begin{cases} \mathbf{\Gamma}(0)^{-1/2}\mathbf{Y}_1 + \mathbf{\Gamma}(0)^{-1/2}\mathbf{a} & \text{for } t=1 \\ \mathbf{\Sigma}^{-1/2}\boldsymbol{\varepsilon}_t + \mathbf{\Sigma}^{-1/2}(\mathbf{I}-\mathbf{\Phi})\mathbf{a} & \text{for } t \geq 2 \end{cases}$$

if (1) holds with $q=1$. This representation shows that because of the starting value problem the shift influences the residuals in a different way. In principle this is the reason why the residuals are not directionally invariant.

It is possible to prove a result similar to Theorem 1 for the residual charts. If the coefficients of a VARMA(1,1) process fulfill (14) then the MED is a function of $\lambda_{\mathbf{\Gamma}(0)}$. The proof of this result is given in BODNAR (2005) and follows the arguments given in Theorem 1. Moreover, the residual charts are not directionally invariant if (14) is not satisfied (cf. section 5).

5 A comparison study

The aim of this section is to compare the control charts introduced in section 3 and section 4 with the CUSUM chart of THEODOSSIOU (1993), the MEWMA scheme of KRAMER and SCHMID (1997) based on the asymptotic variance, and the control procedures of BODNAR and SCHMID (2006). In Appendix B we briefly describe the chart of THEODOSSIOU (1993) and the approach of KRAMER and SCHMID (1997).

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As a measure of the performance of a control chart the MED is taken. The expected delay of a stopping time N is defined as

$$\mathrm{ED}_{\mathbf{a},q}(N) = E_{\mathbf{a},q}(N - q + 1 | N \geq q) \tag{20}$$

provided $E_{\mathbf{a},q}(N) < \infty$. Pollak and Siegmund (1975) proposed the use of the maximum expected delay

$$\mathrm{MED} = \sup_{q \geq 1} \mathrm{ED}_{\mathbf{a},q}(N) \tag{21}$$

which can be considered as a worst-case criterion.

All charts are calibrated in the same way. The control limits are determined such that the in-control ARLs of all charts are equal. Here we choose the in-control ARL equal to $\xi = 200$. Using these control limits the MEDs are determined and compared with each other. Note that for modified charts the control limits depend in general on the parameter matrices of the underlying target process.

Because no explicit formulas for the ARLs and MEDs are available, a Monte Carlo study is used to estimate these quantities. The estimators are obtained by averaging the corresponding sample values. In our simulation study 10^5 independent realizations of the target process are generated to estimate the in-control ARLs. The control limits of all charts are determined by applying the regula falsi (see, e.g. Conte and de Boor, 1981) to the estimated in-control ARLs. For the estimation of the MEDs 10^6 realizations are taken.

In our comparison study the target process is always a zero-mean stationary VAR(1) process. The variables $\{\boldsymbol{\varepsilon}_t\}$ are assumed to be independent and normally distributed with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$.

A significant problem in dealing with multivariate time series is the huge number of model parameters. To simplify the situation we take the coefficient matrix $\boldsymbol{\Phi}$ as a diagonal matrix. Here we present our results for three processes. The first process is a two-dimensional VAR(1) process with

$$\boldsymbol{\Phi} = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.6 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \tag{22}$$

the second process is a two-dimensional VAR(1) process with

$$\boldsymbol{\Phi} = \begin{pmatrix} -0.4 & 0 \\ 0 & 0.6 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \tag{23}$$

and the third process is a 10-dimensional VAR(1) process given by

$$\boldsymbol{\Phi} = 0.7\mathbf{I} \quad \text{and} \quad \boldsymbol{\Sigma} = (0.5^{|i-j|})_{i,j=1,\ldots,10}. \tag{24}$$

The actual process is related to the target process as described in (1).

The control schemes depend on further design parameters, the CUSUM charts on the reference value k and the EWMA charts on the smoothing matrix $\mathbf{R}$ (see Appendix B).

In our study k takes values within the set $\{0.0, 0.1, \ldots, 1.0, 1.1\}$. Values larger than 1.1 are not considered because for larger values of k determination of the MED

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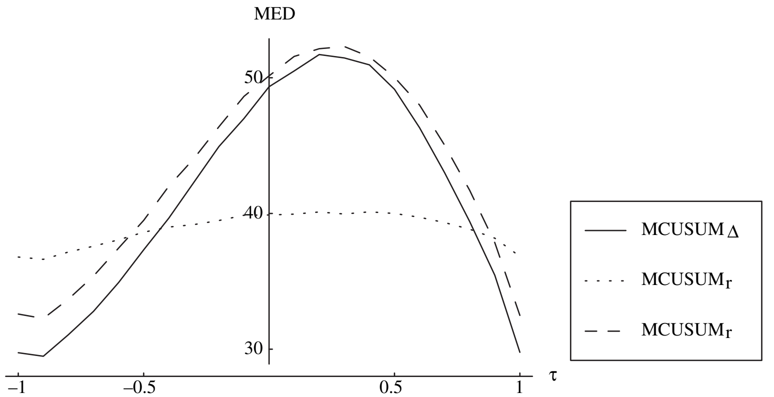

Fig. 1. Maximum expected delays of the modified MCUSUM control charts based on the norms $\|\cdot\|_{\boldsymbol{\Delta}_t}$ and $\|\cdot\|_{\boldsymbol{\Gamma}(0)}$ (see Section 3), and of the MCUSUM residual chart (cf. Section 4) as a function of τ (cf. (25)) for a two-dimensional VAR(1) process with $\boldsymbol{\Phi} = \text{diag}(0.4, 0.6)$ ($k = 0.5$, in-control ARL = 200).

may be difficult. To facilitate the comparison study the smoothing matrix is taken equal to $\mathbf{R} = r\mathbf{I}$ with $r \in \{0.1, 0.2, \ldots, 1.0\}$.

The results of our simulation study are given in Figures 1–3. As a result of the space constraints we present graphs for some selected charts. (Comprehensive tables of the MEDs of all charts can be obtained from the authors on request.)

5.1 Results for the two-dimensional processes

Note that the process (22) does not fulfill the sufficient condition (15) of Theorem 1. Consequently it is not clear whether it is directionally invariant or not. We show here that it is not. We choose various shifts for which the noncentrality parameter is the same and show that the quantity MED changes with the selected shifts.

First, we put $\lambda_{\Gamma(0)} = \mathbf{a}'\boldsymbol{\Gamma}(0)^{-1}\mathbf{a} = 0.5$. A solution to this equation is given by $\mathbf{a} = (a_1, a_2)$ with

$$a_1 = \tau\sqrt{\gamma_{11}(0)/2}, \quad a_2 = \frac{1}{\sqrt{2\gamma_{11}(0)}}\left(\tau\gamma_{12}(0) + \sqrt{1-\tau^2}\sqrt{\gamma_{11}(0)\gamma_{22}(0) - \gamma_{12}^2(0)}\right) \tag{25}$$

for $|\tau| \leq 1$.

Using simulations we estimated the quantity MED for different values of τ. Figure 1 shows the results for the modified MCUSUM charts and the MCUSUM residual scheme. In the description of the figure the notation $\text{MCUSUM}_{\boldsymbol{\Delta}}$ ($\text{MCUSUM}_{\boldsymbol{\Gamma}}$) refers to the modified vector-valued CUSUM chart based on the norm $\|\cdot\|_{\boldsymbol{\Delta}_t}$. ($\|\cdot\|_{\boldsymbol{\Gamma}(0)}$). The index r refers to the residual charts. The plots show that the modified designs as well as the residual charts are not directionally invariant. It can be seen that the control charts based on $\|\cdot\|_{\boldsymbol{\Delta}_t}$ and the residual charts react more sensibly to changes in τ than the control schemes based on $\|\cdot\|_{\boldsymbol{\Gamma}(0)}$. Note that this behavior

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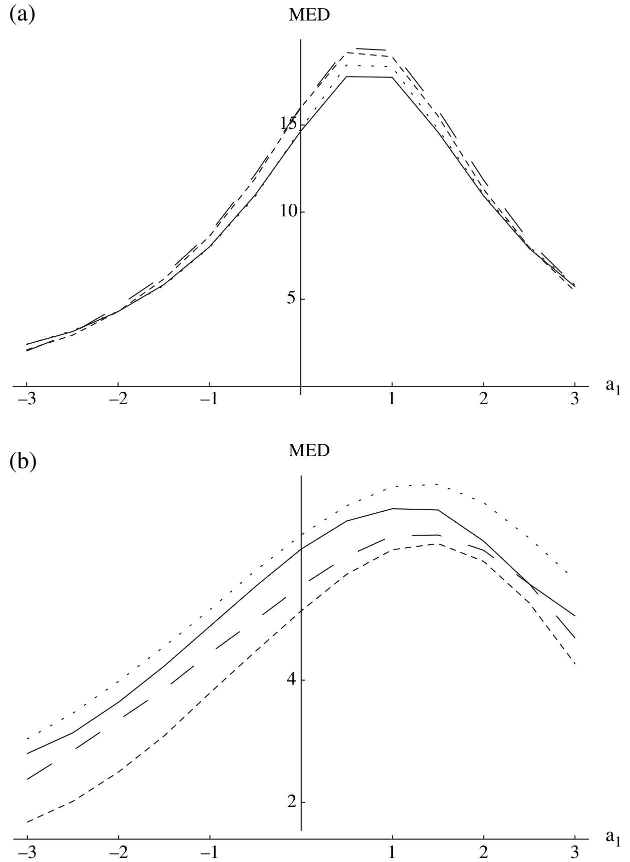

Fig. 2. Maximum expected delays (MEDs) of the modified MCUSUM (joint line), the modified PPCUSUM (dots) control charts based on the norm $\|\cdot\|_{\mathbf{\Delta}_t}$, the MEWMA (larger dashing line) chart, and the MC1 (smaller dashing line) residual control chart as a function of the shift a_1 ($a_2 = 1.0$) for a two-dimensional VAR(1) process [$\mathbf{\Phi} = \text{diag}(0.4, 0.6)$ in 'a' and $\mathbf{\Phi} = \text{diag}(-0.4, 0.6)$ in 'b']. The figure shows the smallest MED over $k \in \{0.0, 0.1, \ldots, 1.0, 1.1\}$. The in-control ARL is 200.

can be observed for the other control charts as well. The charts based on $\|\cdot\|_{\mathbf{\Gamma}(0)}$ turn out to be more robust with respect to τ.

Because for this process the considered control schemes are not directionally invariant, we present the MED as a function of $\mathbf{a}$. The results of our Monte Carlo study are summarized in Figure 2. For a fixed shift the smallest MEDs over r and k, are presented.

We also analyzed at which positions the maximum value of the expected delay is taken. It is interesting that for nearly all control charts of our study the maximum is already taken at $q = 1$. The only exceptions are the modified MCUSUM and PPCUSUM schemes based on the norm $\|\cdot\|_{\mathbf{\Delta}_t}$, the modified MC1 charts, and the MC1 scheme applied to residuals. For these control charts the MEDs are obtained

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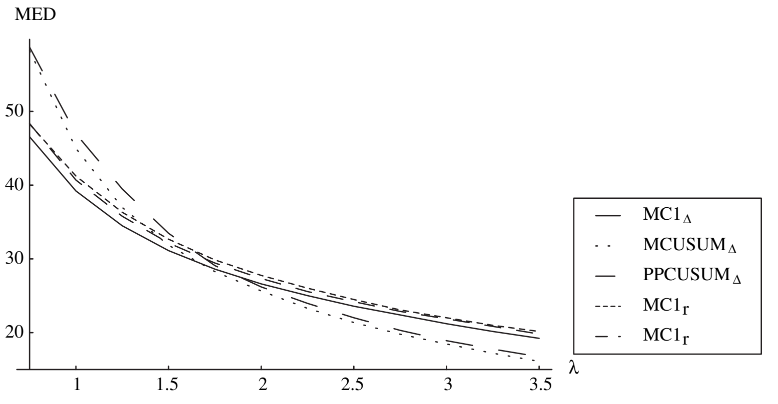

Fig. 3. Maximum expected delays (MEDs) of the modified MC1, MCUSUM, PPCUSUM control charts based on the norm $\|\cdot\|_{\mathbf{\Delta}_t}$, the modified MC1 control chart based on the norm $\|\cdot\|_{\mathbf{\Gamma}(0)}$, and the residual MC1 control chart as a function of the noncentrality parameter λ for the 10-dimensional VAR(1) process with $\mathbf{\Phi}=0.7\mathbf{I}$. The figure shows the smallest MEDs over $k\in\{0.0, 0.1, \ldots, 1.0, 1.1\}$. The in-control ARL is 200.

for different values of q. The first result is in line with the observation of Kramer and Schmid (1998) who obtained a similar performance in case of univariate schemes applied to time series. The second result corresponds to the findings of Woodall and Mahmoud (2005). The authors showed that the MC1 scheme for independent observation can built a large amount of inertia.

In Figure 2a,b graphs of the MEDs are presented for some selected charts. The figures show the behavior of the modified MCUSUM scheme and the modified PPCUSUM chart based on the norm $\|\cdot\|_{\Delta_t}$, and the MEWMA chart and the MC1 chart based on the residuals. On the horizontal axis a_1 is plotted. The quantity a_2 is always equal to 1.0. In Figure 2a the results for process (22) are given while in Figure 2b the underlying target process is (23).

Figure 2a shows that for small and moderate shifts the best results are obtained for the modified MCUSUM and the modified PPCUSUM chart based on the norm $\|\cdot\|_{\Delta_t}$. The residual charts have a slightly smaller MED for larger shifts.

Taking into account all our simulations, i.e. the results for all charts and for various values of a_2, it turns out that the best results are obtained for the charts presented in Figure 2 and for the MEWMA residual charts. For small and moderate shifts the modified approaches overperform the residual schemes. The modified schemes based on the norm $\|\cdot\|_{\Delta_t}$ perform better for shifts of moderate size, while for smaller shifts the MC1$_\Gamma$ approach is the best. In particular the modified MCUSUM and the modified PPCUSUM scheme based on the norm $\|\cdot\|_{\Delta_t}$ behave very well. The MEWMA residual chart has the smallest MEDs for larger shifts.

In Figure 2b another target process is considered. One coefficient of the parameter matrix is negative. The ranking of the charts changes completely. Now the

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best scheme is the MC1 residual chart followed by the MEWMA residual scheme. The modified CUSUM charts behave worse.

An overall comparison shows that in almost all cases the MC1 residual chart and the MEWMA residual chart have the smallest MEDs. In our study we observe a very good performance of the CUSUM chart of Thedossiou in case of positive shifts. There are two possible explanations for this result. First, the CUSUM chart of Theodossiou is a residual scheme, which performs rather good for negatively correlated variables. Second, the appropriate direction of the expected change $\mathbf{a}_0$ is chosen for the selected parameters (23).

5.2 Results for the 10-Dimensional Process

Note that the process (24) fulfills the condition (15). Thus all charts considered in sections 3 and 4 are directionally invariant. Consequently it is much easier to compare the control procedures although the process has a higher dimension. The out-of-control behavior is studied as a function of the noncentrality parameter which is equal to $\lambda_{\Gamma(0)} = 0.51\mathbf{a}'\mathbf{\Sigma}^{-1}\mathbf{a}$. To simulate such a change we put $\mathbf{a} = (0, \ldots, 0, a_1)$ and took $a_1 = 1.212678\sqrt{\lambda_{\Gamma(0)}}$.

Because there is no proof of the invariance property for the multivariate EWMA chart and the CUSUM design of Theodossiou (1993), we do not consider these charts in this section.

In Figure 3 the MEDs of some selected CUSUM charts are given. For small values of λ the $MC1_\Delta$ chart provides the best results ($\lambda \leq 1.5$) followed by the $MC1_\Gamma$ and $MC1_r$ approaches. Behind these charts we find the $MCUSUM_\Delta$ and $PPCUSUM_\Delta$ schemes. For moderate and larger values of λ the $MCUSUM_\Delta$ chart has the smallest MEDs. On the second place we rank the $PPCUSUM_\Delta$ scheme followed by the $MC1_\Delta$ approach.

Summarizing the results of our comparison study, we can state that it is difficult to make recommendations about the choice of the charts. This is due to the fact that the relationship between the charts depends on the parameter matrices. In the multivariate case many parameters are present. The situation gets better if we focus on coefficient matrices satisfying (14) because the MEDs are functions of the noncentrality parameter.

Because the control limits of the residual charts do not depend on the coefficient matrices it is much easier to determine their control designs. This is an advantage in the multivariate case. Moreover, the residual charts behave quite well in case of negative autocorrelation, i.e. if one coefficient is negative [cf. process (23)]. If the process is positively autocorrelated we recommend the application of the modified MC1 schemes for shifts of small sizes and the $MCUSUM_\Delta$ and the $PPCUSUM_\Delta$ chart in case of moderate and larger shifts.

This is in line with a well-known fact (e.g. Knoth and Schmid, 2004) that the residual schemes behave better than the modified charts for univariate autoregressive

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processes of order 1 with a negative coefficient, while for a positive coefficient the modified charts should be preferred. Kramer and Schmid (1997) made the same observation for VAR(1) processes in case of MEWMA charts. Our results show that for the introduced CUSUM schemes a similar behavior is observed. In order to improve the understanding of the charts further research is necessary. It would be extremely useful to have statements about the behavior of the charts as a function of the parameter matrices.

There are many possibilities to compare control charts. In Woodall and Mahmoud (2005) the inertia behavior of control charts is discussed. They show for independent multivariate observations that the PPCUSUM chart has a much better worst-case signal resistance than the MC1 chart. We have not discussed inertia measures in our analysis.

6 Conclusion

A number of studies have already pointed out the complexity of controlling multivariate processes. The situation becomes even more difficult in the time-series context because the number of parameters is huge.

In this paper we introduce several CUSUM control charts for a multivariate stationary process. Our control schemes are extensions of the control charts of Crosier (1988), Pignatiello and Runger (1990), and Ngai and Zhang (2001) who only considered the case of independent multivariate observations. It is shown that for time series the charts are in general not directionally invariant. A sufficient condition is derived for several control charts assuming that the target process is a VARMA(1,1) process.

Based on the MED we compare all control schemes proposed in the paper. In our comparison study we include the CUSUM chart of Theodossiou (1993), the multivariate EWMA chart of Kramer and Schmid (1997) and the control procedures of Bodnar and Schmid (2006). In our comparison study we discuss the behavior of the charts for two two-dimensional VAR(1) processes which do not fulfill the invariance condition and a 10-dimensional VAR(1) model satisfying the property. It turns out that there is no best chart. Because the control limits of the residual charts do not depend on the coefficient matrices of the target process and as the same limits can be used as in the i.i.d. case they can be much easier applied than the modified charts. This is an advantage of the residual approach, especially in the present case of multivariate time series. Thus a practitioner will favor the residual approach. Moreover, in our study it has turned out that they provide in nearly all cases good results if the process is negatively correlated. In case of a positive correlation, however, we recommend the use of one of the modified MC1 charts for small shifts, the modified MCUSUM chart based on the varying Mahalanobis distance for moderate shifts, and the modified MCUSUM or the modified PPCUSUM approaches based on the varying Mahalanobis distance for larger deviations.

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Appendix

Appendix A

In the appendix the proof of Theorem 1 is given. We follow the procedure of CROSIER (1988) who considered an independent sample. We provide the proof only for the MCUSUM chart based $\|\cdot\|_{\Delta_t}$, i.e. $\text{MCUSUM}_{\Delta,t}$. For the other control schemes the result can be derived in similar way. Finally, without loss of generality it is assumed that $\boldsymbol{\mu}_0 = 0$.

First, we show that the statistics $C_{\Delta,t}$ and $\text{MCUSUM}_{\Delta,t}$ are invariant with respect to the non-degenerate transformations of the process $\{\mathbf{X}_t\}$, i.e. they are the same for the processes $\{\mathbf{X}_t\}$ and $\{\tilde{\mathbf{X}}_t\}$, where $\tilde{\mathbf{X}}_t = \mathbf{M}\mathbf{X}_t$ for all $t \geq 1$ and $\mathbf{M}$ is a regular $p \times p$ matrix.

In the following all quantities with the index '$\sim$' are calculated with respect to the process $\{\tilde{\mathbf{X}}_t\}$.

LEMMA 1. *Let $\{\mathbf{Y}_t\}$ be a stationary Gaussian process with cross-covariance function $\boldsymbol{\Gamma}(h)$ at lag h. Suppose that* (1) *holds. Let $\mathbf{M}$ be a regular $p \times p$ matrix and $\tilde{\mathbf{X}}_t = \mathbf{M}\mathbf{X}_t$ for $t \geq 1$. Then it follows that $C_{\Delta,t} = \tilde{C}_{\Delta,t}$, $\tilde{\mathbf{S}}_{\Delta,t} = \mathbf{M}\mathbf{S}_{\Delta,t}$, and $MCUSUM_{\Delta,t} = \widetilde{MCUSUM}_{\Delta,t}$.*

PROOF OF LEMMA 1. For $t < q$ it holds that $E(\tilde{\mathbf{X}}_t) = \mathbf{0}$ and for $t \geq q$ we have that $E(\tilde{\mathbf{X}}_t) = \mathbf{M}\mathbf{a}$. Furthermore, for $t \geq 1$ we obtain $\text{cov}(\tilde{\mathbf{X}}_{t+h}, \tilde{\mathbf{X}}_t) = \mathbf{M}\boldsymbol{\Gamma}(h)\mathbf{M}'$. We get for $h \geq 0$ that $\tilde{\boldsymbol{\Delta}}_h = \mathbf{M}\boldsymbol{\Delta}_h\mathbf{M}'$ and $\boldsymbol{\Delta}_h$ as in (4).

To prove the result we use the principle of induction. For $t = 1$ we see that $\tilde{C}_{\Delta,1} = \sqrt{\tilde{\mathbf{X}}_1'\tilde{\boldsymbol{\Delta}}_1^{-1}\tilde{\mathbf{X}}_1} = C_{\Delta,1}$ and for $\tilde{C}_{\Delta,1} \leq k$ that

$$\tilde{\mathbf{S}}_{\Delta,1} = \tilde{\mathbf{X}}_1\left(1 - \frac{k}{\tilde{C}_{\Delta,1}}\right) = \mathbf{M}\mathbf{X}_1\left(1 - \frac{k}{C_{\Delta,1}}\right) = \mathbf{M}\mathbf{S}_{\Delta,1}.$$

Consequently, $\widetilde{\text{MCUSUM}}_{\Delta,1} = \text{MCUSUM}_{\Delta,1}$.

It is assumed that the hypothesis holds for $t-1$ and we prove it for t. It holds

$$\begin{aligned}\tilde{C}_{\Delta,t} &= \sqrt{(\tilde{\mathbf{S}}_{\Delta,t-1} + \tilde{\mathbf{X}}_t)'\tilde{\boldsymbol{\Delta}}_t^{-1}(\tilde{\mathbf{S}}_{\Delta,t-1} + \tilde{\mathbf{X}}_t)}\\ &= \sqrt{(\mathbf{S}_{\Delta,t-1} + \mathbf{X}_t)'\mathbf{M}'\mathbf{M}'^{-1}\boldsymbol{\Delta}_t^{-1}\mathbf{M}^{-1}\mathbf{M}(\mathbf{S}_{\Delta,t-1} + \mathbf{X}_t)} = C_{\Delta,t}\end{aligned}$$

and for $\tilde{C}_{\Delta,t} \leq k$

$$\tilde{\mathbf{S}}_{\Delta,t} = \mathbf{M}(\mathbf{S}_{\Delta,t-1} + \mathbf{X}_t)\left(1 - \frac{k}{C_{\Delta,t}}\right) = \mathbf{M}\mathbf{S}_{\Delta,t}.$$

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Thus,

$$\widetilde{\mathrm{MCUSUM}}_{\Delta,t} = \sqrt{\mathbf{S}'_{\Delta,t}\mathbf{M}'\mathbf{M}'^{-1}\boldsymbol{\Delta}_t^{-1}\mathbf{M}^{-1}\mathbf{M}\mathbf{S}_{\Delta,t}} = \mathrm{MCUSUM}_{\Delta,t}.$$

This completes the proof of the lemma. □

Proof of Theorem 1. Suppose that $\mathbf{a}_1'\boldsymbol{\Gamma}(0)^{-1}\mathbf{a}_1 = \mathbf{a}_2'\boldsymbol{\Gamma}(0)^{-1}\mathbf{a}_2$. It is now proved that then the distribution of $(\mathrm{MCUSUM}_{\Delta,1}, \dots, \mathrm{MCUSUM}_{\Delta,i})$ for a shift of size $\mathbf{a}_1$ at position q is equal to its distribution for a shift of size $\mathbf{a}_2$ at position q, i.e.

$$\begin{aligned} &P_{\mathbf{a}_1,q}((\mathrm{MCUSUM}_{\Delta,1}, \dots, \mathrm{MCUSUM}_{\Delta,i}) \in \mathbf{B}) \\ &\quad = P_{\mathbf{a}_2,q}((\mathrm{MCUSUM}_{\Delta,1}, \dots, \mathrm{MCUSUM}_{\Delta,i}) \in \mathbf{B}) \end{aligned}$$

for all Borel sets $\mathbf{B} \subset \mathbb{R}^i$.

Let $\boldsymbol{\upsilon}_1 = \boldsymbol{\Gamma}(0)^{-1/2}\mathbf{a}_1$ and $\boldsymbol{\upsilon}_2 = \boldsymbol{\Gamma}(0)^{-1/2}\mathbf{a}_2$. Then it holds that $\boldsymbol{\upsilon}_1'\boldsymbol{\upsilon}_1 = \boldsymbol{\upsilon}_2'\boldsymbol{\upsilon}_2$. There exists an orthogonal matrix $\mathbf{Q}$ (e.g. Crosier, 1988) such that $\boldsymbol{\upsilon}_2 = \mathbf{Q}\boldsymbol{\upsilon}_1$. Hence,

$$\mathbf{a}_2 = \boldsymbol{\Gamma}(0)^{1/2}\mathbf{Q}\boldsymbol{\Gamma}(0)^{-1/2}\mathbf{a}_1. \tag{26}$$

Define $\mathbf{M} = \boldsymbol{\Gamma}(0)^{1/2}\mathbf{Q}\boldsymbol{\Gamma}(0)^{-1/2}$ and let $\tilde{\mathbf{X}}_t = \mathbf{M}\mathbf{X}_t$. Then $E(\tilde{\mathbf{X}}_t) = \mathbf{M}\mathbf{a}_1 = \mathbf{a}_2$ for $t \geq q$ and $E(\tilde{\mathbf{X}}_t) = \mathbf{0} = E(\mathbf{X}_t)$ for $t < q$. It holds for $t \geq 1$ that

$$\mathrm{cov}(\tilde{\mathbf{X}}_{t+h}, \tilde{\mathbf{X}}_t) = \mathbf{M}\boldsymbol{\Gamma}(h)\mathbf{M}' = d_h\boldsymbol{\Gamma}(0)^{1/2}\mathbf{Q}\boldsymbol{\Gamma}(0)^{-1/2}\boldsymbol{\Gamma}(0)\boldsymbol{\Gamma}(0)^{-1/2}\mathbf{Q}'\boldsymbol{\Gamma}(0)^{1/2} = \boldsymbol{\Gamma}(h).$$

Because the underlying processes are Gaussian, it follows that the distribution of $\{\tilde{\mathbf{X}}_t\}$ with respect to $P_{\mathbf{a}_1,q}$ is equal to the distribution of $\{\mathbf{X}_t\}$ with respect to $P_{\mathbf{a}_2,q}$, i.e. $P_{\mathbf{a}_1,q}(\{\tilde{\mathbf{X}}_t\} \in \mathbf{A}) = P_{\mathbf{a}_2,q}(\{\mathbf{X}_t\} \in \mathbf{A})$ for all $\mathbf{A}$. Using Lemma 1 it follows that for $i \in \mathbb{N}$

$$\begin{aligned} &P_{\mathbf{a}_1,q}((\mathrm{MCUSUM}_{\Delta,1}, \dots, \mathrm{MCUSUM}_{\Delta,i} \in \mathbf{B}) \\ &\quad = P_{\mathbf{a}_1,q}(\widetilde{\mathrm{MCUSUM}}_{\Delta,1}, \dots, \widetilde{\mathrm{MCUSUM}}_{\Delta,i}) \in \mathbf{B}) \\ &\quad = P_{\mathbf{a}_2,q}((\mathrm{MCUSUM}_{\Delta,1}, \dots, \mathrm{MCUSUM}_{\Delta,i}) \in \mathbf{B}) \end{aligned}$$

and thus the theorem follows. □

Proof of the Equivalence in the Example. It is clear that (14) implies (15). We will show that the contrary is valid as well.

Suppose that (15) holds for all positive-definite matrices $\boldsymbol{\Sigma}$. We take $\boldsymbol{\Sigma} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$ with an orthogonal matrix $\mathbf{U}$ and a diagonal matrix $\boldsymbol{\Lambda}$ with positive diagonal elements. Then $\mathbf{U}'\boldsymbol{\Phi}\mathbf{U}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\mathbf{U}'\boldsymbol{\Phi}'\mathbf{U}$ for all $\boldsymbol{\Lambda}$. This implies that $\mathbf{U}'\boldsymbol{\Phi}\mathbf{U} = \lambda\mathbf{I}$ with $\lambda \in \mathbb{R}$. Thus (14) follows. □

Appendix B

In this section we briefly present the control schemes of Theodossiou (1993) and Kramer and Schmid (1997).

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Theodossiou (1993) derived his chart by applying the sequential probability ratio test to a VARMA process. It is necessary to fix a direction $\mathbf{a}_0$ of the shift. Assuming that $\boldsymbol{\mu}_0 = 0$, the chart gives a signal as soon as

$$TC_t = \min\{TC_{t-1} + Z_t - k, 0\} < -L \quad \text{for } k, L > 0, \tag{27}$$

where

$$Z_t = \beta_0 + \boldsymbol{\beta}_1(\mathbf{X}_t - \boldsymbol{\Phi}\mathbf{X}_{t-1} + \boldsymbol{\Theta}\boldsymbol{\varepsilon}_{t-1}) \tag{28}$$

with $\beta_0 = D/2$, $\boldsymbol{\beta}_1 = -\mathbf{a}_0'(\mathbf{I}-\boldsymbol{\Phi})'\boldsymbol{\Sigma}^{-1}/D$, and $D^2 = \mathbf{a}_0'(\mathbf{I}-\boldsymbol{\Phi})'\boldsymbol{\Sigma}^{-1}(\mathbf{I}-\boldsymbol{\Phi})\mathbf{a}_0$.

In our simulation study we take $\mathbf{a}_0 = (1, (1+\sqrt{3})/2)'$ for the 2-dimensional situation discussed in section 5.1.

The control method of Kramer and Schmid (1997) is based on the multivariate EWMA recursion

$$\mathbf{W}_t = \mathbf{R}\mathbf{X}_t + (\mathbf{I}-\mathbf{R})\mathbf{W}_{t-1}, \quad \text{for } t \geq 1 \tag{29}$$

and $\mathbf{W}_0 = \boldsymbol{\mu}_0$. The control chart signals an alarm as soon as the control statistic

$$Q_t = (\mathbf{W}_t - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_{\mathbf{W}_t}^{-1}(\mathbf{W}_t - \boldsymbol{\mu}_0) \tag{30}$$

exceeds the upper control limit. Here $\boldsymbol{\Sigma}_{\mathbf{W}_t}$ is the covariance matrix of the process $\{\mathbf{W}_t\}$. Although $\mathbf{R}$ is an arbitrary matrix it is usually taken as a diagonal matrix. In order to reduce the number of smoothing parameters we have taken it here as $\mathbf{R} = r\mathbf{I}$, i.e. all components are weighted in the same way.

References

Albers, W. and W. C. M. Kallenberg (2004), Are estimated control charts in control?, *Statistics* **38**, 67–79.

Alwan, L. C. (1989), *Time series modelling for statistical process control*, PhD dissertation, Graduate School of Business, Chicago, IL.

Alwan, L. C. and H. V. Roberts (1988), Time-series modelling for statistical process control, *Journal of Business and Economic Statistics* **6**, 87–95.

Andersson, E., D. Bock and M. Frisen (2004), Detection of turning points in business cycles, *Journal of Business Cycle Measurement and Analysis* **1**, 93–108.

Bodnar, O. (2005), *Control charts for multivariate financial time series*, PhD thesis, European University Viadrina, Frankfurt (Oder), Germany.

Bodnar, O. (2007), Sequential procedures for monitoring covariances of asset returns, in: G. N. Gregoriou (ed.), *Advances in risk management*, Palgrave, London, 241–264.

Bodnar, O. and W. Schmid (2006), CUSUM control schemes for multivariate time series, in: H.-J. Lenz and P. -Th. Wilrich (eds.), *Frontiers of statistical process control*, Vol. 8, Physica, Heidelberg, 55–73.

Brockwell, P. J. and R. A. Davis (1991), *Time series: theory and methods*, Springer, New York.

Conte, S. D. and C. De Boor (1981), *Elementary numerical analysis*, Mc Graw-Hill, London.

Crosier, R. B. (1986), A new two-sided cumulative sum quality control scheme, *Technometrics* **28**, 187–194.

Crosier, R. B. (1988), Multivariate generalizations of cumulative sum quality control schemes, *Technometrics* **30**, 291–303.

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Frisen, M. (1992), Evaluations of methods for statistical surveillance, *Statistics in Medicine* **11**, 1489–1502.
Goldsmith, P. L. and H. Whitfield (1961), Average run lengths in cumulative chart quality control schemes, *Technometrics* **3**, 11–20.
Harris, T. J. and W. H. Ross (1991), Statistical process control procedures for correlated observations, *Canadian Journal of Chemical Engineering* **69**, 48–57.
Harrison, P. J. and O. L. Davies (1964), The use of cumulative sum techniques for the control of routine forecasts of product demand, *Operation Research* **12**, 325–333.
Hawkins, D. M. (1991), Multivariate quality control based on regression-adjusted variables, *Technometrics* **33**, 61–75.
Hawkins, D. M. (1993), Regression adjustment for variables in multivariate quality control, *Journal of Quality Technology* **25**, 170–182.
Healy, J. D. (1987), A note on multivariate CUSUM procedure, *Technometrics* **29**, 409–412.
Hotelling, H. (1947), Multivariate quality control – illustrated by the air testing of sample bombsights, in: C. Eisenhart, M. W. Hastay and W. A. Wallis (eds.), *Techniques of statistical analysis*, McGraw-Hill, New York, 111–184.
Jiang, W., K.-L. Tsui and W. H. Woodall (2000), A new SPC monitoring method: the ARMA chart, *Technometrics* **42**, 399–410.
Juran, J. M. (1997), Early SPC: a historical supplement, *Quality Progress* **30**, 73–81.
Knoth, S. and W. Schmid (2004), Control charts for time series: a review, in: H.-J. Lenz and P.-Th. Wilrich (eds.), *Frontiers of statistical process control*, Vol. 7, Physica, Heidelberg, 210–236.
Kramer, H. and W. Schmid (1997), EWMA charts for multivariate time series, *Sequential Analysis* **16**, 131–154.
Kramer, H. and W. Schmid (1998), On the average delay of control charts, in: P. Kahle, E. von Collani, J. Franz and U. Jensen (eds.), *Advance in stochastic models for relability, quality and safety*, 341–360.
Kramer, H. and W. Schmid (2000), The influence of parameter estimation on the ARL of Shewhart-type charts for time series, *Statistical Papers* **41**, 173–196.
Lawson, A. B. and K. Kleinman (2005), *Spatial & syndromic surveillance*, Wiley, New York.
Lowry, C. A., W. H. Woodall, C. W. Champ and S. E. Rigdon (1992), A multivariate exponentially weighted moving average control chart, *Technometrics* **34**, 46–53.
Lu, C. W. and M. R. Reynolds (1999), EWMA control charts for monitoring the mean of autocorrelated processes, *Journal of Quality and Technology* **31**, 166–188.
Montgomery, D. C. and C. M. Mastrangelo (1991), Some statistical process control methods for autocorrelated data, *Journal of Quality Technology* **23**, 179–204.
Moustakides, G. V. (1986), Optimal stopping times for detecting changes in distributions, *Annals of Statistics* **14**, 1379–1387.
Ngai, H.-M. and J. Zhang (2001), Multivariate cumulative sum control charts based on projection pursuit, *Statistica Sinica* **11**, 747–766.
Page, E. S. (1954), Continuous inspection schemes, *Biometrika* **41**, 100–114.
Pignatiello, J. J. Jr and G. C. Runger (1990), Comparison of multivariate CUSUM charts, *Journal of Quality Technology* **22**, 173–186.
Pollak, M. (1985), Optimal detection of a change in distribution, *Annals of Statistics* **13**, 206–227.
Pollak, M. and D. Siegmund (1975), Approximations to the expected sample size of certain sequential tests, *Annals of Statistics* **3**, 1267–1282.
Roberts, S. W. (1959), Control charts tests based on geometric moving averages, *Technometrics* **1**, 239–250.
Schipper, S. and W. Schmid (2001), Sequential methods for detecting changes in the variance of economic time series, *Sequential Analysis* **20**, 235–262.
Schmid, W. (1995), On the run length of a Shewhart chart for correlated data, *Statistical Papers* **36**, 111–130.

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Schmid, W. (1997a), On EWMA charts for time series, in: H.-J. Lenz and P.-Th. Wilrich (eds.), *Frontiers of statistical process control*, Vol. 5, Physica, Heidelberg, 115–137.

Schmid, W. (1997b), CUSUM control schemes for Gaussian processes, *Statistical Papers* **38**, 191–217.

Schmid, W. and D. Tzotchev (2004), Sequential monitoring of the parameters of a one-factor Cox-Ingersoll-Ross model, *Sequential Analysis* **23**, 1–40.

Shewhart, W. A. (1926), Quality control charts, *Bell System Technical Journal* **3**, 593–603.

Shewhart, W. A. (1931), *Economic control of quality of manufactured product*, D. van Nostrand Company, Toronto.

Sonesson, C. and D. Bock (2003), A review and discussion of prospective statistical surveillance in public health, *Journal of the Royal Statistical Society* A **166**, 5–21.

Theodossiou, P. T. (1993), Predicting shifts in the mean of a multivariate time series process: an application in predicting business failures, *Journal of the American Statistical Association* **88**, 441–449.

Vasilopoulos, A. V. and A. P. Stamboulis (1978), Modification of control chart limits in the presence of data correlation, *Journal of Quality Technology* **20**, 20–30.

Wald, A. (1947), *Sequential analysis*, Wiley, New York.

Wardell, D. G., H. Moskowitz and R. D. Plante (1994a), Run length distributions of residual control charts for autocorrelated processes, *Journal of Quality Technology* **26**, 308–317.

Wardell, D. G., H. Moskowitz and R. D. Plante (1994b), Run length distributions of special-cause control charts for correlated processes, *Technometrics* **36**, 3–27.

Wierda, S. J. (1994), Multivariate statistical process control - recent results and directions for future research, *Statistica Neerlandica* **48**, 147–168.

Woodall, W. H. and M. A. Mahmoud (2005), The inertial properties of quality control charts, *Technometrics* **47**, 425–436.

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