Distribution under elliptical symmetry of a distance-based multivariate coefficient of variation

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Abstract

In the univariate setting, the coefficient of variation is widely used to measure the relative dispersion of a random variable with respect to its mean. Several extensions of the univariate coefficient of variation to the multivariate setting have been introduced in the literature. In this paper, we focus on a distance-based multivariate coefficient of variation. First, some real examples are discussed to motivate the use of the considered multivariate dispersion measure. Then, the asymptotic distribution of several estimators is analyzed under elliptical symmetry and used to construct approximate parametric confidence intervals that are compared with nonparametric intervals in a simulation study. Under normality, the exact distribution of the classical estimator is derived. As this natural estimator is biased, some bias corrections are proposed and compared by means of simulations.

Keywords: Bias reduction, Decentralized F-distribution, Elliptical symmetry, Multivariate coefficient of variation, Sharpe Ratio

1. Introduction and motivation

By definition, the coefficient of variation (CV) is the ratio of the standard deviation to the mean. It is a dimensionless quantity and as such, it allows to compare the variability of populations characterized by variables expressed in different units or having really different means.

The use of the CV is widespread in many applications in science, medicine, engineering, economics, etc. It is often advocated to control the repeatability of assay techniques,for instance in External Quality Assessment (EQA) schemes, on the basis of the following principle : the lower the relative variability in measurements, the better the technique. In psychology, Babkoff et al. [2] use the CV to compare the effectiveness of stimulants on cognitive and psychomotor performance during long-term sleep deprivation. They compute the CV of reaction time of the participants as a measure of performance stability. In quality control engineering, some authors (see e.g.[4]) have proposed various control charts, i.e. statistical tools used to determine whether a manufacturing process is in-control, based on the coefficient of variation.

All these examples refer to univariate data but when the comparison of intra-population variability is based on several features, i.e. when the data are intrinsically multivariate, the widespread approach consists in performing comparisons on marginal CV's (see for instance [22]), which may lead to contradictory results. This potential problem has already been highlighted by many authors, in the past or nowadays. For instance, in biology, Van Valen [24] noticed the inadequacy of the univariate CV when dealing with skeletal measurements on mammals. In the context of EQA analyses, Albert and Zhang [1] and Zhang et al. [28] also pointed out this particular issue. Another common approach consists in averaging the marginal CV's to

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obtain an index of global relative dispersion (as in [2]). In both procedures, the intra-population correlation structure is ignored.

To extend the CV to the multivariate setting while taking into account the correlation structure, some authors (e.g. Bennett [3]) suggest to work with a $p \times p$ matrix called the *coefficient of variation matrix*, with element (i, j) given by $\boldsymbol{\Sigma}_{ij}/(\boldsymbol{\mu}_i\boldsymbol{\mu}_j), i, j = 1, ..., p$, assuming $\boldsymbol{\mu}_i \neq 0$ for all i. However, as it is not easy to compare $p \times p$ matrices, the aim of this paper is to further develop the theory related to multivariate extensions of the coefficient of variation (MCV) that summarize multivariate relative variability in one single index. Several propositions have already been introduced in the literature (see [1] for a review). In this paper, focus will be on Voinov and Nikulin's MCV [26], which is a distance-based measure defined by the inverse of the square root of the Mahalanobis distance between the (non-null) mean vector and the origin of the design space, i.e.

$$\gamma = \frac{1}{\sqrt{\boldsymbol{\mu}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}} \tag{1}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are respectively the non null mean vector and the covariance matrix of the underlying multivariate distribution (assumed to be symmetric and positive definite). It is interesting to note that γ reduces to the univariate CV, σ/μ, when the dimension p equals 1 and $\boldsymbol{\mu} > 0$.

As regards the above-mentioned applications, this coefficient would provide a more appropriate way to compare the global relative variability of populations based on several traits (e.g. in [22]); or to assess repeatability of measurement devices on the basis of several samples in EQA programs. In [2], this coefficient would allow to compare stimulants performance on the basis of several distinct cognitive tasks instead of only one, while a multivariate control chart based on Voinov & Nikulin's MCV would provide a statistical control of complex multivariate manufacturing processes. Furthermore, it can also be used in finance as a measure of the performance of a portfolio in the Markowitz optimal portfolio theory framework ([15]), as will be illustrated on a real example in Section 2.

In the univariate case, many authors have developed inferential results for the univariate CV. However, under normality, the focus was almost exclusively on small values of the CV (lower than 0.33). This hypothesis, originating from McKay [16], ensures that the probability of obtaining a negative sample mean is negligible (≤ 0.001) for all n. In the multivariate setting, the computation of the classical estimator of γ, obtained by plugging the sample mean and the sample covariance matrix in (1), only has sense when the sample mean is non-null. Under normality $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, imposing, for example, that the origin falls outside the tolerance ellipsoid of mass 0.99 for the sample mean is in fact equivalent to the following condition on Voinov & Nikulin's MCV:

$$\gamma < \sqrt{\frac{n}{q_{\chi^2_p;0.99}}}$$

where $q_{\chi^2_p;0.99}$ denotes the 0.99 quantile of a chi-square distribution with p degrees of freedom. Table 1 gives the value of this upper bound, which can be interpreted as an upper bound for values of γ that can be reasonably expected under this assumption on the data, for several sample sizes and dimensions. However, in practice, higher values of γ are not uncommon in 1 dimension as in p dimensions, as will be illustrated in Section 2. The purpose of this article is therefore to study the properties of MCV estimators under more general conditions by extending the scope to the family of elliptical distributions and by considering larger values of γ.

The paper is organized as follows. First, two illustrations on real data are given in Section 2 in order to highlight the usefulness of the MCV in practice. The first one is about assessing the performance of several protein electrophoresis techniques in the context of EQA schemes, while the second one uses Voinov & Nikulin's MCV as an aid to investment decision in finance. In Section 3, the asymptotic distribution of estimators of (1) will be studied under elliptical symmetry. Under normality, the finite-sample distribution of the classical estimator is derived, which allows to compute exact confidence intervals. Unfortunately, the classical estimator is biased, as illustrated in Section 4. Therefore, several possible parametric and non-parametric bias reduction techniques are outlined in Sections 5 to 7 and compared at finite samples by

	p		
n	3	7	15
20	1.33	1.07	0.89
50	2.09	1.65	1.28
100	2.97	2.33	1.81
200	4.20	3.29	2.56
1000	9.38	7.36	5.72

Table 1: Upper bound for reasonable values of γ under the assumption that the origin falls outside the tolerance ellipsoid of mass 0.99 for the mean $\bar{\mathbf{X}}_n$

means of simulations in Section 8. An empirical comparison of exact and approximate confidence intervals is provided in Section 9. Some concluding remarks are outlined in Section 10.

2. Illustration on real datasets

In this section, the usefulness of Voinov & Nikulin's MCV is illustrated in two real applications from distinct areas. First, the coefficient is used, as in [28], to evaluate the repeatability of protein electrophoresis techniques. Secondly, in the financial context, its applicability in decision making as regards investing into two stocks portfolios is shown. The range of values taken by the MCV is distinct from one example to the other.

2.1. Comparing protein electrophoresis techniques

For many years, External Quality Assessmen (or Proficiency Testing) programs have been organized worldwide, either by healthcare agencies or by external companies, in order to control the performance of assay techniques used by clinical laboratories. In these programs, all participating laboratories are requested to assay the same control sample and to send their result back to the EQA center. For a single analyte, an univariate CV is then computed for each technique as a measure of its inter-laboratory repeatability. Among the controlled laboratory test, serum protein electrophoresis is a test used to determine the relative proportions of albumin and $\alpha_1, \alpha_2, \beta$ and *gamma* -globulins in the blood. When expressed in percent of total serum protein, these five fractions sum up to 100%. As an illustration, let us consider the subset of results from the 2004 French and Belgian EQA programs sent by laboratories using the following four techniques: HT Cellulose acetate, HT Agarose gel(Acid blue), HT Agarose gel (Amido black) and BCP Capillary zone electrophoresis. For each technique and fraction, the CV estimates obtained after removing severe outliers, are listed in Table 2.

Electrophoretic technique	n	CV of electrophoretic fraction				
		Albumin	α_1	α_2	β	*gamma*
HT Cellulose acetate	133	.053	.309	.149	.094	.091
HT Agarose gel (Acid blue)	112	.058	.219	.110	.156	.097
HT Agarose gel (Amido black)	74	.063	.269	.110	.136	.095
BCP Capillary zone	62	.018	.090	.059	.048	.022

Table 2: Sample CV's by electrophoretic technique.

However, as already pointed in the EQA literature [28], comparing the assay techniques using these marginal CV's may lead to contradictory rankings and does not take into account the relationship between the fractions. In order to avoid these two drawbacks, the procedure recommended in [28] to obtain an overall measure of performance consists in computing Voinov & Nikulin's MCV for each technique, after having transformed the five fractions into four non linearly dependent variables by an ILR transformation. The sample MCV estimates for each electrophoretic technique are displayed in Table 3 by increasing order of magnitude. Among the four examined techniques, BCP Capillary zone has the highest repeatability.

In any context where the need for a global measure of relative dispersion facilitating comparisons arises, the multivariate coefficient of variation is the answer.

Electrophoretic technique	n	MCV
BCP Capillary zone	62	.0238
HT Agarose gel (Acid blue)	112	.0558
HT Agarose gel (Amido black)	74	.0616
HT Cellulose acetate	141	.0688

Table 3: Sample marginal CV's (%) by electrophoretic technique.

2.2. Assessing the performance of an investment strategy

In finance, the inverse of the univariate CV, known as the Sharpe Ratio, is one of the most commonly used statistics which was introduced by Sharpe [20] in order to measure the performance of an investment with respect to its risk level. This measure is defined by the ratio of the expected return of the asset to the standard deviation of the returns (representing the risk).

Suppose that an investor wants to invest a capital in p risky assets and let $\mathbf{R}$ denotes the random p-vector representing their respective returns with expected return $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ in the time-interval $[0, T]$ (the p assets may be correlated). According to Markowitz optimal portfolio theory [15], depending on the goal of the investor, there are several ways to determine a vector of optimal allocations $\mathbf{a} = (a_1, ..., a_p)^t$ to invest in each asset. The objective may be either to maximize the expected return $\mathbf{a}^t\boldsymbol{\mu}$ subject to a given risk constraint $\sigma_0^2 = \mathbf{a}^t\boldsymbol{\Sigma}\mathbf{a}$ or to minimize the risk $\mathbf{a}^t\boldsymbol{\Sigma}\mathbf{a}$ subject to a desired expected return $R_0 = \mathbf{a}^t\boldsymbol{\mu}$. Alternatively, the tradeoff between a high expected return and a low risk can be represented by the minimization of the quadratic utility function $\frac{\theta}{2}\mathbf{a}^t\boldsymbol{\Sigma}\mathbf{a} - \mathbf{a}^t\boldsymbol{\mu}$, where θ quantifies the risk-aversion of the investor. These three optimization problems have solutions of the form $\mathbf{a} \propto \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$.

The performance of the obtained optimal portfolio can then be evaluated on the basis of either the value of the objective function reached with the optimal allocation, or the Sharpe Ratio. Let us note that normalizing the allocation vector, so that $\mathbf{a}^t\mathbf{a} = 1$, leads to the same Sharpe Ratio. For each optimization problem, the value of the objective function at its optimum and the Sharpe ratio both correspond, up to an additive or multiplicative constant, to the Voinov & Nikulin's MCV or its inverse. When several independent sets of p assets are considered, one can then compare the performance of the corresponding optimal Markowitz portfolios by computing only their respective MCV's. Therefore, the MCV is very useful in making investment strategies.

As illustration, let us consider two independent sets of $p = 5$ stocks. The first set of stocks comprises five companies belonging to the Technology category of the S&P500 index, which is the most representative index of the US market. More precisely, the considered stocks are Adobe, Apple, Cisco, Intel Corporation and Oracle. The second set includes stocks from the S&P500 related to Health care: Abott Laboratory, Baxter International, Johnson & Johnson, Mylan and Pfizer. The aim is to decide which investment, in the Technology companies or in the Health care firms, has the best performance when using the optimal allocations.

The monthly returns of these sets were collected from March 2009 up to February 2015 in order to compute the respective MCV's. The obtained Voinov & Nikulin's coefficients are respectively 2.021 for the Technology portfolio and 2.134 for the Health care portfolio.

For this example or the previous one, in order to pinpoint any significant difference between the performance of several assay techniques or investment strategies, it would be useful to have confidence intervals and valid statistical tests. The first proposal will be further considered here.

3. Distribution of estimators of Voinov and Nikulin's MCV

Let $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ be a sequence of n independent p-variate random vectors, each distributed according to a distribution $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ with density

$$f_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}) = |\boldsymbol{\Sigma}|^{-1/2}\ h\left((\mathbf{x}-\boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) \tag{2}$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$ with $\boldsymbol{\mu} \neq 0$, $\boldsymbol{\Sigma}$ is assumed to be symmetric and positive-definite, i.e. $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$, and h is a known real-valued function having a strictly negative derivative $h^{'}$. This means that $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ belongs to a parametric class of absolutely continuous and unimodal elliptically symmetric distributions $\mathcal{F}_h := \left\{F_{\boldsymbol{\mu},\boldsymbol{\Sigma}} : \boldsymbol{\mu} \in \mathbb{R}^p \text{ and } \boldsymbol{\Sigma} \in \mathcal{S}_p^+\right\}$ generated by h. In order to properly identify $\boldsymbol{\Sigma}$ and h, let h be defined such that $\boldsymbol{\Sigma}$ represents the covariance matrix of the $\mathbf{X}_i's$. In the remainder of the paper, the notation $\mathbb{E}_{F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}}[f(\mathbf{X})]$ will denote the expectation of a function of the random vector $\mathbf{X}$ computed under the assumption that the distribution of $\mathbf{X}_i$ is $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ for all $i = 1, ..., n$.

In practice, Voinov & Nikulin's MCV can be estimated by plugging any location and covariance estimators, $\mathbf{T}_n(\mathbf{X})$ and $\mathbf{C}_n(\mathbf{X})$, in expression (1). Proposition 1 shows that, under elliptical distributions and for a fixed value of γ, the sample distribution of any estimator $V_n(\mathbf{X})$ of γ computed by means of affine-equivariant location and covariance estimators does not depend on the direction of the mean vector nor on the variance-covariance structure. The proof is given in the Appendix.

Proposition 1. *Let $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ be a sequence of n independent p-variate random vectors each distributed according to $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \in \mathcal{F}_h$ with $\boldsymbol{\mu} \neq \mathbf{0}$. Let $V_n(\mathbf{X})$ be an estimator of Voinov & Nikulin's coefficient computed with affine equivariant estimators of location and covariance, $\mathbf{T}_n(\mathbf{X})$ and $\mathbf{C}_n(\mathbf{X})$. The distribution of $V_n(\mathbf{X})$ depends on the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ only through γ.*

Consequently, for such estimators $V_n(\mathbf{X})$, the above-mentioned notation can be simplified as follows $\mathbb{E}_{F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}}[V_n(\mathbf{X})] = \mathbb{E}_{F_\gamma}[V_n(\mathbf{X})]$ in order to stress that this expectation only depends on the parameter γ.

Proposition 1 applies in particular to the classical estimator obtained by taking as location and covariance estimators respectively the sample mean $\bar{\mathbf{X}}_n = \frac{1}{n}\sum_{i=1}^n \mathbf{X}_i$ and the sample covariance matrix $\mathbf{S}_n = \frac{1}{n}\sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})^t$.

3.1. Asymptotic distribution

In this subsection, the asymptotic distribution of a very wide class of estimators $V_n(\mathbf{X})$ is studied under elliptical symmetry. Let us consider any pair of affine-equivariant location and covariance estimators $(\mathbf{T}_n(\mathbf{X}), \mathbf{C}_n(\mathbf{X}))$ satisfying the three following assumptions under $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \in \mathcal{F}_h$:

(A1) $\mathbf{T}_n(\mathbf{X})$ and $\mathbf{S}_n(\mathbf{X})$ are asymptotically independent

(A2) $\sqrt{n}(\mathbf{T}_n(\mathbf{X}) - \boldsymbol{\mu}) \xrightarrow{\mathcal{L}} N_p(0, \tau\boldsymbol{\Sigma})$

(A3) $\sqrt{n}\ \mathrm{vec}(\mathbf{C}_n(\mathbf{X}) - \boldsymbol{\Sigma}) \xrightarrow{\mathcal{L}} N_{p^2}(0, \boldsymbol{\Xi})$ with $\boldsymbol{\Xi} = \sigma_1(\mathbf{I}_{p^2} + \mathbf{K}_{p^2})(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}) + \sigma_2(\mathrm{vec}\boldsymbol{\Sigma})(\mathrm{vec}\boldsymbol{\Sigma})^t$ where vec and $\otimes$ stand for the usual vectorization operator and Kronecker product respectively, $\mathbf{I}_{p^2}$ is the p^2-dimensional identity matrix and $\mathbf{K}_{p^2}$ is the commutation matrix of dimension $p^2 \times p^2$ defined by $\mathbf{K}_{p^2} := \sum_{i,j=1}^p (\mathbf{e}_i\mathbf{e}_j^t) \otimes (\mathbf{e}_j\mathbf{e}_i^t)$ with $\mathbf{e}_i$ the ith vector in the canonical basis of $\mathbb{R}^p$.

Proposition 2 gives the distribution of any estimator $V_n(\mathbf{X})$ computed using a pair of affine-equivariant location and covariance estimators satisfying conditions (A1), (A2) and (A3). The proof is sketched in the Appendix.

Proposition 2. *Let $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ be a sequence of n independent p-variate random vectors each distributed according to $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \in \mathcal{F}_h$, and $\mathbf{T}_n(\mathbf{X})$ and $\mathbf{C}_n(\mathbf{X})$ be a pair of affine-equivariant location and covariance estimators satisfying the conditions (A1), (A2) and (A3) under $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$. The asymptotic distribution of the estimator $V_n(\mathbf{X}) = (\mathbf{T}_n(\mathbf{X})^t\mathbf{C}_n(\mathbf{X})^{-1}\mathbf{T}_n(\mathbf{X}))^{-1/2}$ is given by*

$$\sqrt{n}(V_n(\mathbf{X}) - \gamma) \xrightarrow{\mathcal{L}} N_p\left(0, \tau\gamma^4 + \frac{\gamma^2}{4}(2\sigma_1 + \sigma_2)\right) \tag{3}$$

In particular, Proposition 2 applies to the classical estimator, provided that $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ admits finite fourth order moments, for which we have $\tau = 1$, $\sigma_1 = 1 + \kappa$ and $\sigma_2 = \kappa$ (see for instance [23]), where κ is defined[1] by

$$\kappa = \frac{p}{p+2}\frac{\mathbb{E}_{F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}}[R^4]}{\mathbb{E}_{F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}}[R^2]^2} - 1, \qquad \text{with } R = \sqrt{(\mathbf{X}-\boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})}. \tag{4}$$

Let us note that the general conditions (A1), (A2) and (A3) are not only satisfied by the classical estimators but also by many robust estimators[2] of location and scatter under elliptical symmetry with finite-fourth order moments, e.g. the M estimators, the S estimators or the MCD estimators. Explicit forms of the constants τ, σ_1 and σ_2 can be found in [23] for the M estimators, in [6] for S estimators and in [5] for MCD estimators.

The asymptotic distribution (3) may be used to construct large sample confidence intervals for the MCV by replacing γ by its estimate. In Section 9, a simulation study investigates the quality of these asymptotic confidence intervals for the classical estimator and compares them with exact confidence intervals, derived under normality in the next subsection, and with nonparametric intervals.

3.2. Finite-sample distribution of the classical estimator under normality

In the remainder of the paper, the focus will be on the classical estimator which will always be denoted by V_n from now on. In the particular case of the p-variate normal distribution $\Phi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ for which the function h in (2) is given by $h(t) = 1/(2\pi)^{p/2}\exp(-t/2)$, the exact finite-sample distribution of the classical estimator V_n can be derived as a straightforward corollary of Theorem 6.7.a.1 [12], as stated in Corollary 1.

Corollary 1. *Let $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ be a sequence of n independent p-variate random vectors each normally distributed with mean vector $\boldsymbol{\mu} \neq \mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$. Then,*

$$\frac{n-p}{p}\frac{1}{V_n^2} \sim F_{p;n-p;n/\gamma^2}, \tag{5}$$

where $F_{p;n-p;n/\gamma^2}$ denotes the non-central F distribution with degrees of freedom p and $n-p$ and noncentrality parameter n/γ^2.

Corollary 1 is the extension to the multivariate setting of the result derived by Johnson and Welch [9], saying that the univariate CV has a noncentral Student distribution.

The expectation and variance of the squared inverse of V_n under normality follow directly from the expectation and variance of the decentralized F distribution, i.e.

$$\mathrm{E}_{\Phi_\gamma}\left[\frac{1}{V_n^2}\right] = \frac{p}{n-p-2}\left(1 + \frac{n}{\gamma^2 p}\right) \tag{6}$$

$$\mathrm{Var}_{\Phi_\gamma}\left[\frac{1}{V_n^2}\right] = 2\frac{(p+n/\gamma^2)^2 + (n-p-2)(p+2n/\gamma^2)}{(n-p-2)^2(n-p-4)}. \tag{7}$$

The specific distribution derived in (5) also allows to construct exact confidence intervals for the true Voinov and Nikulin's MCV. Relying on the confidence interval transformation principle and the inversion confidence interval principle (as detailed in [11] for the univariate case), the construction goes as follows. Let α be the nominal confidence interval coverage and $\beta \in]0;\alpha[$. For n and p being fixed, let $\hat{v}_n$ be a realization of V_n and $f = \frac{n-p}{p}\frac{1}{\hat{v}_n^2}$. Define δ_L to be the solution, if it exists, to the equation for δ

$$\beta = F_{p,n-p,\delta}(f), \tag{8}$$

[1] The parameter κ is a kind of kurtosis measure, which does not reduce to the univariate kurtosis parameter when $p = 1$.
[2] The robustness of the MCV's is studied in a previous paper [**?**].

and δ_U to be the solution, if it exists, to the equation for δ

$$1 - \alpha + \beta = F_{p,n-p,\delta}(f), \tag{9}$$

where $F_{d_1,d_2,\delta}$ denotes the distribution function of a noncentral F distribution with degrees of freedom d_1 and d_2 and noncentrality parameter δ. The proposed confidence interval for the noncentrality parameter $\delta = n/\gamma^2$ is given by $[\delta_U; \delta_L]$. By the transformation principle, $[\gamma_L; \gamma_U] := [\sqrt{n/\delta_L}; \sqrt{n/\delta_U}]$ is the corresponding confidence interval for γ.

Let us more formally state the conditions under which this confidence interval can be computed in practice. The following lemma is useful to derive the conditions of existence of solutions for equations (8) and (9). The proof is sketched in the appendix.

Lemma 1. *For fixed degrees of freedom d_1 and d_2 and for a given $f \in \mathbb{R}_0^+$, the function l defined by $l : \mathbb{R}_0^+ \to [0; 1] : \delta \mapsto F_{d_1,d_2,\delta}(f)$ is strictly decreasing in δ.*

Moreover, it can be shown that $\lim_{\delta \to +\infty} F_{d_1,d_2,\delta}(f) < \beta$ for any fixed d_1, d_2, f and β. In our case, when $d_1 = n - p$ and $d_2 = p$, this implies that a necessary and sufficient condition for the equations (8) and (9) to have solutions is

$$\hat{v}_n \leq C_{p,n-p} \qquad \text{for } C_{p,n-p} = \sqrt{\frac{n-p}{p} \frac{1}{F^{-1}_{p,n-p,0}(1-\alpha+\beta)}} \tag{10}$$

where $F^{-1}_{p,n-p,0}(1-\alpha+\beta)$ denotes the $1-\alpha+\beta$ quantile of the central F distribution with degrees of freedom $n - p$ and p.

The following proposition gives bounds for the exact probability that condition (10) is satisfied and that the constructed interval contains γ. The proof, outlined in the Appendix, follows the lines of that given by Verril [25] in the univariate case.

Proposition 3. *The construction detailed above leads to an interval $I(V_n) = [\gamma_L(V_n); \gamma_U(V_n)]$ satisfying*

$$1 - \alpha - \epsilon \leq \mathbb{P}_\gamma \left[\{V_n \leq C_{p,n-p}\} \cap \{\gamma \in I(V_n)\} \right] \leq 1 - \alpha$$

with $\epsilon = \mathbb{P}_\gamma [V_n > C_{p,n-p}]$ where $C_{p,n-p}$ is defined in (10).

Thus, the proposed construction almost yields an exact confidence interval for γ. The discrepancy is bounded by the probability of not being able to perform the inversion method. For given values of n, p and γ, this probability has been plotted in Figure 1. We can observe that the discrepancy decreases when the sample size n increases but increases with the dimension p. The probability of not being able to compute these exact confidence intervals is more important for large values of γ.

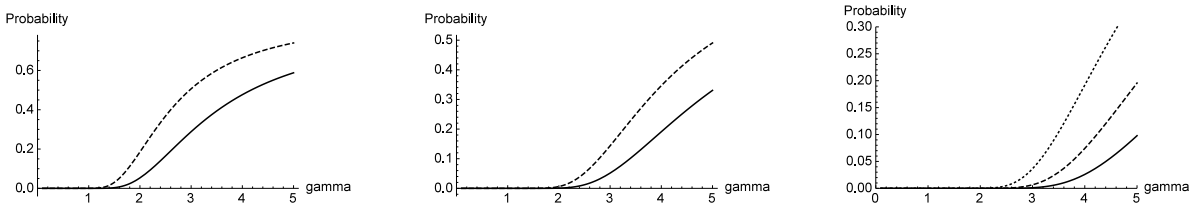

Figure 1: Probability ϵ as a function of γ, for a dimension p equal to 3 (solid line), 7 (dashed line) or 20 (dotted line) and three sample sizes, $n = 50$ (left panel), $n = 100$ (middlepanel) and $n = 200$ (in the right panel).

These intervals are not symmetric about the estimated coefficients of variation. Moreover, one needs to fix β and the common choice of equal tail probabilities $\beta = \alpha/2$ is quite arbitrary. Indeed, since the noncentral

F distribution is asymmetric, it is easy to understand that this choice is not optimal in terms of confidence interval length. In the univariate case (for a noncentral t distribution), Jongphil [10] showed that choosing the value of β leading to the shortest acceptance set or to the acceptance set associated to the uniformly most powerful unbiased test for $H_0 : \delta = \delta_0$ against two-sided alternatives, does not provide uniformly shorter confidence intervals. In [10], a methodology to obtain uniformly shorter CI under non-central t is proposed and validated empirically. This reasoning might be extended to our setting.

4. Bias of the classical estimator of Voinov and Nikulin's MCV

Even though V_n is asymptotically unbiased under elliptical symmetry, its bias may be quite big when dealing with finite samples. In this section, the finite-sample bias of this estimator is first investigated under normality. The bias of V_n under both lighter-tailed and heavier-tailed elliptical distributions than normality is illustrated by means of finite-sample simulations in Section 8.

Under normality, the bias can be computed numerically by means of the expectation $\mathrm{E}\left[1/\sqrt{X}\right]$ where $X \sim F_{p;n-p,n/\gamma^2}$. This is illustrated on Figure 2 where, for some fixed values of γ ($\gamma = 0.5$ in the left panel, 1 in the middle and 2 in the right panel) and three different values of the dimension p, the bias of V_n is represented as a function of n. One can see that the bias is always negative: the classical estimator tends to underestimate the relative dispersion. The absolute value of the bias decreases as n increases and is bigger as the dimension gets bigger. Bigger values of γ lead to bigger biases.

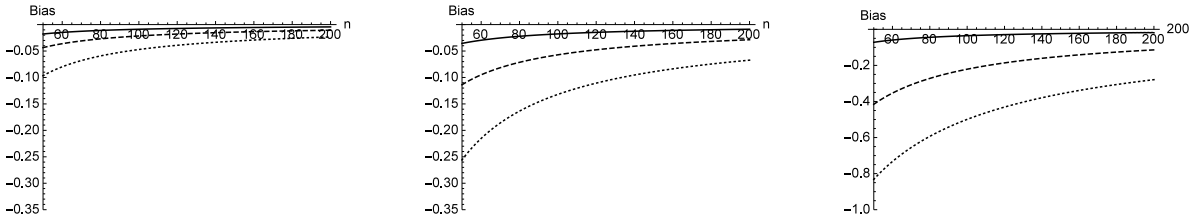

Figure 2: Bias of the estimator V_n w.r.t. the sample size n, for $\gamma = 0.5$ in the left panel, 1 in the middle and 2 in the right panel and for three dimensions p (solid line: $p = 3$, dashed line: $p = 7$ and dotted line: $p = 15$).

For $p = 1$, Sokal and Brauman [21] already reported that the classical estimator of the CV is negatively biased. In the multivariate context, Zhang [27] also empirically noticed the fact that the classical estimators of the multivariate coefficients of variation that Albert and Zhang [1] considered were all biased under normality.

In the following sections, both parametric and non parametric approaches are considered for reducing the bias of the classical estimator of Voinov and Nikulin's MCV. The first one follows Zhang's suggestion and is only valid under normality. The second correction requires an estimator for the bias of V_n. Under normality, as the finite-sample distribution of V_n is known, such an estimator is available. When either the distribution of V_n or the underlying distribution of the random vectors $\mathbf{X}_i$ is unknown, some nonparametric bias-estimators can be obtained via bootstrap or jackknife. Finally, the last correction requires solving a nonlinear equation involving the expectation of V_n. Unfortunately, reducing the bias may increase the variance or even the mean squared error of an estimator. Simulations are carried out in Section 8 to quantify this phenomenon.

5. Bias correction by plugging unbiased estimators

Despite the fact that Zhang [27] had noticed the biasedness of the classical estimators of all the MCV's under normality, only a bias correction for the classical estimator of Reyment's MCV [19] defined by $\gamma_{\mathrm{R}} = \sqrt{(\det \boldsymbol{\Sigma})^{1/p}/\boldsymbol{\mu}^t\boldsymbol{\mu}}$ was proposed. The suggested correction consists in plugging unbiased estimators

separately in the numerator and denominator. Even though this technique does not ensure the unbiasedness of the ratio, simulations showed that this simple idea decreases bias.

For Voinov and Nikulin's coefficient, the expectation of a decentralized F distribution allows to say that $\bar{\mathbf{X}}_n^t \mathbf{S}_n^{-1} \bar{\mathbf{X}}_n$ is biased for the estimation of $\boldsymbol{\mu}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$. However, isolating $1/\gamma^2$ in the right-hand side of (6) allows to construct an unbiased estimator for $1/\gamma^2$. The advocated bias correction consists therefore in taking as estimator the square root of the inverse of

$$\frac{p}{n}\left(\frac{n-p-2}{p}\frac{1}{(V_n)^2}-1\right), \tag{11}$$

estimator denoted by $V_n^{(1)}$ from now on.

However in practice, the estimated value of the estimator $V_n^{(1)}$ can only be computed when the quantity (11) is positive. The probability that it is negative is given by

$$\mathbb{P}\left[X \leq \frac{n-p}{n-p-2}\right] \text{ where } X \sim F_{p,n-p,n/\gamma^2}. \tag{12}$$

The analytical expression (12) and the characteristics of the F distribution allow to say that, as n increases, the probability of not being able to compute $V_n^{(1)}$ decreases. Also, for fixed values of n and p, the probability decreases as γ decreases. Figure 3 plots the probability (12) with respect to varying values of γ, for a dimension p equal to 3, 7 or 15 and three sample sizes, $n = 50$ (left), $n = 100$ (middle) and $n = 200$ (right).

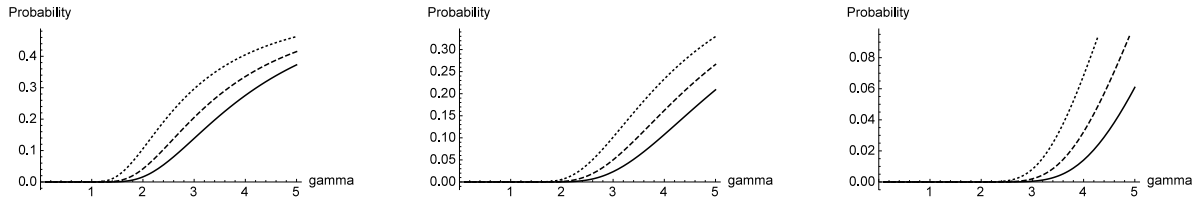

Figure 3: Probability (12) as a function of γ, for a dimension p equal to 3 (solid line), 7 (dashed line) or 15 (dotted line) and three sample sizes, $n = 50$ (left panel), $n = 100$ (middle panel) and $n = 200$ (right panel).

For values of γ smaller than 1, the probability is negligible for all n and p. However, when the parameter gets bigger, it is less and less possible to apply the correction, especially in high dimension and for small sample sizes . Let us note that the correction constructed in a similar way by Zhang [27] for Reyment's coefficient suffers from the same problem as their proposition consists in using the unbiased estimator of $\det \boldsymbol{\Sigma}$ given by $\bar{\mathbf{X}}_n^t \bar{\mathbf{X}}_n - \frac{1}{n} \operatorname{tr} \mathbf{S}$ instead of $\det \mathbf{S}$. This quantity can however takes negative values under normality of the random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$.

Even though correcting the Mahalanobis distance $\bar{\mathbf{X}}_n^t \mathbf{S}_n^{-1} \bar{\mathbf{X}}_n$ such that it becomes an unbiased estimator of $\boldsymbol{\mu}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$ does not ensure the unbiasedness of the square root of its inverse for the parameter of interest γ, this correction allows to reduce the sample bias in several cases, as will be seen in the simulations in Section 8.

6. Bias correction by estimation of the bias

By definition, the bias of the estimator V_n under $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ is given by

$$\operatorname{bias}(V_n; F_\gamma) := \mathrm{E}_{F_\gamma}[V_n(\mathbf{X})] - \gamma. \tag{13}$$

If B_n is an estimator of $\operatorname{bias}(V_n; F_\gamma)$, a natural bias-corrected estimator would be

$$\check{V}_n = V_n - B_n \tag{14}$$

In the following, the notation $\check{}$ will be used to denote all the bias corrections based on that principle. The bias of this estimator will be, in absolute value, smaller than the bias of V_n if B_n satisfies $0 \leq E[B_n] \leq 2\ \text{bias}(V_n; F_\gamma)$ when the initial estimator overestimates γ, or $2\ \text{bias}(V; F_\gamma) \leq E[B_n] \leq 0$ when it underestimates the true parameter. Note however that such a bias correction often yields an increase in variability due to the additional variability of the bias estimator.

6.1. Estimation of bias under normality

Under normality, thanks to distribution (5), the bias of V_n can be computed for any fixed value of $\gamma > 0$, as already illustrated on Figure 2. Replacing the unknown parameter γ by its classical estimator V_n leads to a first bias-correction proposal:

$$\check{V}_n^{(2)} = 2V_n - \mathbb{E}_{\hat{\Phi}_\gamma}\left[V_n(\mathbf{X})\right] \tag{15}$$

where $\mathbb{E}_{\hat{\Phi}_\gamma}[.]$ denotes the expectation computed by using the non-central F-distribution with the unknown non-centrality parameter n/γ^2 replaced by n/V_n^2.

6.2. Nonparametric bootstrap bias estimation

When either the finite-sample distribution of V_n or the underlying distribution of the random vectors $\mathbf{X}_i$ are unknown, the bias $\text{bias}(V_n; F_{\boldsymbol{\mu},\boldsymbol{\Sigma}})$ can be estimated by bootstrap (see [7]), which is a widely applicable method. The *bootstrap estimate* of bias consists in substituting the empirical distribution function $\hat{F}_n$ for $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ in (13), yielding:

$$B_n^{(3)} = \mathbb{E}_{\hat{F}_n}\left[V_n(\mathbf{X})\right] - V_n \tag{16}$$

In many problems, the expectation $\mathbb{E}_{\hat{F}_n}[V_n(X)]$ can only be approximated by Monte-Carlo simulations : let $(\mathbf{X}_1^*, ..., \mathbf{X}_B^*)$ be B independent samples of size n drawn with replacement from the empirical distribution function $\hat{F}_n$, i.e. each $\mathbf{X}_i^*$ is of size $n \times p$. For each bootstrap sample, an estimator $V_{n;b}^* = V_n(\mathbf{X}_b^*)$ is computed $(b = 1, ..., B)$. The bootstrap expectation can then be approximated by the average $\overline{V}_n^* := \frac{1}{B}\sum_{b=1}^{B} V_{n;b}^*$ to obtain the following approximated bias corrected bootstrap estimator:

$$\check{V}_n^{(3)} := 2V_n - \overline{V}_n^* \tag{17}$$

There exist ways to improve the convergence of the bootstrap bias estimator (see [7]). However, in our context, the improvement is negligible and therefore, the possible adaptations are not further developped.

6.3. Nonparametric Jackknife bias estimation

Jackknifing is a general non-parametric method for reducing the bias of an estimator by exploiting the dependance of the bias on the sample size. The jackknife estimator of the bias is defined by

$$B_n^{(4)} = (n-1)\left(\frac{1}{n}\sum_{i=1}^{n} V_{n-1,(-i)} - V_n\right),$$

where $V_{n-1,(-i)}$ is the classical estimator evaluated on the subsample with the ith observation omitted, which leads to the first order jackknife corrected estimator

$$\check{V}_n^{(4)} = nV_n - \frac{n-1}{n}\sum_{i=1}^{n} V_{n-1,(-i)}$$

The interested reader can find more information on the theoretical justification of the bias reduction when using the jackknife in [18] and [7].

7. Bias correction by inversion

Under normality, the discrepancy between the expectation and the target γ can be illustrated as in Figure 4 where, for all values of γ (and for $n = 50$ and $p = 7$), the solid-line curve represents the expectation of V_n and the dashed-line is the bissector (which yields the target value). Let g denotes the function that associates to any positive value of γ the expectation $\mathrm{E}_{\Phi_\gamma}[V_n(\mathbf{X})]$ computed from the non-central F distribution with non-centrality parameter $1/\gamma^2$.

When observing a value $\hat{v}_n$, one may consider that it corresponds to the expectation computed under the assumption that the true value of the parameter is $\gamma_1 = g^{-1}(\hat{v}_n)$. The bias is therefore

$$\mathrm{bias}(V_n, \Phi_{\gamma_1}) = \mathrm{E}_{\Phi_{\gamma_1}}[V_n(\mathbf{X})] - \gamma_1 = \hat{v}_n - \gamma_1.$$

Thus, the estimator corrected for bias, denoted by $\check{V}_n^{(5)}$, yields an estimation given by $\hat{v}_n - \mathrm{bias}(V_n, \Phi_{\gamma_1}) = \gamma_1$.

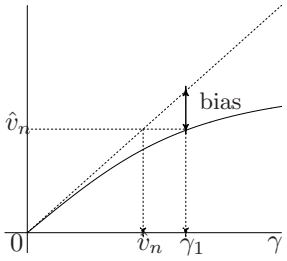

Figure 4: Expectation of V_n for all γ, for $n = 50$ and $p = 7$.

In other words, as explained in [13], this method can be seen as a variant of equation (14) since it consists in taking as corrected estimator $\check{V}_n^{(5)}$ the initial estimator V_n minus an estimate of the bias but this time evaluated at $\check{V}_n^{(5)}$. The proposed bias-corrected estimator is the solution, if it exists, of the following non-linear equation :

$$\begin{aligned} \check{V}_n^{(5)} &= V_n - (\mathbb{E}_{\hat{\Phi}_\gamma^{(5)}}[V_n(\mathbf{X})] - \check{V}_n^{(5)}) \\ \Leftrightarrow V_n &= \mathbb{E}_{\hat{\Phi}_\gamma^{(5)}}[V_n(\mathbf{X})] \end{aligned} \tag{18}$$

where $\mathbb{E}_{\hat{\Phi}_\gamma^{(5)}}[V_n(\mathbf{X})]$ denotes the expectation computed using the noncentral F distribution with the unknown non-centrality parameter n/γ^2 replaced by $n/(\check{V}_n^{(5)})^2$. Assuming the existence of the solution of (18), any technique for finding the roots of a non-linear equation may be used, such as the discretized Newton-Raphson procedure or the one proposed by MacKinnon and Smith [13] for instance.

Regarding the conditions of existence of a solution for (18), Proposition 4 gives the condition for equation (18) to have a unique solution (the proof is sketched in the appendix).

Proposition 4. *Under normality, for fixed values of n and p and provided that $1 < p < n$, the bias correction based on equation (18) has a unique solution in $]0; +\infty[$ if and only if the observed value $\hat{v}_n$ of the initial estimator satisfies*

$$0 < \hat{v}_n < \frac{\Gamma\left(\frac{n-p+1}{2}\right)\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{n-p}{2}\right)\Gamma\left(\frac{p}{2}\right)}$$

Accordingly, the probability of not being able to compute $\check{V}_n^{(5)}$ is given by

$$\mathbb{P}\left[X \leq \frac{n-p}{p}\left(\frac{\Gamma\left(\frac{n-p}{2}\right)\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{n-p+1}{2}\right)\Gamma\left(\frac{p-1}{2}\right)}\right)^2\right] \qquad \text{where } X \sim F_{p,n-p,n/\gamma^2} \tag{19}$$

and can be computed for several values of n, p and γ. This probability is represented in Figure 5. The probability of not being able to apply this correction increases with the value of γ and the dimension p while it decreases when the sample size n gets bigger.

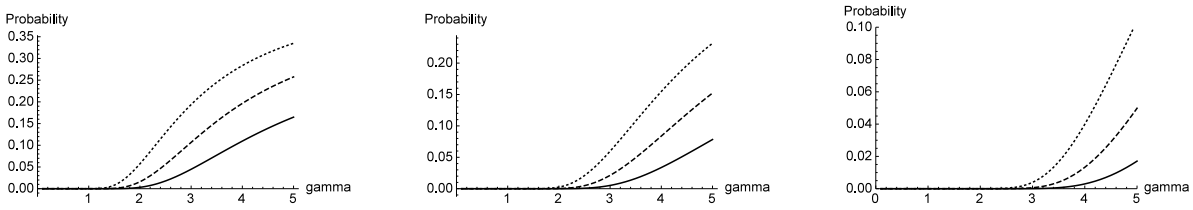

Figure 5: Probability (19) as a function of γ, for a dimension p equal to 3 (solid line), 7 (dashed line) or 15 (dotted line) and three sample sizes, $n = 50$ (left panel), $n = 100$ (middle panel) and $n = 200$ (in the right panel).

This bias correction can also be used for other underlying distributions, provided the fact that $g(\gamma) = E_{F_\gamma}[V_n(\mathbf{X})]$ can be numerically computed and that it is a one-to-one transformation.

8. Simulation study - Bias

In Sections 5, 6 and 7, as the classical estimator was found to be biased in finite samples under normality, several bias-corrected estimators have been proposed. In practice, when multivariate normality can be accepted for the data, as in the EQA and financial applications presented in Section 2, all these estimators can be computed easily. For these datasets, the distinct estimators proposed to measure the performance of each electrophoretic technique or portfolio are given in Tables 4 and 5. As a reminder, V_n denotes the classical estimator, while V_n^1 uses an unbiased estimator of γ^{-2} under normality. All other estimators, based on the same principle, use an estimator of the bias obtained from the expectation-curve under normality for $\check{V}_n^{(2)}$, by jackknife for $\check{V}_n^{(3)}$, by bootstrap for $\check{V}_n^{(4)}$ and by inversion of the expectation-curve under normality for $\check{V}_n^{(5)}$. We can observe that all the obtained corrected estimates are indeed higher than the classical estimates. for small values of γ, the parametric and non-parametric bias corrections seem to give similar results. For bigger values of γ, as encountered in the financial application, the jackknife and the plug-in estimates seem to yield higher values than the other corrections.

Electrophoretic technique	n	V_n classical	$V_n^{(1)}$ plug-in	$\check{V}_n^{(2)}$ bias estim.	$\check{V}_n^{(3)}$ bootstrap	$\check{V}_n^{(4)}$ jackknife	$\check{V}_n^{(5)}$ inversion
BCP Capillary zone	62	.0238	.0248		.0248	.0248	
HT Agarose gel (Acid blue)	112	.0558	.0573		.0566	.0570	
HT Agarose gel (Amido black)	74	.0616	.0643		.0640	.0642	
HT Cellulose acetate	141	.0688	.0704		.0700	.0702	

Table 4: Bias-corrected estimates - EQA application ($p = 4$)

Portfolio	n	V_n classical	$V_n^{(1)}$ plug-in	$\check{V}_n^{(2)}$ bias estim.	$\check{V}_n^{(3)}$ bootstrap	$\check{V}_n^{(4)}$ jackknife	$\check{V}_n^{(5)}$ inversion
Technology	72	2.021	2.568	2.210	2.230	3.212	2.263
Health care	72	2.134	2.786	2.347	2.404	3.481	2.411

Table 5: Bias corrected estimates - Financial application ($p = 5$)

In this section, in order to pinpoint the best estimator to use in practice, under normality or not, some Monte Carlo simulations are performed to compare the biases and mean squared errors of the bias-corrections under several elliptical distributions. Since the distribution of V_n only depends on the mean vector $\boldsymbol{\mu}$ and the scatter matrix $\boldsymbol{\Sigma}$ through the theoretical MCV, it is sufficient to study the performance of the bias-corrected estimators under $F_{\boldsymbol{\mu}_0,\boldsymbol{\Sigma}_0}$, with $\mu_0 = 1/\gamma e_1$ and $\boldsymbol{\Sigma}_0 = I_{p\times p}$, where e_1 denotes the first canonical vector of $\mathbb{R}^p$ and $I_{p\times p}$ is the identity matrix, for several values of γ. Several sample sizes and dimensions were investigated: $n = 20, 50, 100$ and 200 and $p = 3, 7$ and 15. Three values of γ were also considered, i.e. $\gamma = 0.5$, $\gamma = 1$ and $\gamma = 2$. For each distribution, dimension, sample size and value of γ, $m = 500$ samples were generated. The number of bootstrap replications was set to 500 for $\check{V}_n^{(2)}$. As summary measures, the bias and mean squared error of the bias-corrected estimators were computed.

Let us note that the two parametric bias-corrections using the exact distribution (5) of V_n, i.e. $\check{V}_n^{(2)}$ and $\check{V}_n^{(5)}$, are quite computationally cumbersome and time-consuming. For each of the 500 initial estimates v_n^i, the correction $\check{V}_n^{(2)}$ consists in computing the expected value $E_{\check{\Phi}_{v_n^i}}[V_n(\mathbf{X})]$. For correction $\check{V}_n^{(5)}$, for each of the 500 initial estimates v_n^i, it requires to find the solution of the equation $E_{\Phi_\gamma}[V_n(\mathbf{X})] = v_n^i$ in γ. To do this, we have used the discretized Newton Raphson method (or secant method) with the classical and the jackknife estimates as initial values. The procedure stops when either (i) the maximum number of iterations (say 10) is reached, or (ii) when the difference between the expectation and the target is in absolute value lower than 10^{-10}, or (iii) when the difference between two consecutive values is lower than 10^{-8}. However, contrary to what might be thought, the long computation time is not due to a low convergence of this method (the worst average number of iterations among all considered combinations of n, p and γ was 4.6) but to the computations of the expectation by means of numerical integration, especially when the sample size n is high. For this reason, correction $\check{V}_n^{(5)}$ under the sample size $n = 200$ was not computed.

As seen in Sections 5 and 7, the two parametric corrections $V_n^{(1)}$ and $\check{V}_n^{(5)}$ are not always possible to compute, especially for high values of γ and when n/p is small. In the following, results are therefore not reported when the calculation was impossible for more than 10% of cases.

Results obtained under normality are displayed in Table 6. The sample bias of the classical estimator is in line with trends presented in Section 4: V_n underestimates γ and the bias increases with γ and p and decreases as the sample size n gets bigger. For small values of γ (≤ 1), the bias-corrected estimator $V_n^{(1)}$ seems to have a lower bias in absolute value and similar MSE's than the classical estimator. However, for bigger values of γ, the correction seems to be too strong and this estimator highly overestimates γ. Moreover, it yields higher MSE's, especially when n/p is small. Indeed, in these cases, the probability that expression (11) takes values close to zero is higher and the estimator can thus take arbitrary large values under these settings. The two parametric corrections based on the normality assumption, $\check{V}_n^{(2)}$ and $\check{V}_n^{(5)}$, lead to an effective reduction in bias and their MSE's remain similar to those of V_n except when $\gamma = 2$ and n/p is small. The two non-parametric corrections, i.e. the non-parametric bootstrap and the jackknife, are also performant in terms of bias reduction. As expected, these methods also lead to bigger MSE's when $\gamma = 2$ and for small values of n/p.

As seen in Section 3, under normality, $1/V_n^2$ has a skewed noncentral distribution. In order to examine the empirical distribution of the corrected estimators, Figure 6 shows the corresponding boxplots when $\gamma = 1$ and $\gamma = 2$ and for several values of n and p. The horizontal line represents the target γ value while the star represents the empirical mean. All these empirical distributions are skewed. These boxplots allow to analyze the median bias, i.e. the difference between the median and the target value. Correction $V_n^{(1)}$ improves the median bias but its variability is, in most cases, more important than the variability of other estimators. As for the other corrections, their medians are closer to the target value than the median of the classical estimator while keeping a similar variability.

Despite the numerous properties and procedures developed under multivariate normality, it appears that in many applications, modeling the data using a multivariate distribution with heavier tails would be more appropriate. A class of elliptical distributions incorporating distributions with either heavier-tails or lighter-

		V_n classical		$V_n^{(1)}$ plug-in		$\check{V}_n^{(2)}$ bias estim.		$\check{V}_n^{(3)}$ bootstrap		$\check{V}_n^{(4)}$ jackknife		$\check{V}_n^{(5)}$ inversion	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
n	p	$\gamma = 0.5$											
20	3	-.045	.011	.038	.016	-.003	.011	-.004	.011	.003	.011	.001	.011
50	3	-.022	.004	.008	.004	-.005	.004	-.005	.004	-.004	.004	.004	.004
	7	-.040	.005	.018	.006	.000	.005	.000	.005	.006	.005	.005	.005
100	3	-.006	.002	.009	.002	.002	.002	.002	.002	.003	.002	.003	.002
	7	-.019	.002	.009	.002	.001	.002	.001	.002	.003	.002	.003	.002
200	3	-.005	.001	.003	.001			.000	.001	.000	.001		
	7	-.011	.001	.003	.001			.000	.001	.000	.001		
	15	-.032	.002	.002	.001			-.004	.001	-.001	.001		
n	p	$\gamma = 1$											
20	3	-.101	.077	.199	.682	-.018	.082	-.019	.082	.004	.081	-.008	.087
50	3	-.038	.030	.050	.043	-.004	.031	-.005	.031	.005	.031	-.003	.031
	7	-.111	.036	.066	.056	-.014	.034	-.013	.034	-.005	.036	.009	.039
100	3	-.013	.015	.029	.018	.004	.015	.004	.015	.009	.015	.004	.015
	7	-.054	.015	.030	.018	-.001	.015	-.001	.015	.004	.015	.005	.015
200	3	-.004	.007	.016	.008			.004	.007	.004	.007		
	7	-.032	.008	.009	.008			-.005	.007	-.003	.007		
	15	-.094	.016	.009	.011			-.017	.010	-.007	.010		
n	p	$\gamma = 2$											
20	3	-.324	.786	-	-	-.083	1.471	-.088	1.473	-.106	3.139	-	-
50	3	-.065	.476	.559	4.722	.026	.660	.025	.670	.004	.476	.079	2.025
	7	-.446	.370	.452	2.378	-.174	.370	-.172	.374	-.096	.496	.054	1.162
100	3	-.023	.208	.201	.442	.012	.217	.011	.216	.012	.215	.013	.218
	7	-.213	.193	.259	.805	-.027	.232	-.027	.232	.000	.254	0.055	0.361
200	3	-.017	.085	.077	.113			.002	.088	.001	.086		
	7	-.105	.084	.093	.135			.000	.093	.007	.095		
	15	-.356	.174	.131	.233			-.107	.095	-.042	.105		

Table 6: Sample Bias and MSE for the classical estimator and five bias-corrected estimator of γ under multivariate normality.

tails than under normality is the family of *multivariate exponential power distributions* (see [8]), which are generalizations of the univariate exponential distribution, for which the function h in (2) is given by

$$h(t) = K \exp\left(-at^{\beta}\right)$$

where K is a normalizing factor ensuring that the density integrates to one and a is a positive constant such that $\mathbf{\Sigma}$ is the covariance matrix. The parameter β can be seen as a measure of kurtosis: when β increases, the sharpness of the distribution decreases. In this simulation study, the values $\beta = 0.5$, which corresponds to the multivariate generalization of the double exponential distribution, and $\beta = 5$, which corresponds to a platykurtic distribution, were investigated.

The empirical distributions of the estimators under these distributions are shown (only for $\gamma = 2$) in Figure 7. Results for the estimators $V_n^{(1)}$, $\check{V}_n^{(2)}$ an $\check{V}_n^{(5)}$, which are only valid under normality, are no longer displayed. Similar conclusions about the classical estimator can be drawn from these plots: it underestimates the true parameter γ, especially for small n, great γ and in high dimension p. Under the platykurtic (top panel) and leptokurtic (bottom panel) exponential distribution, the bootstrap and the jackknife estimators seem to effectively reduce both the bias and the median bias without excessively increasing the variability. Let us note that the distribution is leptokurtic (top panel), the empirical distributions of the initial or corrected estimators are more skewed and take slightly greater range of values, especially when the ratio n/p is small. Indeed, the asymptotic variance of this estimator is an increasing function of the kurtosis parameter. As an indication, for $p = 3$ and $\gamma = 2$, the ASV of V_n is 18 under normality, 17.22 under the platykurtic power exponential distribution and 19.50 under the exponential power distribution with $\beta = 0.5$. Similar conclusions are reached under the Student distribution (not shown here).

In view of the above results, we strongly advocate the use of a corrected estimator, since they effectively

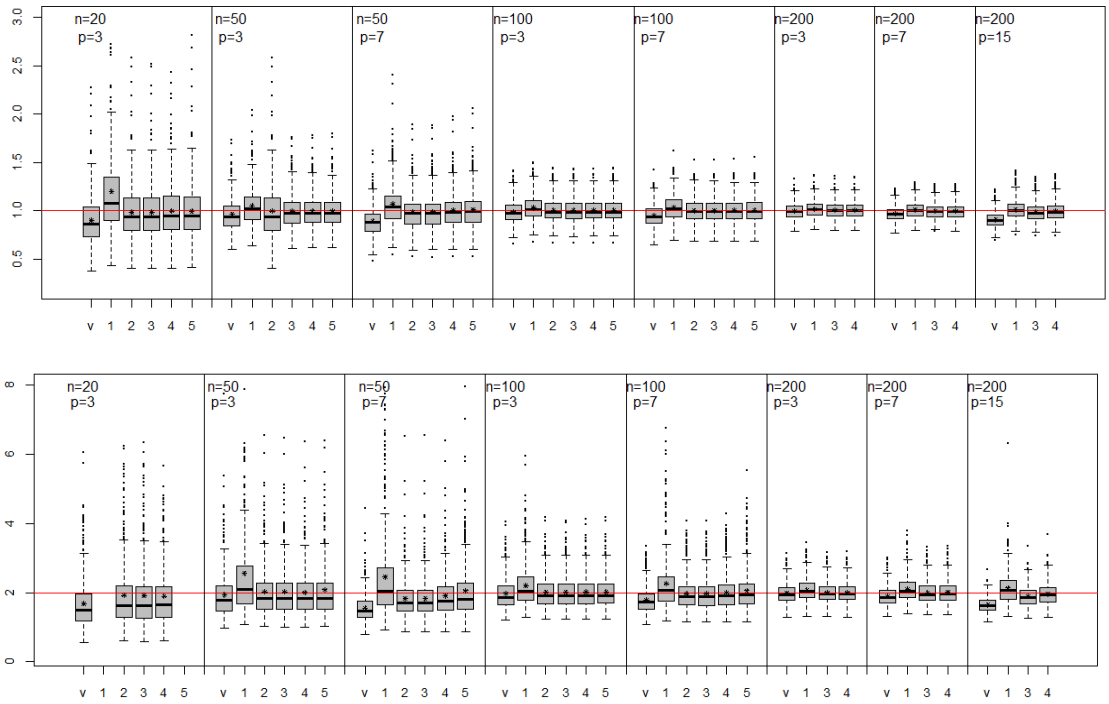

Figure 6: Empirical distribution of the initial and corrected estimators for $\gamma = 1$ (top) and $\gamma = 2$ (bottom) under normality.

reduce the bias in all cases. Under normality, the parametric correction $\check{V}_n^{(2)}$ and $\check{V}_n^{(5)}$ and the non-parametric estimators yield very similar results. As already mentioned in the introduction, many results in 1D rely on the assumption that γ is small. Under such an assumption, for $\gamma < 1$, our corrections are good enough for an efficient use in practice. Since the jackknife is less time-consuming and performs well in any case, we would recommend its use in practice both under Gaussian distributions or lighter- and heavier-tailed ones. For bigger values of γ, which are not unnatural in practice, as illustrated in Section 2), even if the proposed corrections effectively reduce the bias, they remain quite biased when n/p is small. Further research would be worth considering to improve estimation under this setting.

9. Simulation study - Confidence intervals

In the univariate context, many authors have aimed to find some good interval estimators for the univariate population CV. In this section, our goal is to compare by Monte Carlo simulations several parametric and non parametric confidence intervals for the multivariate population CV. The parametric confidence intervals under consideration are the exact confidence interval, denoted by I from now on, using distribution (5) (only valid under normality) and the inversion principle, and the asymptotic confidence interval, denoted by I_{as} from now on, using Proposition (2). We also considered the following non parametric intervals:

- The *bootstrap percentile interval*, denoted by $I_{\text{boot,p}}$, which uses the percentiles of the bootstrap distribution and is defined by
$$I_{\text{boot,p}} = [V^*_{n,(\beta)}; V^*_{n,(1-\alpha+\beta)}] \tag{20}$$
where $V^*_{n,(\beta)}$ denotes the βth percentile of the bootstrapped coefficients.

- The *normal bootstrap interval*, denoted by I_{boot}, which assumes asymptotic normality and is given by:
$$I_{\text{boot}} = [V_n - q_{\alpha/2}\ \widehat{\text{se}}_{\text{boot}}; V_n + q_{\alpha/2}\ \widehat{\text{se}}_{\text{boot}}] \tag{21}$$

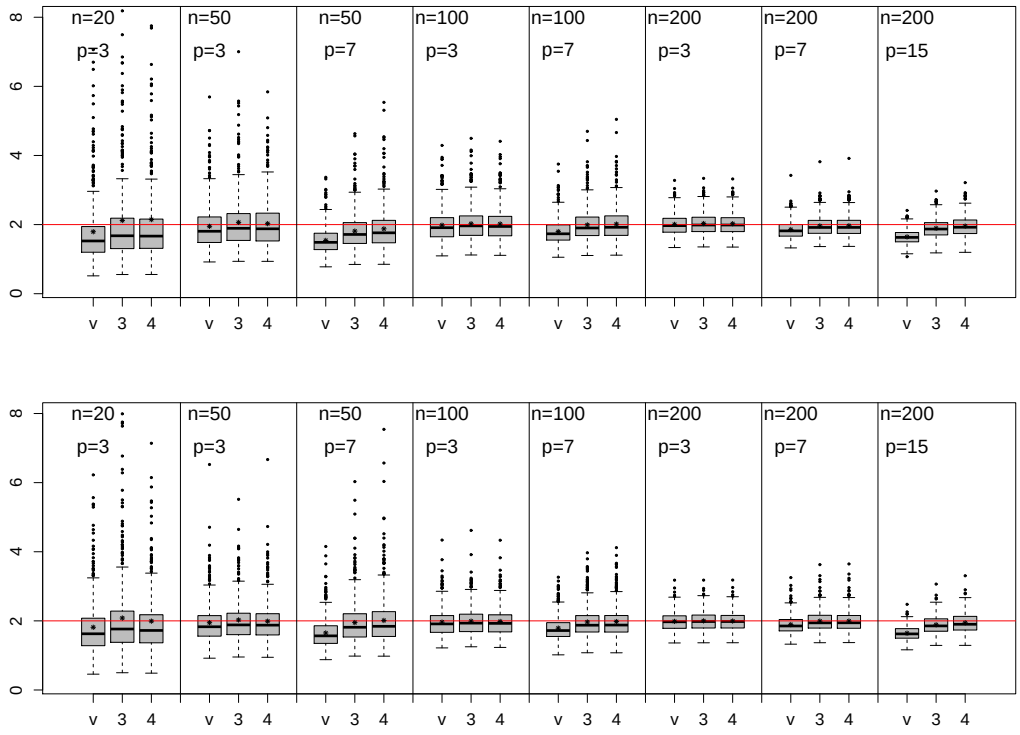

Figure 7: Empirical distribution of the initial and corrected estimators for $\gamma = 2$ under power exponential distribution with $\beta = 0.5$ (top) and $\beta = 5$ (bottom)

where V_n is the classical estimator, $\widehat{\text{se}}_{\text{boot}}$ is the bootstrap standard error estimator and $q_{\alpha/2}$ is the quantile $\alpha/2$ of the standard normal distribution.

- The *bias-corrected normal bootstrap interval*, denoted by I'_{boot} which consists in replacing V_n by the corrected estimator $\check{V}_n^{(3)}$ based on the bias-estimation principle (14), or equivalently in translating I_{boot}:

$$I'_{\text{boot}} = I_{\text{boot}} - B_n^{(3)} \tag{22}$$

where $B_n^{(3)}$ is the bootstrap estimator of the bias of V_n.

- The *Jackknife interval*, denoted by I_{jack}, which assumes asymptotic normality and is given by

$$I_{\text{jack}} = [V_n - q_{\alpha/2}\ \widehat{\text{se}}_{\text{jack}}; V_n + q_{\alpha/2}\ \widehat{\text{se}}_{\text{jack}}] \tag{23}$$

where $\widehat{\text{se}}_{\text{jack}}$ is the jackknife standard error estimator.

- The *bias-corrected Jackknife interval*, denoted by I'_{jack} which consists in replacing V_n by the corrected estimator $\check{V}_n^{(4)}$ based on the bias-estimation principle (14), or equivalently in translating I_{jack}:

$$I'_{\text{jack}} = I_{\text{jack}} - B_n^{(4)} \tag{24}$$

where $B_n^{(4)}$ is the jackknife estimator of the bias of V_n.

In practice, all these intervals can be computed easily. Tables 7 and 8 give the distinct 95% confidence intervals for the EQA and financial applications presented in Section 2. For the EQA application, whatever the confidence interval, the intervals of the assay techniques do not overlap, which suggest that there is a clear ranking of the electrophorectic techniques in terms of repeatability. This should however be assessed significantly by a statistical test. The asymptotic interval is close to the non-parametric uncorrected intervals, while the exact interval is similar to their corrected versions. For the financial application, whatever the interval technique, the intervals of the two portfolios overlap. As expected, for big values of γ, the exact interval is longer and strongly asymmetric.

Techn.	n	I	I_{as}	I_{boot}	I'_{boot}	I_{jack}	I'_{jack}	$I_{\text{boot,p}}$
1	62	[.0240; .0255]	[.0231; .0245]	[.0231; .0245]	[.0241; .0254]	[.0231; .0245]	[.0241; .0255]	[.0222; .0245]
2	112	[.0558; .0583]	[.0546; .0570]	[.0546; .0570]	[.0557; .0581]	[.0545; .0571]	[.0558; .0583]	[.0534; .0567]
3	74	[.0620; .0655]	[.0600; .0633]	[.0599; .0634]	[.0623; .0657]	[.0598; .0634]	[.0624; .0661]	[.0576; .0611]
4	141	[.0687; .0715]	[.0675; .0701]	[.0675; .0702]	[.0689; .0716]	[.0674; .0703]	[.0687; .0716]	[.0659; .0686]

Table 7: Confidence intervals ($\alpha = 0.05$) - EQA application ($p = 4$) with techniques sorted by decreasing performance : BCP Capillary zone (1), HT Agarose gel (Acid blue) (2), HT Agarose gel (Amido black) (3) and HT Cellulose acetate (4)

Portf.	n	I	I_{as}	I_{boot}	I'_{boot}	I_{jack}	I'_{jack}	$I_{\text{boot,p}}$
Techn.	72	[1.487; 8.468]	[1.021; 3.020]	[1.077; 2.964]	[1.253; 3.140]	[0.957; 3.084]	[2.148; 4.276]	[1.188; 3.001]
H. care	72	[1.555; 14.668]	[1.026; 3.242]	[1.244; 3.024]	[1.496; 3.276]	[1.054, 3.214]	[2.401; 4.561]	[1.225; 2.973]

Table 8: Confidence intervals ($\alpha = 0.05$) - Financial application ($p = 5$)

In order to compare these intervals by Monte Carlo simulations, the settings are the same as in Section 8, except that we added the sample size value $n = 1000$. For each combination of an elliptical distribution and fixed values of n, p and γ, 500 samples of size n were generated from this distribution and the distinct confidence intervals were computed. The number of bootstrap replications was set to 1000. The most common 5% level of significance ($\alpha = 0.05$) is used. For the exact confidence interval and the bootstrap percentile interval, the common choice of equal probability tails was made. As already explained in Section 3, the bounds of the exact interval are not always possible to compute, especially for big values of γ, p and small values of n. Results are therefore not reported when the computation of one of the two bounds was impossible for more than 10% of the cases. Concerning the last four non-parametric confidence intervals and the asymptotic interval, let us note that those may have negative bounds. Since Voinov & Nikulin's coefficient can only take positive values, these values were replaced by zero.

The comparison criteria we consider in this section are the average width of the interval and the empirical coverage probability, i.e. the proportion of intervals containing the true value of the parameter. Under normality, Figure 8 represents, for several values of n, p and γ, the average bounds of the exact CI (N$^{\text{o}}$ 1), the asymptotic CI (N$^{\text{o}}$ 2), the normal bootstrap interval and its corrected version (N$^{\text{os}}$ 3 and 4), the jackknife interval and its corrected version (N$^{\text{os}}$ 5 and 6) and the bootstrap percentile interval (N$^{\text{o}}$ 7). Their respective empirical coverage probabilities are listed in Table 9.

We can observe that when the sample size is big (= 1000), all confidence intervals have average bounds close to those of the exact interval. In other cases, on the basis of these results, we do not recommend the use of the bootstrap percentile interval since it always yields the lowest coverage probabilities. The bounds of this interval are, in average, lower than those of other CI's in most cases. The asymptotic confidence interval, as expected, attains a coverage probability close to the nominal one for large sample sizes and provided that the ratio n/p is not too small.

For the other non-parametric intervals, their corrected versions, although quite naive, indeed seem more powerful than the original versions, especially for the jackknife interval. When n/p decreases and for low values of γ, it is, together with the exact confidence interval, the only satisfactory intervals in terms of coverage probabilities. However, this correction does not suffice when γ takes bigger values: in this case, and especially for low values of n/p, the bias of the corrected versions, although smaller than the one of the initial estimator, remain big and the corresponding confidence interval bounds are therefore too low,

		I	I_{as}	I_{boot}	I'_{boot}	I_{jack}	I'_{jack}	$I_{\text{boot;p}}$
n	p	$\gamma = 0.5$						
20	3	95.4	80.2	78.0	84.2	82.6	88.6	66.4
50	3	93.8	87.4	85.6	88.2	87.4	90.4	82.6
	7	94.2	78.8	78.2	85.0	82.4	90.0	61.8
100	3	95.0	90.8	90.2	92.4	91.4	92.8	88.2
	7	94.8	87.8	86.8	91.6	90.2	93.6	74.8
200	3	94.6	93.8	93.8	93.4	94.0	93.2	93.0
	7	96.0	91.2	91.4	92.8	91.6	94.0	86.0
	15	93.8	84.2	83.2	90.4	86.0	92.8	62.8
1000	3	94.8	95.4	94.6	94.4	95.0	94.6	95.0
	7	94.4	95.0	94.0	94.6	95.0	94.6	93.6
	15	95.2	91.4	90.6	94.6	91.4	95.2	85.8
n	p	$\gamma = 1$						
20	3	-	80.6	79.6	84.4	82.6	87.4	72.8
50	3	94.4	89.8	89.6	92.0	90.6	92.4	88.8
	7	93.8	75.4	71.8	80.8	79.6	88.6	59.8
100	3	94.6	91.0	90.0	91.4	90.8	92.4	89.0
	7	95.6	82.6	81.0	87.0	84.6	90.8	74.2
200	3	93.8	93.4	93.8	94.0	93.8	94.6	93.4
	7	94.4	90.2	88.8	92.2	91.0	93.6	86.4
	15	93.8	78.4	75.0	86.4	80.6	90.6	58.6
1000	3	95.6	95.2	95.4	95.2	95.4	95.4	95.2
	7	94.0	94.4	94.2	94.6	94.6	94.8	94.0
	15	94.0	92.2	92.0	93.4	92.6	93.6	87.2
n	p	$\gamma = 2$						
20	3	-	74.4	73.4	76.8	74.6	80.4	75.0
50	3	-	84.8	85.8	86.6	85.0	87.4	88.0
	7	-	60.2	51.8	66.2	66.4	79.0	43.2
100	3	95.6	87.2	88.8	90.2	88.4	89.2	91.2
	7	94.4	77.6	72.0	77.6	78.4	84.6	68.8
200	3	94.2	92.2	92.2	92.8	92.2	93.2	91.8
	7	96.0	85.6	84.2	87.6	86.40	91.0	82.0
	15	96.4	65.2	57.8	75.8	68.4	86.2	35.0
1000	3	94.8	92.6	92.6	92.8	92.6	93.0	93.6
	7	95.0	93.0	92.4	93.2	93.0	94.2	92.2
	15	94.6	87.0	85.6	90.6	88.0	91.2	80.4

Table 9: Empirical coverage probabilities (%) under normality

yielding poor coverage probabilities.

Finally, the exact confidence interval has the best coverage probabilities. For large values of γ and a small ratio n/p, this interval is the only reliable one in terms of coverage probabilities. However, it is also the widest and the most asymmetric around the target value. Let us note that, as mentioned in Section 3, the choice of equal-tailed probabilities does not yield uniformly shorter intervals. To obtain a good trade-off between performance and width, it would therefore be interesting to investigate on efficient methods to choose β or in other ways of constructing an exact and uniformly shorter interval, following for example and as cited before, the idea of Jongphil [10].

In order to compare the asymptotic CI with the non-parametric ones under both heavier-tailed and lighter-tailed distributions, results for the power exponential distribution with parameters $\beta = 0.5$ and $\beta = 5$ are displayed in Figure 9 and Table 10 for $\gamma = 1$. The results are quite similar to those obtained under normality. Both under the leptokurtic case (top panel) and the platykurtic case (bottom panel), the bootstrap percentile interval systematically has lower coverage probabilities because its bounds are too low. Both the asymptotic CI and the non-parametric ones yields satisfactory results when the ratio n/p is big. All of them have higher coverage probabilities under the platykurtic distribution than under the leptokurtic one, under which the distribution of the corresponding estimators is more skewed. Again, the corrected versions of the non-parametric CI's allow to systematically improve the coverage probabilities and

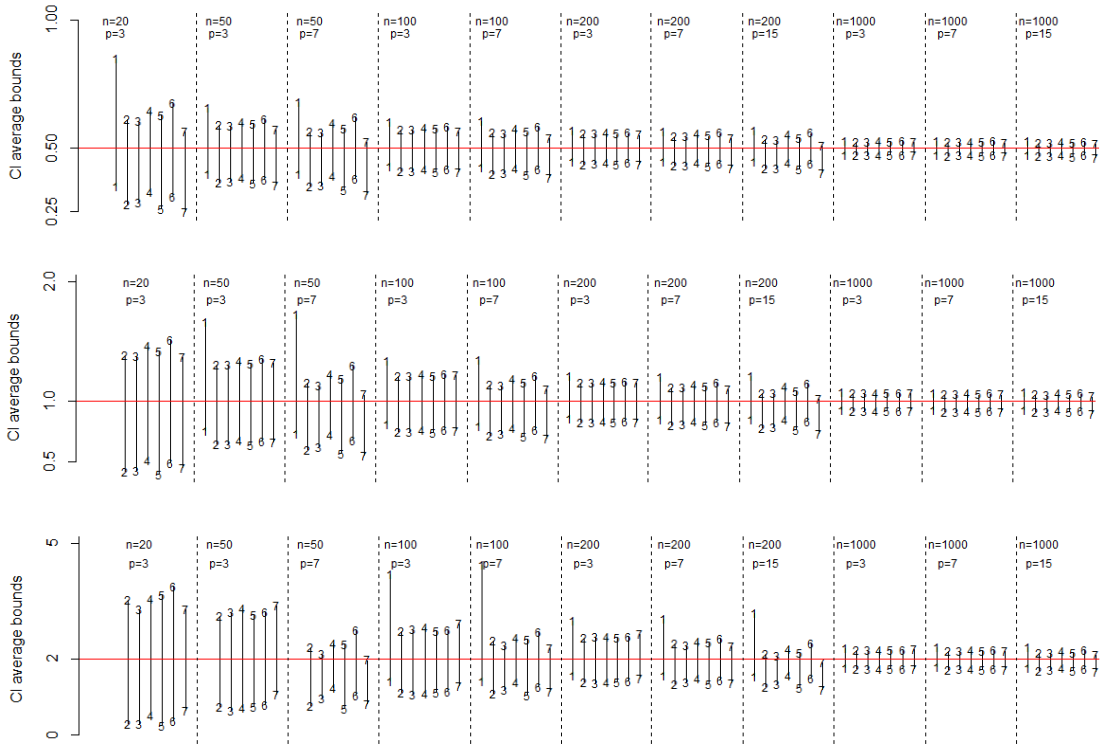

Figure 8: Average confidence interval width for $\gamma = 0.5$ (top panel), $\gamma = 1$ (middle panel) and $\gamma = 2$ (bottom panel) under normality.

the jackknife corrected interval has the best results. In small sample, the asymptotic CI under-performs the other CI's and should therefore be used with caution.

In conclusion, under normality, we strongly recommend the use of the exact confidence interval and the construction of uniformly shorter exact intervals would be worth investigating. Under non Gaussian distributions, either heavier- or lighter-tailed ones, for which the exact interval as constructed above is not valid anymore, we advocate the use of the corrected jackknife interval or the asymptotic one, at least when the ratio n/p is big and γ is small (the coverage probabilities indeed decreases with the value of γ - not shown here). In small samples, high dimension or for big values of γ, although the corrected versions of the non-parametric intervals are better than uncorrected ones, their coverage probabilities remain quite low and further research is therefore needed to construct more efficient and shorter intervals.

10. Conclusion

In many areas, one wishes to compare the variability of a variable in several populations that may have really different means. The advocated statistic for that purpose is the univariate CV. When the number of variables is greater than one, measuring variability only marginally may lead to contradictory results. Although some generalizations of the coefficient of variation have already been introduced by several authors (Reyment [19], Van Valen [24], Voinov and Nikulin [26] and Albert and Zhang [1]), the literature on the subject is scarce. In this paper, focus is on the distance based extension due to Voinov & Nikulin.

The sample distribution and the asymptotic distribution of several estimators have been studied under elliptical symmetry. Under normality, the exact distribution of the squared inverse of the classical estimator, a decentralized F, is known, which allows not only to build exact confidence intervals but also to measure the bias. Indeed, this estimator tends to underestimate the relative dispersion, especially when the dimension or the theoretical coefficient are big.

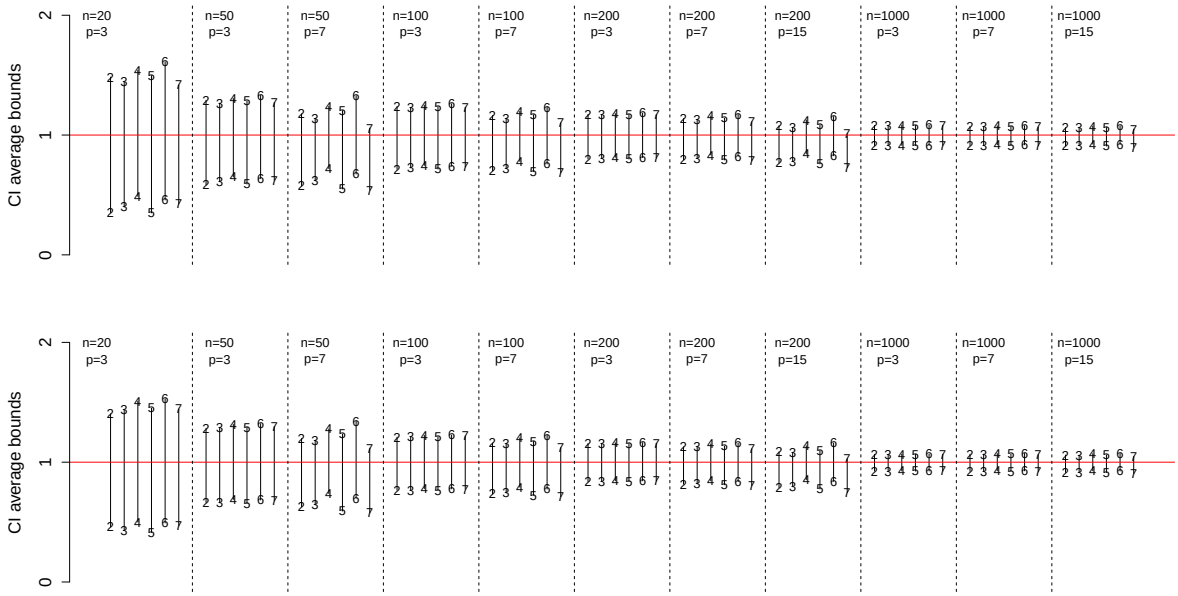

Figure 9: Average confidence interval width for $\gamma = 1$ under power exponential distribution with parameter $\beta = 0.5$ (top panel) and $\beta = 5$ (bottom panel).

		I_{as}	I_{boot}	I'_{boot}	I_{jack}	I'_{jack}	$I_{\text{boot;p}}$
n	p	Exponential distribution $\beta = 0.5$, $\gamma = 1$					
20	3	79.4	72.6	78.4	77.2	83.2	68.2
50	3	84.2	80.4	83.4	81.8	86.0	79.0
	7	74.2	69.4	79.4	76.8	86.4	55.8
100	3	90.6	87.4	88.6	88.6	90.0	85.8
	7	85.0	80.0	87.2	84.0	90.4	71.8
200	3	92.8	90.8	92.4	91.0	92.4	90.6
	7	88.4	86.0	88.4	88.0	92.6	82.0
	15	75.8	72.0	84.2	77.8	88.4	52.4
1000	3	94.6	94.8	94.8	94.8	95.2	94.8
	7	92.4	91.8	93.4	91.8	93.8	90.8
	15	92.0	91.2	93.2	92.0	93.8	87.4
n	p	Exponential distribution $\beta = 5$, $\gamma = 1$					
20	3	81.0	83.0	86.6	84.8	90.0	78.8
50	3	87.4	88.8	90.2	89.6	91.4	86.6
	7	83.0	81.0	87.4	85.6	92.4	69.6
100	3	92.0	92.8	93.8	93.2	94.0	92.6
	7	87.4	86.6	90.8	89.2	93.2	80.0
200	3	93.2	92.8	92.4	92.8	92.8	92.2
	7	88.0	87.6	91.0	89.2	91.4	84.4
	15	81.6	79.8	87.8	83.2	91.0	61.0
1000	3	94.0	94.2	94.4	94.0	94.4	93.4
	7	94.2	94.0	94.4	94.0	94.8	93.2
	15	92.8	92.4	94.4	92.8	94.6	88.0

Table 10: Empirical coverage probabilities (%) under power exponential distribution with parameter $\beta = 5$ (top panel) and $\beta = 0.5$ (bottom panel)

Several parametric and non-parametric methods for correcting this bias have been proposed. The first one, which is only valid under normality, is based on a suggestion made by Zhang [27] and directly uses the properties of the F-distribution. The second approach needs an estimate of the bias, which can be obtained using parametric or non-parametric methods. Finally, the third correction, only valid under normality, requires the numerical computation of the expectation of the classical estimator.

Under normality, we strongly advocate the use of these corrections in practice, at least for small values of γ, e.g. for the EQA example proposed in Section 2. Indeed, except for the first correction, they significantly reduce the bias without increasing the variability excessively. The findings are similar under platykurtic and leptokurtic elliptical distributions. The literature about the univariate CV mainly focuses on small values of the CV. In the multivariate context, we wanted to extend our study to larger values of γ, since they are not so rare in practice, as encountered in the financial application outlined in Section 2. However, in small samples, for large values of γ and p, although the corrected versions are more performant than the classical estimator, they still remain biased. Further research is needed to obtain satisfying estimators in those cases.

The theoretical results from Section 3 also allowed to build two parametric confidence intervals: the exact one (only valid under normality) and the asymptotic one. Their performance, in terms of coverage probabilities and width, was compared with that of non-parametric intervals, corrected for bias or not. Under normality, the exact interval has the best coverage probabilities and further research to find a uniformly shorter exact interval deserves to be conducted. The asymptotic interval gives very satisfactory results provided that n/p is not too small and γ is not too big, even under platykurtic and leptokurtic exponential power distributions.

For large values of γ and when n/p is small, even if the corrected version improve the coverage probabilities, the results remain below the nominal level. Further research should focus on this aspect in order to provide confidence intervals for portfolio's performance for instance. Let us note that the fact that the results are not satisfactory when n/p is small is not surprising: with the MCV, one attempts to reduce p dimensions in one single index. Therefore, the larger the dimension, the more difficult it is to accurately summarize all the information.

The MCV should be used for comparison purpose, as illustrated in the two real-data examples. Of course, in practice, one wants to go one step further than simply looking for the technique with the best repeatability or the portfolio with the highest performance. "Significant" differences are of interest. In the univariate context, financiers, in parallel with statisticians, have quickly been interested in testing procedures for comparing several Sharpe ratios or univariate CV's. For instance, Nairy and Rao [17] provide a review of the existing tests comparing two or several CV's. Further research is required to adapt some of these approaches to construct a test comparing the values of the estimations of Voinov & Nikulin's MCV.

Appendix

Proof of Proposition 1:

Let $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ be a sequence of n independent p-variate random vectors and $g_{\mathbf{A}}$ be a transformation on $\mathbf{X}$ defined by $g_{\mathbf{A}} : \mathbf{X} \mapsto (\mathbf{A}\mathbf{X}_1, ..., \mathbf{A}\mathbf{X}_n)$ where $\mathbf{A}$ is a $p \times p$ non-singular matrix.

First, since the location and covariance estimators $\mathbf{T}_n(\mathbf{X})$ and $\mathbf{C}_n(\mathbf{X})$ are affine equivariant, the estimator $V_n(\mathbf{X})$ is invariant under $g_{\mathbf{A}}$ as detailed below:

$$V_n(g_{\mathbf{A}}(\mathbf{X})) = \frac{1}{\sqrt{\mathbf{T}_n(\mathbf{X})^t \mathbf{A}^t (\mathbf{A}^t)^{-1} \mathbf{C}_n(\mathbf{X}) \mathbf{A}^{-1} \mathbf{A} \mathbf{T}_n(\mathbf{X})}} = V_n(\mathbf{X})$$

Now, let $F_{\boldsymbol{\mu},\boldsymbol{\Sigma}}$ and $F_{\boldsymbol{\mu}',\boldsymbol{\Sigma}'}$ be two distributions belonging to $\mathcal{F}_h$ and having the same theoretical coefficient of variation $\gamma = \gamma'$. As $\sqrt{\boldsymbol{\mu}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} = \|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}\|$ where $\|.\|$ is the Euclidean norm, the equality $\gamma = \gamma'$ implies the equality of the two norms $\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}\| = \|\boldsymbol{\Sigma}'^{-1/2}\boldsymbol{\mu}'\|$. Therefore, there exists an orthogonal matrix $\mathbf{B}$ such that $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\Sigma}'^{-1/2}\boldsymbol{\mu}'$.

Take as matrix $\mathbf{A}$ the non-singular matrix $\mathbf{\Sigma}^{1/2}\mathbf{B}\mathbf{\Sigma}'^{-1/2}$. It follows directly that $F_{\boldsymbol{\mu},\mathbf{\Sigma}} = F_{\mathbf{A}\boldsymbol{\mu}',\mathbf{A}\mathbf{\Sigma}'\mathbf{A}^t}$. From there, it comes

$$\mathrm{G}_{F_{\boldsymbol{\mu},\mathbf{\Sigma}}}\left[V_n(\mathbf{X})\right] = \mathrm{G}_{F_{\mathbf{A}\boldsymbol{\mu}',\mathbf{A}\mathbf{\Sigma}'\mathbf{A}^t}}\left[V_n(\mathbf{X})\right] = \mathrm{G}_{F_{\boldsymbol{\mu}',\mathbf{\Sigma}'}}\left[V_n(g_{\mathbf{A}}(\mathbf{X}))\right] = \mathrm{G}_{F_{\boldsymbol{\mu}',\mathbf{\Sigma}'}}\left[V_n(\mathbf{X})\right]$$

where $\mathrm{G}_F\left[.\right]$ corresponds to the distribution of $[.]$ computed under the assumption that $\mathbf{X}_i \sim F$ for $i = 1, ..., n$. This concludes the proof. □

Proof of Proposition 2:

For affine-equivariant estimators $\mathbf{T}_n := \mathbf{T}_n(\mathbf{X})$ and $\mathbf{C}_n := \mathbf{C}_n(\mathbf{X})$ satisfying (A1), (A2) and (A3), their joint asymptotic distribution is given by $N_{p^2+p}((\boldsymbol{\mu}, \mathrm{vec}\mathbf{\Sigma})^t, V)$ with

$$V = \begin{pmatrix} \tau\mathbf{\Sigma} & 0_{p\times p^2} \\ 0^t_{p\times p^2} & \mathbf{\Xi} \end{pmatrix}$$

.

The delta method for the function f defined by $f : \mathbb{R}^{p+p^2} \to \mathbb{R} : W = (\mathbf{T}_n, \mathrm{vec}\mathbf{C}_n) \mapsto (\mathbf{T}_n^t\mathbf{C}_n^{-1}\mathbf{T}_n)^{-1/2} = V_n$ allows to say that

$$\sqrt{n}(V_n - \gamma) \xrightarrow{\mathcal{L}} N\left(0, \nabla f(\boldsymbol{\mu}, \mathrm{vec}\mathbf{\Sigma})^t \; V \; \nabla f(\boldsymbol{\mu}, \mathrm{vec}\mathbf{\Sigma})\right)$$

where ∇f denotes the vector of partial derivatives of f.

The following identities can be derived from properties of the vec operator and the Kronecker product (see for instance [14]):

$$\left.\frac{\partial f}{\partial \mathbf{T}_n}\right|_{\boldsymbol{\mu},\mathrm{vec}\mathbf{\Sigma}} = -\gamma^3\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$$
$$\left.\frac{\partial f}{\partial \mathrm{vec}\mathbf{C}_n}\right|_{\boldsymbol{\mu},\mathrm{vec}\mathbf{\Sigma}} = \frac{\gamma^3}{2}(\boldsymbol{\mu}^t \otimes \boldsymbol{\mu}^t)(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}),$$

which allows to obtain the following expression for the asymptotic variance of V_n:

$$\gamma^6\tau\boldsymbol{\mu}^t\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \frac{\gamma^6}{4}(\boldsymbol{\mu}^t \otimes \boldsymbol{\mu}^t)(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1})^t \; \mathbf{\Xi} \; (\boldsymbol{\mu}^t \otimes \boldsymbol{\mu}^t)(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}). \tag{25}$$

Since the asymptotic distribution of V_n depends on $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ only through γ, it suffices to compute expression (25) for any parameters satisfying $(\boldsymbol{\mu}^t\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^{-1/2} = \gamma$. Taking for instance $\boldsymbol{\mu}_0 = (1/\gamma)\mathbf{e}_1$ and $\mathbf{\Sigma}_0 = \mathbf{I}_p$ allows to conclude. □

Proof of Lemma 1: The noncentral F distribution function with degrees of freedom d_1 and d_2 and noncentrality parameter δ evaluated in a fixed x can be expressed as a function of $t = \delta/2$

$$G(t) = e^{-t}\sum_{j=0}^{+\infty}\frac{t^j}{j!}C_j \tag{26}$$

where $C_j := I\left(d_1x/(d_2 + d_1x)\middle|\, d_1/2 + j, d_2/2\right)$ with I the regularized incomplete Beta function. Since the series converges uniformly on any compact of $]0; +\infty[$, the function G is continuous in t. The idea is to examine the sign of the derivative of G with respect to t. As a power series in t with convergence domain $[0, +\infty[$, G can be differentiated easily to obtain

$$G'(t) = e^{-t}\sum_{j=0}^{+\infty}\frac{t^j}{j!}\left(C_{j+1} - C_j\right)$$

Using properties of the regularized incomplete Beta function, this derivative can be shown to be strictly negative, which concludes the proof. □

Proof of Proposition 3: Let $\mathcal{A}$ be the event

$$\left\{F^{-1}_{p,n-p,\delta}(\beta) \leq T \leq F^{-1}_{p,n-p,\delta}(1-\alpha+\beta)\right\},$$

where $T = \frac{n-p}{p}\frac{1}{V_n^2}$ is a random variable following a non-central F distribution with degrees of freedom p and $n-p$ and non-centrality parameter $\delta = \frac{n}{\gamma^2}$.

As a consequence of Lemma 1 and by definition of $I(V_n)$, the following events are equivalent

$$\mathcal{A} \cap \{V_n \leq C_{p,n-p}\} = \{\gamma \in I(V_n)\} \cap \{V_n \leq C_{p,n-p}\}.$$

The proof can then be concluded thanks to the inequalities

$$\begin{aligned} 1-\alpha = \mathbb{P}_\gamma[\mathcal{A}] &\geq \mathbb{P}_\gamma[\mathcal{A} \cap \{V_n \leq C_{p,n-p}\}] \\ &\geq \mathbb{P}_\gamma[\mathcal{A}] - \epsilon = 1-\alpha-\epsilon. \end{aligned}$$

□

Proof of Proposition 4: First, let us show that the function g is strictly increasing in γ. This function simplifies as follows, provided that $0 < p < n$,

$$g(\gamma) = \frac{\Gamma\left(\frac{n-p}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-p}{2}\right)} e^{-\frac{n}{2\gamma^2}} \sum_{j=0}^{+\infty} \left(\frac{n}{2\gamma^2}\right)^j \frac{1}{j!} \frac{\Gamma\left(\frac{p}{2}+j-\frac{1}{2}\right)}{\Gamma\left(\frac{p}{2}+j\right)}. \tag{27}$$

Since the series is uniformly convergent on any compact of $]0;+\infty[$, this function is continuous. The idea is to examine the sign of the derivative of g with respect to γ. As a power series in γ whose convergence domain is $]0;+\infty[$, the series in (27) can be easily differentiated to obtain

$$g'(\gamma) = \frac{\Gamma\left(\frac{n-p+1}{2}\right)}{\Gamma\left(\frac{n-p}{2}\right)} \frac{n}{\gamma^3} e^{-\frac{n}{2\gamma^2}} \sum_{j=0}^{+\infty} \left(\frac{n}{2\gamma^2}\right)^j \frac{1}{j!} \frac{\Gamma\left(\frac{p-1}{2}+j\right)}{\Gamma\left(\frac{p}{2}+j\right)} \left(1 - \frac{\frac{p+1}{2}+j}{\frac{p}{2}+j}\right),$$

which is strictly positive for every $\gamma \in]0;+\infty[$. Moreover, as

$$\begin{aligned} \lim_{\gamma \to +\infty} g(\gamma) &= \frac{\Gamma\left(\frac{n-p+1}{2}\right)\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{n-p}{2}\right)\Gamma\left(\frac{p}{2}\right)} \\ \lim_{\gamma \to 0^+} g(\gamma) &= 0 \end{aligned}$$

this concludes the proof. □

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