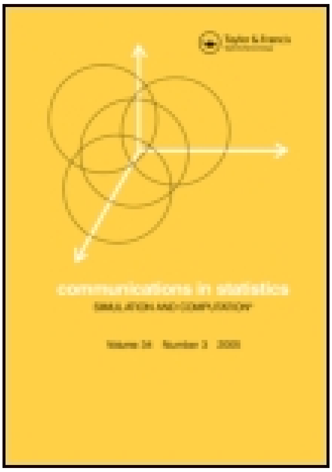

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Quality Control

Optimal Statistical Design of a Multivariate EWMA Chart Based on ARL and MRL

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Statistical design is applied to a multivariate exponentially weighted moving average (MEWMA) control chart. The chart parameters are control limit H and smoothing constant r. The choices of the parameters depend on the number of variables p and the size of the process mean shift δ. The MEWMA statistic is modeled as a Markov chain and the Markov chain approach is used to determine the properties of the chart. Although average run length has become a traditional measure of the performance of control schemes, some authors have suggested other measures, such as median and other percentiles of the run length distribution to explain run length properties of a control scheme. This will allow a thorough study of the performance of the control scheme. Consequently, conclusions based on these measures would provide a better and comprehensive understanding of a scheme. In this article, we present the performance of the MEWMA control chart as measured by the average run length and median run length. Graphs are given so that the chart parameters of an optimal MEWMA chart can be determined easily.

Keywords Average run length (ARL); Markov chain; Median run length (MRL); Multivariate exponentially weighted moving average (MEWMA) control chart.

Mathematics Subject Classification 62P30; 60J22.

1. Introduction

Control charts are used in the monitoring of the quality of products from manufacturing processes. Over the past 20 years, multivariate extensions of the univariate cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) schemes have been developed. Crosier (1988) and Pignatiello and Runger (1990) present several multivariate CUSUM (MCUSUM) charts. Lowry et al. (1992) extend the univariate EWMA statistic to the multivariate case.

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The methods that are usually used to evaluate the performance of control schemes are simulation, integral equation, and Markov chain approaches. The Markov chain method is introduced by Brook and Evans (1972) for evaluating the performance of the one-sided CUSUM chart. Lucas and Saccucci (1990) then consider the Markov chain approximation to examine the performance of a two-sided EWMA chart and provide design recommendations for the chart. Crowder (1987a) derives an integral equation for the EWMA chart and a computer program is presented by Crowder (1987b) to calculate the average run length (ARL) of the EWMA chart using integral equation. Champ and Rigdon (1991) show that if the product midpoint rule is used to approximate the integral equation, the integral equation approach, and the Markov chain approach yield the same approximation for the ARL. Calzada and Scariano (2003) also study the integral equation and Markov chain approaches for computing the ARL of a two-sided EWMA chart. Recently, Fu et al. (2003) introduced a general unified framework on the Markov chain embedding technique which is based either on a simple boundary crossing rule, or on a compound rule. Runger and Prabhu (1996) describe a two-dimensional Markov chain approach to determine the run length performance of a MEWMA control chart. Prabhu and Runger (1997) use the Markov chain method to provide design recommendations for the MEWMA chart. Molnau et al. (2001b) then present a computer program that calculates the ARL for the MEWMA control chart using the Markov chain approximation. Rigdon (1995a,b) considers an integral equation and a double integral equation to calculate the in-control and out-of-control ARLs for the MEWMA scheme, respectively. Bodden and Rigdon (1999) provide a computer program for the in-control ARL approximation of the MEWMA chart that can be expressed as the solution of an integral equation.

Control charts have been developed to fulfill statistical requirements, economic requirements, or both. Woodall (1985) presents a method for designing the control charts on the basis of their statistical performance. Montgomery (1980) provides an economic design of control charts to minimize the expected total cost. Saniga (1989) proposes an economic statistical design. Linderman and Love (2000a,b) apply the economic statistical design approach to a MEWMA chart. Molnau et al. (2001a) show that the economic statistical design can provide for better statistical properties without significantly increasing the optimal total cost. Noorossana et al. (2002) offer an alternative procedure that can improve the ARL properties and the overall performance of a multivariate control chart using economic design. A review and comparison of different design strategies of the MEWMA chart is provided by Testik and Borror (2004).

Reviews of optimal design of control charts based on statistical performance have appeared in several statistical literatures. Lucas and Saccucci (1990) provide a table of chart parameters for an optimal EWMA chart. Crowder (1989) gives plots of optimal smoothing parameters and control limit constants which make the design of the EWMA chart simple. Plots are also given by Gan (1991) which enable the chart parameters of an optimal CUSUM chart to be determined easily. Prabhu and Runger (1997) present a table of an optimal MEWMA control scheme that provides the optimum choices for r and the corresponding control limits H for selected number of variables, in-control ARLs, and sizes of shifts. Recently, Aparisi and García-Díaz (2004) present a program for the optimal design of EWMA and MEWMA chart parameters using genetic algorithms.

Prabhu and Runger (1997) show that the run length distribution of a MEWMA chart is highly skewed using in-control ARL of 200 and the distribution is also

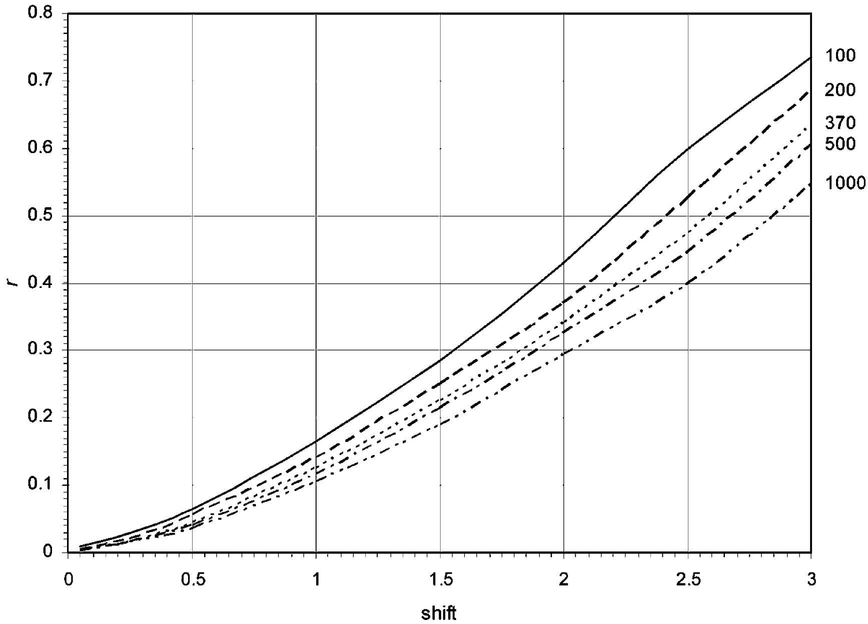

Figure 1. Optimal chart parameter r's for $p = 2$ with in-control ARLs of 100, 200, 370, 500, and 1,000.

highly skewed when the process is out-of-control with the process mean shifted by $\delta = 2$ for $p = 4$, 10, and 20. For highly skewed run length distributions, other properties of the run length distributions, such as the median, can provide important information that can be used to study and design a control chart. Gan (1992, 1993a) uses median run length (MRL) in the design of a CUSUM and EWMA chart to determine the optimal chart parameters, respectively. In this article, the design strategy of the MEWMA chart developed by Prabhu and Runger (1997) is employed to obtain the optimal values of the chart parameters. The performance of the MEWMA chart is evaluated by both ARL and MRL. Besides simplifying the existing procedure in designing an optimal MEWMA chart which is based mainly on the ARL, this article introduces the design strategy of the chart using the MRL. Due to space constraints, we only give the graphs of optimal parameters for $p = 2$ and 4. However, the graphs for $p = 3, 5, 8, 10$, and 20 for the in-control ARLs or MRLs of 100, 200, 300, 370, 400, 500, 600, 700, 800, and 1000 (including the in-control ARLs or MRLs of 300, 400, 600, 700, and 800 for $p = 2$ and 4 that are not given in Figs. 1–8) are available from the authors upon request.

2. MEWMA Control Chart

The MEWMA vector proposed by Lowry et al. (1992) is defined as

$$\boldsymbol{Z}_t = r\boldsymbol{X}_t + (1 - r)\boldsymbol{Z}_{t-1} \quad t = 1, 2, 3, \ldots \tag{1}$$

where $\mathbf{Z}_0 = \mathbf{0}$ and $0 < r \leq 1$. $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$ are assumed to be independent multivariate normal random vectors with p components. To study the performance

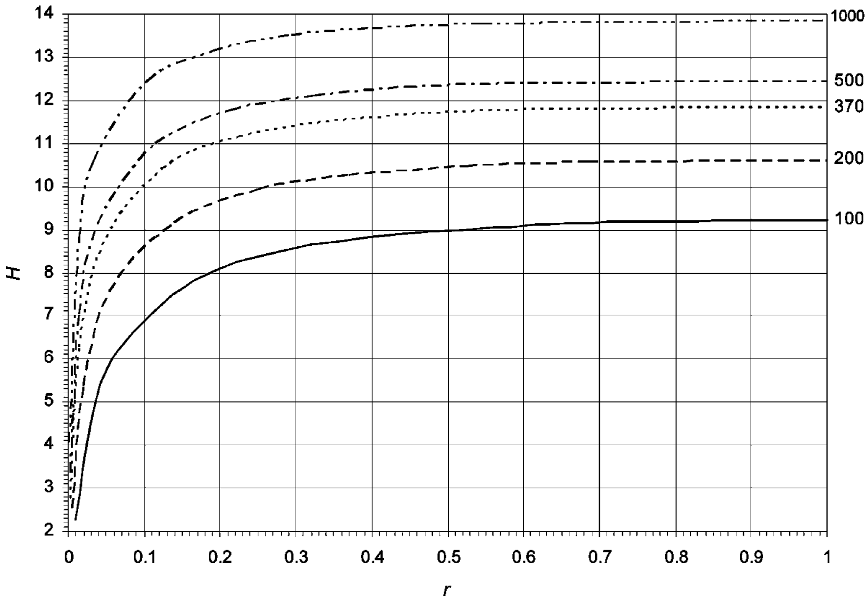

Figure 2. Combinations of r and H for $p = 2$ with in-control ARLs of 100, 200, 370, 500, and 1,000.

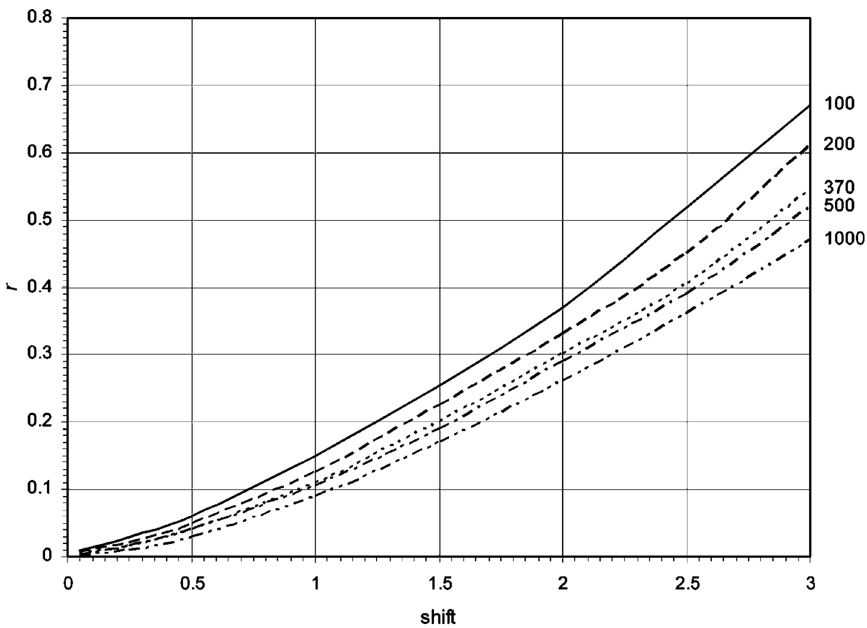

Figure 3. Optimal chart parameter r's for $p = 4$ with in-control ARLs of 100, 200, 370, 500, and 1,000.

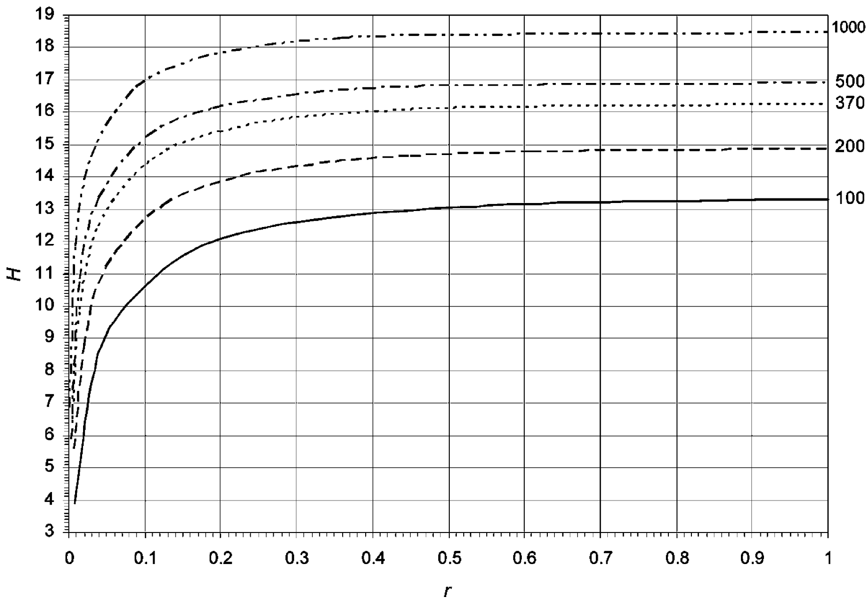

Figure 4. Combinations of r and H for $p = 4$ with in-control ARLs of 100, 200, 370, 500, and 1,000.

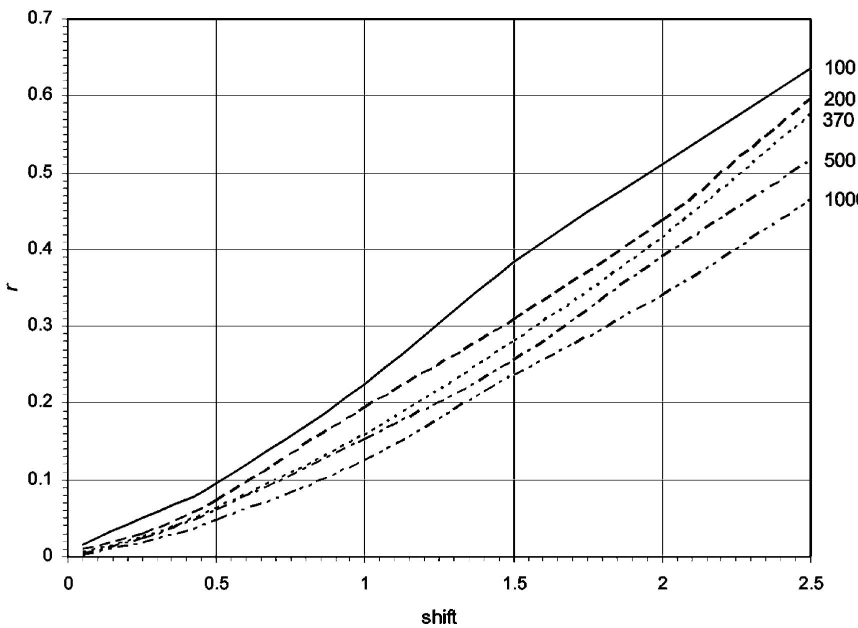

Figure 5. Optimal chart parameter r's for $p = 2$ with in-control MRLs of 100, 200, 370, 500, and 1,000.

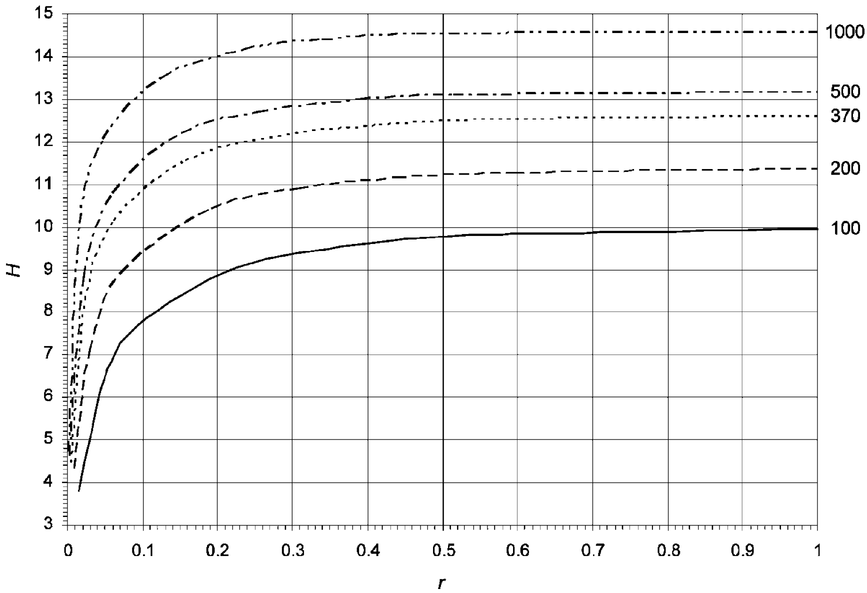

Figure 6. Combinations of r and H for $p = 2$ with in-control MRLs of 100, 200, 370, 500, and 1,000.

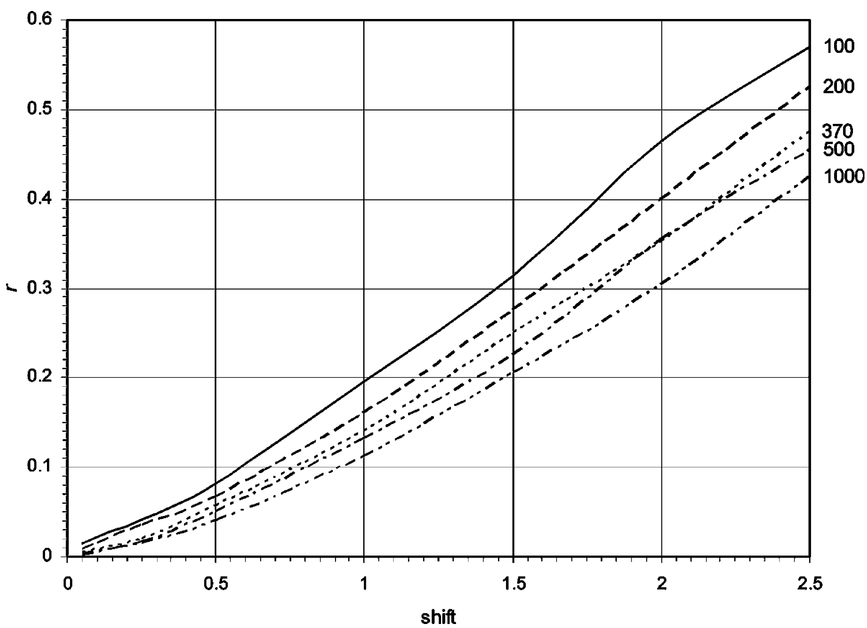

Figure 7. Optimal chart parameter r's for $p = 4$ with in-control MRLs of 100, 200, 370, 500, and 1,000.

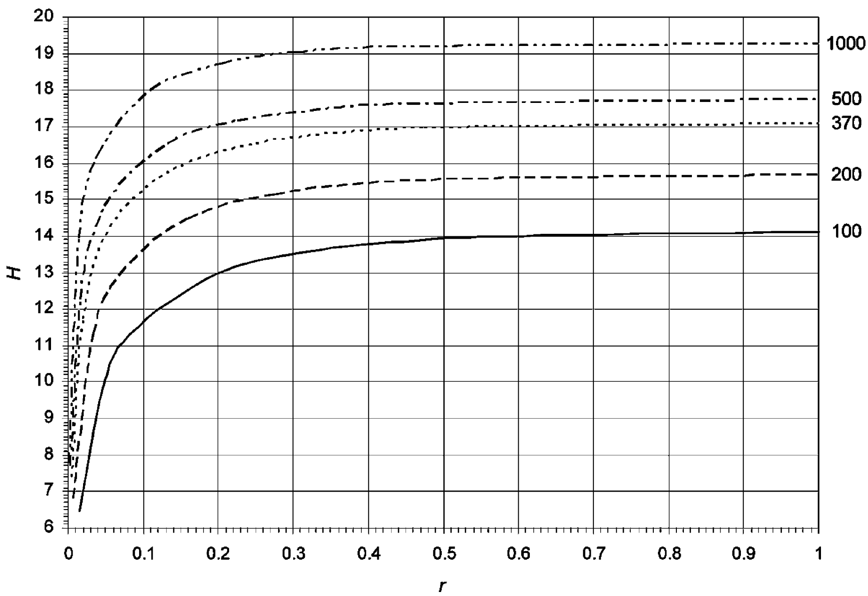

Figure 8. Combinations of r and H for $p = 4$ with in-control MRLs of 100, 200, 370, 500, and 1,000.

of the MEWMA chart, it is also assumed without loss of generality that the on-target process mean vector $\boldsymbol{\mu}_0$ is a vector of zeros.

The plotted values on the control chart are $T_t^2 = \boldsymbol{Z}_t' \sum_{\boldsymbol{Z}_t}^{-1} \boldsymbol{Z}_t$. The chart signals that the process is out-of-control if $T_t^2 > H$ where the control limit H is a constant chosen to achieve the desired in-control ARL and $\sum_{\boldsymbol{Z}_t} = \frac{r}{2-r}[1-(1-r)^{2t}] \sum_{\mathbf{X}}$ is the covariance matrix of $\mathbf{Z}_t$.

Lowry et al. (1992) have shown that the run length performance of the MEWMA chart depends on the off-target mean vector $\boldsymbol{\mu}$ and covariance matrix of $\mathbf{X}_t$, $\sum_{\mathbf{X}}$ only through the value of the non centrality parameter $\delta = (\boldsymbol{\mu}' \sum_{\mathbf{X}}^{-1} \boldsymbol{\mu})^{1/2}$.

3. Markov Chain Approach

The two-dimensional Markov chain for evaluating the ARL performance of the MEWMA chart, which was developed by Runger and Prabhu (1996), is used in this article. The Markov chain approximation is implemented in the PROC IML of SAS 8.0.

The asymptotic form of the covariance matrix of $\mathbf{Z}_t$ is $\sum_{\mathbf{Z}} = \lim_{t\to\infty} \sum_{\mathbf{Z}_t} = \frac{r}{2-r} \sum_{\mathbf{X}}$ where $\sum_{\mathbf{X}}$ is assumed to be an identity covariance matrix. Then T_t^2 can be modified to $T_t^2 = b\|\mathbf{Z}_t\|^2$ where $b = (2-r)/r$. The statistic $q_t = \|\mathbf{Z}_t\|$ is plotted with q_t being a measure of distance in p-dimensional space. Then $\mathrm{UCL} = \mathrm{H}^{\frac{1}{2}}/b^{\frac{1}{2}}$.

3.1. *In-Control Case* ($\boldsymbol{\delta} = \mathbf{0}$)

The ARL analysis for the in-control case can be approximated by one dimensional Markov chain (Runger and Prabhu, 1996). For $i = 0, 1, 2, \ldots, m$ and j not equal to 0, the probability of a transition from state i to state j is denoted by

$$p(i, j) = P\left[\frac{(j-0.5)^2 g^2}{r^2} < \chi^2(p, c) < \frac{(j+0.5)^2 g^2}{r^2}\right] \tag{2}$$

where $\chi^2(p, c)$ denotes a non central chi-square random variable with p degrees of freedom and noncentrality parameter $c = [(1-r)ig/r]^2$ where the width of the states $g = 2\text{UCL}/(2m+1)$. For $j = 0$,

$$p(i, 0) = P\left[\chi^2(p, c) < \frac{(0.5)^2 g^2}{r^2}\right]. \tag{3}$$

The zero-state ARL of a MEWMA control scheme modeled as the Markov chain is

$$\text{ARL} = \mathbf{s}'(\mathbf{I} - \mathbf{P})^{-1}\mathbf{1} \tag{4}$$

where $\mathbf{1}$ is the $m+1$ column vector with each element unity, $\mathbf{I}$ is the $m+1$ dimensional identity matrix, and $\mathbf{s}$ is the $m+1$ initial probability vector with a single element equals one, corresponding to the starting state of the chain and zeros elsewhere. For the in-control case, the 1st element of $\mathbf{s}$ is the starting state of the chain, i.e., $\mathbf{s}' = [1\ 0\ 0\ 0\ \ldots\ 0\ 0\ 0]$.

3.2. *Out-of-Control Case* ($\delta \neq 0$)

The MEWMA control scheme can be analyzed as a two dimensional Markov chain with $\mathbf{Z}_t$ partitioned into Z_{t1} and $\mathbf{Z}_{t2}$. Then $q_t = \|\mathbf{Z}_t\| = (Z_{t1}^2 + \mathbf{Z}'_{t2}\mathbf{Z}_{t2})^{1/2}$ (Runger and Prabhu, 1996).

The transition probability of Z_{t1} from state i to state j is denoted by $h(i, j)$. For $i, j = 1, 2, \ldots, 2m_1 + 1$,

$$\begin{aligned} h(i, j) = {} & \Phi\left[\frac{-\text{UCL} + jg_1 - (1-r)c_i}{r} - \delta\right] \\ & -\Phi\left[\frac{-\text{UCL} + t(j-1)g_1 - (1-r)c_i}{r} - \delta\right] \end{aligned} \tag{5}$$

where $\Phi(\cdot)$ represents the cumulative standard normal distribution function with $c_i = -\text{UCL} + (i - 0.5)g_1$ be the centerpoint of state i and $g_1 = 2\text{UCL}/(2m_1 + 1)$ is the width of each state.

The transition probability of $\|\mathbf{Z}_{t2}\|$ from state i to state j is denoted by $v(i, j)$. This transition probability has the same expression as in the in-control case except that $p - 1$ replaces p. For $i = 0, 1, 2, \ldots, m_2$ and $j = 1, 2, \ldots, m_2$,

$$v(i, j) = P\left[\frac{(j-0.5)^2 g_2^2}{r^2} < \chi^2(p-1, c) < \frac{(j+0.5)^2 g_2^2}{r^2}\right] \tag{6}$$

where $c = [(1-r)ig_2/r]^2$ and the width of the states, $g_2 = 2\text{UCL}/(2m_2 + 1)$. For $j = 0$,

$$v(i, 0) = P\left[\chi^2(p-1, c) < \frac{0.5^2 g_2^2}{r^2}\right]. \tag{7}$$

Let $\mathbf{P}$ denote the transition probability matrix of the two dimensional Markov chain. Then $\mathbf{P} = \mathbf{H} \otimes \mathbf{V}$ where $\mathbf{H}$ is the transition probability matrix of Z_{t1}, $\mathbf{V}$ is the transition probability matrix of $\|\mathbf{Z}_{t2}\|$, and $\otimes$ denotes the Kronecker product of the matrices. The transition probability matrix $\mathbf{P}$ consists of the transition probabilities between all transient and some absorbing states of the Markov chain.

The bivariate Markov chain is formed by the ordered pair (α, β) where $\alpha = 1, 2, \ldots, 2m_1 + 1$ and $\beta = 0, 1, \ldots, m_2$. The ordered pair is a transient state if $[\alpha - (m_1 + 1)]^2 g_1^2 + \beta^2 g_2^2 < \mathrm{UCL}^2$. See Runger and Prabhu (1996) for more detailed explanation about the two-dimensional Markov chain.

Let $\mathbf{T}$ denote a $(2m_1 + 1) \times (m_2 + 1)$ dimensional matrix with a one in each component corresponding to a transient state and zero otherwise, i.e.,

$$\mathbf{T}(\alpha, \beta) = \begin{cases} 1 & \text{if state } (\alpha, \beta) \text{ is transient} \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{P}_t$ be the transition probability matrix that contains all transient states of the chain. Then $\mathbf{P}_t = \mathbf{T}(\alpha, \beta) \odot \mathbf{P}$ where the symbol $\odot$ indicates elementwise multiplication of the matrices.

The ARL of a MEWMA control scheme is defined as $\mathrm{ARL} = \mathbf{s}'(\mathbf{I} - \mathbf{P}_t)^{-1}\mathbf{1}$ where $\mathbf{1}$ is the vector of all ones and $\mathbf{s}$ is the initial probability vector with all components equal to zero except the component corresponding to the state $\alpha = m_1 + 1$ and $\beta = 0$ equals one. For the out-of-control case, the component equals one is the $\{m(m+1)+1\}$th component where $m_1 = m_2 = m$. By using greater number of states, the approximation of ARL will become more precise. A discussion about the effect of the number of states on the ARL values can be found in Molnau et al. (2001b).

4. Probability Distribution of Run Length

Two methods for computing the percentiles of the run length distribution of a MEWMA scheme are given here. Let N be the run length of the scheme, then the cumulative probability distribution of the run length given by Brook and Evans (1972) is determined as

$$P(\mathrm{N} \leq n) = \mathbf{s}'(\mathbf{I} - \mathbf{P}_t^n)\mathbf{1}. \tag{8}$$

The percentile of the run length distribution can be obtained from the cumulative distribution function of Eq. (8). Palm (1990) calculated the 100γth percentile of the run length distribution as the smallest integer value N_γ such that $\mathrm{P}(\mathrm{N} \leq \mathrm{N}_\gamma) > \gamma$. The 100γth percentile can also be computed using Eq. (9) which is given by Gan (1993b):

$$P(\mathrm{N} \leq \mathrm{N}_\gamma - 1) \leq \gamma \quad \text{and} \quad P(\mathrm{N} \leq \mathrm{N}_\gamma) > \gamma. \tag{9}$$

By letting $\gamma = 0.5$ in Eq. (9), we obtain $\mathrm{MRL} = \mathrm{N}_{0.5}$. Approximation (10) below can be used as an alternative method to find the value of the upper percentage point of the run length distribution. This approximation is introduced by Brook and Evans (1972) for a CUSUM chart and can be extended to the MEWMA procedure. Let $\mathrm{N}_i(\alpha)$ be the value of run length for an upper tail probability of α when starting from state i, then

$$N_i(\alpha) = 1 + \frac{1}{\log \lambda} \log\left\{\frac{\alpha \sum xy}{x_i \sum y}\right\} = 1 + \frac{1}{\log \lambda} \log\left(\frac{\alpha}{c_i}\right) \tag{10}$$

and since $\alpha = 1 - \gamma$, then the 100γth percentile of the scheme can be expressed as

$$\mathrm{N}_i(\gamma) = 1 + \frac{1}{\log \lambda} \log\left(\frac{1-\gamma}{c_i}\right) \tag{11}$$

since $c_i = \frac{x_i \sum y}{\sum xy}$, where $\sum xy = \mathbf{x}'\mathbf{y}$ ($\mathbf{x}'$ and $\mathbf{y}$ are the right-hand and left-hand eigenvectors, respectively, corresponding to the eigenvalue λ where $\mathbf{P}_t\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}'\mathbf{P}_t = \lambda\mathbf{y}'$), $\sum y$ is the sum of all elements of $\mathbf{y}$, x_i is the ith element of $\mathbf{x}$, λ is the largest eigenvalue of the transition probability matrix $\mathbf{P}_t$. By rounding the value of $N_i(\gamma)$ to the next largest integer, N_γ (i.e., the 100γth percentile of the run length distribution) can be approximated. For $\delta = 0$, i equals 0, whereas for $\delta \neq 0$, $i = m(m+1) + 1$ correspond to the starting state for the out-of-control case where $\alpha = m + 1$ and $\beta = 0$ for $m_1 = m_2 = m$.

Table 1 shows the values of the 5th, 10th, 50th and 75th percentiles of the run length distribution for various sizes of shifts, δ, calculated using Eqs. (8) and (9) for

Table 1
A comparison of the percentage points of the run length distribution computed using the exact (Exact) and approximated (Approx.) methods for in-control MRL = 100, $r = 0.1$, $p = 2$ and 10, and for various sizes of shifts, δ

			Percentiles							
			5th		10th		50th		75th	
p	H	δ	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
2	7.80	0.00	14	14	21	21	100	105	192	198
		0.10	12	14	18	20	78	79	149	150
		0.25	10	12	13	15	44	45	78	79
		0.50	7	10	8	11	20	21	31	32
		1.00	4	8	5	8	8	10	11	13
		2.00	3	6	3	6	4	6	5	6
		3.00	2	4	2	4	3	5	3	5
		4.00	2	4	2	4	2	4	2	4
		5.00	1	3	2	3	2	3	2	3
10	21.35	0.00	17	17	24	23	100	99	189	189
		0.10	15	16	21	22	83	84	156	157
		0.25	13	16	18	19	59	60	106	107
		0.50	10	14	13	15	30	31	48	49
		1.00	7	11	8	11	13	15	17	19
		2.00	4	8	4	8	6	8	7	9
		3.00	3	6	3	6	4	6	5	6
		4.00	2	5	3	5	3	5	3	5
		5.00	2	4	2	4	3	4	3	5

the exact method and Eq. (11) for the approximation. An in-control MRL of 100 for $p = 2$ and 10 based on $r = 0.1$ is considered. Numerical results obtained indicate that the percentiles of the run length computed using the exact method are very close to the approximation for a wide range of shifts. This result is consistent with the univariate EWMA chart of Palm (1990). An explanation about the accuracy of the approximation is also given by Brook and Evans (1972) for the univariate CUSUM chart.

5. Design of Optimal MEWMA Control Chart

The selection of parameters of the MEWMA control chart using Markov chain analysis is based on the recommendation of Prabhu and Runger (1997). ARL and MRL are used to evaluate the performance of the chart. Typically, MRL is less than ARL. For example, the in-control ARL and MRL of a MEWMA chart with number of variables $p = 4$, smoothing constant $r = 0.04$, and control limit $H = 12.48$ calculated using Markov chain approximation is given as 396.94 and 263, respectively. This indicates that half of the run lengths will be equal to or less than 263 although the ARL is 396.94.

A MEWMA chart is optimal in detecting a shift if the MEWMA chart has the smallest out-of-control ARL or MRL among all MEWMA charts with the same in-control ARL or MRL for a particular shift. In-control ARL is the expected number of observations that must be plotted before the control chart sounds an alarm given that the process is in-control. It is desirable to have this value as large as possible to reduce the false alarm rate. Out-of-control ARL is the expected number of observations that must be plotted before the control chart signals that the process is out-of-control given that the process is out-of-control. It is desirable to have a small value of the out-of-control ARL so that the control chart can detect the process shift as quickly as possible.

MRL is a discrete quantity. There can be more than one optimal chart for an interval of r that gives the minimum MRL at a particular shift. For example, using Markov chain approximation, MEWMA charts with $p = 2$, in-control MRL of 200, and r in the range of [0.17, 0.44] are all optimal in detecting a shift of size $\delta = 1.5$. The smallest MRL for this shift is 5. The center of the interval is selected as the optimal value of r. Then, the optimal value of r for the range of [0.17, 0.44] is $\frac{0.17+0.44}{2} = 0.305$.

The optimal design approach developed here for the multivariate case is an extension of the optimal design of an univariate EWMA chart provided by Crowder (1989). The design specifies the optimal selection of parameters for the MEWMA chart based on statistical criteria. In order to obtain the chart parameters, r and H for an optimal MEWMA chart, the following four step procedure is recommended.

Step 1. Specify the smallest acceptable in-control ARL or MRL.

Step 2. Decide on the smallest shift (noncentrality parameter, δ) considered important enough to be detected quickly. Determine the value of r that yields the minimum ARL or MRL (i.e., optimal r) for the shift decided based on the in-control ARL or MRL specified in Step 1.

Step 3. Based on the optimal r obtained in Step 2, determine the appropriate H that satisfies the in-control ARL or MRL specified in Step 1.

Step 4. Perform a sensitivity analysis by comparing the out-of-control ARL or MRL of the optimal combination of r and H to other choices of r and H that produce the same in-control ARL or MRL. From these choices, the combination of r and H with the most desirable performance in terms of out-of-control ARL or MRL is selected for implementation.

To simplify the design procedure, plots of optimal r versus the size of a shift where a shift is expressed in units of the non centrality parameter are constructed using the Markov chain approach discussed in Secs. 3 and 4. Optimal r of the MEWMA charts for $p = 2$ and $p = 4$ based on the ARL are given in Figs. 1 and 3, respectively. Figures 5 and 7 show the optimal r of the control charts based on the MRL for $p = 2$ and $p = 4$, respectively. All of these MEWMA charts have approximated in-control ARLs or MRLs of 100, 200, 370, 500, and 1,000 with shifts ranging from $\delta = 0.05$ to $\delta = 3.00$ for the ARL and $\delta = 0.05$ to $\delta = 2.50$ for the MRL. The larger shifts are not considered here because for large shifts, the values of ARLs or MRLs are very small and the differences among the ARL values or among the MRL values are less significant for a wide range of the values of r. For this case, the multivariate Hotelling chart is recommended since the multivariate Hotelling chart is more sensitive to large shifts.

As the shift increases, the optimal r also increases for both the ARL and MRL. The plots are consistent with the results provided by Crowder (1989) and Gan (1993a) for the optimal design of univariate EWMA charts based on the ARL and MRL respectively. Given these plots, one can easily find the optimal value of r with two decimal accuracy for a specified size of a shift.

With the in-control ARL or MRL and optimal r specified, H can be quickly obtained in step 3 from Figs. 2, 4, 6 or 8. All of these plots are constructed using Markov chain approximation to find the combination of r and H giving the desired in-control ARL or MRL and having the smallest ARL or MRL for the specified shift. The approximated value of H with one decimal place can be read off easily from the curve. The optimal combination of r and H in Step 3 will give the minimum ARL or MRL in detecting a specified shift in Step 2. However, other combinations which give larger out-of-control ARL or MRL for that particular shift may be more sensitive for other shifts which are important. A sensitivity analysis is performed to further investigate the performance of the optimal combination and other combinations of the chart parameters with the same in-control ARL or MRL in order to decide which combination of the chart parameters is to be implemented. The graphical approach for an optimal design of a MEWMA chart based on ARL and MRL introduced here is consistent with currently available univariate EWMA and CUSUM approaches of Crowder (1989), Gan (1991, 1994, 1998), and Crowder and Hamilton (1992).

6. Examples

The following examples calculate the optimal parameters numerically, followed by illustrating the use of graphs to obtain the optimal parameters. Then the numerical and graphical results are compared. Although the graphs are constructed under the assumptions that the mean vector $\boldsymbol{\mu}_0 = \mathbf{0}$ and covariance matrix $\sum_{\mathbf{X}} = \mathbf{I}$, the graphs can be used for any process parameters, $\boldsymbol{\mu}_0$ and $\sum_{\mathbf{X}}$, provided the mean vector and covariance matrix are known.

Table 2
ARLs and MRLs of MEWMA control schemes for $\delta = 1.37$ and $\delta = 1.09$, respectively, with both in-control ARL and MRL = 200 where $p = 4$

r	0.05	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18
H_1	11.22	12.73	12.90	13.06	13.20	13.33	13.44	13.54	13.63	13.72
ARL	9.44	8.06	7.92	7.81	7.72	7.65	7.59	7.55	7.52	7.50
H_2	12.21	13.68	13.85	14.00	14.14	14.26	14.37	14.47	14.55	14.63
MRL	12	11	10	10	10	10	10	10	10	10
r	0.19	0.20	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28
H_1	13.79	13.86	13.93	13.99	14.04	14.10	14.14	14.19	14.23	14.27
ARL	7.49	7.49	7.50	7.51	7.53	7.57	7.60	7.65	7.69	7.75
H_2	14.70	14.77	14.83	14.89	14.94	14.99	15.04	15.07	15.12	15.15
MRL	10	10	10	10	10	10	10	10	11	11
r	0.29	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
H_1	14.3	14.34	14.48	14.58	14.65	14.71	14.75	14.78	14.81	14.82
ARL	7.81	7.88	8.31	8.87	9.57	10.45	11.47	12.68	14.11	15.72
H_2	15.18	15.22	15.34	15.44	15.50	15.55	15.59	15.62	15.64	15.66
MRL	11	11	12	13	14	16	17	19	22	24

Consider Montgomery's (2005) example of chemical process data (Table 10.6, p. 516). The covariance matrix $\sum_{\mathbf{X}}$ in correlation form is given as

$$\sum\nolimits_{\mathbf{X}} = \begin{bmatrix} 1.000 & 0.9302 & 0.2060 & 0.3595 \\ 0.9302 & 1.000 & 0.1669 & 0.4502 \\ 0.2060 & 0.1669 & 1.000 & 0.3439 \\ 0.3595 & 0.4502 & 0.3439 & 1.000 \end{bmatrix}$$

and suppose that the noncentrality parameter $\delta = \left[(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)' \sum_{\mathbf{X}}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)\right]^{\frac{1}{2}} = 1.37$ is important enough for a quick detection with respect to a shift in the process mean from $\boldsymbol{\mu}_0' = [9.955\ 20\ 14.68\ 15.765]$ to $\boldsymbol{\mu}_1' = [10.387\ 20\ 15.48\ 15.55]$.

Assume that we want to design an optimal MEWMA chart for detecting the specified shift with an approximate in-control ARL of 200. The approximation of ARL is considered acceptable as long as the value is between 199.5 and 200.5 (Margavio and Conerly, 1995). The ARL is determined numerically with control limit H_1 chosen to give the desired in-control ARL for a wide range of r. From Table 2, the optimal chart parameters are the values of r ranging from 0.19 to 0.20 and H_1 between 13.79 and 13.86 with minimum ARL of approximately 7.49.

Suppose we want to find the optimal chart parameters graphically. Then the four step procedure discussed in Sec. 5 is used as follows:

Step 1. The smallest acceptable in-control ARL is 200.

Step 2. The smallest shift considered important enough to be detected quickly is 1.37. Using Fig. 3 and the curve associated with in-control ARL = 200, the optimal value of r corresponding to the specified shift is determined to be approximately 0.2.

Step 3. With optimal $r = 0.2$, the corresponding H_1 is read from Fig. 4. The appropriate H_1 is approximately 13.8.

Step 4. Table 3 gives the sensitivity analysis.

Table 3 contains the sensitivity analysis, giving the out-of-control ARLs of MEWMA charts with values of $r = 0.10, 0.15, 0.20, 0.25$, and 0.30 for comparison. The values of H_1 of these charts are determined using the Markov chain approximation so that the in-control ARL is approximately 200 and these values are found to be 12.73, 13.44, 13.86, 14.14, and 14.34, respectively. After the chart parameters of these charts are determined, the ARLs for the different shift size are then computed.

From Table 3, it is decided that the optimal chart with combination of $r = 0.2$ and $H_1 = 13.86$ is preferred because it is only less sensitive at very small shifts and slightly less sensitive at larger shifts in the range of [1.75, 4.00] when compared to the other charts.

Using the same data and the same covariance matrix, the following section discusses the design of the MEWMA chart based on MRL. Assume that a quick detection is important for a shift in the process mean from $\boldsymbol{\mu}_0' = [9.955\ 20\ 14.68\ 15.765]$ to $\boldsymbol{\mu}_1' = [9.75\ 20.2\ 14.51\ 15.925]$ with shift $\delta = [(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)' \sum_{\mathbf{X}}^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)]^{\frac{1}{2}} = 1.09$.

Suppose we consider using an optimal MEWMA control chart with an in-control MRL of 200. Table 2 shows that the MEWMA charts with values of r in the

Table 3

Sensitivity analysis: Values of r and H_1 for in-control ARL of 200, and values of r and H_2 for in-control MRL of 200 (with values of ARL and MRL when process is out-of-control) where $p = 4$

	ARL(r/H_1)						MRL(r/H_2)				
δ	(0.10/ 12.73)	(0.15/ 13.44)	(0.20/ 13.86)	(0.25/ 14.14)	(0.30/ 14.34)	δ	(0.14/ 14.26)	(0.16/ 14.47)	(0.18/ 14.63)	(0.20/ 14.77)	(0.22/ 14.89)
0.00	200.49	200.04	200.46	199.78	200.33	0.00	200	200	200	200	200
0.25	93.40	106.20	117.26	126.03	134.11	0.25	98	103	108	113	116
0.50	35.13	40.17	46.27	52.70	59.26	0.50	35	37	39	42	44
0.75	18.49	19.65	21.69	24.24	27.30	0.75	18	18	19	20	20
1.00	12.17	12.13	12.67	13.57	14.81	1.00	12	12	12	12	12
1.25	9.05	8.65	8.66	8.92	9.39	1.09	10	10	10	10	10
1.37	8.06	7.59	7.49	7.60	7.88	1.25	8	8	8	8	8
1.50	7.22	6.71	6.54	6.54	6.68	1.50	7	7	6	6	6
1.75	6.03	5.51	5.26	5.13	5.15	1.75	6	5	5	5	5
2.00	5.19	4.69	4.42	4.27	4.20	2.00	5	5	4	4	4
2.25	4.57	4.09	3.82	3.65	3.56	2.25	4	4	4	4	4
2.50	4.10	3.65	3.38	3.21	3.10	2.50	4	4	4	3	3
2.75	3.72	3.30	3.04	2.87	2.76	2.75	3	3	3	3	3
3.00	3.42	3.02	2.77	2.61	2.50	3.00	3	3	3	3	3
3.25	3.17	2.78	2.55	2.40	2.30	3.25	3	3	3	3	3
3.50	2.96	2.59	2.38	2.24	2.14	3.50	3	3	3	2	2
3.75	2.77	2.42	2.23	2.11	2.01	3.75	3	2	2	2	2
4.00	2.61	2.28	2.12	2.01	1.90	4.00	2	2	2	2	2

range of [0.11, 0.26] are optimal in detecting the shift of $\delta = 1.09$ with the smallest out-of-control MRL of 10. The center of the interval is determined to be the optimal value of r and thus the optimal r is $\frac{0.11+0.26}{2} = 0.185$ with control limit H_2 between 14.63 and 14.7.

Suppose we also want to determine the optimal chart parameters using graphs. The four step procedure is used to design the optimal MEWMA chart.

Step 1. The in-control MRL is 200.

Step 2. The smallest shift considered important enough to be detected quickly is 1.09. From Fig. 7, the optimal r is approximately 0.18 at the shift specified.

Step 3. Using Fig. 8 and $r = 0.18$, the H_2 corresponding to in-control MRL of 200 is read off as 14.7.

Step 4. A sensitivity analysis is performed.

In order to perform a sensitivity analysis, MEWMA charts of $r = 0.14, 0.16, 0.18, 0.20$, and 0.22 are considered. The values of H_2 for these charts are determined using Markov chain approximation so that these charts have an in-control MRL of 200. These values of H_2 are found to be 14.26, 14.47, 14.63, 14.77, and 14.89, respectively. The MRL of these charts are computed once the chart parameters are determined. Table 3 is used to select the MEWMA chart to be employed.

Table 3 reveals that the performance of the charts in the range of $\delta \in [0.75, 4.00]$ is almost the same. Note that the MEWMA chart with $r = 0.14$ is more sensitive in detecting small shifts. If the shifts in the range of [0.75, 4.00] are of major concern and it is considered to be important to detect a shift of size 1.09 quickly, then the chart combination of $r = 0.18$ and $H_2 = 14.63$ is preferred.

Both numerical and graphical results give the optimal chart parameters that are approximately the same for the design of the chart based on ARL and MRL. This indicates that the graphical approach can be employed to determine the optimal parameters of a MEWMA chart. The graphs will make the design of a MEWMA chart simple, less complicated and user friendly without having to refer to an ARL or ARL table.

7. Conclusion

The use of ARL as a primary criterion for designing the MEWMA chart is recommended and an optimal design of the MEWMA chart based on ARL is given. The design of a chart based solely on ARL consideration does not contain enough information about the actual run length distribution of the chart. For example, for an in-control process, though a big value of the sum of run lengths, or alternatively a big value of ARL is desired so that the chart gives higher protection against the occurrence of false alarm, ARL alone may not give a complete overall picture of the in-control process over some long horizon. Interpretation based on ARL alone may confuse practitioners, causing them to have a false impression that the run lengths of an in-control process are generally of the same length, where very often they are not since the in-control run length distribution is highly skewed. On the contrary, designing an optimal statistical MEWMA chart based on MRL will give assurance to practitioners that there is a 50% probability that an in-control process will have a run length of not exceeding a specified value of the in-control MRL. In many cases, more information may be needed and a practitioner may be concerned with the probabilities of the occurrence of early false alarms for a given MEWMA control scheme. These probabilities are reflected in the lower percentiles of the in-control run length distribution which give information on the early false alarm rates. These percentiles can be evaluated from the formulas proposed by Brook and Evans (1972) which are discussed in Sec. 4 of this article. The lower percentiles of the in-control run length distribution may give important information about the early false alarm rates, but computing the in-control MRL also provides equally useful information. Thus, an optimal design based on MRL (i.e., the 50th percentile of the run length distribution of the scheme) is also given as MRL is more meaningful as a measure of centrality with respect to the highly skewed run length distribution. In addition to ARL, MRL can be used as a secondary criterion for evaluating the performance of a MEWMA chart. This allows a more complete study of the chart's performance. Plots of optimal smoothing constant r and plots of optimal control limit H are provided to make the design of the MEWMA chart simple and less complicated. These graphs can give immediate approximation of the optimal r and its corresponding H for a given in-control ARL or MRL.

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