Exercises

Exercice 1.

Compute the following matrix products, if possible :

1. $\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix}$

Exercice 2.

Find the set S of all solutions in x of the following inhomogeneous linear systems $Ax = b$, where A and b are defined as follows :

1. $$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$$

2. $$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 12 \\ 4 \end{bmatrix}$$

3.

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

Exercice 3.

Find all solutions in $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $Ax = 12x$ where,

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} x_i = 1$.

Exercice 4.

Consider the set G of 3×3 matrices defined as follows :

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \middle| x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ as the standard matrix multiplication.
Is $(G,\cdot)$ a group ? If yes, is it Abelian ? Justify your answer.

Exercice 5.

Which of the following sets are subspaces of $\mathbb{R}^3$?

1. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$
2. $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}\}$
3. Let γ be in $\mathbb{R}$
   $C = \{(\zeta_1, \zeta_2; \zeta_3) \in \mathbb{R}^3 | \ \zeta_1 - 2\zeta_2 + 3\zeta_3 = \gamma\}$
4. $D = \{(\zeta_1, \zeta_2; \zeta_3) \in \mathbb{R}^3 | \ \zeta_2 \ \in \mathbb{Z}\}$

Exercice 6.

Are the following sets of vectors linearly independent ?

1.
$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad , \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad , \quad x_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

2.
$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad , \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Exercice 7.

Let $F = \{(x, y, z) \in R^3 | x + y - z = 0\}$ and $G = \{(a - b, a + b, a - 3b) | a, b \in \mathbb{R}\}$

1. Show that F and G are subspaces of $\mathbb{R}^3$.
2. Calculate $F \cap G$ without resorting to any basis vector.
3. Find one basis for F and one for G, calculate $F \cap G$ using the basis vectors previously found and check your result with the previous question.

Exercice 8.

Are the following mappings linear ?

1.
$$\Phi : C^1 \to C^0$$
$$f \mapsto \Phi(f) = f',$$

where for $k \geq 1, C^k$ denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.

2.
$$\Phi : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \Phi(x) = cos(x)$$

3.
$$\Phi : \mathbb{R}^3 \to \mathbb{R}^2$$
$$x \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$$

4. Let θ be in $[0, 2\pi[$ and
$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$$
$$x \mapsto \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} x$$

Exercice 9.

Consider the linear mapping

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ 3x_1 + x_2 + x_3 \\ x_1 - 3x_3 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

— Find the transformation matrix A_Φ.

— Determine $\mathrm{rk}(A_\Phi)$.

— Compute the kernel and image of Φ. What are $\dim(\ker(\Phi))$ and $\dim(\mathrm{Im}(\Phi))$?