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INVESTIGATION AND CHARACTERIZATION OF A CONTROL SCHEME FOR MULTIVARIATE QUALITY CONTROL

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SUMMARY

A new scheme for multivariate statistical quality control is investigated and characterized. The control scheme consists of three steps and it will identify any out-of-control samples, select the subset of variables that are out of control, and diagnose the out-of-control variables. A new control variable selection algorithm, the backward selection algorithm, and a new control variable diagnosis method, the hyperplane method, are proposed. It is shown by simulation that the control scheme is useful in cases where the process variables are correlated and where they are uncorrelated.

KEY WORDS Multivariate quality control Elliptical control chart Backward selection algorithm Hyperplane

INTRODUCTION

Recently, a control scheme which is useful for multivariate statistical quality control has been proposed by Chua.[1] However, the control scheme was studied under the assumption of statistically independent process variables. Since most multivariate quality control applications involve correlated variables, it is essential to investigate the effects of process variable dependence on the control scheme. The objective of this study is to report the performance of the control scheme under both correlated and uncorrelated process variable cases.

As shown in Figure 1, the control scheme contains three major steps or components. They are (1) the multivariate exponentially weighted moving average (MEWMA) control chart, (2) the backward selection algorithm (BSA) and (3) the hyperplane method. The MEWMA control chart is a type of multivariate quality control that is used to detect any process shifts under the case of multiple variables. The BSA is a newly proposed algorithm to identify the true out-of-control variables as soon as the process under study is detected to be out of control. The hyperplane method is a newly proposed control variable diagnosis method which provides a set of elliptical control charts as a tool for process variable diagnosis.

The operations of the control scheme, as shown in Figure 1, can be described as follows:

1. A multivariate observation is fed into the MEWMA control chart
2. If the observation is in control, the control operation loops back to the beginning and checks for the next observation. Otherwise, it checks for the number of process variables.
3. If the number of process variables is greater than five, bypass the BSA and use the hyperplane method directly. Otherwise, feed the observation into the BSA.
4. The BSA will select the out-of-control variable set and feed it into the hyperplane method.
5. The hyperplane method will generate the necessary elliptical control charts for diagnosis.
6. Based on the diagnoses, corrective actions are then taken and the control operation loops back to the beginning and checks for the next observation.

The following sections provide additional information about each of the components in the control scheme. Then, a report of a simulation study of this control scheme is presented. For ease of presentation, four assumptions are initially made in this study:

1. The target mean vector is a zero vector.
2. The sample size is equal to one.
3. The process is a multivariate normally distributed process.
4. The observations are obtained from an independent and identically distributed process.

MEWMA CONTOL CHART

The MEWMA control chart, proposed by Lowry *et al.*[2] has been shown to be effective in detecting small process shifts. Besides the MEWMA control chart, the other two popular multivariate control charts

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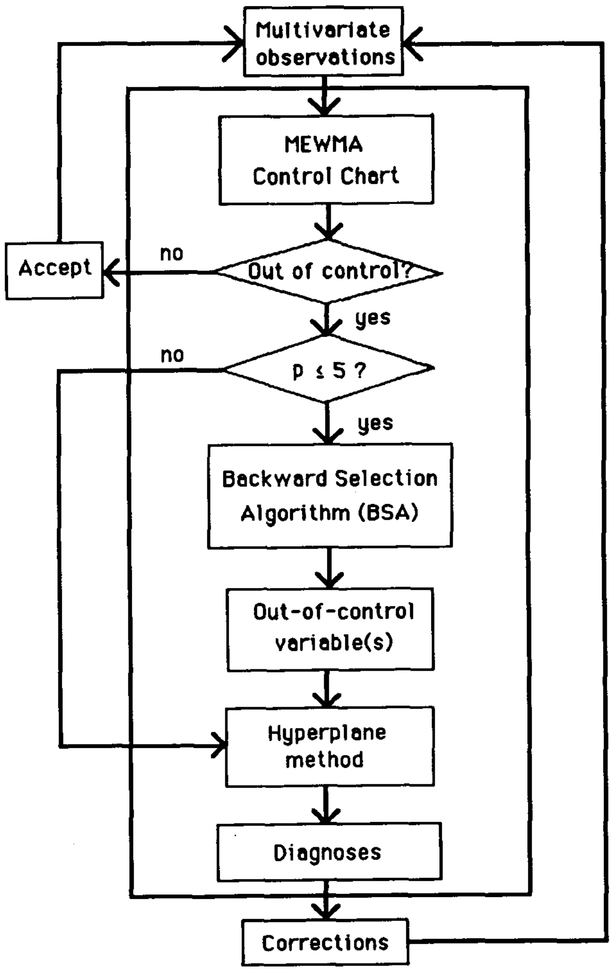

Figure 1. Framework of the control scheme

are Hotelling's T^2 control chart[3] and the multivariate cumulative sum (MCUSUM) control chart. However, the Hotelling T^2 control chart is insensitive to small or moderate process shifts since only the most current observation information is used. To detect small or moderate process shifts more effectively, the MCUSUM or MEWMA control chart is normally suggested. The differences of the sensitivity of these control charts are described by Hunter.[4]

In this study, the MEWMA control chart is preferred over the MCUSUM control chart, as it is shown by Lowry *et al.*[2] that the average run length (ARL) performance of the MEWMA control chart is better than that of the MCUSUM control chart as proposed by Crosier[5] or Pignatiello and Runger.[6] Furthermore, the design of the MEWMA control chart is simpler[7] and it can also be used as a process forecasting tool.[4]

To use the MEWMA control chart, $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are assumed to be independent observation vectors and the MEWMA vectors $\mathbf{Z}_i$ are defined as

$$\mathbf{Z}_i = \mathbf{R}\,\mathbf{X}_i + (\mathbf{I} - \mathbf{R})\,\mathbf{Z}_{i-1} \qquad (1)$$

where $\mathbf{Z}_i$ is the $(p \times 1)$ MEWMA vector, $i = 1, 2, \ldots, n$, with $\mathbf{Z}_0 = 0$, $\mathbf{R} = \text{diag}\,(r_1, r_2, \ldots, r_p)$, $0 < r_j < 1$, $j = 1, 2, \ldots, p$, $\mathbf{I}$ is the $p \times p$ identity matrix and $\mathbf{X}_i$ is the observation vector $(p \times 1)$.

For the convenience of comparisons between different control schemes, Lowry *et al.*[2] showed that the ARL performance of the MEWMA control chart depends only on the value of the non-centrality parameter with the additional assumption that $r_1 = r_2 = \ldots = r_p = r$. Then, the MEWMA vectors can be redefined as

$$\mathbf{Z}_i = r\mathbf{X}_i + (1 - r)\,\mathbf{Z}_{i-1} \qquad (2)$$

and the sample covariance matrix for the MEWMA can be shown to be

$$\mathbf{S}_{\mathbf{Z}_i} = \frac{r[1 - (1 - r)^{2i}]}{2 - r}\mathbf{S} \qquad (3)$$

where r is the smoothing constant or EWMA parameter, $\mathbf{S}$ is the sample covariance matrix of the observation $\mathbf{X}_i$ and $\mathbf{S}_{\mathbf{Z}_i}$ is the sample covariance matrix for MEWMA.

Hence, any observation vector $\mathbf{X}i$ will be deemed out of control if the test statistic

$$T^2 = \mathbf{Z}_i^{\mathrm{T}}\mathbf{S}_{\mathbf{Z}_i}^{-1}\,\mathbf{Z}_i \qquad (4)$$

is greater than the constant limit H, where $H > 0$ and can be obtained from a simulation based on a specified ARL. On the other hand, the process is in control if the test statistic is less than H.

BACKWARD SELECTION ALGORITHM

The BSA is a newly proposed algorithm, and it incorporates the features that are found in the all subsets selection algorithm (ASSA) and the forward selection algorithm (FSA) which were proposed by Murphy.[8] All the algorithms use the log-odds approach as a discriminant function[8–10] for classifying the in-control or out-of-control variable set, and their performances are based on the following two criteria:

1. The number of tests should be as few as possible.
2. The algorithm should be conservative.

The first criterion implies that the response time of the algorithm can be minimized when the number of tests is small. The second criterion ensures that the selected out-of-control variable set will not contain any redundant variables which are actually in control. The mathematical background of the BSA is as follows:

The algorithm divides the p variables into two subsets of sizes p_1 and p_2, where $p_1 + p_2 = p$. Also, the sample covariance matrix is divided into two parts with p_1 and p_2 variables, respectively. For p variables, the T_p^2 value is

$$T_p^2 = (\mathbf{X} - \bar{\mathbf{X}})^{\mathrm{T}}\,\mathbf{S}^{-1}\,(\mathbf{X} - \bar{\mathbf{X}}) \qquad (5)$$

where $\mathbf{X}$ is the sample observation vector, $\bar{\mathbf{X}}$ is the sample mean vector and $\mathbf{S}^{-1}$ is the inverse of the sample covariance matrix.

Similarly, for the subset of p_1 variables, the $T^2_{p_1}$ value is

$$T^2_{p_1} = (\mathbf{X}_{p_1} - \bar{\mathbf{X}}_{p_1})^{\mathrm{T}} \mathbf{S}^{-1}_{p_1} (\mathbf{X}_{p_1} - \bar{\mathbf{X}}_{p_1}) \qquad (6)$$

where $\mathbf{X}_{p_1}$ is the sample observation vector for p_1 variables, $\bar{\mathbf{X}}_{p_1}$ is the sample mean vector for p_1 variables and $\mathbf{S}^{-1}_{p_1}$ is the inverse of the sample covariance matrix for p_1 variables.

A difference value D is then defined as

$$D = T^2_p - T^2_{p_1} \qquad (7)$$

which can be shown to follow a chi-square distribution with p_2 degrees of freedom.[8] D is then compared to a control limit K_i which is obtained from computer simulation based on a specified ARL and number of variables i.

The selection process proceeds according to the magnitude of the D value. If the D value is large, the hypothesis that the subset of p_1 variables is causing the out-of-control abnormality is rejected. The reason is that this particular set of variables does not contribute significantly to the out-of-control T^2_p value. Therefore, this set of variables will not contain the out-of-control variables. On the other hand, if the D value is small, the hypothesis should be accepted and this variable set will contain the out-of-control variables. The operations of the BSA are summarized in the Appendix.

Using the union of out-of-control variable sets as a criterion, the BSA involves a maximum of

$$\sum_{j=1}^{p-1} \binom{p}{j}$$

tests. Since the size of the out-of-control variable set after each step as described in the BSA may be less than p, the number of tests in general requires less than this maximum number of tests.

HYPERPLANE METHOD

It is well known that the elliptical control chart is an effective diagnostic tool for bivariate quality control.[11] Therefore, it would be beneficial if this control chart could be extended to the multivariate case. However, the elliptical control chart will suffer some large type I and type II errors if it is used repeatedly.[11,12] In addition, the use of two-dimensional control charts will also suffer the supplemental type I and type II errors which are incurred from the misclassifications of samples under cases of multiple control charts. Thus, the objective is to find a diagnostic method which incorporates the features of an elliptical control chart but is free of any supplemental type I and type II errors while preserving the traditional type I and type II errors.

Owing to the large supplemental type II errors in the projection method,[1] a new diagnostic method named the hyperplane method has been proposed. The hyperplane method can be shown to be free of any supplemental type I and type II errors and is especially useful when the process shifts are small. The mathematical background of the hyperplane method is as follows.

Recall that the equation for the hyper-ellipsoid control region is

$$\mathbf{X}^{\mathrm{T}} \mathbf{S}^{-1} \mathbf{X} = T_{\mathrm{ref}} \qquad (8)$$

where $\mathbf{X}$ is the observation vector, $\mathbf{S}^{-1}$ is the inverse of the sample covariance and T_{ref} is the Hotelling T reference value. Obviously, if the left-hand side of the equation is less than T_{ref}, the observation will be inside the hyper-ellipsoid, and thus in control. On the other hand, if it is greater than T_{ref}, the observation will be outside the hyper-ellipsoid and out of control.

The hyperplane method can be described graphically as in Figure 2. Figure 2 shows an ellipsoid E in three-dimensional space and two planes P1 and P2 perpendicular to each other. These two planes are parallel to the co-ordinate planes YZ and XZ, respectively. Suppose there exists an out-of-control point A, and the plane P1 passes through this point. Then there will be an elliptical-shaped control region formed by the intersection of this plane with the ellipsoid. This elliptical control region can then be used as an elliptical control chart. The key feature of this method is that an out-of-control data point will always be out of control. The same process applies to the P2 plane, whence another elliptical control chart is obtained. Thus, for p variables there will be $p(p-1)/2$ elliptical control charts.

Recall also that the vector equation of a hyperplane is

$$\mathbf{d}\cdot(\mathbf{x}-\mathbf{b}) = 0$$

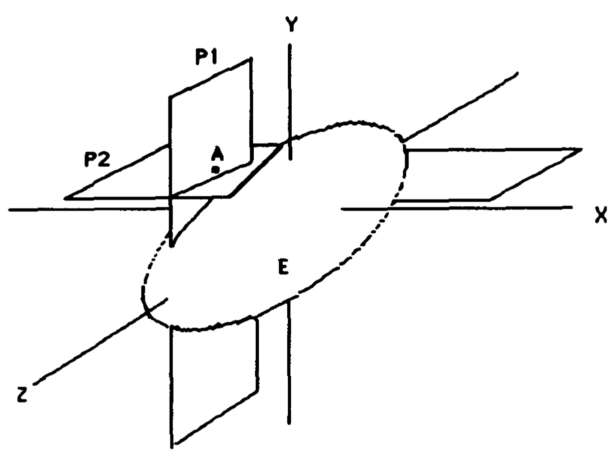

Figure 2. Hyperplane method

where **d** is the direction vector of the hyperplane, **x** is the observation vector, and **b** is any known observation which lies in the hyperplane. In order to obtain a hyperplane which has a direction vector parallel to one of the co-ordinate axes, it is necessry that the direction vector **d** be

$$\mathbf{d} = [0{,}0, \ldots, i, \ldots, 0{,}0]$$

where $i=1$ denotes the desired co-ordinate axis.

To cut the hyper-ellipsoid with the hyperplane, it is equivalent to solve the $p-1$ simultaneous equations

$$\begin{gathered}\mathbf{X}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{X} = T_{\mathrm{ref}} \\ \mathbf{d}\cdot(\mathbf{x}-\mathbf{b}) = 0\end{gathered}$$

for $i = 1, 2, \ldots, p$, excluding the two targeted control variables. For instance, to cut a hyperplane through the ellipsoid in three-dimensional space with direction vector parallel to the x_1 axis, the following are the required equations, assuming that $\mathbf{b} = (b_1, b_2, b_3)$ is the out-of-control observation:

$$[x_1, x_2, x_3]\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33}\end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix} = T_{\mathrm{ref}} \quad (9)$$

and

$$[1\,0\,0]\cdot[x_1\,x_2\,x_3] = [b_1\,b_2\,b_3]$$

After the substitution, the remaining equation is a second-degree polynomial equation with two unknown variables. In fact, the whole process here is to reduce the $p-1$ simultaneous equations into a second-degree polynomial equation. Then the polynomial equation is diagonalized, so that the cross-product term is eliminated. The remaining equation is further reduced to the standard equation of an ellipse by completing the squares. The three required steps are summarized as follows:

Step 1. Substitute $p-2$ hyperplane equations into the hyperellipsoid equation, with the targeted direction vectors parallel to the desired co-ordinate axes.

Step 2. Diagonalize the quadratic portion of the equation from step 1 with a transformation matrix **C**, where $\det(\mathbf{C}) = 1$.

Step 3. Reduce the equation from step 2 into a standard ellipse equation by completing the squares.

Under normal conditions, the hyperplane method will provide $p(p-1)/2$ elliptical control charts. Nevertheless, it is possible that the ellipse equation is degenerate or empty, which means that the out-of-control observation is too far away from the hyperellipsoid control region. In that case, the number of elliptical control charts will be reduced to some extent, depending on how far away the observation is located.

SIMULATION RESULTS

To facilitate presentation, the simulations are conducted using the following parameters:

1. The process shift (λ) occurs with magnitudes of 0·0, 0·5, 1·0, 1·5, 2·0, 2·5 and 3·0 standard deviation units, and the shift is equal for all variables.
2. The in-control ARL is targeted at 100, based on 10,000 simulations.
3. The smoothing constant (r) is chosen to be 0·05 for the MEWMA control scheme.
4. The number of control variables (p) is given the values of 2, 3, 4 and 5.
5. The diagonal elements of the correlation matrix is assumed to be 1·0, and all off-diagonal elements are assumed to be 0·0, 0·2, 0·4, 0·6 and 0·8, respectively.

Since all variables are assumed to be incremented by an amount λ, a process shift means a shift in the direction of equal angles with each co-ordinate axis. The equal angle process shift represents the worst process shift case, as shown by Chua.[1] This particular selection is important because if a control scheme can function well under the worst case, it can function no worse in other cases. Hence, reliable information about the performance of the control scheme can be obtained.

It is also assumed that every test conducted in this study has a type I error probability of 0·01. It is important that this type I error be kept at the minimal value. For instance, the potential number of tests involved in the BSA for the case $p = 5$ can be as high as 30, and the overall type I error under this case will be 0·2603. This is also why the BSA is recommended only for the case where $p \leq 5$.

Average run lengths

The ARL average values of the control scheme are shown in Table I. The values in parentheses beneath the average values are the standard errors. The average values can be concluded as stable since the standard errors are small. From Table I, it is shown that the control scheme is very efficient in detecting any range of process shifts. However, as illustrated by Lowry *et al.*[2] the efficiency of the MEWMA control chart in the large process shift case is nevertheless an artefact of the simulation settings.

In addition, when the correlation between variables increases, the ARLs of the control scheme also increase. Furthermore, compared to the ARLs of the Hotelling T^2 control scheme,[1,2] all the ARLs in Table I are relatively small. It can then be concluded that the control scheme performs well in both the independent and dependent variable cases.

Table I. Average run lengths for the control scheme

	$p = 2$					$p = 3$				
λ	$r = 0{\cdot}0$	$r = 0{\cdot}2$	$r = 0{\cdot}4$	$r = 0{\cdot}6$	$r = 0{\cdot}8$	$r = 0{\cdot}0$	$r = 0{\cdot}2$	$r = 0{\cdot}4$	$r = 0{\cdot}6$	$r = 0{\cdot}8$
0·0	98·751	99·258	101·770	100·033	101·549	99·355	99·979	99·976	98·715	100·487
	(1·127)	(1·119)	(1·173)	(1·156)	(1·142)	(1·122)	(1·135)	(1·135)	(1·125)	(1·136)
0·5	9·332	10·684	11·993	13·322	14·576	7·661	9·834	11·980	13·847	15·577
	(0·074)	(0·088)	(0·099)	(0·013)	(0·124)	(0·058)	(0·078)	(0·098)	(0·116)	(0·131)
1·00	3·272	3·725	4·225	4·717	5·138	2·729	3·454	4·151	4·855	5·543
	(0·021)	(0·025)	(0·029)	(0·033)	(0·037)	(0·016)	(0·022)	(0·028)	(0·034)	(0·039)
1·5	1·890	2·107	2·333	2·572	2·809	1·582	1·963	2·299	2·668	3·011
	(0·010)	(0·012)	(0·014)	(0·016)	(0·017)	(0·007)	(0·010)	(0·013)	(0·016)	(0·018)
2·0	1·357	1·483	1·624	1·759	1·904	1·183	1·380	1·605	1·811	2·019
	(0·006)	(0·007)	(0·008)	(0·009)	(0·010)	(0·004)	(0·006)	(0·008)	(0·009)	(0·011)
2·5	1·116	1·193	1·283	1·375	1·448	1·036	1·132	1·267	1·395	1·529
	(0·003)	(0·004)	(0·005)	(0·006)	(0·007)	(0·002)	(0·004)	(0·005)	(0·006)	(0·007)
3·00	1·027	1·066	1·112	1·165	1·209	1·004	1·032	1·099	1·169	1·264
	(0·002)	(0·002)	(0·003)	(0·004)	(0·004)	(0·001)	(0·002)	(0·003)	(0·004)	(0·005)

Table I (continued). Average run lengths for the control scheme

	$p = 4$					$p = 5$				
λ	$r = 0{\cdot}0$	$r = 0{\cdot}2$	$r = 0{\cdot}4$	$r = 0{\cdot}6$	$r = 0{\cdot}8$	$r = 0{\cdot}0$	$r = 0{\cdot}2$	$r = 0{\cdot}4$	$r = 0{\cdot}6$	$r = 0{\cdot}8$
0·0	101·285	100·769	100·769	102·285	100·073	101·007	100·872	98·996	100·497	99·302
	(1·152)	(1·159)	(1·150)	(1·153)	(1·131)	(1·159)	(1·154)	(1·122)	(1·127)	(1·128)
0·5	6·744	9·573	12·298	14·621	16·633	5·984	9·410	12·422	15·203	17·790
	(0·048)	(0·073)	(0·098)	(0·121)	(0·139)	(0·041)	(0·072)	(0·099)	(0·124)	(0·151)
1·0	2·397	3·355	4·270	5·068	5·939	2·170	3·293	4·325	5·330	6·248
	(0·013)	(0·020)	(0·028)	(0·035)	(0·041)	(0·011)	(0·020)	(0·028)	(0·036)	(0·044)
1·5	1·435	1·916	2·349	2·805	3·209	1·317	1·871	2·397	2·898	3·385
	(0·006)	(0·010)	(0·013)	(0·016)	(0·020)	(0·005)	(0·009)	(0·013)	(0·017)	(0·020)
2·0	1·108	1·347	1·620	1·875	2·148	1·058	1·331	1·640	1·945	2·233
	(0·003)	(0·006)	(0·008)	(0·010)	(0·012)	(0·002)	(0·005)	(0·008)	(0·010)	(0·012)
2·5	1·011	1·113	1·268	1·437	1·611	1·003	1·099	1·281	1·466	1·660
	(0·001)	(0·003)	(0·005)	(0·006)	(0·007)	(0·001)	(0·003)	(0·005)	(0·006)	(0·008)
3·0	1·001	1·023	1·094	1·197	1·312	1·000	1·018	1·097	1·212	1·359
	(0·000)	(0·001)	(0·003)	(0·004)	(0·005)	(0·000)	(0·001)	(0·003)	(0·004)	(0·006)

Average number of variables

From Table II, it is observed that the number of out-of-control variables is in the range 1–2 in most cases. This is a significant result since it implies that on the average, the out-of-control variable set can be reduced from as high as five variables to one or two variables. Hence, the tasks for the control variable diagnosis and interpretation can be simplified to a great extent.

Table II also indicates the fact that it is always beneficial to detect a process shift as soon as possible because the number of variables becomes large when the process shift is large for cases where the correlation between variables is low. Furthermore, it is observed that the number of variables in the high correlated variable case is in general smaller than that of the case with low correlation between variables, and the differences are more prominent in the large process shift case.

It is also found that when the correlation between the variables is low, the number of variables increases exponentially when the process shift increases. However, when the process variables are highly correlated, the number of variables decreases exponentially when the process shift is large. All these features are expected, as when the variables are highly correlated it is always true that a small subset of the process variables is sufficient to represent the whole set of variables.

Average number of tests

The simulation results for the number of tests are shown in Table III. It is observed that the number of tests when $p = 2$ is always equal to two. This is obvious, as two tests are needed for a $p = 2$ process.

It is found that when the correlations are low, the number of tests decreases in general for large pro-

Table II. Average number of variables for the control scheme

λ	$p = 2$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$p = 3$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$
0·0	1·012	1·032	1·094	1·168	1·252	1·009	1·020	1·061	1·157	1·288
0·5	1·044	1·010	1·013	1·025	1·049	1·053	1·011	1·013	1·031	1·051
1·0	1·135	1·033	1·008	1·011	1·023	1·195	1·031	1·011	1·013	1·026
1·5	1·240	1·086	1·019	1·006	1·011	1·434	1·104	1·015	1·008	1·015
2·0	1·394	1·190	1·045	1·005	1·008	1·727	1·227	1·040	1·006	1·011
2·5	1·559	1·303	1·101	1·011	1·005	2·134	1·429	1·098	1·008	1·007
3·0	1·735	1·471	1·194	1·031	1·003	2·519	1·703	1·212	1·020	1·006

Table II (continued). Average number of variables for the control scheme

λ	$p = 4$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$p = 5$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$
0·0	1·011	1·022	1·059	1·158	1·308	1·012	1·021	1·052	1·144	1·306
0·5	1·077	1·010	1·016	1·030	1·062	1·085	1·015	1·014	1·029	1·060
1·0	1·272	1·041	1·011	1·014	1·027	1·342	1·038	1·011	1·016	1·027
1·5	1·598	1·112	1·014	1·007	1·019	1·782	1·119	1·013	1·011	1·016
2·0	2·094	1·259	1·037	1·007	1·010	2·462	1·270	1·029	1·007	1·010
2·5	2·745	1·490	1·089	1·008	1·008	3·381	1·527	1·076	1·007	1·010
3·0	3·329	1·837	1·199	1·014	1·007	4·172	1·909	1·171	1·011	1·006

cess shift case. However, the number of tests increases in general when the process shift is large for the high correlated variable case. Once again, these features are expected, as when the variables are highly correlated, all the tests must be performed for all variables if any one of the variables is out of control. Hence, the number of tests will be increased accordingly where variables are highly correlated.

Percentage of empty elliptical control charts

From Table IV, it is found that the percentage of empty elliptical control charts is always equal to zero in the case $p = 2$. This is anticipated, as there are no hyperplanes involved in this case.

It is also found that the percentages of empty elliptical control charts deteriorate exponentially in

Table III. Average number of tests for the control scheme

λ	$p = 2$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$p = 3$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$
0·0	2·000	2·000	2·000	2·000	2·000	5·933	5·890	5·838	5·784	5·735
0·5	2·000	2·000	2·000	2·000	2·000	5·856	5·965	5·970	5·951	5·939
1·0	2·000	2·000	2·000	2·000	2·000	5·734	5·931	5·977	5·976	5·963
1·5	2·000	2·000	2·000	2·000	2·000	5·611	5·870	5·970	5·982	5·973
2·0	2·000	2·000	2·000	2·000	2·000	5·540	5·771	5·947	5·986	5·980
2·5	2·000	2·000	2·000	2·000	2·000	5·552	5·660	5·896	5·981	5·987
3·0	2·000	2·000	2·000	2·000	2·000	5·657	5·542	5·826	5·970	5·987

Table III (continued). Average number of tests for the control scheme

λ	$p = 4$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$p = 5$ $r = 0.0$	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$
0·0	13·926	13·878	13·787	13·691	13·620	29·909	29·850	29·750	29·628	29·528
0·5	13·753	13·956	13·950	13·931	13·896	29·585	29·928	29·935	29·916	29·876
1·0	13·335	13·904	13·963	13·959	13·947	28·674	29·873	29·957	29·940	29·930
1·5	12·776	13·789	13·958	13·975	13·959	27·069	29·679	29·945	29·953	29·948
2·0	12·161	13·555	13·926	13·979	13·967	24·862	29·297	29·909	29·966	29·961
2·5	11·833	13·188	13·864	13·970	13·974	23·199	28·613	29·830	29·962	29·954
3·0	12·257	12·599	13·734	13·962	13·978	24·299	27·445	29·660	29·954	29·965

Table IV. Percentage of empty elliptical control charts for the control scheme

λ	$p=2$ $r=0{\cdot}0$	$r=0{\cdot}2$	$r=0{\cdot}4$	$r=0{\cdot}6$	$r=0{\cdot}8$	$p=3$ $r=0{\cdot}0$	$r=0{\cdot}2$	$r=0{\cdot}4$	$r=0{\cdot}6$	$r=0{\cdot}8$
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0002	0·0077	0·1305	0·3573
0·5	0·0000	0·0000	0·0000	0·0000	0·0000	0·0004	0·0021	0·0555	0·4239	0·7910
1·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0070	0·0134	0·1400	0·5661	0·8844
1·5	0·0000	0·0000	0·0000	0·0000	0·0000	0·0489	0·0418	0·2424	0·6386	0·9272
2·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·1447	0·1021	0·3744	0·7410	0·9559
2·5	0·0000	0·0000	0·0000	0·0000	0·0000	0·3298	0·2354	0·5114	0·8112	0·9725
3·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·5353	0·4508	0·6553	0·8726	0·9859

Table IV (continued) Percentage of empty elliptical control charts for the control scheme

λ	$p=4$ $r=0{\cdot}0$	$r=0{\cdot}2$	$r=0{\cdot}4$	$r=0{\cdot}6$	$r=0{\cdot}8$	$p=5$ $r=0{\cdot}0$	$r=0{\cdot}2$	$r=0{\cdot}4$	$r=0{\cdot}6$	$r=0{\cdot}8$
0·0	0·0000	0·0014	0·0381	0·1729	0·4155	0·0005	0·0020	0·0545	0·2029	0·4340
0·5	0·0015	0·0201	0·2652	0·6481	0·8638	0·0022	0·0595	0·4037	0·7291	0·8875
1·0	0·0283	0·0884	0·4185	0·7836	0·9422	0·0634	0·1761	0·5860	0·8574	0·9540
1·5	0·1585	0·1938	0·5514	0·8604	0·9687	0·2996	0·3436	0·7121	0·9135	0·9777
2·0	0·4154	0·3471	0·6786	0·9066	0·9836	0·6289	0·5310	0·8066	0·9491	0·9914
2·5	0·6966	0·5550	0·7827	0·9459	0·9914	0·8798	0·7378	0·8853	0·9744	0·9954
3·0	0·8877	0·7872	0·8733	0·9717	0·9973	0·9787	0·9080	0·9447	0·9869	0·9989

the large process shift case. This is obvious, since when the data points are too far away from the control region, the chances for the hyperplanes to cut the control region will become small, and thus the elliptical control charts may not always be formed.

Comparing the percentages of empty elliptical control charts under low and high levels of correlation, it is observed that the percentage is too high when the variables are highly correlated, even when the process shift is small. This indicates that the hyperplane method will not perform well when the variables are highly correlated.

Fortunately, multivariate quality control is found most useful in cases where the process variables are moderately correlated. Under the highly-correlated variable case, the variables under investigation obviously contain too many redundant variables. In this situation, the appropriate action is to select a smaller set of variables for process control. On the other hand, under the low and moderately correlated variable cases, it is beneficial to detect the process shifts as soon as possible so that the percentage of the empty elliptical control charts can be minimized.

CONCLUSION

Some of the major conclusions from the simulation studies of the control scheme presented in this paper are

1. The ARLs are relatively small even in the high correlated variable case.
2. The average number of variables can be reduced from 2–5 to 1–2 in most cases.
3. The average number of tests can be reduced slightly in most cases.
4. The percentage of the empty elliptical control charts is relatively small for the low and moderately correlated variable cases.
5. The process shifts should be detected as soon as possible for the best performance of the control scheme.

Thus, this multivariate quality control scheme can be concluded to be useful for cases with either correlated or uncorrelated process variables. More importantly, this control scheme provides a framework which integrates the quality functions of process variable detection, process variable selection and process variable diagnosis into one unique context. Consequently, this control scheme could be regarded as a benchmark for any future development of multivariate quality control schemes.

APPENDIX: BACKWARD SELECTION ALGORITHM (BSA)

Step 1. Perform the T_p^2 test for p variables and compare with K_p using p variables as the criterion.

If $T_p^2 < K_p$, the process is in control; exit.

If $T_p^2 \geq K_p$, the process is out of control; continue to the next step.

Step 2. Perform T_{p-1}^2 tests and calculate the differences $D_{p-1} = T_p^2 - T_{p-1}^2$.

If $D_{p-1} \leq K_{p-1}$, denote NS (non-significant).

If $D_{p-1} > K_{p-1}$, denote S (significant).

The union of variable sets for all the tests which are denoted as NS is the new criterion. Otherwise, keep the criterion from step 1. Denote the minimum NS D value as $D(p)$ and continue to the next step.

Step 3. Perform T_{p-2}^2 tests and calculate the differences $D_{p-2} = T_p^2 - T_{p-2}^2$.
If $D_{p-2} \leq K_{p-2}$ denote NS (non-significant).
If $D_{p-2} > K_{p-2}$, denote S (significant).
The union of variable sets for all the tests which are denoted as NS is the new criterion. Otherwise, keep the criterion from step 2. Denote the minimum NS D value as $D(p-1)$ and continue to next step.

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Step p. Perform T_1^2 tests.
If $D_1 \leq K_1$, denote NS (non-significant).
If $D_1 > K_1$, denote S (significant).
Denote the minimum NS D value as $D(1)$ and continue to the next step.

Step $p+1$. If all the tests from steps 2 to p are S, exit. All p variables are out of control. If $D(i) \leq D(j)$ where $i < j$, $i = 1, \ldots, p - 1$ and $j = i + 1, \ldots, p$ and the variable set for $D(i)$ is a subset of the variable set for $D(j)$, then the variable set for $D(i)$ is the out-of-control variable set. Otherwise, increment i by 1 and repeat this condition check.

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