1 Problems

We consider a vector of features describing the behaviour of a process. So the data are given as a time series $X_1, X_2, \ldots, X_T$, $X_t \in \mathbb{R}^d$.

The process works non-perturbated if X_t lies in the neighbourhood of the centre $\mu_0 \in \mathbb{R}^d$, and the distribution of X_t keeps the same.

How to identify the out-of-control modus? What indicates abnormal behaviour of the process?

We distinguish several cases:

(1) change of $\mathbb{E}(X_t)$

The distance from the center μ_0 is an indicator.

(2) the marginal distributions of X_t changes.

(3.1) the copula changes

(3.2) The covariance matrix changes

Multivariate Models/Approaches

(a) elliptical distributions/star-shaped distributions

(b) copulas

(c) data depth

(d) nonparametric statistics

2 Ideas

2.1 Topic (1)(a)

• Measure the distance from the center μ_0

Mahalanobis distance and $X_t \sim \mathcal{N}(\mu_0, \Sigma)$

for single process value

$$T^2 = (X - \mu_0)^T \Sigma^{-1} (X - \mu_0)$$

Let us consider the average

$$\bar{X} = \frac{1}{n} \sum_{t=1}^{n} X_t.$$

$$T^2 = (\bar{X} - \mu_0)^T \Sigma^{-1} (\bar{X} - \mu_0)$$

The process is out-of-control if

$$(x - \mu_0)^T \Sigma^{-1} (x - \mu_0) \geq C \text{ with appropriate } C$$

• MCUSUM chart

k, h reference values

$$C_t = \|S_{t-1} + X_t - \mu_0\|$$

$$S_t = \begin{cases} 0 \ \text{ for } C_t \leq k \\ (S_{t-1} + X_t - \mu_0)\left(1 - \frac{k}{C_t}\right) \ \text{ for } C_t > k \end{cases}$$

out-of-control signal if $C_t - k > h$.

• MEWMA chart

$$\tilde{X}_t = \lambda X_t + (1-\lambda)\tilde{X}_{t-1}$$

(λ smoothing parameter) or

$$\tilde{X}_t = \Lambda X_t + (I-\Lambda)\tilde{X}_{t-1}$$

with diagonal matrix Λ

out-of-control signal if

$$T^2 = \left(\tilde{X}_t - \mu_0\right)^T \Sigma^{-1}\left(\tilde{X}_t - \mu_0\right) > h,$$

h reference value

References:

Petros Maravelakis, Sotiris Bersimis, John Panaretos, Stelios Psarakis (2002). Identifying the Out of Control Variable in a Multivariate Control Chart. *Communications in Statistics, A (Theory and Methods)* **31**, 2391-2408.

multivariate normal distribution, principal component analysis (for detecting variables caused the problem)

Aparisi, F., Champ, C. W., and Garcia-Diaz, J. C. (2004). A performance analysis of Hotelling■s T^2 control chart with supplementary runs rules, *Quality Engineering*, 16(3), pp. 359-368

T^2 statistic incorporating run rules

Robert L. Mason, Nola D. Tracy & John C. Young (1997) A Practical Approach for Interpreting Multivariate T2 Control Chart Signals. *Journal of Quality Technology*, 29, 396-406. DOI: 10.1080/00224065.1997.11979791

interpretation of the T^2 statistic

Robert L. Mason, Nola D. Tracy & John C. Young (1995) Decomposition of T2 for Multivariate Control Chart Interpretation, Journal of Quality Technology, 27:2, 99-108, DOI: 10.1080/00224065.1995.11979573

interpretation of the T^2 statistic

Runger, George C.; Alt, Frank B.; Montgomery, Douglas C. (1996). Contributors to a multivariate statistical process control chart signal. Communications in Statistics - Theory and Methods, 25(10), 2203–2213. doi:10.1080/03610929608831832

interpretation of the T^2 statistic

R.L. Mason, J.C. Young. (2002) Multivariate Statistical Process Control with Industrial Applications. SIAM. DOI:10.1137/1.9780898718461

Ronald B. Crosier (1988) Multivariate Generalizations of Cumulative Sum Quality-Control Schemes. *Technometrics*, Vol. 30, 291-303

MCUSUM chart

Bodnar, Schmid (2007) Surveillance of the mean behavior of multivariate time series. *Statistica Neerlandica* (2007) Vol. 61, 383-406

MCUSUM chart

Cynthia A. Lowry, William H. Woodall, Charles W. Champ and Steven E. Rigdon (1992) A Multivariate Exponentially Weighted Moving Average Control Chart. *Technometrics*, Vol. 34, 46-53

MEWMA chart

Cynthia A. Lowry, D. C. Montgomery (1995): A review of multivariate control charts, IIE Transactions, 27:6, 800-810

review multivariate charts

Lee, M. H. and Khoo, M. (2006). Optimal statistical design of a multivariate EWMA chart based on ARL and MRL, Communications in Statistics: Simulation and Computation, Vol. 35, 831-847.

MEWMA chart,

H. G. Kramer & L.V. Schmid (1997) Ewma charts for multivariate time series, Sequential Analysis: Design Methods and Applications, 16:2, 131-154, DOI: 10.1080/07474949708836378

MEWMA chart with a matrix as smoothing parameter

- Mahalanobis distance $\rightarrow$ elliptical distribution

density of X_t:

$$f(x) = g\left((x-\mu_0)^T \Sigma^{-1}(x-\mu_0)\right)$$

where g is the generator satisfying

$$\int_0^\infty r^{d/2-1} g(r)dr = \frac{\Gamma(d/2)}{\pi^{d/2}}.$$

The process is out-of-control if

$$(x-\mu_0)^T \Sigma^{-1}(x-\mu_0) \geq C \text{ with appropriate } C$$

Define $\tilde{\Sigma}$:=diag$(\sigma_1^2, \ldots, \sigma_d^2)$,

$$\tilde{X}_t = \tilde{\Sigma}^{-1/2}(X_t-\mu_0)\left\|\tilde{\Sigma}^{-1/2}(X_t-\mu_0)\right\|^{-1}.$$

Contribution of variable j to the out-of-control state ($e_j = (0, \ldots, 1_j, 0, \ldots, 0)^T$):

$$c_j = \left(\tilde{X}_t^T e_j\right)^2, \quad \sum_{i=1}^{d} c_j = 1.$$

$\triangleright$ consider average $\bar{X}$

$X_t \sim EC(\mu_0, \Sigma, g) \Longrightarrow$

$$\begin{aligned} \bar{X} &= \frac{1}{n}\sum_{t=1}^{n} X_t \sim EC(\mu_0, \Sigma, \bar{g}) \\ \Sigma^{-1/2}(\bar{X} - \mu_0) &\sim EC(0, I, \bar{g}) = S_d(\bar{g}) \text{ spherical} \end{aligned}$$

References:

Pairoj Khawsithiwong & Nihal Yatawara (2007) Monitoring Process Variability with Individual Measurements Following Elliptically Contoured Distributions, *Communications in Statistics - Simulation and Computation*, 36:3, 699-718, DOI: 10.1080/03610910701208833

MEWMA chart

- star-shaped distributions

density of X_t:

$$f(x) = (\kappa \det(D))^{-1} \; g\left(h_K(D^{-1}(x - \mu))\right)$$

where D =diag$(\lambda_1, \ldots, \lambda_d)$,

$$h_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$$

is the Minkowski functional of star body K.

The process is out-of-control if

$$h_K(D^{-1}(x - \mu)) \geq C$$

no references!

2.2 Topic (1)(b)

- (1)(b) copula C of X_t

density of X_t:

$$f(x) = c\left(F_1(x_1), \ldots, F_d(x_d)\right) f_1(x_1), \ldots, f_d(x_d)$$

where $x = (x_1, \ldots, x_d)^T$, F_j is the marginal distribution function of the j-th component, f_j its density. Define $\tilde{X}_t = (\hat{F}_{n1}(X_t^{(1)}), \ldots, \hat{F}_{nd}(X_t^{(d)})^T$, where $\hat{F}_{nj}$ is the empirical marginal distribution function of $X_t^{(j)}$.

The process is out-of-control if..

References

Krupskii, P., Harrou, F., Hering, A. S., & Sun, Y. (2019). Copulabased monitoring schemes for non-Gaussian multivariate processes. *Journal of Quality Technology*, 52(3), 219–234. doi:10.1080/00224065.2019.1571339

2.3 Topic (1)(c)

data items x_{ij}, data depth $D(x_{ij}, F_N)$, F_N empirical distribution function
R_{ij} ranks of the data depth values

$$\bar{R}_i = \frac{1}{N}\sum_{j=1}^{N} R_{ij}, \quad Z_i = \frac{\bar{R}_i - \frac{N+1}{2}}{\sqrt{\frac{(N-n)(N+1)}{12n}}}$$

The process is out-of-control if $Z > h_2$ or $Z < h_1$. h_1, h_2 are computed empirically.

References

Richard C. Bell (Jr.), L. Allison Jones-Farmer & Nedret Billor (2014) A Distribution-Free Multivariate Phase I Location Control Chart for Subgrouped Data from Elliptical Distributions, *Technometrics*, 56:4, 528-538, DOI: 10.1080/00401706.2013.879264

data depth

2.4 Topic (1)(d)

References

Chen, N., Zi, X., Zou, C. (2016) A Distribution-Free Multivariate Control Chart. *Technometrics* Vol. 58, 448-459

distribution-free method based on ranks, componentwise - dependence structure is not taken into account

2.5 Topic (3.1)(b)

- Comparison of copulas
  C_1 copula for $t = 1 \ldots T$, C_2 copula for $t = T+1, T+2 \ldots$
  measure for the deviation between C_1 and C_2 :

$$\int_{[0,1]^d} (C_1(u) - C_2(u))^2 \, du$$

References

Y. He and A. Kusiak (2018). Performance Assessment of Wind Turbines: Data-Derived Quantitative Metrics," in IEEE Transactions on Sustainable Energy, vol. 9, no. 1, pp. 65-73, Jan. 2018, doi: 10.1109/TSTE.2017.2715061.

change of copula parameters

Libin Liu, Xihui Liang, Ming J. Zuo, (2018) A dependence-based feature vector and its application on planetary gearbox fault classification, Journal of Sound and Vibration, Volume 431, 192-211. https://doi.org/10.1016/j.jsv.2018.06.015.

change of copula parameters

S. Gill, B. Stephen and S. Galloway, (2012) Wind Turbine Condition Assessment Through Power Curve Copula Modeling," in IEEE Transactions on Sustainable Energy, vol. 3, no. 1, pp. 94-101, Jan. 2012, doi: 10.1109/TSTE.2011.2167164.

change of copula in time

2.6 Topic (3.2)

References

Wenjuan Liang, Dongdong Xiang, Xiaolong Pu, Yan Li & Lingzhu Jin (2019) A robust multivariate sign control chart for detecting shifts in covariance matrix under the elliptical directions distributions, Quality Technology & Quantitative Management, 16:1, 113-127, DOI: 10.1080/16843703.2017.1372852

EWMA chart of the covariance matrix