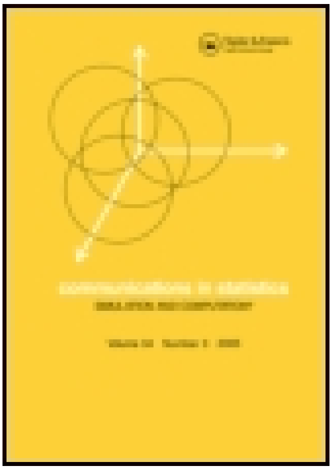

Communications in Statistics - Simulation and Computation
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lssp20

Monitoring Process Variability with Individual Measurements Following Elliptically Contoured Distributions

Pairoj Khawsithiwong [a] & Nihal Yatawara [b]

[a] Department of Statistics , Silpakorn University , Nakorn Phatom, Thailand

[b] Department of Mathematics and Statistics , Curtin University of Technology , Perth, Western Australia

Published online: 07 May 2007.

Communications in Statistics—Simulation and Computation®, 36: 699–718, 2007

ISSN: 0361-0918 print/1532-4141 online
DOI: 10.1080/03610910701208833

Taylor & Francis
Taylor & Francis Group

Quality Control

Monitoring Process Variability with Individual Measurements Following Elliptically Contoured Distributions

PAIROJ KHAWSITHIWONG[1] AND
NIHAL YATAWARA[2]

[1]Department of Statistics, Silpakorn University,
Nakorn Phatom, Thailand
[2]Department of Mathematics and Statistics,
Curtin University of Technology, Perth, Western Australia

An exponentially weighted moving average (EWMA) control chart of squared distance is developed by means of a double EWMA approach to monitor process dispersion with individual measurements distributed within the class of elliptically symmetric distributions. Several examples highlighting possible extensions of the control chart to multivariate processes are provided. In particular, for multivariate normal processes, an investigation on the detection power of the chart is carried out through Monte Carlo studies. The results show that the proposed control chart performs well, especially when a process has a small or moderate shift.

Keywords Control chart; Double exponentially weighted moving average; Elliptically contoured distribution; Individual observation; Process variability.

Mathematics Subject Classification 62N; 62N10.

1. Introduction

Several multivariate control charts for monitoring process dispersion have been developed and presented for subgroup data in the literature. These are extensions to the preliminary control charts based on the likelihood ratio test statistic or the sample generalized variance such as the ones proposed by Alt (1985) and Alt and Smith (1988). More recently introduced, alternative control charts associated with cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) techniques show improved performance when a process has small to moderate changes in dispersion. This is exemplified by the EWMA control chart based on

Received November 2, 2005; Accepted January 12, 2006
Address correspondence to Nihal Yatawara, Department of Mathematics and Statistics, Curtin University of Technology, Perth 6102, Western Australia; E-mail: N.Yatawara@curtin.edu.au

the logarithm of the sample generalized variance introduced by Bernard (2001) and the chart based on the cumulative probability of the sample generalized variance proposed by Yeh et al. (2003). Chan and Zhang (2001), on the other hand, used a projection pursuit method with the CUSUM technique to detect changes in the process dispersion for both individual and subgroup observations.

The other established control charts for monitoring process variability use individual observations to estimate the covariance matrix with a moving window technique. In the multivariate situation, chi-squared type control charts based on a squared standardized distance of the difference between successive pairs of observation vectors, $(\boldsymbol{X}_i - \boldsymbol{X}_{i-1})'\boldsymbol{\Sigma}^{-1}_{X_i - X_{i-1}}(\boldsymbol{X}_i - \boldsymbol{X}_{i-1})$ (see Khoo and Quah, 2003) and multivariate EWMA charts (see Yeh et al., 2005) have been proposed. The development of the squared distance chart has been motivated by its connection to the variance. However, it has low detection power if the variance shifts are small to moderate. The efficiency of this chart is also affected by serial correlation between successive values as in the moving range chart.

In the current work, a control chart known as "exponentially weighted moving average of distance" (EWMAD), based on a double EWMA approach is proposed. This general procedure can be adopted to monitor process variability with individual measurement vectors generated from various distributions in the class of elliptically symmetric distributions. The article is organized as follows. In Section 2, a brief introduction to this class of elliptically symmetric distributions which consists of several useful multivariate distributions such as the multivariate normal distribution, multivariate generalized Laplace distribution, multivariate t distribution, and the multivariate Pearson Type II distribution, is provided. Section 3 gives the details of the development of the chart. Section 4 presents the results from a performance investigation based on Monte-Carlo simulations. Section 5 provides a summary of the work and the conclusions.

2. Elliptically Contoured Distributions

An elliptically contoured (EC) distribution forms a class of symmetric distributions with varying tails relative to a multivariate normal distribution. Let $\boldsymbol{X}$ be a $p \times 1$ random vector, $\boldsymbol{\mu}$ be a $p \times 1$ vector of constants, and $\boldsymbol{\Sigma}$ be a $p \times p$ non-negative definite matrix. A random vector $\boldsymbol{X}$ following an EC distribution with parameters $\boldsymbol{\mu}, \boldsymbol{\Sigma}$, and g, is denoted by $\boldsymbol{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. The characteristic function of $\boldsymbol{X} - \boldsymbol{\mu}$, $\phi_{X-\mu}(\mathbf{t})$, is a function of the quadratic form, $\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}$, such that

$$\phi_X(\mathbf{t}) = \exp(i\mathbf{t}'\boldsymbol{\mu})g(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}), \tag{1}$$

where $g(\cdot)$ is a one-dimensional, real-valued function of a scalar variable. Furthermore, an EC random vector $\boldsymbol{X}$ can be defined in terms of a stochastic representation as

$$\boldsymbol{X} \equiv \boldsymbol{\mu} + RCU,$$

where "$\equiv$" means equality in distribution, $\boldsymbol{C}$ is any $p \times p$ matrix satisfying $\boldsymbol{\Sigma} = \mathbf{CC}'$, $\boldsymbol{U}$ has a uniform distribution on the p-dimensional sphere and

$R = \sqrt{(\boldsymbol{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu})}$ is independent of $\boldsymbol{U}$. Suppose that the joint density of $\boldsymbol{X}$ is given by

$$p_X(\mathbf{x}) = k_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}}g((\boldsymbol{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu})), \tag{2}$$

where k_p is a normalizing constant. Then, the density of the random variable R is defined in relation to function $g(\cdot)$ by,

$$p_R(r) = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} r^{p-1} g(r^2). \tag{3}$$

Clearly, the contours of constant density in (2) are ellipsoids. In addition, the marginal and conditional distributions of EC distributions are elliptically contoured. This is an exceptional property of the distributions belonging to this class. The mean vector and covariance matrix of a random vector with this distribution are, respectively, given by

$$E(\boldsymbol{X}) = \boldsymbol{\mu},$$

$$\text{and} \quad E(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})' = \frac{ER^2}{p}\boldsymbol{\Sigma}.$$

(See more details on EC distributions in Cambanis et al., 1981; Fang et al., 1990; Johnson, 1987, Ch. 6).

3. An EWMA Control Chart of Squared Distance

The construction of an EWMA control chart based on the multivariate generalized distance from an observation $\boldsymbol{X}_i$ to a process mean $\boldsymbol{\mu}_0$ at time i, $(\boldsymbol{X}_i - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{X}_i - \boldsymbol{\mu}_0)$, is proposed as a generalized algorithm for EC distributed processes. For a sequence of random vectors

$$\boldsymbol{X}_i = \boldsymbol{\mu}_0 + \boldsymbol{\varepsilon}_i, \quad i = 1, 2, \ldots, \tag{4}$$

where $\boldsymbol{\varepsilon}_i$ is an independent and identically distributed (i.i.d.) random error vector with an EC distribution, denoted by $\boldsymbol{\varepsilon}_i \sim EC_p(\mathbf{0}, \boldsymbol{\Sigma}_0; g)$, and $\boldsymbol{\mu}_0$ is unknown. The double EWMA procedure, introduced by Sweet (1986), yields the multivariate EWMA (MEWMA) statistic defined by

$$\boldsymbol{Y}_i = \mathbf{A}\boldsymbol{X}_i + (\mathbf{I} - \mathbf{A})\boldsymbol{Y}_{i-1}, \quad i = 1, 2, \ldots,$$

where $\boldsymbol{A}$ is a diagonal matrix consisting of weights $\{a_1, a_2, \ldots, a_p\}$, and $0 < a_j \leq 1, \ \ j = 1, 2, \ldots, p$ (see Lowry et al., 1992). Without loss of generality, it is possible to set $a_1 = a_2 = \cdots = a_p = a$ to obtain a simplified form for the statistic as

$$\boldsymbol{Y}_i = a\boldsymbol{X}_i + (1 - a)\boldsymbol{Y}_{i-1}, \quad i = 1, 2, \ldots. \tag{5}$$

The asymptotic mean vector and the covariance matrix of the MEWMA in (5) are, respectively,

$$\boldsymbol{\mu}_{\boldsymbol{Y}} = \lim_{t\to\infty} E(\boldsymbol{Y}_i) = \boldsymbol{\mu}_0$$

$$\text{and}\quad \boldsymbol{\Sigma}_{\boldsymbol{Y}} = \lim_{t\to\infty} VAR(\boldsymbol{Y}_i) = \frac{a}{2-a}\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}},$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} = \frac{E(\mathbf{R}_{\varepsilon}^2)}{p}\boldsymbol{\Sigma}_0$ and $R_{\boldsymbol{\varepsilon}}$ is a random variable with the density as defined in (3).

Let $D_i = (\boldsymbol{X}_i - \boldsymbol{Y}_{i-1})'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{X}_i - \boldsymbol{Y}_{i-1})$ be a measure of deviation of a measurement from the estimated geometric mean depending on the past data. Then, as in the traditional approach, an EWMA of the squared distance D_i, known as EWMAD, is easily constructed by

$$V_i = bD_i + (1-b)V_{i-1}, \quad 0 < b \leq 1. \tag{6}$$

For i sufficiently large, ignoring the effect of V_0, the Eq. (6) can be reformulated as

$$V_i = b\sum_{j=0}^{\infty}(1-b)^j D_{i-j}. \tag{7}$$

From (7), the asymptotic mean and the asymptotic variance of EWMAD statistic can be obtained as

$$\mu_V = \lim_{i\to\infty} E(V_i) = \mu_D \tag{8}$$

$$\text{and}\quad \sigma_V^2 = \lim_{i\to\infty} VAR(V_i) = \frac{b}{2-b}\left(\gamma_{D0} + 2\sum_{j=1}^{\infty}(1-b)^j\gamma_{Dj}\right), \tag{9}$$

where γ_{Dj}, $j = 0, 1, 2, \ldots$, are autocovariance functions of the limiting EWMAD statistic D. Then, using (8) and (9), the limiting EWMAD control chart can be established with asymptotic control limits:

$$\begin{aligned} UCL &= \mu_V + k_U\sigma_V \\ CL &= \mu_V \\ LCL &= \max(0, \mu_V - k_L\sigma_V) \end{aligned}$$

where UCL, CL, and LCL are the upper control limit, center line, and lower control limit, respectively, and k_U and k_L are chosen so that the control chart achieves a preferred in-control ARL.

For practical implementation of the EWMAD control chart, the required mean and the variance are derived by expanding MEWMA in (5) in an asymptotic form. It follows from (4) that

$$\mathbf{C}_0^{-1}(\boldsymbol{X}_i - \boldsymbol{Y}_{i-1}) = \sum_{j=0}^{\infty}\alpha_j\mathbf{C}_0^{-1}\boldsymbol{\varepsilon}_{i-j}, \quad \boldsymbol{\Sigma}_0 = \mathbf{C}_0\mathbf{C}_0'.$$

When the effect of $\boldsymbol{Y}_0$ is ignored, the expression for D_i in terms of random errors $\boldsymbol{\varepsilon}_i$ can be written as

$$D_i = \sum_{j=0}^{\infty} \alpha_j^2 \boldsymbol{\varepsilon}_{i-j}' \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\varepsilon}_{i-j} + 2\sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \alpha_j \alpha_k \boldsymbol{\varepsilon}_{i-j}' \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\varepsilon}_{i-k}, \tag{10}$$

where

$$\alpha_j = \begin{cases} 1, & j = 0 \\ -a(1-a)^{j-1}, & j \geq 0 \end{cases}.$$

Now from Equation (10), the asymptotic mean of EWMAD statistic can be obtained as,

$$\begin{aligned} \mu_D &= E(R_{\boldsymbol{\varepsilon}}^2) \sum_{j=0}^{\infty} \alpha_j^2 \\ &= \frac{2}{2-a} E(R_{\boldsymbol{\varepsilon}}^2), \end{aligned} \tag{11}$$

where $R_{\boldsymbol{\varepsilon}} = \sqrt{\boldsymbol{\varepsilon}' \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\varepsilon}}$. Also, the *zero*th-lag autocovariance function takes the form,

$$\begin{aligned} \gamma_{D_0} &= E\left(R_{\boldsymbol{\varepsilon}}^4\right) \sum_{j=0}^{\infty} \alpha_j^4 + 6\left(E(R_{\boldsymbol{\varepsilon}}^2)\right)^2 \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \alpha_j^2 \alpha_k^2 - \mu_D^2 \\ &= \left(1 + \frac{a^4}{1-(1-a)^4}\right) E\left(R_{\boldsymbol{\varepsilon}}^4\right) + \frac{2}{2-a}\left[3a\left(1 + \frac{a^2(1-a)^2}{1-(1-a)^4}\right) - \frac{2}{2-a}\right]\left(E(R_{\boldsymbol{\varepsilon}}^2)\right)^2, \end{aligned} \tag{12}$$

and the general jth-lag autocovariance functions can be given by

$$\begin{aligned} \gamma_{D_j} &= E(R_{\boldsymbol{\varepsilon}}^4) \sum_{i=j}^{\infty} \alpha_j^2 \alpha_{i-j}^2 \\ &\quad + 6\left(E(R_{\boldsymbol{\varepsilon}}^2)\right)^2 \left[\left(\sum_{i=0}^{\infty} \alpha_i^2\right)^2 - \sum_{i=j}^{\infty} \alpha_j^2 \alpha_{i-j}^2 + 4 \sum_{i=j}^{\infty} \sum_{k=1}^{\infty} \alpha_{i-j} \alpha_{i-j+k} \alpha_i \alpha_{i+k}\right] - \mu_D^2 \\ &= a^2(1-a)^{2(j-1)}\left(1 + \frac{a^2(1-a)^2}{1-(1-a)^4}\right)\left(E\left(R_{\boldsymbol{\varepsilon}}^4\right) - \left(E\left(R_{\boldsymbol{\varepsilon}}^2\right)\right)^2\right) \\ &\quad + 4\left(\frac{a^2(1-a)^{2j-1}}{2-a}\right)\left(\frac{a(1-a)^3}{1-(1-a)^4} - 1\right)\left(E\left(R_{\boldsymbol{\varepsilon}}^2\right)\right)^2, \quad j = 1, 2, \ldots \end{aligned} \tag{13}$$

Combining (9), (12), and (13), the asymptotic variance of EWMAD can be derived as

$$\sigma_V^2 = \frac{b}{2-b}\left[\beta_1 VAR\left(R_{\boldsymbol{\varepsilon}}^2\right) + \beta_2\left(E(R_{\boldsymbol{\varepsilon}}^2)\right)^2\right], \tag{14}$$

where

$$\beta_1 = 1 + \frac{a^4}{1-(1-a)^4} + \frac{2a^2(1-b)}{1-(1-b)(1-a)^2}\left(1 + \frac{a^2(1-a)^2}{1-(1-a)^4}\right),$$

$$\beta_2 = 1 + \frac{a^4}{1-(1-a)^4} + \frac{2}{2-a}\left[3a\left(1 + \frac{a^2(1-a)^2}{1-(1-a)^4}\right) - \frac{2}{2-a}\right] + \frac{8a^2(1-a)(1-b)}{(2-a)\big(1-(1-b)(1-a)^2\big)}\left(\frac{a(1-a)^3}{1-(1-a)^4} - 1\right).$$

These results, together with Eqs. (11) and (14), yield the EWMAD control chart with the asymptotic control limits:

$$UCL = \frac{2}{2-a}E(R_{\varepsilon}^2) + k_U\sqrt{\frac{b}{2-b}\left[\beta_1 VAR(R_{\varepsilon}^2) + \beta_2\big(E(R_{\varepsilon}^2)\big)^2\right]}$$

$$CL = \frac{2}{2-a}E\big(R_{\varepsilon}^2\big)$$

$$LCL = \max\left(0, \frac{2}{2-a}E\big(R_{\varepsilon}^2\big) - k_L\sqrt{\frac{b}{2-b}\left[\beta_1 VAR\big(R_{\varepsilon}^2\big) + beta_2\big(E(R_{\varepsilon}^2)\big)^2\right]}\right),$$

where the constants k_U and k_L are evaluated to satisfy specific in-control ARL's. Note that the values of these constant factors can vary from one distribution to another. In addition, since R_{ε} is a standardized distance of an i.i.d. EC distributed random vector, this makes the EWMAD control chart suitable to monitor a process with measurements following any distribution in this class. In some practical applications, the in-control distribution would be approximately known. However, if it is unknown and the wrong distribution is assumed the control limits of the chart must be fairly robust to guard against an excessive false alarm rate. The robustness of the chart and the effect of smoothing parameters on the performance of the chart are examined in the next section.

4. Performance Study

This section presents a Monte Carlo performance analysis of the EWMAD control chart when the measurements come from p-variate normal processes with $p = 2, 5, 10$, and 20. In the in-control state, the process is assumed to have a zero mean vector and a covariance matrix

$$\begin{pmatrix} 1 & 0.5 & \cdots & 0.5 \\ 0.5 & 1 & \cdots & 0.5 \\ \vdots & \vdots & \ddots & \vdots \\ 0.5 & 0.5 & \cdots & 1 \end{pmatrix}.$$

In the monitoring period, the variance of the first variable is assumed to shift according to the relation, $\delta_{SD} = \frac{\sigma_1}{\sigma_0}$, where σ_0 and σ_1 are the respective standard deviations corresponding to an in-control and a monitoring state. Both cases of increasing and decreasing process variability are considered by setting the values of δ_{SD} equal to 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 2.0, 2.5, 3.0, and 4.0. Furthermore, to understand the effect of the smoothing parameters on the performance of the EWMAD control chart, the values of the smoothing parameters of MEWMA and EWMAD, a and b are set to be any combination of values chosen from 0.05, 0.2,

Table 1
Examples of EWMAD control charts

Distribution	Distribution and moments of $R_\varepsilon^2 = \varepsilon'\Sigma^{-1}\varepsilon$	Control limits
Multivariate Normal		
$f(\varepsilon) = \frac{1}{(2\pi)^{\frac{p}{2}}}\lvert\Sigma\rvert^{-\frac{1}{2}} e^{-\frac{1}{2}\varepsilon'\Sigma^{-1}\varepsilon}$	$R_\varepsilon^2 \sim \chi_p^2$	$UCL = \frac{2}{2-a}p + k_U^{(1)}\sqrt{\frac{2}{2-b}(2\beta_1 p + \beta_2 p^2)}$
	$E(R_\varepsilon^2) = p$	$CL = \frac{2}{2-a}p$
	$VAR(R_\varepsilon^2) = 2p$	$LCL = \max\left(0, \frac{2}{2-a}p - k_L^{(1)}\sqrt{\frac{2}{2-b}(2\beta_1 p + \beta_2 p^2)}\right)$
Multivariate Generalized Laplace		
$f(\varepsilon) = \frac{\lambda\Gamma(\frac{p}{2})}{2\pi^{\frac{p}{2}}\Gamma(\frac{p}{\lambda})}\lvert\Sigma\rvert^{-\frac{1}{2}} e^{-\left(\varepsilon'\Sigma^{-1}\varepsilon\right)^{\frac{\lambda}{2}}}$,	$R_\varepsilon^2 \sim GGamma(\frac{p}{\lambda}, 1, \frac{\lambda}{2})$	$UCL = \frac{2\Gamma(\frac{p+2}{\lambda})}{(2-a)\Gamma(\frac{p}{\lambda})} + k_U^{(2)}\sqrt{\frac{2}{2-b}\left(\beta_1\frac{\Gamma(\frac{p+4}{\lambda})\Gamma(\frac{p}{\lambda})-(\Gamma(\frac{p+2}{\lambda}))^2}{(\Gamma(\frac{p}{\lambda}))^2} + \beta_2\left(\frac{\Gamma(\frac{p+2}{\lambda})}{\Gamma(\frac{p}{\lambda})}\right)^2\right)}$
$\lambda > 0$	$E(R_\varepsilon^2) = \frac{\Gamma(\frac{p+2}{\lambda})}{\Gamma(\frac{p}{\lambda})}$	$CL = \frac{2\Gamma(\frac{p+2}{\lambda})}{(2-a)\Gamma(\frac{p}{\lambda})}$
	$VAR(R_\varepsilon^2) = \frac{\Gamma(\frac{p+4}{\lambda})\Gamma(\frac{p}{\lambda})-(\Gamma(\frac{p+2}{\lambda}))^2}{(\Gamma(\frac{p}{\lambda}))^2}$	$LCL = \max\left(0, \frac{2\Gamma(\frac{p+2}{\lambda})}{(2-a)\Gamma(\frac{p}{\lambda})} - k_L^{(2)}\sqrt{\frac{2}{2-b}\left(\beta_1\frac{\Gamma(\frac{p+4}{\lambda})\Gamma(\frac{p}{\lambda})-(\Gamma(\frac{p+2}{\lambda}))^2}{(\Gamma(\frac{p}{\lambda}))^2} + \beta_2\left(\frac{\Gamma(\frac{p+2}{\lambda})}{\Gamma(\frac{p}{\lambda})}\right)^2\right)}\right)$
Multivariate t		
$f(\varepsilon) = \frac{\Gamma(\frac{p+m}{2})}{(m\pi)^{\frac{p}{2}}\Gamma(\frac{m}{2})}\lvert\Sigma\rvert^{-\frac{1}{2}}$	$R_\varepsilon^2 \sim BetaII(\frac{p}{2}, \frac{m}{2})$	$UCL = \frac{2mp}{(2-a)(m-2)} + k_U^{(3)}\sqrt{\frac{2}{2-b}\left(\beta_1\frac{2pm^2(5p-mp+m-2)}{(m-4)(m-2)^2} + \beta_2(\frac{mp}{m-2})^2\right)}$
$\times(1 + \frac{\varepsilon'\Sigma^{-1}\varepsilon}{m})^{-(\frac{p+m}{2})}$,	$E(R_\varepsilon^2) = \frac{mp}{m-2}$	$CL = \frac{2mp}{(2-a)(m-2)}$
$m > 0$	$VAR(R_\varepsilon^2) = \frac{2pm^2(5p-mp+m-2)}{(m-4)(m-2)^2}$	$LCL = \max\left(0, \frac{2mp}{(2-a)(m-2)} - k_L^{(3)}\sqrt{\frac{2}{2-b}\left(\beta_1\frac{2pm^2(5p-mp+m-2)}{(m-4)(m-2)^2} + \beta_2(\frac{mp}{m-2})^2\right)}\right)$
Multivariate Pearson Type II		
$f(\varepsilon) = \frac{\Gamma(\frac{p}{2}+m+1)}{\pi^{\frac{p}{2}}\Gamma(m+1)}\lvert\Sigma\rvert^{-\frac{1}{2}}$	$R_\varepsilon^2 \sim Beta(\frac{p}{2}, m+1)$	$UCL = \frac{2p}{(2-a)(p+2(m-1))} + k_U^{(4)}\sqrt{\frac{2}{2-b}\left(\beta_1\frac{4p(m+1)}{(p+2(m+2))(p+2(m+1))^2} + \beta_2(\frac{p}{p+2(m-1)})^2\right)}$
$\times(1 + \varepsilon'\Sigma^{-1}\varepsilon)^m$, $m > -1$	$E(R_\varepsilon^2) = \frac{p}{p+2(m-1)}$	$CL = \frac{2p}{(2-a)(p+2(m-1))}$
	$VAR(R_\varepsilon^2) = \frac{4p(m+1)}{(p+2(m+2))(p+2(m+1))^2}$	$LCL = \max\left(0, \frac{2p}{(2-a)(p+2(m-1))} - k_L^{(4)}\sqrt{\frac{2}{2-b}\left(\beta_1\frac{4p(m+1)}{(p+2(m+2))(p+2(m+1))^2} + \beta_2(\frac{p}{p+2(m-1)})^2\right)}\right)$

Notes: χ^2, *G Gamma, BetaII*, and *Beta* denote the chi-square, generalized gamma, beta Type *II*, and beta distributions, respectively. $E(\cdot)$ and $VAR(\cdot)$ are the expected value and the variance, and $\Gamma(\cdot)$ is the gamma function. *UCL*, *LCL*, and *CL* denote the upper and lower control limits and center line. The multivariate generalized Laplace distribution was presented by Ernst (1998) as a class of multivariate models consisting of Laplace, normal, and uniform distributions.

Table 2

ARL of the EWMAD control charts with various smoothing parameters for a p-Variate normal distribution; $p = 2, 5, 10$, and 20

	Shift in standard deviation														
a	0.05					0.5					1.0				
b	0.05	0.2	0.5	0.7	1.0	0.05	0.2	0.5	0.7	1.0	0.05	0.2	0.5	0.7	1.0
$p = 2$															
4.0	1.02	1.02	1.21	1.55	2.28	1.03	1.03	1.25	1.59	2.28	1.04	1.05	1.36	1.71	2.29
3.0	1.05	1.07	1.64	2.31	3.33	1.06	1.10	1.73	2.38	3.33	1.09	1.17	1.97	2.61	3.37
2.5	1.09	1.20	2.35	3.42	4.75	1.11	1.27	2.50	3.52	4.75	1.15	1.45	2.90	3.89	4.84
2.0	1.19	1.76	4.58	6.69	8.74	1.25	1.98	4.93	6.87	8.71	1.35	2.50	5.82	7.56	8.95
1.5	2.26	7.13	19.94	26.73	31.59	2.70	8.44	21.08	26.75	30.84	3.48	11.55	24.26	29.70	32.52
1.4	3.46	12.39	32.81	41.86	47.70	4.32	14.64	34.51	41.99	46.90	5.88	19.72	39.40	46.63	49.67
1.3	7.06	26.10	59.50	72.99	79.90	9.12	29.74	61.89	72.15	78.26	12.26	38.74	69.96	80.08	82.06
1.2	21.26	67.16	122.75	140.85	142.73	28.80	76.30	127.80	141.05	144.97	37.66	96.88	143.57	158.22	156.35
1.1	100.68	205.45	275.13	290.28	275.90	126.48	222.63	279.84	284.72	281.93	144.20	243.18	295.89	311.31	298.60
1.0	503.08	500.00	498.98	503.60	505.68	498.98	500.51	500.51	500.00	504.12	503.08	500.51	501.54	501.02	502.05
0.9	141.29	259.53	403.62	470.70	599.02	167.01	278.25	393.26	414.55	589.65	194.29	279.36	391.37	469.80	632.26
0.8	26.65	88.29	211.21	289.60	584.73	31.41	91.57	198.62	289.26	574.44	44.38	102.83	226.54	299.51	553.67
0.7	8.02	28.23	95.89	155.36	477.58	9.38	30.94	94.50	148.26	493.95	12.56	35.11	105.97	167.35	501.02
0.6	3.70	10.18	34.81	64.60	342.66	4.22	10.99	34.72	62.42	387.66	5.44	12.82	39.86	69.86	360.29
0.5	2.42	4.93	10.95	15.02	28.29	2.68	5.23	10.94	14.28	28.12	3.24	5.75	11.55	15.47	27.94
$p = 5$															
4.0	1.03	1.02	1.23	1.58	2.33	1.04	1.03	1.27	1.62	2.33	1.05	1.05	1.37	1.73	2.33
3.0	1.06	1.08	1.72	2.41	3.48	1.07	1.12	1.81	2.48	3.47	1.10	1.20	2.02	2.68	3.47
2.5	1.10	1.24	2.48	3.62	5.03	1.13	1.32	2.64	3.72	5.01	1.18	1.50	2.99	3.99	5.02
2.0	1.25	1.96	5.08	7.34	9.61	1.31	2.19	5.41	7.53	9.62	1.45	2.72	6.16	8.01	9.60
1.5	2.89	9.11	24.14	31.83	37.64	3.39	10.47	25.20	32.06	37.42	4.60	13.51	27.56	33.18	37.56
1.4	5.02	16.83	41.28	52.03	59.61	5.99	19.09	42.53	51.92	58.54	8.35	24.19	45.81	53.72	58.94
1.3	11.06	36.49	78.15	92.28	101.39	13.48	40.77	79.71	92.56	99.98	18.64	49.61	83.50	94.96	101.30
1.2	35.32	95.42	166.33	188.24	194.83	43.04	105.17	172.66	189.19	192.23	56.72	118.04	170.32	187.02	193.14
1.1	155.56	256.95	332.88	341.70	351.76	171.93	276.21	331.53	341.70	327.98	201.23	289.09	322.16	341.46	349.50
1.0	501.54	498.98	500.00	498.98	503.08	502.56	501.54	501.02	501.02	500.00	500.51	502.05	501.54	498.98	500.00

0.9	189.34	323.01	424.24	457.09	553.05	196.55	324.29	425.35	439.07	503.60	225.18	317.98	426.46	460.09	560.64
0.8	48.48	135.43	249.34	311.11	492.96	54.91	144.46	256.01	306.83	458.80	74.72	168.91	271.17	331.98	513.09
0.7	19.00	60.16	135.25	181.15	359.50	22.36	67.27	143.23	189.63	339.81	30.73	78.21	156.50	207.45	374.90
$p = 10$															
4.0	1.03	1.03	1.31	1.71	2.51	1.05	1.05	1.35	1.75	2.51	1.06	1.07	1.47	1.88	2.51
3.0	1.08	1.13	1.91	2.71	3.88	1.10	1.17	2.01	2.78	3.86	1.13	1.28	2.28	3.01	3.86
2.5	1.14	1.37	2.91	4.22	5.82	1.17	1.48	3.07	4.32	5.77	1.24	1.73	3.54	4.64	5.77
2.0	1.37	2.47	6.46	9.21	11.96	1.46	2.77	6.77	9.32	11.83	1.68	3.53	7.87	9.96	11.79
1.5	4.59	14.81	35.17	44.68	52.76	5.35	16.62	35.70	44.64	52.18	7.40	21.39	39.51	46.95	52.36
1.4	8.63	27.61	60.34	74.13	85.16	10.27	31.12	61.04	74.03	83.96	14.45	39.06	67.00	77.24	83.90
1.3	20.29	58.26	112.23	132.40	144.29	23.29	63.22	110.16	130.08	140.93	31.67	76.74	118.44	133.55	141.01
1.2	59.81	138.85	211.30	231.68	239.84	65.98	147.10	207.10	232.23	240.20	86.94	166.72	224.46	241.98	239.73
1.1	228.97	332.88	376.06	387.97	385.83	249.36	336.54	377.79	386.13	379.85	274.51	354.82	382.81	393.89	379.55
1.0	497.97	503.60	497.46	500.00	499.49	498.47	501.54	503.60	501.02	501.54	499.49	499.49	500.51	499.49	501.02
0.9	268.35	383.71	432.86	441.04	498.98	279.36	380.73	432.48	442.24	470.70	300.80	431.72	441.04	495.95	501.02
0.8	98.18	235.92	327.98	369.25	468.90	112.49	226.33	332.20	385.52	437.11	129.66	258.58	322.58	394.84	460.09
0.7	43.77	126.88	214.16	256.14	348.26	52.12	129.56	229.08	271.32	336.54	63.23	150.31	227.91	291.67	378.96
$p = 20$															
4.0	1.05	1.05	1.45	1.94	2.83	1.06	1.07	1.51	2.01	2.83	1.09	1.12	1.66	2.15	2.84
3.0	1.11	1.23	2.36	3.35	4.68	1.14	1.31	2.49	3.50	4.71	1.19	1.48	2.84	3.75	4.75
2.5	1.21	1.70	3.88	5.60	7.54	1.26	1.85	4.19	5.86	7.51	1.37	2.25	4.79	6.24	7.64
2.0	1.68	3.77	9.74	13.58	17.04	1.83	4.23	10.57	14.09	17.19	2.25	5.47	11.83	14.77	17.47
1.5	8.59	28.13	62.56	74.46	85.07	10.01	30.79	66.29	76.08	86.28	13.94	39.11	67.05	78.82	87.33
1.4	18.33	56.94	104.12	126.48	136.72	19.94	61.12	110.11	131.12	134.14	30.01	75.75	119.63	135.21	148.62
1.3	41.23	111.41	178.70	209.67	222.22	44.98	116.69	186.17	214.07	218.07	66.50	140.93	197.02	215.67	228.01
1.2	109.57	223.64	321.52	331.98	333.79	108.24	234.67	321.31	332.43	333.33	154.43	261.19	330.41	346.53	347.27
1.1	244.88	374.33	432.10	434.01	430.20	238.10	370.65	438.28	445.86	454.55	286.22	379.55	441.04	463.14	460.09
1.0	501.54	498.47	498.47	499.49	500.51	502.56	499.49	500.00	500.51	500.51	499.49	499.49	500.51	500.00	499.49
0.9	345.07	401.31	498.47	492.46	547.49	355.07	439.07	464.90	479.45	492.46	370.65	449.95	478.98	495.45	542.04
0.8	168.73	297.33	376.06	420.60	504.63	169.43	312.90	410.39	436.33	457.52	215.57	344.10	411.07	444.65	504.12
0.7	92.26	189.56	293.77	339.10	406.98	97.34	219.24	296.43	342.18	401.31	116.36	228.23	323.65	354.30	420.60

0.5, 0.7, and 1, respectively. For comparison purposes, an in-control ARL value of 500 was maintained based on 10,000 iterations.

4.1. *Run Length Study*

The results in Table 2 show that the EWMAD chart with two-sided control limits rapidly detects shifts in process variability when it either increases or decreases, especially, when the change is small or moderate. Moreover, it is also seen that smaller values of a and b yield faster detection. When b is near 1, the EWMAD chart hardly gives any out of control signal for a small decrease in the process variance.

The analysis also suggests that the detection capability of the EWMAD chart mainly depends upon the value of b rather than both of them. This is contrary to our expectations as a is an integral part of the MEWMA used to estimate the mean of a process. This is illustrated in Fig. 1, which shows that the EWMAD charts constructed with various values of a do not impact on the detection capability. However, as Fig. 2 shows, the magnitude of b has a strong effect on the detection power of the chart and, clearly, small values of b are preferred in the establishment of the EWMAD chart.

Suppose now that the process follows a multivariate t distribution but the Gaussian-based EWMAD control chart is used. This leads to an investigation on the sensitivity of the EWMAD chart to incorrect specification of the distribution. Following the preceding simulation investigation, zero mean bivariate t random vectors are generated with 5, 10, 50, 100, and 1,000 degrees of freedom.

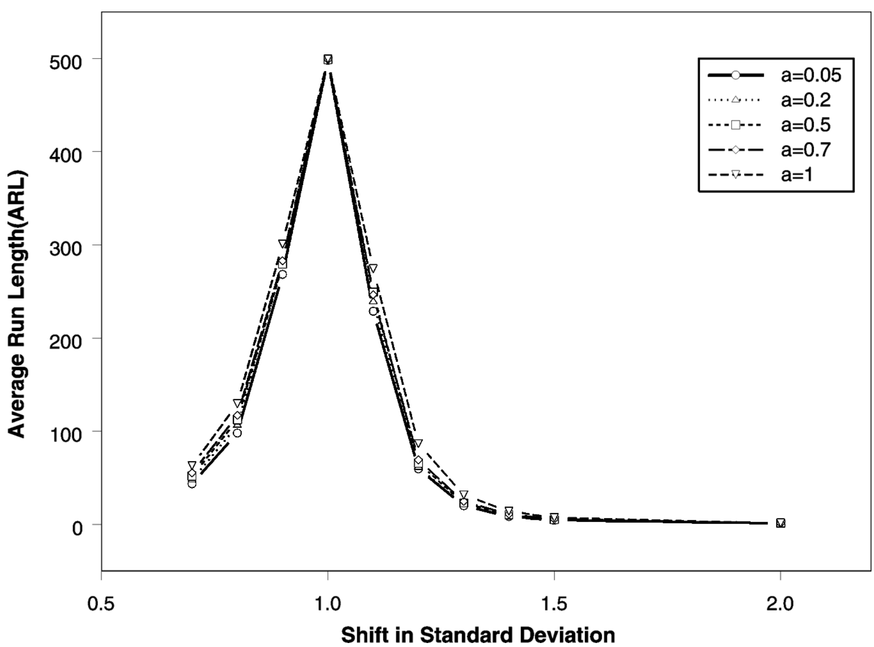

Figure 1. The effect of smoothing parameter a on the performance of the EWMAD control chart as the variance changes; $a = 0.05$, 0.2, 0.5, 0.7, and 1.

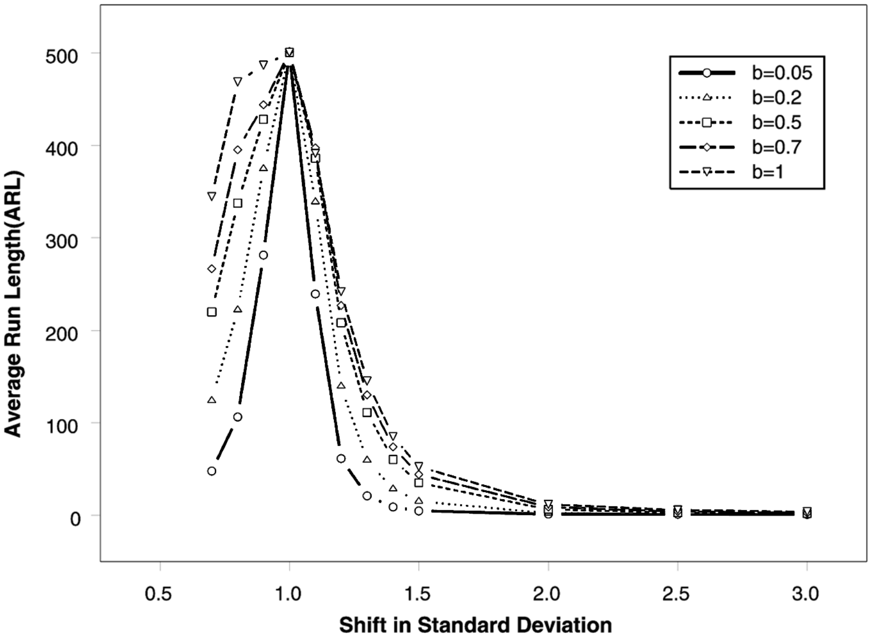

Figure 2. The effect of smoothing parameter b on the performance of the EWMAD control chart as the variance changes; $b = 0.05$, 0.2, 0.5, 0.7, and 1.

The results in Table 3 show that the Gaussian-based EWMAD charts do not maintain the in-control ARL except when the degrees of freedom becomes large. This is because the multivariate t distribution with large degrees of freedom approaches the multivariate normal distribution. Conversely, when the degrees of freedom is small, the multivariate t distribution is flatter and thicker tailed than a multivariate Gaussian distribution. It means that the data from such a multivariate t distribution have more variability than those from the Gaussian distribution, even though the covariance matrices of the distributions are the same. Therefore, when the Gaussian-based EWMAD chart is used for monitoring t distributed data-therefore, the ARL values in case of no change in covariance matrix become less than 500 which is the in-control ARL for the Gaussian case.

Since the EC distribution tails are expressed relative to the multivariate normal distribution, the variances of these EC distributions are also proportional to that of the normal distribution. For example, suppose that a covariance matrix of the multivariate normal distribution is given by $\mathbf{\Sigma}$. The covariance matrices of a multivariate generalized Laplace distribution, a multivariate t distribution, and a multivariate Pearson Type II distribution are, respectively, given by $\frac{\Gamma(\frac{p+2}{\lambda})}{p\Gamma(\frac{p}{\lambda})}\mathbf{\Sigma}$, $\frac{m}{m-2}\mathbf{\Sigma}$, and $\frac{1}{p+2m+2}\mathbf{\Sigma}$ which are proportional to $\mathbf{\Sigma}$. Therefore, if the distribution of data changes from one to another in this class, as a result of a variance change in the process, then the EWMAD chart could be successfully used to detect these variance changes.

In addition, the effectiveness of the EWMAD chart, when variables are subject to different correlations, is also an important issue and this is investigated using a bivariate normal process with the above in-control covariance matrix.

Table 3
ARL of the Gaussian-based EWMAD control charts with various smoothing parameters for data generated through bivariate t distributions with degrees of freedom 5,10,50,100, and 1,000

	Shift in standard deviation														
a	0.05					0.5					1.0				
b	0.05	0.2	0.5	0.7	1.0	0.05	0.2	0.5	0.7	1.0	0.05	0.2	0.5	0.7	1.0
$df = 5$															
4.0	1.02	1.01	1.13	1.37	2.06	1.02	1.02	1.14	1.38	1.96	1.03	1.03	1.21	1.46	1.97
3.0	1.03	1.04	1.36	1.82	2.76	1.04	1.05	1.40	1.82	2.60	1.06	1.08	1.52	1.95	2.61
2.5	1.05	1.06	1.68	2.35	3.54	1.07	1.11	1.73	2.35	3.32	1.09	1.17	1.93	2.53	3.33
2.0	1.10	1.25	2.47	3.57	5.17	1.12	1.31	2.55	3.54	4.87	1.16	1.47	2.89	3.83	4.90
1.5	1.27	2.07	5.00	7.24	9.72	1.34	2.20	5.27	7.23	9.51	1.40	2.77	6.09	7.92	9.02
1.4	1.37	2.48	6.14	8.68	11.47	1.45	2.73	6.41	8.72	11.39	1.62	3.40	7.42	9.52	11.49
1.3	1.53	3.11	7.66	10.70	13.73	1.66	3.46	8.06	10.83	13.99	1.90	4.36	9.38	11.94	14.16
1.2	1.63	4.11	9.91	13.52	16.90	2.04	4.63	10.56	13.95	17.64	2.43	5.94	12.39	15.47	18.19
1.1	2.38	5.70	13.06	17.41	21.26	2.71	6.43	13.97	18.21	22.82	3.32	8.25	16.42	20.25	23.35
1.0	3.45	8.19	17.28	22.39	26.42	4.02	9.30	18.93	24.10	29.60	5.06	12.14	22.36	27.04	30.82
0.9	5.59	12.27	23.61	29.72	34.44	6.63	14.03	26.04	32.34	39.20	8.39	18.25	30.91	36.41	41.23
0.8	9.40	17.25	29.84	36.60	42.02	11.20	19.75	33.04	40.00	49.04	14.01	25.37	39.28	44.99	51.33
0.7	13.76	20.82	33.66	41.47	50.50	16.20	23.71	37.33	45.68	59.91	20.26	29.76	43.34	50.76	61.57
0.6	12.17	16.93	26.74	37.05	55.21	14.05	19.06	31.12	40.58	65.28	17.84	23.55	36.97	46.36	69.41
0.5	6.84	8.96	13.26	15.90	21.62	7.64	9.71	13.82	16.34	23.48	9.39	10.98	15.03	17.40	23.73

$df = 10$															
4.0	1.02	1.02	1.16	1.45	2.16	1.03	1.02	1.19	1.47	2.11	1.04	1.04	1.27	1.57	2.12
3.0	1.04	1.05	1.49	2.04	3.02	1.05	1.07	1.54	2.06	2.93	1.07	1.11	1.71	2.24	2.95
2.5	1.07	1.12	1.96	2.81	4.07	1.08	1.16	2.05	2.84	3.93	1.11	1.27	2.33	3.10	3.97
2.0	1.13	1.42	3.23	4.72	6.50	1.17	1.53	3.38	4.75	6.32	1.23	1.82	3.94	5.21	6.41
1.5	1.51	3.28	8.76	12.24	15.18	1.66	3.72	9.35	12.52	15.57	1.95	4.88	11.07	13.98	16.20
1.4	1.79	4.43	11.65	15.81	19.17	2.03	5.10	12.46	16.48	20.14	2.46	6.79	14.85	18.46	21.03
1.3	2.34	6.55	16.46	21.71	25.56	2.75	7.63	17.96	23.13	27.59	3.52	10.31	21.57	26.19	29.32
1.2	3.65	10.44	23.99	30.59	34.56	4.47	12.33	26.92	33.57	39.06	5.94	16.91	32.55	38.51	42.30
1.1	6.91	18.75	37.49	46.05	49.86	8.74	22.20	42.40	50.96	58.33	11.77	30.15	51.29	58.56	63.19
1.0	15.73	34.28	57.43	66.14	69.52	20.44	41.45	68.07	78.01	86.28	27.68	56.45	81.50	91.18	96.80
0.9	43.84	63.46	87.36	97.36	102.64	58.64	78.09	105.47	116.92	130.77	75.84	102.02	129.05	138.42	149.34
0.8	61.34	73.29	99.07	112.33	128.74	75.65	85.00	114.09	131.33	165.26	100.43	104.19	137.14	152.13	186.81
0.7	23.99	43.90	80.10	102.25	146.14	27.84	49.56	87.11	112.23	185.68	37.13	58.40	101.93	127.94	206.75
0.6	8.31	16.94	38.92	59.62	145.40	9.50	18.72	42.38	64.07	175.75	12.52	22.11	49.67	73.10	191.86
0.5	4.02	7.05	12.88	16.34	25.69	4.48	7.54	13.11	16.39	27.37	5.64	8.53	14.16	17.40	27.84
$df = 50$															
4.0	1.02	1.02	1.20	1.53	2.25	1.03	1.03	1.24	1.56	2.24	1.04	1.05	1.34	1.68	2.26
3.0	1.05	1.06	1.62	2.26	3.28	1.06	1.09	1.70	2.32	3.26	1.08	1.16	1.92	2.54	3.29
2.5	1.06	1.17	2.24	3.26	4.56	1.10	1.24	2.38	3.35	4.53	1.14	1.40	2.76	3.69	4.62
2.0	1.18	1.67	4.22	6.17	8.19	1.22	1.85	4.49	6.28	8.07	1.32	2.31	5.30	6.93	8.27
1.5	2.00	5.86	16.34	22.13	26.03	2.34	6.88	17.46	22.39	26.15	3.00	9.35	20.52	24.96	27.48
1.4	2.92	9.66	25.38	33.08	37.93	3.59	11.46	27.29	34.11	39.04	4.83	15.71	32.14	38.09	40.91
1.3	5.22	18.34	43.04	54.38	60.43	6.63	21.42	46.09	54.92	60.64	9.10	28.65	53.16	61.30	64.13
1.2	12.51	40.46	79.44	93.65	96.90	16.38	47.00	87.19	98.61	103.73	22.17	62.08	100.14	110.21	111.74
1.1	47.16	106.59	163.33	176.58	176.45	63.32	121.56	176.39	187.52	188.10	78.15	149.80	197.98	210.84	204.59
1.0	245.61	286.72	309.73	323.01	313.10	305.49	317.15	334.24	340.75	356.10	327.54	350.00	369.25	386.13	384.01
0.9	204.59	263.40	353.03	376.92	410.39	200.90	288.57	350.00	376.34	492.46	247.47	298.78	371.21	407.65	508.83
0.8	34.97	95.40	205.45	274.36	467.11	40.06	101.34	208.96	268.94	516.88	54.05	115.32	224.67	301.54	507.77
0.7	9.94	32.08	98.14	156.50	413.85	11.62	35.28	98.79	152.32	451.61	16.20	41.75	112.51	166.38	451.61
0.6	4.24	11.25	37.00	66.59	324.72	4.85	12.17	36.77	64.29	319.63	6.39	14.43	42.10	71.84	331.75
0.5	2.04	5.25	11.29	15.34	26.00	2.93	5.61	11.37	14.99	26.35	3.00	0.36	12.21	15.85	26.31

(continued)

Table 3
Continued

	Shift in standard deviation														
a	0.05					0.5					1.0				
b	0.05	0.2	0.5	0.7	1.0	0.05	0.2	0.5	0.7	1.0	0.05	0.2	0.5	0.7	1.0
$df = 100$															
4.0	1.02	1.02	1.20	1.54	2.27	1.03	1.03	1.24	1.57	2.25	1.04	1.05	1.34	1.69	2.27
3.0	1.05	1.06	1.63	2.28	3.30	1.06	1.09	1.71	2.34	3.28	1.09	1.16	1.94	2.57	3.32
2.5	1.06	1.18	2.30	3.35	4.68	1.11	1.25	2.44	3.44	4.66	1.14	1.42	2.84	3.79	4.74
2.0	1.18	1.71	4.38	6.39	8.41	1.23	1.90	4.69	6.54	8.34	1.33	2.40	5.55	7.25	8.59
1.5	2.13	6.49	18.03	24.24	28.53	2.52	7.64	19.20	24.48	28.43	3.27	10.46	22.58	27.22	29.94
1.4	3.15	10.90	28.59	36.86	41.91	3.92	12.95	30.95	37.92	42.87	5.31	17.73	35.88	42.22	45.50
1.3	6.06	21.67	50.20	62.17	67.54	7.87	25.48	53.57	63.39	68.82	10.70	33.93	61.96	69.96	72.86
1.2	16.17	51.56	98.95	115.65	120.16	21.33	59.23	106.75	118.47	122.53	28.44	76.35	121.41	131.90	131.47
1.1	71.67	155.56	209.94	218.07	214.44	91.18	167.87	223.23	228.54	235.58	109.11	194.75	247.60	260.50	251.02
1.0	316.74	359.77	378.38	385.83	394.21	367.32	380.73	388.89	391.06	409.70	404.96	412.46	420.60	436.72	429.82
0.9	166.83	276.73	379.85	426.46	520.72	181.89	272.22	360.56	411.07	558.08	234.56	289.77	378.96	455.39	583.33
0.8	31.72	95.37	213.88	294.47	533.77	37.10	96.85	206.58	268.20	573.77	51.24	106.61	219.24	291.49	553.67
0.7	8.73	29.51	98.10	163.01	458.37	10.21	32.18	96.74	157.46	474.35	14.29	38.19	109.13	172.35	465.34
0.6	3.92	10.62	35.83	64.98	335.39	4.46	11.45	35.57	63.09	350.50	5.83	13.50	41.06	71.57	370.65
0.5	2.51	5.02	11.04	15.06	28.20	2.79	5.33	10.97	14.60	28.30	3.47	6.03	11.81	15.46	28.13

$df = 1000$															
4.0	1.02	1.02	1.21	1.55	2.28	1.03	1.03	1.25	1.59	2.27	1.05	1.05	1.36	1.71	2.29
3.0	1.05	1.07	1.64	2.30	3.32	1.06	1.10	1.73	2.37	3.31	1.09	1.17	1.97	2.61	3.36
2.5	1.08	1.19	2.34	3.40	4.74	1.11	1.26	2.49	3.50	4.73	1.15	1.44	2.90	3.86	4.81
2.0	1.19	1.75	4.54	6.64	8.68	1.24	1.96	4.86	6.79	8.66	1.35	2.48	5.76	7.49	8.88
1.5	2.26	7.12	19.90	26.50	31.29	2.69	8.41	21.08	26.73	30.76	3.52	11.47	24.69	29.54	32.54
1.4	3.44	12.34	32.04	41.23	46.53	4.29	14.56	34.39	41.68	46.29	5.84	19.83	39.48	45.80	49.11
1.3	6.85	25.15	57.69	70.91	76.72	8.95	29.45	61.01	71.57	77.14	12.28	39.07	71.22	81.21	84.00
1.2	19.52	63.79	119.11	137.45	139.80	25.85	72.67	123.58	136.00	140.04	34.03	90.07	139.84	153.36	151.47
1.1	103.84	203.66	262.59	280.64	276.37	132.97	216.05	261.33	271.62	258.58	150.77	245.74	277.94	294.47	278.73
1.0	456.24	451.61	481.34	478.05	496.45	486.11	439.07	455.39	456.66	504.12	450.78	441.44	456.24	492.96	514.71
0.9	134.62	275.59	400.00	470.25	602.71	148.22	273.90	386.74	433.63	649.01	182.77	285.05	420.60	480.86	652.46
0.8	28.17	88.34	214.72	299.88	595.38	32.03	90.32	205.88	281.61	593.94	43.66	99.88	219.53	305.87	597.56
0.7	8.00	28.02	94.54	159.66	511.48	9.38	30.21	94.12	149.89	518.52	13.07	35.46	107.98	167.64	513.63
0.6	3.70	10.23	34.97	66.63	355.07	4.22	11.02	35.38	63.50	340.51	5.54	12.94	40.17	71.12	352.77
0.5	2.43	4.90	10.88	14.86	28.43	2.67	5.19	10.74	14.40	28.06	3.31	5.87	11.51	15.09	27.93

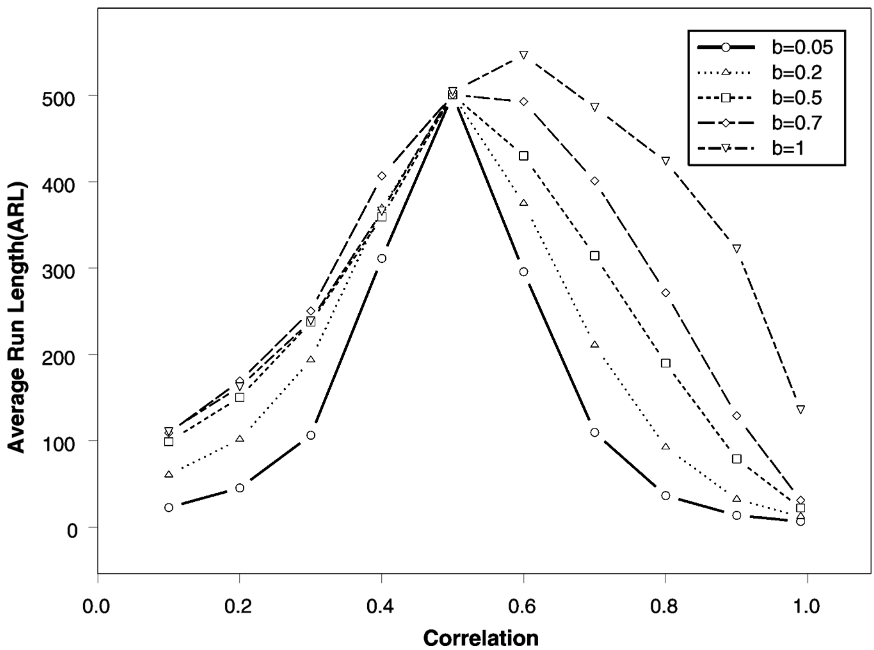

Figure 3. The effect of smoothing parameter b on the performance of the EWMAD control chart as the correlation changes; $b = 0.05, 0.2, 0.5, 0.7$, and 1.

With variances of all variables being 1, correlations between variables assume values, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99. It is clear from Fig. 3 that the EWMAD chart is capable of detecting changes in correlations between variables. Notice that when the correlation is increasing and b is nearly equal to 1, the EWMAD chart has difficulty in detecting an out of control situation since the ARL values are slowly decreasing. Thus, as in the previous study of variance change detection, small values of b are preferred also in this case.

4.2. *Choice of Smoothing Parameters*

The EWMAD chart is very sensitive to process mean shifts and as the unknown actual mean is estimated by MEWMA in the implementation, this could lead to serious misinterpretations. This means that a relatively small change in the smoothing parameter can have a significant impact on the outcome.

This fact is transparent in Fig. 4 as, when the smoothing parameter, a is small, a large transient period is caused by a shift in the process mean, which gradually decays over time. This also means that if the mean shift is large enough, an excessive number of out of control signals will appear on the control chart, particularly if a small smoothing parameter, a is employed. However, the effect of the mean shift can be weakened by using a larger value of the smoothing parameter, a. Unfortunately, this adversely affects the independence between consecutive EWMAD values making them correlated. Consequently, the variance of EWMAD increases and the interpretation of the pattern visible in the EWMAD chart would be rather difficult.

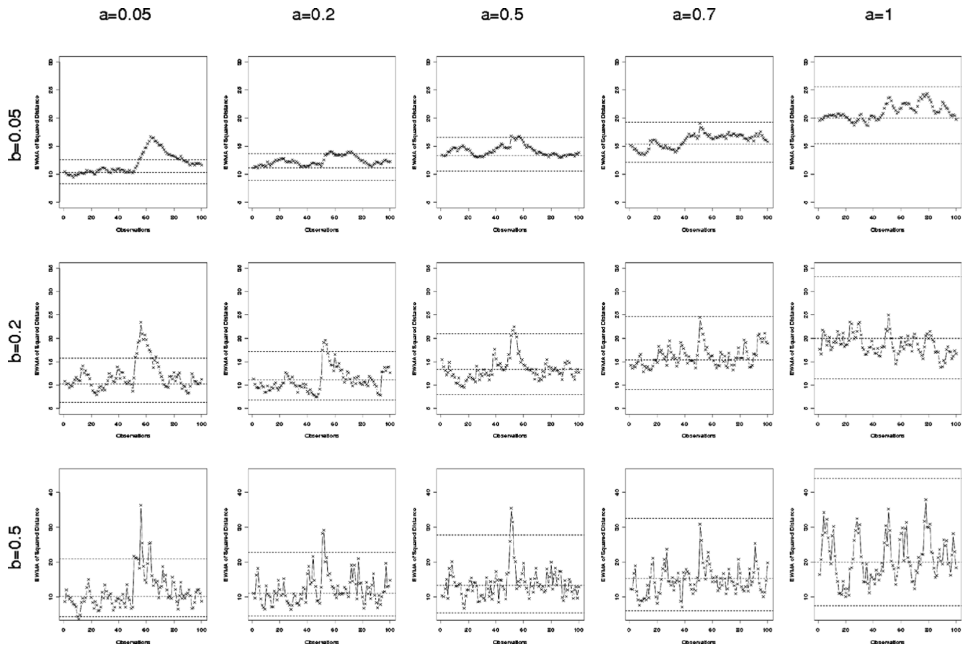

Figure 4. The effect of process mean shifts on the performance of the EWMAD control chart where the y- and the x- axes represent EWMA of squared distance and order of observation, respectively.

The ARL values in Table 4 demonstrate the sensitivity of the EWMAD chart to mean shifts. The results indicate that when the smoothing parameters a and b approach zero, the EWMAD chart is quite sensitive to mean shifts of even small magnitudes. When a and b both approach one, the EWMAD chart becomes more tolerant. To compensate for mean shifts, large values of the smoothing parameters a and b are required. However, this does not lead to improvements in detection capability. Consequently, the efficiency of the chart can be increased by choosing a possible combination of large MEWMA and small EWMAD smoothing parameters.

Note that when the smoothing parameter a is one, the estimated process mean at the current time t is represented by the observation at the previous time $t-1$. Then, the EWMAD statistic based on differences between successive pairs of observations can be defined as:

$$V_i = b(\boldsymbol{X}_i - \boldsymbol{X}_{i-1})'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{X}_i - \boldsymbol{X}_{i-1}) + (1-b)V_{i-1}, \quad 0 < b \leq 1.$$

As the preceding discussion shows, this statistic might have a high tolerance to mean shifts but the corresponding control chart may not detect changes in variability fast because of the small values of smoothing parameters used.

According to the previous results, it is clear that the smoothing parameter of the EWMAD chart is directly related to the efficiency of the control chart. In contrast, the smoothing parameter of MEWMA significantly affects the variance of the EWMAD. These observations suggest further studies based on the joint effect

Table 4
Sensitivity of the EWMAD control chart to mean shifts

		Magnitude of mean shift					
a	b	0.5	1	2	3	4	5
0.05	0.05	461.39	131.16	3.17	1.18	1.01	1.00
	0.2	463.14	220.52	10.88	2.88	2.06	1.72
	0.5	472.06	296.79	37.14	7.35	3.83	2.85
	0.7	469.80	327.10	59.55	12.17	5.35	3.62
	1	445.05	315.52	83.89	20.73	8.43	5.02
0.5	0.05	505.68	479.45	494.45	239.37	42.79	8.58
	0.2	502.56	460.09	462.26	209.13	44.01	15.67
	0.5	498.98	465.34	462.26	304.73	98.61	40.27
	0.7	501.02	470.70	466.67	361.62	174.07	69.26
	1	511.48	478.98	475.27	393.26	291.32	196.79
1	0.05	504.12	473.89	581.95	499.49	315.11	146.66
	0.2	500.00	441.84	491.97	441.04	336.31	159.71
	0.5	503.60	471.15	499.49	475.27	437.50	316.33
	0.7	508.30	508.83	534.93	513.09	496.45	446.67
	1	502.05	521.28	533.77	489.02	520.17	490.00

of smoothing parameters on both charts are needed. Let us suppose that the actual process mean is known, then the variance of EWMAD is

$$\sigma^2_{V(0)} = \frac{b}{2-b} VAR(R^2_\varepsilon), \tag{15}$$

and the ratio between variances of EWMAD defined by (14) and (15) is given by

$$\begin{aligned} Ratio &= \frac{\sigma^2_V}{\sigma^2_{V(0)}} \\ &= \beta_1 + \beta_2 \frac{(E(R^2_\varepsilon))^2}{VAR(R^2_\varepsilon)}. \end{aligned} \tag{16}$$

This is plotted against a and b in Fig. 5. Clearly, the variance of EWMAD gradually increases when both smoothing parameters increase, particularly, for a small value of the MEWMA smoothing parameter. This provides a pathway to choose desirable values of a and b to design the EWMAD chart so that its variability is controlled.

5. Summary

To monitor process variability, an EWMA chart of squared distance is developed for individual observations distributed according to an EC symmetric distribution. It is clear that the EWMAD chart can be used not only for detecting changes in the covariance matrix of a specified distribution but it could be also used for detecting changes in process variability. By means of a double EWMA procedure, an MEWMA is constructed to estimate the unknown actual process mean. It appears

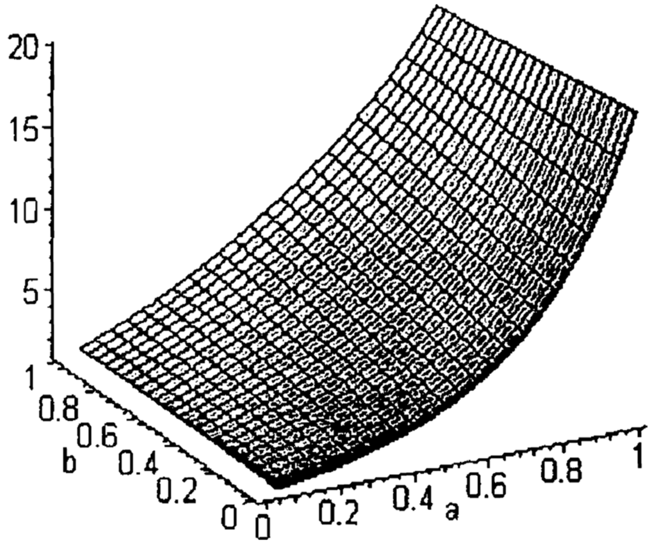

Figure 5. Ratio of the EWMAD variances plotted against smoothing parameters *a* and *b*.

that higher values of the MEWMA smoothing parameter tend to increase the EWMAD variance rather than improve its efficiency.

Simulation studies reveal that the EWMAD smoothing parameter influences the detection capability of the control chart, and by using small values, it is possible to obtain faster detection of shifts in process dispersion. However, use of a combination of small values of smoothing parameters for both MEWMA and EWMAD statistics are recommended to establish an efficient EWMAD chart.

The sensitivity analysis of the EWMAD control chart with respect to process mean shifts imply that a combination of large MEWMA and small EWMAD smoothing parameters might be employed to shorten the transient period manifested by the estimation of a local process mean. However, this can affect the detection capability and the values must be chosen not to weaken the performance of the control chart.

References

Alt, F. B. (1985). Multivariate quality control. In: Kotz, S., Johnson, N., eds. *Encyclopedia of Statistical Sciences Volume 6*. New York: John Wiley & Sons, pp. 110–122.

Alt, F. B., Smith, N. D. (1988). Multivariate process control. In: Krishnaiah, P., Rao, C. R., eds. *Handbook of Statistics Volume 7*. Elsevier Science Publishers B.V, pp. 333–351.

Bernard, S. A. (2001). *A Multivariate EWMA Approach to Monitor Process Dispersion [dissertation]*. College Park, MD: University of Maryland.

Cambanis, S., Huang, S., Simons, G. (1981). On the theory of elliptically contoured distributions. *Journal of Multivariate Analysis* 11:368–385.

Chan, L. K., Zhang, J. (2001). Cumulative sum control charts for the covariance matrix. *Statistica Sinica* 11:767–790.

Ernst, M. D. (1998). A multivariate generalized Laplace distribution. *Computational Statistics* 13:227–232.

Fang, K.-T., Kotz, S., Ng, K. W. (1990). *Symmetric Multivariate and Related Distributions*. New York: Chapman and Hall.

Johnson, M. E. (1987). *Multivariate Statistical Simulation*. New York: John Wiley & Sons.
Khoo, M. S. C., Quah, S. H. (2003). Multivariate control chart for process dispersion based on individual observations. *Quality Engineering* 15:639–642.
Lowry, C. A., Woodall, W. H., Champ, C. W., Rigdon, S. E. (1992). A multivariate exponentially weighted moving average control chart. *Technometrics* 34:46–53.
Sweet, A. L. (1986). Control charts using coupled exponentially weighted moving averages. *IIE Transactions* 18:26–33.
Yeh, A. B., Lin, D. K. J., Zhou, H., Venkataramani, C. (2003). A multivariate exponentially weighted moving average control chart for monitoring process variability. *Journal of Applied Statistics* 74:537–552.
Yeh, A. B., Huwang, L., Wu, C.-W. (2005). A multivariate EWMA control chart for monitoring process variability with individual observations. *IIE Transactions* 7:1023–1035.