The MAX-CUSUM Chart

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Summary. Control charts have been widely used in industries to monitor process quality. We usually use two control charts to monitor the process. One chart is used for monitoring process mean and another for monitoring process variability, when dealing with variables data. A single Cumulative Sum (CUSUM) control chart capable of detecting changes in both mean and standard deviation, referred to as the Max-CUSUM chart is proposed. This chart is based on standardizing the sample means and standard deviations. This chart uses only one plotting variable to monitor both parameters. The proposed chart is compared with other recently developed single charts. Comparisons are based on the average run lengths. The Max-CUSUM chart detects small shifts in the mean and standard deviation quicker than the Max-EWMA chart and the Max chart. This makes the Max-CUSUM chart more applicable in modern production process where high quality goods are produced with very low fraction of nonconforming products and there is high demand for good quality.

1 Introduction

Control charts are basic and most powerful tools in statistical process control (SPC) and are widely used for monitoring quality characteristics of production processes. The first types of control charts were developed by Shewhart in the 1920s and ever since, several new charts have been developed in an effort to improve their capability to quickly detect a shift of the process from a target value. The statistical control chart, generally with 3σ action limits and 2σ warning limits, is the longest established statistical form of graphical tool. The control chart statistics are plotted by simply plotting time on the horizontal axis and a quality characteristic on the vertical axis. A quality characteristic is regarded to be in an in-control state if the statistic falls within the action limits of the chart and out-of-control if the statistic plots outside the action limits.

The disadvantage of the Shewhart control charts is that they only use the information about the process contained in the last plotted point and

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information given by the entire sequence of points. This has led to various additional rules about runs of points above the mean. Attempts to incorporate the information from several successive results have resulted in charts based on some form of weighted mean of past results. In particular the Arithmetic Running Mean has been used in some instances by Ewan (1963).

One of the charts developed as an improvement to the Shewhart chart is the cumulative sum (CUSUM) chart developed by Page in 1954. This technique plots the cumulative sums of deviations of the sample values from a target value against time. The Shewhart control chart is effective if the magnitude of the shift is 1.5σ to 2σ or larger (Montgomery (2001)). The CUSUM charts are highly recommended by Marquardt (1984) for use in the U.S industry since they can detect small changes in the distribution of a quality characteristic and thus maintain tight control over a process. An important feature of the CUSUM chart is that it incorporates all the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value. If there is no assignable cause of variation, the two-sided CUSUM chart is a random walk with a drift zero. If the process statistic shifts, a trend will develop either upwards or downwards depending on the direction of the shift and thus if a trend develops, a search for the assignable causes of variation should be taken. The magnitude of a change can be determined from the slope of the CUSUM chart and a point at which a change first occurred is the point where a trend first developed. The ability to show a point at which changes in the process mean began makes the CUSUM chart a viable chart and also helps to quickly diagnose the cause of changes in the process. There are two ways to represent CUSUM charts: the tabular (or algorithmic) CUSUM chart and the V-mask form of the CUSUM chart. We shall discuss the tabular CUSUM chart in this article.

The CUSUM chart for the change in the mean and the standard deviation for variables data have been extensively studied and two separate plots generated to assess for a shift from the targeted values. Constructing individual charts for the mean and standard deviation is very cumbersome and sometimes tedious and Hawkins (1993) suggested plotting the two statistics on the same plot using different plotting variables. This produces a chart that is somewhat complicated to interpret and is congested with many plotting points on the same chart. Other charts have been developed with an effort to propose single charts to monitor both the mean and the standard deviation of the process such as those suggested by Cheng and Spiring (1998), Domangue and Patch (1991), Chao and Cheng (1996), Chen and Cheng (1998) and Cheng and Xie (1999).

The major objective of this paper is to develop a single CUSUM chart that simultaneously monitors both the process mean and variability by using a single plotting variable. This chart is capable of quickly detecting both small and large shifts in the process mean and/or standard deviation and is also capable of handling cases of varying sample sizes.

2 The New Control Chart

Let $X_i = X_{i1}, \ldots, X_{in}, i = 1, 2, \ldots$ denote a sequence of samples of size n taken on a quality characteristic X. It is assumed that, for each i, $X_{i1}, \ldots, X_{in}$ are independent and identically distributed observations following a normal distribution with means and standard deviations possibly depending on i, where i indicates the i^{th} group. Let μ_0 and σ_0 be the nominal process mean and standard deviation previously established. Assume that the process parameters μ and σ can be expressed as $\mu = \mu_0 + a\sigma_0$ and $\sigma = b\sigma_0$, where $a = 0$ and $b = 1$ when the process is in control, otherwise the process has changed due to some assignable causes. Then a represents a shift in the process mean and b a shift in the process standard deviation and $b > 0$.

Let $\bar{X}_i = (X_{i1} + \ldots + X_{in})/n$ and $S_i^2 = \frac{\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2}{n-1}$ be the mean and variance for the i^{th} sample respectively. The sample mean $\bar{X}_i$ and sample variance S_i^2 are the uniformly minimum variance unbiased estimators for the corresponding population parameters. These statistics are also independently distributed as do the sample values. These two statistics follow different distributions. The CUSUM charts for the mean and standard deviation are based on $\bar{X}_i$ and S_i respectively.

To develop a single CUSUM chart, we define the following statistics:

$$Z_i = \sqrt{n}\frac{(\bar{X}_i - \mu_0)}{\sigma_0} \tag{1}$$

$$Y_i = \Phi^{-1}\left\{H\left[\frac{(n-1)S_i^2}{\sigma_0^2}; n-1\right]\right\}, \tag{2}$$

where $\Phi(z) = P(Z \leq z)$, for $Z \sim$N (0, 1) the standard normal distribution. Φ^{-1} is the inverse of the standard normal cumulative distribution function, and $H(w; p) = P(W \leq w|p)$ for $W \sim \chi_p^2$ the chi-square distribution with p degrees of freedom. The functions Z_i and Y_i are independent and when the process variance is at its nominal value, Y_i follows the standard normal distribution. The CUSUM statistics based on Z_i and Y_i are given by

$$C_i^+ = \max[0, Z_i - k + C_{i-1}^+], \tag{3}$$

$$C_i^- = \max[0, -Z_i - k + C_{i-1}^-], \tag{4}$$

and

$$S_i^+ = \max[0, Y_i - v + S_{i-1}^+], \tag{5}$$

$$S_i^- = \max[0, -Y_i - v + S_{i-1}^-], \tag{6}$$

respectively, where C_0 and S_0 are starting points. Because Z_i and Y_i follow the same distribution, a new statistic for the single control chart can be defined as

$$M_i = \max[C_i^+, C_i^-, S_i^+, S_i^-]. \tag{7}$$

The statistic M_i will be large when the process mean has drifted away from μ_0 and/or when the process standard deviation has drifted away from σ_0. Small values of M_i indicate that the process is in statistical control. Since M_i's are non-negative, they are compared with the upper decision interval only.

The average run length (ARL) of a control chart is often used as the sole measure of performance of the chart. The ARL of the chart is the average number of points that must be plotted before a point plots above or below the decision interval. If this happens, an out-of-control signal is issued and a search for an assignable cause(s) of variation must be mounted. A chart is considered to be more efficient if its ARL is smaller than those of all other competing charts when the process is out of control and the largest when the process is in control.

The out-of-control signal is issued when either the mean or the standard deviation or both have shifted from their target values. Therefore the plan (the sample size and control limits) is chosen so that the ARL is large, when the process is in control and small when the process is out of control. Cox (1999) suggested that the criteria for a good chart are acceptable risks of incorrect actions, expected average quality levels reaching the customer and expected average inspection loads. Therefore the in-control ARL should be chosen so as to minimize the frequency of false alarms and to ensure adequate response times to genuine shifts.

For a predetermined in-control ARL, for quickly detecting shifts in the mean and variability, an optimal combination of h and k is determined which will minimize the out-of-control ARL for a specified change in the mean and standard deviation, where h is the decision interval and k is the reference value of the chart. The proposed chart is sensitive to changes in both mean and standard deviation when there is an increase in the standard deviation and is less sensitive when the standard deviation shifts downwards. This phenomenon has been observed for other charts based on the standardized values Domangue and Patch (1991).

3 Design of a Max-CUSUM chart

We use the statistic M_i to construct a new control chart. Because M_i is the maximum of four statistics, we call this new chart the Max-CUSUM chart. Monte Carlo simulation is used to compute the in control ARL for our Max-CUSUM chart. For a given in-control ARL, and a shift for the mean and/or standard deviation intended to be detected by the chart, the reference value k is computed as half the shift. For these values (ARL, k), the value of the decision interval (h) follows. For various changes in the process mean and/or standard deviation, each ARL value is also obtained by using 10 000 simulations.

Table 1 gives the combinations of k and h for an in-control ARL fixed at 250. We assume that the process starts in an in-control state with mean zero ($\mu_0 = 0$) and standard deviation of one ($\sigma_0 = 1$) and thus the initial value of the CUSUM statistic is set at zero. For example if one wants to have in-control ARL of 250 and to guard against $3\sigma_0$ increase in the mean and $1.25\sigma_0$ increase in the standard deviation, i.e., $a = 3$ and $b = 1.25$, the optimal parameter values are $h = 1.215$ and $k = 1.500$. These shifts can be detected on the second sample, i.e., the ARL is approximately two. A good feature of the Max-CUSUM chart is that smaller shifts in the process mean are detected much faster than in the single Shewhart chart (Max chart) as seen in the next section.

Table 1 shows that small values of k with large values of h result in quick detection of small shifts in mean and/or standard deviations. If one wants to guard against $3\sigma_0$ increase in the mean and $3\sigma_0$ increase in the standard deviation, the value of $h = 1.220$ and the value of $k = 1.500$. But for a $1\sigma_0$ increase in mean and $1.25\sigma_0$ increase in standard deviation, $h = 4.051$ and the value of k decreases to $k = 0.500$.

The Max-CUSUM scheme is sensitive to both small and large shifts in both mean and standard deviation. A $0.25\sigma_0$ increase in the process mean reduces the ARL from 250 to about 53 and a $1.25\sigma_0$ increase in the process standard deviation with a $0.25\sigma_0$increase in the process mean reduces the ARL from 250 to about 41 runs. If both parameters increase by large values, the ARL is reduced to 2. Thus the increase will be detected within the second sample. For example, a $3\sigma_0$ increase in both parameters will be detected within the second sample.

Another alternative method of assessing the performance of the CUSUM chart is to fix the values of h and k and calculate the ARL's for various shifts in the mean and/or standard deviation. This is displayed in Table 2. The value of $k = 0.5$ and thus we want to detect a $1\sigma_0$ shift in the mean and $h =$ 4.051. This combination gives an in-control ARL = 250. From Table 2 it can be concluded that, even when the chart is designed to detect a $1\sigma_0$ shift in the process, it is sensitive to both small and large shifts in the mean and/or standard deviation.

4 Comparison with other Procedures

In this section, the performance of the Max-CUSUM chart is compared with those of several other charts used for quality monitoring. Most of the CUSUM charts developed are designed to monitor the mean and standard deviation separately, even the combined CUSUM charts developed monitor these parameters separately in the same plots. This is done by plotting the charts using different plotting variables for the means and standard deviations, and then calculating ARLs separately for each parameter. The ARL for the chart will be taken as the minimum of the two. The new chart (Max-CUSUM) is

Table 1. (k, h) combinations and the corresponding ARL for the Max-CUSUM chart with $\mathrm{ARL}_0 = 250$

$ARL_0 = 250$									
		a							
b	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	250.21	53.21	22.04	7.99	4.56	2.76	1.96	1.50
0.5	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	91.30	67.33	25.00	8.24	4.43	2.62	1.84	1.30
1.25	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	82.42	41.01	18.98	7.40	4.21	2.73	1.91	1.47
1.50	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	41.84	34.92	16.37	6.93	4.11	2.72	1.89	1.43
2.00	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	18.81	23.97	12.75	5.95	3.77	2.61	1.87	1.43
2.50	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	11.78	18.16	10.22	5.11	3.37	2.55	1.86	1.42
3.00	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	8.63	14.32	8.49	4.53	3.15	2.47	1.84	1.39
4.00	h	4.051	8.572	6.161	4.051	2.981	2.103	1.554	1.220
	k	0.500	0.125	0.250	0.500	0.750	1.000	1.250	1.500
	ARL	5.85	10.16	6.38	3.80	2.87	2.36	1.81	1.37

Table 2. ARL's for the Max-CUSUM chart with $h = 4.051$ and $k = 0.500$

	a							
b	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	250.21	69.66	29.33	7.99	4.95	3.44	2.68	2.24
1.25	82.42	36.97	19.58	7.40	4.93	3.44	2.64	2.21
1.50	41.84	24.08	15.20	6.93	4.86	3.42	2.61	2.17
2.00	18.81	13.72	10.39	5.95	4.59	3.37	2.59	2.12
2.50	11.78	9.55	7.88	5.11	4.26	3.33	2.55	2.04
3.00	8.63	7.40	6.41	4.53	3.96	3.28	2.47	1.94
4.00	5.85	5.31	4.84	3.80	3.49	3.25	2.44	1.88

compared with the omnibus CUSUM chart proposed by Domanque and Patch (1991), the Max chart by Chen and Cheng (1998) and the Max-EWMA chart by Cheng and Xie (1999).

Table 4 shows the ARL's for the Max-CUSUM chart and the omnibus CUSUM chart developed by Domangue and Patch (1991) for shifts shown in Table 3. For various changes in the mean and/or standard deviation, we have calculated the ARL's for the Max CUSUM chart and compared them with those given by Domangue and Patch (1991) in Table 4. The Max-CUSUM chart performs better than the omnibus CUSUM chart for all shifts since its ARL's are smaller than those of the omnibus chart. The Max-CUSUM chart is also easy to plot and read as compared to the omnibus CUSUM chart since it plots only one plotting variable for each sample.

Table 3. Level of shifts in mean and standard deviation considered

Label	μ	σ
S_1	0.75	1.0
S_2	1.5	1.0
S_3	0	1.2
S_4	0	1.4
S_5	0.75	1.3
S_6	1.0	1.2

Table 4. ARL's of the Max-CUSUM chart and the omnibus CUSUM chart

$k = 1$ $h = 1.279$ $\alpha = 0.5$ $n = 1$						
Scheme	S_1	S_2	S_3	S_4	S_5	S_6
Omnibus CUSUM	37.0	7.0	50.4	21.5	15.7	13.0
Max-CUSUM	9.2	3.1	26.1	15.5	6.3	5.0

In Table 5 we compare the Max-CUSUM chart with the Max chart. The Max-CUSUM chart is more sensitive for small shifts in the mean than the Max chart and there is no significant difference in the performance of these charts at larger shifts even though the Max chart has slightly lower ARLs for large shifts. This is a major improvement in the CUSUM scheme as existing CUSUM charts are less sensitive to large shifts in the process mean and/or standard deviation.

Table 6 shows the performance of the Max-CUSUM chart and Max-EWMA chart for in-control ARL = 250. Both charts are sensitive to small and large shifts in the mean and/or standard deviation with the Max-EWMA chart performing better than the Max- CUSUM chart for both small and large shifts. These two charts use only one plotting variable for each sample and have good procedures of indicating the source and direction of shifts in the process.

5 Charting Procedures

The charting procedure of a Max-CUSUM chart is similar to that of the standard upper CUSUM chart. The successive CUSUM values, M_i's are plotted against the sample numbers. If a point plots below the decision interval, the process is said to be in statistical control and the point is plotted as a dot point. An out-of-control signal is given if any point plots above the decision interval and is plotted as one of the characters defined below. The Max-CUSUM chart is a combination of two two-sided standard CUSUM charts. The following procedure is followed in building the CUSUM chart:

1. Specify the following parameters; h, k, δ and the in-control or target value of the mean μ_0 and the nominal value of the standard deviation σ_0.
2. If μ_0 is unknown, use the sample grand average $\overline{\overline{X}}$ of the data to estimate it, where $\overline{\overline{X}} = (\bar{X}_1 + \cdots + \bar{X}_m)/m$. If σ_0 is unknown, use $\overline{R}/d_2$ or $\overline{S}/c_4$ to estimate it, where $\overline{R} = (R_1 + \cdots + R_m)/m$ is the average of the sample ranges and $\overline{S} = (S_1 + \cdots + S_m)/m$ is the average of the sample standard deviations, and d_2 and c_4 are statistically determined constants.
3. For each sample compute Z_i and Y_i.
4. To detect specified changes in the process mean and standard deviation, choose an optimal (h, k) combination and calculate C_i^+, C_i^-, S_i^+ and S_i^-.
5. Compute the M_i's and compare them with h, the decision interval.
6. Denote the sample points with a dot and plot them against the sample number if $M_i \leq h$.
7. If any of the M_i's is greater than h, the following plotting characters should be used to show the direction as well as the statistic(s) that is plotting above the decision interval:
   a) If $C_i^+ > h$, plot $C+$. This shows an increase in the process mean.
   b) If $C_i^- > h$, plot $C-$. This indicates a decrease in the process mean.
   c) If $S_i^+ > h$, plot $S+$. This shows an increase in the process standard deviation.
   d) If $S_i^- > h$, plot $S-$. This shows a decrease in the process standard deviation.
   e) If both $C_i^+ > h$ and $S_i^+ > h$, plot $B++$. This indicates an increase in both the mean and the standard deviation of the process.
   f) If $C_i^+ > h$ and $S_i^- > h$, plot $B+-$. This indicates an increase in the mean and a decrease in the standard deviation of the process.
   g) If $C_i^- > h$ and $S_i^+ > h$, plot $B-+$. This indicates a decrease in the mean and an increase in the standard deviation of the process.
   h) If $C_i^- > h$ and $S_i^- > h$, plot $B--$. This shows a decrease in both the mean and the standard deviation of the process.

Table 5. ARL for Max-CUSUM chart and the Max chart

$ARL_0 = 250$							$n = 4$					
	Max-CUSUM						Max Chart					
	a						a					
b	0.00	0.25	0.50	1.00	2.00	3.00	0.00	0.25	0.50	1.00	2.00	3.00
1.00	250.21	69.66	29.33	7.99	3.44	2.24	250.0	143.8	49.3	7.2	1.2	1.0
1.25	82.42	36.97	19.58	7.40	3.44	2.21	34.3	27.2	15.9	4.9	1.3	1.0
1.50	41.84	24.08	15.20	6.93	3.42	2.17	9.8	8.9	6.9	3.5	1.3	1.0
2.00	18.81	13.72	10.39	5.95	3.37	2.04	2.9	2.8	2.6	2.1	1.3	1.1
3.00	8.63	7.40	6.41	4.53	3.28	1.94	1.4	1.4	1.4	1.3	1.2	1.1

Table 6. ARL for Max-CUSUM chart and the Max-EWMA chart

$ARL_0 = 250$												
	Max-CUSUM						Max-EWMA					
	a						a					
b	0.00	0.25	0.50	1.00	2.00	3.00	0.00	0.25	0.50	1.00	2.00	3.00
1.00	250.21	69.66	29.33	7.99	3.44	2.24	250.0	24.6	8.6	2.9	1.1	1.0
1.25	82.42	36.97	19.58	7.40	3.44	2.21	17.8	12.3	7.1	2.9	1.2	1.0
1.50	41.84	24.08	15.20	6.93	3.42	2.17	6.3	5.7	4.5	2.5	1.2	1.0
2.00	18.81	13.72	10.39	5.95	3.37	2.04	2.5	2.5	2.3	1.8	1.2	1.0
3.00	8.63	7.40	6.41	4.53	3.28	1.94	1.7	1.6	1.6	1.5	1.2	1.1

8. Investigate the cause(s) of the shift for each out-of-control point in the chart and carry out the remedial measures needed to bring the process back into an in-control state.

6 An Example

A Max-CUSUM chart is applied to real data obtained from DeVor, Chang and Sutherland (1992). The data is for measurements of the inside diameter of the cylinder bores in an engine block. The measurements are made to 1/10,000 of an inch. Samples of size $n = 5$ are taken roughly every half hour, and the first 35 samples are given in Table 7. The actual measurements are of the form 3.5205, 3.5202, 3.5204 and so on. The entries given in Table 7 provide the last three digits in the measurements.

Suppose based on past experience, an operator wants to detect a 1σ shift in the mean, that is $a = 1$ and a 2σ shift in the standard deviation, that is $b = 2$ with an in-control ARL = 250, the corresponding decision interval from Table 1 is $h = 2.475$ and the reference value is $k = 0.500$. The chart is developed as follows: The nominal mean μ_0 is estimated by $\overline{\overline{X}}$ and σ_0 is estimated by $\bar{S}/c_4$. The sample produced the following estimates $\overline{\overline{X}} = 200.25$ and $\bar{S}/c_4 = 3.31$.

The Max-CUSUM chart in Figure 1 which plots all the 35 observations shows that several points plot above the decision interval. Sample number 6 shows an increase in the standard deviation. After this sample, samples 7 and 8 also plot above the decision interval. However these points show that the standard deviation is decreasing towards the in-control region. Due to very high value of the CUSUM statistic for the standard deviation in sample 6, the successive cumulative values at samples 7 and 8 show a value above the decision interval even though the standard deviation values corresponding to these samples are in control. We therefore investigate the cause of higher variability at sample number 6. According to DeVor, Chang and Sutherland (1992), this sample was taken when the regular operator was absent, and a relief, inexperienced operator was in charge of the production line and thus could have affected the process.

Sample number 11 also plots above the decision interval. This point corresponds to an increase in the mean. This corresponds to a sample taken at 1:00 P.M. when production had just resumed after lunch break. The machines were shut down at lunch time for tool changing and thus these items were produced when the machines were still cold. Once the machines warmed up, the process settled to an in-control state. This shows that the shift in the mean was caused by the machine tune-up problem.

Sample 16 also plots above the decision interval; this shift shows an increase in the standard deviation. According to DeVor, Chang and Sutherland (1992), this sample corresponds to a time when an inexperienced operator was in control of the process. In addition to the above mentioned points which also

Table 7. Cylinder diameter data

Sample	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}
1	205	202	204	207	205
2	202	196	201	198	202
3	201	202	199	197	196
4	205	203	196	201	197
5	199	196	201	200	195
6	203	198	192	217	196
7	202	202	198	203	202
8	197	196	196	200	204
9	199	200	204	196	202
10	202	196	204	195	197
11	205	204	202	208	205
12	200	201	199	200	201
13	205	196	201	197	198
14	202	199	200	198	200
15	200	200	201	205	201
16	201	187	209	202	200
17	202	202	204	198	203
18	201	198	204	201	201
19	207	206	194	197	201
20	200	204	198	199	199
21	203	200	204	199	200
22	196	203	197	201	194
23	197	199	203	200	196
24	201	197	196	199	207
25	204	196	201	199	197
26	206	206	199	200	203
27	204	203	199	199	197
28	199	201	201	194	200
29	201	196	197	204	200
30	203	206	201	196	201
31	203	197	199	197	201
32	197	194	199	200	199
33	200	201	200	197	200
34	199	199	201	201	201
35	200	204	197	197	199

plotted above the control limit in the Shewhart chart, Max chart and EWMA chart, sample 34 plots above the decision interval. This point corresponds to a decrease in the standard deviation. The Shewhart S chart plotted this value close to the lower control limit but within the acceptable area. Table 1 show that the Max-CUSUM chart is very sensitive to small shift and thus signals for this small decrease in the standard deviation.

When these four samples are removed from the data, new estimates for the mean and standard deviation were computed, giving the following; $\overline{\overline{X}} =$

200.08 and $\bar{S}/c_4 = 3.02$. The revised chart is shown in Figure 2. The chart plots only one point above the decision interval. This point corresponding to sample 1 shows an increase in the mean. It corresponds to a sample that was taken at 8:00 A.M., roughly the start up of the production line in the morning, when the machine was cold. Once the machine warmed up, the production returns to an in-control state.

When sample 1 is removed from the data, we re-calculate the estimates and obtain $\overline{\overline{X}} = 199.93$ and $\bar{S}/c_4 = 3.06$. The Max-CUSUM chart for this new data is shown in Figure 3. All the points plot within the decision interval showing that the process is in-control.

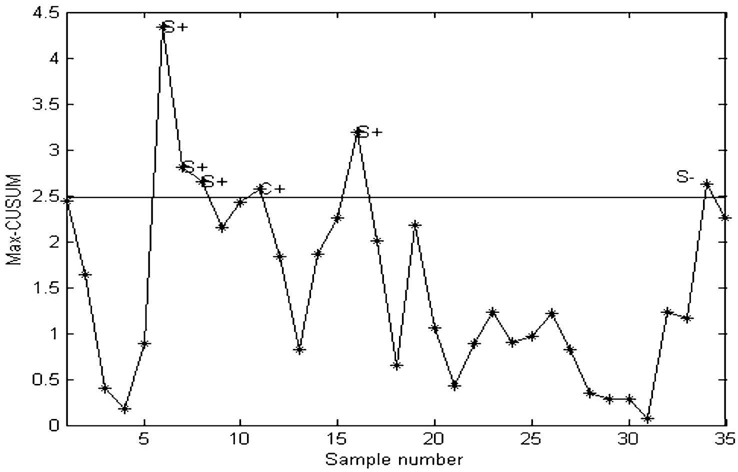

Fig. 1. The first Max-CUSUM control chart for the cylinder diameter data

7 Conclusion

The ARL for this chart reduces as the shift increases. One disadvantage of the standard CUSUM chart is that it does not quickly detect a large increase in the process parameters and thus is not recommended for large increase in both mean and variability.

A good feature of the Max-CUSUM chart developed here is its ability to quickly detect both small and large changes in both the process mean and the process variability. Another advantage of the Max-CUSUM is that we are able to monitor both the process center and spread by looking at one chart. The performance of the proposed Max-CUSUM is very competitive in comparison with the Max chart and the Max-EWMA chart.

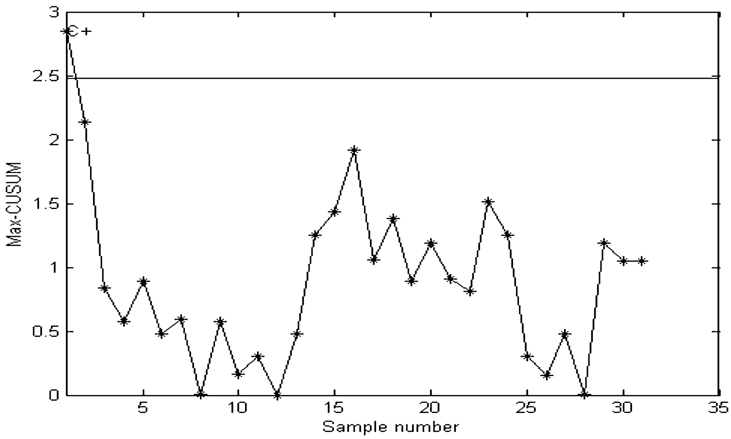

Fig. 2. The second Max-CUSUM control chart for the cylinder diameter data

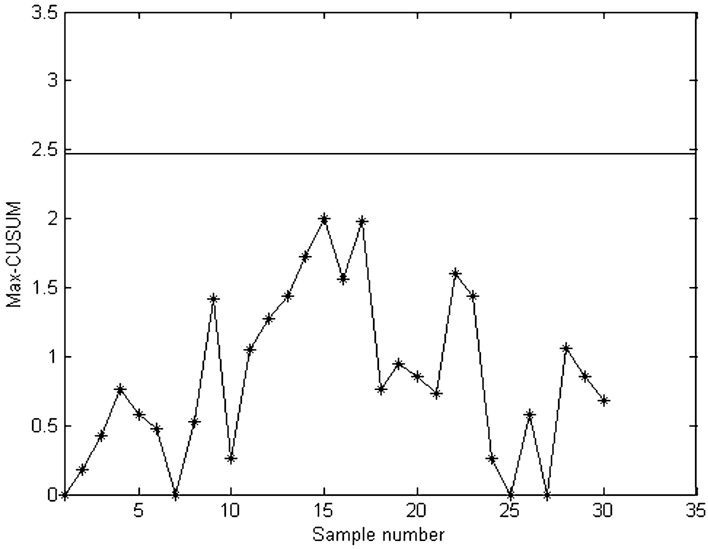

Fig. 3. The third Max-CUSUM control chart for the cylinder diameter data

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