# Probability Distributions

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We can then associate a number to each element of $\Omega$, representing the number of heads.

## Introductory Example

In this way, we define a function denoted as $X$ from $\Omega$ to the set of numbers $\{0, 1, 2, 3\}$, and we can calculate the probability of each of these numbers by evaluating the probabilities of events $A_x$: "Having exactly $x$ heads, $x \in \{0, 1, 2, 3\}$.

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- $P(A_2) = P(X = x = 2) = P(PPF, PFP, FPP) = \frac{3}{8}$.
- $P(A_3) = P(X = x = 3) = P(PPP) = \frac{1}{8}$.

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In this case, we say that the function $X$ is a **random variable** defined on the sample space $\Omega$, which takes its values in the set $D_X = \{0, 1, 2, 3\}$.

# Random Variable

### Definition

A random variable $X$ is a numerical function associated with a random experiment. $X$ assigns a real value to each element of the sample space $\Omega$. It can be discrete or continuous.

# Support of a Random Variable or Domain of Definition

The study of each random variable $X$ begins with the determination and description of the set $X(\Omega)$ of values taken by $X$. This set is called the **support** of the random variable $X$ and is denoted as $D_X$ if it is discrete and $C_X$ if it is continuous.

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- We roll two dice, and we call $X$ the random variable equal to the sum of the points obtained; in this case, we have $X(\Omega) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
- We roll two dice, and we define $X$ as the random variable that gives the higher of the two points obtained; in this case, $X(\Omega) = \{1, 2, 3, 4, 5, 6\}$.

# Cumulative Distribution Function

**Definition**

Let $X$ be a random variable defined on a probability space $(\Omega, T, P)$. The **cumulative distribution function** of $X$, denoted as $F_X$, is defined on $\mathbb{R}$ by $F_X(x) = P(X \leq x)$.

# Properties of the Cumulative Distribution Function

The cumulative distribution function $F_X$ of a random variable $X$ has the following properties:

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1. $F_X$ is a non-decreasing function.
2. $F_X$ is right-continuous at every point in $\mathbb{R}$.
3. $\lim_{x \longrightarrow -\infty} F_X(x) = 0$ and $\lim_{x \longrightarrow +\infty} F_X(x) = 1$.

# Application of the Cumulative Distribution Function

Let $X$ be a random variable, and let $F_X$ be its cumulative distribution function. We have:

i) For any $a, b \in \mathbb{R}$ with $a < b$,
$P(a < X \leq b) = F_X(b) - F_X(a)$.

# Application of the Cumulative Distribution Function

Let $X$ be a random variable, and let $F_X$ be its cumulative distribution function. We have:

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ii) For any $a \in \mathbb{R}$, $P(X = a) = F_X(a) - F_X(a^-)$. Here, $F_X(a^-) = P(X < a)$.

# Probability Distribution of a Random Variable

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Let $X$ be a real random variable.

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Let $X$ be a real random variable.

The **probability distribution** of the variable $X$ is the assignment of probabilities for $X$ at every point in $\mathbb{R}$.

## Summary Example

Consider the random experiment of observing the upper faces of two dice. Let $X$ be the random variable that represents the total number of points obtained. Determine:

- the support of $X$,
- the probability distribution of $X$,
- the cumulative distribution function $F_X(x)$,

# Discrete Random Variable

**Definition**

A real random variable $X$ is called a **discrete random variable** if its support $D_X$ is countable, finite, or infinite.

## Examples

- If $X$ represents the value shown on the upper face of a 6-sided dice, $X$ is a discrete random variable with the support $D_X = \{1, 2, 3, 4, 5, 6\}$.

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- If $X$ represents the number of cars passing a bridge from 8 am to 9 am, then $D_X = \mathbb{N}$.

# Probability Distribution of a Discrete Random Variable or Probability Mass Function

## Probability Mass Function of a Discrete Random Variable

Let $X$ be a discrete random variable with the support $D_X$. The **probability mass function** of $X$ assigns probabilities to each value $x \in D_X$, denoted as $p(x) = P(X = x)$.

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- $\sum\limits_{x \in D_X} p(x) = 1$

# Cumulative Distribution Function of a Discrete Random Variable

Let $X$ be a discrete random variable with support $D_X$ and probability mass function $p(x)$.
The **cumulative distribution function** of $X$ is given by

$$F_X(x) = P(X \leq x) = \sum_{t \leq x} P(X = t)$$

# Probability Attached to an Interval

Let $X$ be a discrete random variable with probability distribution $p(x)$ and cumulative distribution function $F_X(x)$. Let $P(]a, b])$ denote the probability of the event $(a < X \leq b)$. We can then deduce the following properties:

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- $P(]b,+\infty[) = 1 - F_X(b)$.

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# Mathematical Expectation (Mean) of a Discrete Random Variable

# Mathematical Expectation

Let $X$ be a discrete random variable with support $D_X$ and probability mass function $p(x)$.
The **expectation** of $X$ ($E(X)$ or $\mu$) is defined as:

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i p(x_i)$$

# Variance of a Discrete Random Variable

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Let $X$ be a discrete random variable with support $D_X$, probability mass function $p(x)$, and expectation $E(X)$. The **variance** of $X$ ($V(X) \in \mathbb{R}^+$) is defined as:

$$V(X) = \sum_{x \in D_X} (x - E(X))^2 p(x)$$

In this case, the **standard deviation** of $X$ is defined as:

$$\sigma(X) = \sqrt{V(X)}$$

# Variance of a Discrete Random Variable

The variance can also be expressed as:

$$V(X) = E(X^2) - [E(X)]^2.$$

# Properties of $E(X)$ and $V(X)$

Let $X$ be a random variable, and $a, b$ are two real numbers. We have:

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- $E(aX + b) = aE(X) + b$.
- $V(aX + b) = a^2 V(X)$.
- $V(X) = E(X(X-1)) + E(X) - [E(X)]^2$.

# Common Discrete Probability Distributions

In real-life situations, several random phenomena can be modeled using the same probability distribution.

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To do this, you need to identify the probabilistic structure that applies to a given problem. In this course, we introduce the most commonly used models.

# Bernoulli Distribution

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# Bernoulli Distribution

A Bernoulli trial is an experiment with two possible outcomes, often referred to as "success (1)" and "failure (0)," where $\Omega = \{S, F\}$.
We define the Bernoulli variable $X$ with support $D_X = \{1, 0\}$. Let's assume that $P(X = 1) = p$, so $P(X = 0) = q = 1 - p$. The probability mass function is then given by:

# Bernoulli Distribution

$$p(x) = P(X = x) = \begin{cases} p & \text{if } x = 1 \\ q = 1 - p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

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We say that the random variable $X$ follows the Bernoulli distribution with parameter $p$, denoted as $X \sim B(p)$.

# Bernoulli Distribution: Mean and Variance

Let $X$ be a random variable following the Bernoulli distribution $B(p)$. Then we have:

$$E(X) = p \text{ and } V(X) = pq = p(1-p).$$

# Binomial Distribution

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We consider a Bernoulli trial repeated $n$ times independently. So, the sample space is $\Omega = \{S, F\}^n$, and an element of $\Omega$ is an $n$-tuple composed of $S$ and $F$.
We define $X$ as the random variable counting the **number of successes** obtained in the $n$ trials.

# Binomial Distribution

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We then say that the random variable $X$ follows the binomial distribution with parameters $n$ and $p$, denoted as $X \sim \mathfrak{B}(n, p)$.

# Binomial Distribution: Example Application

We estimate that there is a 40% probability that a student has COVID-19.

This Wednesday, there are 50 students in the classroom, and we want to know what is the probability that the number of students with COVID-19 is equal to 5?

What is the average number of students affected?

## Binomial Distribution: Mean and Variance

So, if $X$ follows the binomial distribution $\mathfrak{B}(n, p)$, we have:

$$E(X) = np \text{ and } V(X) = np(1-p).$$

# Geometric Distribution

# Geometric Distribution

**Definition**

We repeat a Bernoulli trial infinitely many times, where the probability of success is $p$, and the probability of failure is $1 - p$. Let $X$ be the random variable that counts the number of trials needed to obtain the first success.

# Geometric Distribution

## Definition

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We then say that the random variable $X$ follows the *geometric distribution* with parameter $p$, denoted as $X \sim \mathcal{G}(p)$.

## Geometric Distribution: Example

In a game consisting of a large number of independent trials, if the player fails in a single trial, they are eliminated from the game.
Assuming that the probability of success for all trials is the same and is equal to 0.6, what is the probability that the player is eliminated during the third trial?

## Geometric Distribution: Mean and Variance

If $X$ is a random variable following the geometric distribution $\mathcal{G}(p)$ with $p > 0$, then:

$$E(X) = \frac{1}{p} \text{ and } V(X) = \frac{1-p}{p^2}.$$

# Hypergeometric Distribution

# Hypergeometric Distribution

## Definition

Consider an urn containing $M$ balls, of which $B$ are white and $M - B = N$ are black. We draw $k$ balls **in a single draw**, and we define the random variable $X$ as the number of white balls drawn among the $k$.
In this case, we have $D_X \subset [[0, k]]$, specifically
$D_X = [[\max(0, k - N), \min(k, B)]]$.

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$$p(x) = P(X = x) = \begin{cases} \dfrac{\binom{B}{x} \times \binom{N}{k-x}}{\binom{M}{k}} & \text{if } x \in D_X \\ 0 & \text{otherwise} \end{cases}$$

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We say that the random variable $X$ follows the *hypergeometric distribution* with parameters $(M, B, k)$, denoted as $X \sim \mathcal{H}(M, B, k)$.

# Hypergeometric Distribution: Example

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In a batch of 10 coins, 4 of them are defective. We randomly select 3 coins, and we want to calculate the probability of having 2 defective coins among those chosen.

By denoting $X$ as the number of defective coins among the 3 selected, the support of $X$ is then $D = [0, 3]$. The probability mass function of $X$ is given by

$$p(x) = P(X = x) = \begin{cases} \dfrac{\binom{4}{x} \times \binom{6}{3-x}}{\binom{10}{3}} & \text{if } x \in D_X \\ 0 & \text{otherwise} \end{cases}$$

# Hypergeometric Distribution: Example

We want to calculate $P(X = 2) = \dfrac{\binom{4}{2} \times \binom{6}{1}}{\binom{10}{3}}$.

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$$p(x) = P(X = x) = \begin{cases} \dfrac{\binom{4}{x} \times \binom{6}{7-x}}{\binom{10}{7}} & \text{if } x \in D_X \\ 0 & \text{otherwise} \end{cases}$$

### Hypergeometric Distribution: Mean and Variance

So, if $X$ is a random variable following the hypergeometric distribution $\mathcal{H}(M, B, k)$, we have:

$$E(X) = k\frac{B}{M} \text{ and } V(X) = k\frac{B \times N \times (M-k)}{M^2 \times (M-1)}.$$

# Poisson Distribution

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## Definition

We say that a random variable $X$ follows *the Poisson distribution* with parameter $\lambda > 0$ if the support is $D_X = \mathbb{N}$, and the probability distribution is given by:

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We say that a random variable $X$ follows *the Poisson distribution* with parameter $\lambda > 0$ if the support is $D_X = \mathbb{N}$, and the probability distribution is given by:

$$p(x) = P(X = x) = \begin{cases} e^{-\lambda} \dfrac{\lambda^x}{x!} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

We denote this as $X \sim \mathcal{P}(\lambda)$.

# Poisson Distribution

In practice, the Poisson distribution is used to approximate certain phenomena, such as:

- The number of phone calls received during a time interval.
- The number of cars passing over a bridge in a given time interval.
- The number of arrivals at a counter within a time interval.

## Poisson Distribution: Example

The average number of phone calls received in one minute is 1.8.

What is the probability of receiving 2 calls between 10:54 AM and 10:55 AM?

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# Poisson Distribution: Example

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$$p(x) = P(X = x) = \begin{cases} \mathrm{e}^{-1.8}\,\dfrac{1.8^x}{x!} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

# Poisson Distribution: Example

So, to calculate the probability of receiving 2 calls, we will compute $p(2) = P(X = 2) = \mathrm{e}^{-1.8}\,\dfrac{1.8^2}{2!} \approx 0.134.$

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If we now want to know the probability of receiving a call between 10:56 AM and 11:00 AM (4 minutes in total),

# Poisson Distribution: Example

So, to calculate the probability of receiving 2 calls, we will compute $p(2) = P(X = 2) = e^{-1.8}\dfrac{1.8^2}{2!} \approx 0.134$.
If we now want to know the probability of receiving a call between 10:56 AM and 11:00 AM (4 minutes in total), we need to consider a random variable $Y$ that follows a Poisson distribution with a parameter of $1.8 \times 4 = 7.2$.

### Poisson Distribution: Mean and Variance

If $X$ follows a Poisson distribution with parameter $\lambda$, then $E(X) = V(X) = \lambda$.

# Absolutely Continuous Random Variable

# Absolutely Continuous Random Variable

**Definition**

*A random variable $X$ is called absolutely continuous if its cumulative distribution function $F_X(x)$ is continuous (from the right and left) and has right and left derivatives.*
*In this case, the support of $X$ is a union of intervals denoted as $C_X$.*

## Examples

- Consider a circular target with a radius of $R_0 > 0$. If shots hit the target, and if $X$ represents the distance from the point of impact to the center of the target, we can consider $X$ as a random variable with a support of $[0, R_0]$.
- Let $X$ be the random variable representing the lifespan of a certain type of engine. In this case, $X$ is an absolutely continuous random variable over $C = \mathbb{R}^+$.

# Important Note

Consider $X$ as an absolutely continuous random variable and $F_X(x)$ as its cumulative distribution function.
According to the properties of the cumulative distribution function, we have

$$p(a) = P(X = a) = F_X(a) - F_X(a^-),$$

where $F_X(a^-) = P(X < a)$ for any point $a$.

# Important Note

However, since $X$ is an absolutely continuous variable, $F_X(x)$ is a continuous function on both the right and left, which implies

$$F_X(a^-) = P(X < a) = \lim_{x \to a^-} P(X \leq a) = F_X(a),$$

Consequently, we have

$$p(a) = P(X = a) = 0.$$

# Probability Density of an Absolutely Continuous Random Variable

We have seen that in the case of an absolutely continuous random variable, the probability attached to a point is zero. In this case, we define the concept of a *probability density function*, denoted as $f(x)$, which is obtained by differentiating the cumulative distribution function of $X$.

Consider $X$ as an absolutely continuous random variable with a cumulative distribution function $F_X(x)$. We define $f(x)$, the probability density of $X$, if and only if $f(x)$ satisfies the following conditions:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{+\infty} f(t)dt = 1$.
3. $F_X$ is related to $f$ by the equation $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$.

# Probability Attached to an Interval

When the random variable $X$ is absolutely continuous, and due to the fact that for any $a \in \mathbb{R}$, we have $p(a) = 0$, and thus $P([a,b]) = P((a,b)) = F_X(b) - F_X(a)$.
This allows us to write the important relation:

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f(t)dt - \int_{-\infty}^{a} f(t)dt = \int_{a}^{b} f(t)d$$

# Expectation and Variance of an Absolutely Continuous Random Variable

# Expectation and Variance of an Absolutely Continuous Random Variable

The expectation of $X$ is given by

$$E(X) = \int_{-\infty}^{+\infty} tf(t)dt.$$

If $X$ has an expectation, the variance of $X$ is calculated as follows:

$$V(X) = \int_{-\infty}^{+\infty} (t - E(X))^2 f(t)dt = E(X^2) - [E(X)]^2.$$

# Common Continuous Distributions

# Uniform Distribution

**Definition**

*A random variable $X$ is said to follow a uniform distribution on the interval $[a, b]$ if its probability density function is constant over this interval. We denote it as $X \sim \mathcal{U}[a, b]$.*

## Definition

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*The probability density function of the uniform distribution is written as*

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

# Uniform Distribution: Cumulative Distribution Function

The cumulative distribution function of the uniform random variable is calculated as

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So we have

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } x \in [a,b] \\ 1 & \text{if } x \geq 1 \end{cases}$$

## Uniform Distribution: Expectation and Variance

Let $X$ be a random variable following a uniform distribution on the interval $[a, b]$, then we have

$$E(X) = \frac{a+b}{2} \text{ and } V(X) = \frac{(b-a)^2}{12}.$$

# Uniform Distribution: Example

The waiting time in the emergency room of a given hospital can be up to three hours. We want to calculate the probability of waiting for at most one hour.

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# Uniform Distribution: Example

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We define a random variable $X$ that represents the waiting time in the emergency room. $X$ then follows a uniform distribution on the interval $[0, 3]$.

We are looking to calculate $P(X \leq 1) = F_X(1) = \dfrac{1}{3}$.

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If we now want to calculate the probability that the waiting time is between 15 and 30 minutes,

## Uniform Distribution: Example

If we now want to calculate the probability that the waiting time is between 15 and 30 minutes, we calculate
$P(\frac{1}{4} \leq X \leq \frac{1}{2}) = F_X(\frac{1}{2}) - F_X(\frac{1}{4}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{3} dx = \frac{1}{12}$.

# Exponential Distribution

### Definition

*Let X be a continuous random variable with a probability density function defined as:*

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x} & \textit{if } x \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

*Where $\lambda > 0$. We then say that X follows the exponential distribution with parameter $\lambda$, and it is denoted as $\mathcal{E}(\lambda)$.*

# Exponential Distribution: Cumulative Distribution Function

The cumulative distribution function of the exponential random variable $X$ is given by

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} \lambda\, \mathrm{e}^{-\lambda t}\, dt = \lambda \left[ \frac{-1}{\lambda}\, \mathrm{e}^{-\lambda t} \right]_0^x = 1 - \mathrm{e}^{-\lambda x}\,.$$

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So we have

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

# Exponential Distribution: Expectation and Variance

Let $X$ be a random variable following the exponential distribution, then

$$E(X) = \frac{1}{\lambda} \text{ and } V(X) = \frac{1}{\lambda^2}.$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_{0}^{+\infty} \lambda x \, \mathrm{e}^{-\lambda x} \, dx$$

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{+\infty} x f(x) dx = \int_{0}^{+\infty} \lambda x \,\mathrm{e}^{-\lambda x}\, dx \\
&= \lambda \left( \left[ -\frac{1}{\lambda} x \,\mathrm{e}^{-\lambda x} \right]_0^{+\infty} + \frac{1}{\lambda} \left[ -\frac{1}{\lambda} \,\mathrm{e}^{-\lambda x} \right]_0^{+\infty} \right)
\end{aligned}
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&= \frac{1}{\lambda}.
\end{aligned}
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$$= \frac{2}{\lambda^2}.$$

So $V(X) = \dfrac{2}{\lambda^2} - \dfrac{1}{\lambda^2} = \dfrac{1}{\lambda^2}$.

# Remark

The exponential distribution is one of the distributions used to model random variables describing the **lack of memory or no aging** phenomenon.

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For example, for a crystal glass, the probability of it breaking within five years does not depend on its date of manufacture.
For such phenomena, the probability that the lifetime of an object is more than $t + s$ given that the object is in good condition at time $t$ does not depend on $t$.

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Because the event "X is greater than $t + s$" is included in the event "X is greater than $t$".

Therefore

$$P(X \geq s + t / X \geq t) = \frac{P(\text{''}X \geq s + t\text{''} \cap \text{''}X \geq t\text{''})}{P(X \geq t)}$$

Therefore

$$P(X \geq s+t/X \geq t) = \frac{P(\text{"}X \geq s+t\text{"} \cap \text{"}X \geq t\text{"})}{P(X \geq t)}$$
$$= \frac{P(X \geq s+t)}{P(X \geq t)}$$

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&= \mathrm{e}^{-\lambda(s+t-t)} = 1-F_X(s) \\
&= P(X \geq s).
\end{aligned}
$$

## Example

The lifetime of a component is a random variable, expressed in days, following an exponential distribution with a rate parameter of 0.004. We want to calculate the probability that the component's lifetime exceeds three hundred days.

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$P(X \geq 300) = 1 - F_X(300) = 1 - (1 - \mathrm{e}^{-0.004 \times 300}) \approx 0.301.$

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If we now want to calculate the probability that the lifetime is at least 400 given that it has exceeded 300, we then calculate the probability $P(X \geq 400/X \geq 300) = P(X \geq 300 + 100/X \geq 300) = P(X \geq 100) = 1 - F_X(100) = e^{-0.004 \times 100} \approx 0.02.$

# Gamma Distribution

## Definition

*We say that a real random variable $X$ follows the Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, denoted as $\Gamma(\alpha, \lambda)$, if its probability density function is defined as:*

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} & \text{if } x \in \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases}$$

*Where $\Gamma(\alpha)$, called the Gamma function, is defined as*

$$\Gamma(\alpha) = \int_0^{+\infty} \mathrm{e}^{-x} x^{\alpha-1} dx.$$

# Properties of the Gamma Function

The Gamma function $\Gamma(\alpha)$ has the following properties:

- $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi},$

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- $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$,
- if $\alpha \in \mathbb{N}^*, \Gamma(\alpha) = (\alpha - 1)!$,
- if $\alpha > 1, \Gamma(\alpha) = \alpha\Gamma(\alpha - 1)$.

## Remarks

- The Gamma distribution is also used to describe certain lifetime phenomena. Note that for $\alpha = 1$, it reduces to the exponential distribution with rate parameter $\lambda$.

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# Remarks

- The Gamma distribution is also used to describe certain lifetime phenomena. Note that for $\alpha = 1$, it reduces to the exponential distribution with rate parameter $\lambda$.
- When $\alpha \in \mathbb{N}^*$, this distribution is called the Erlang distribution with parameters $\alpha$ and $\lambda$.
- In the special case of the Gamma distribution with $\alpha = \frac{n}{2}$ and $\lambda = \frac{1}{2}$, this is known as the *Chi-squared distribution* with $n$ degrees of freedom, denoted as $\chi_n^2$. This distribution is commonly used in goodness-of-fit tests and constructing confidence intervals.

# Expectation and Variance

Let $X \sim \Gamma(\alpha, \lambda)$, then

$$E(X) = \frac{\alpha}{\lambda} \text{ and } V(X) = \frac{\alpha}{\lambda^2}.$$

# Cumulative Distribution Function

In the general case, it is difficult to explicitly express the cumulative distribution function of the Gamma distribution. Its values are given in tables or computed using specialized software or programming environments (MatLab, Scilab, Mathematica).

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In the general case, it is difficult to explicitly express the cumulative distribution function of the Gamma distribution. Its values are given in tables or computed using specialized software or programming environments (MatLab, Scilab, Mathematica).

But in the case where $\alpha \in \mathbb{N}^*$, the cumulative distribution function of the Erlang distribution is given by

$$F_X(x) = 1 - \sum_{n=0}^{\alpha-1} \mathrm{e}^{-\lambda x} \frac{(\lambda x)^n}{n!}.$$

# Loi normale

## Definition

*We say that $X$ follows the Laplace-Gaussian or normal distribution with parameters $m$ and $\sigma$ (where $m \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$) if $X$ is an absolutely continuous random variable with the probability density function given by*

**Definition**

*We say that $X$ follows the Laplace-Gaussian or normal distribution with parameters $m$ and $\sigma$ (where $m \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$) if $X$ is an absolutely continuous random variable with the probability density function given by*

$$\forall x \in \mathbb{R}, \varphi_{m,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{\dfrac{-(x-m)^2}{2\sigma^2}}.$$

*We denote this as $X \sim \mathcal{N}(m, \sigma)$.*

## Study of the probability density $\varphi_{m,\sigma}(x)$.

We can observe that:

- $\varphi_{m,\sigma}(m+x) = \varphi_{m,\sigma}(m-x)$ (the graphical representation of $\varphi_{m,\sigma}(x)$ is symmetric with respect to the line $x = m$).

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- $\lim\limits_{x \longrightarrow -\infty} \varphi_{m,\sigma}(x) = \lim\limits_{x \longrightarrow +\infty} \varphi_{m,\sigma}(x) = 0.$
- For all $x \in \mathbb{R}$, $\varphi'_{m,\sigma}(x) = -\dfrac{x-m}{\sigma^2} \dfrac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\dfrac{-(x-m)^2}{2\sigma^2}}$.

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- For all $x \in \mathbb{R}$,

$$\varphi''_{m,\sigma}(x) = \frac{(x-m)^2 - \sigma^2}{\sigma^4} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\dfrac{-(x-m)^2}{2\sigma^2}}.$$

We have the variation table :

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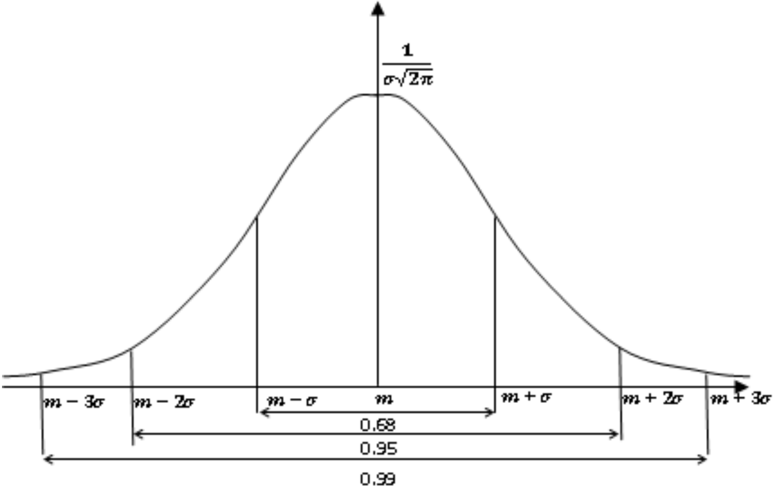
| $x$ | $-\infty$ | | $m-\sigma$ | | $m$ | | $m+\sigma$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $\varphi''_{m,\sigma}(x)$ | | $+$ | $0$ | $-$ | | $-$ | $0$ | $+$ | |
| $\varphi'_{m,\sigma}(x)$ | | | $+$ | | $0$ | | $-$ | | |
| $\varphi_{m,\sigma}(x)$ | $0$ | $\nearrow$ | | | $\dfrac{1}{\sigma\sqrt{2\pi}}$ | | | $\searrow$ | $0$ |

And the graphical representation:



# Cumulative Distribution Function.

It is not easy to explicitly express the cumulative distribution function denoted by $\Phi_{m,\sigma}$, defined as

$$\Phi_{m,\sigma} = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \, \mathrm{e}^{\frac{-(x-m)^2}{2\sigma^2}} \, dx$$

using standard functions.

# Expectation and Variance

Let $X \sim \mathcal{N}(m, \sigma)$. We have $E(X) = m$ and $V(X) = \sigma^2$.

# Standard Normal Distribution

## Definition

*A random variable X follows a standard normal distribution if its probability density is given by*

$$f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-x^2}{2}}, \forall x \in \mathbb{R}.$$

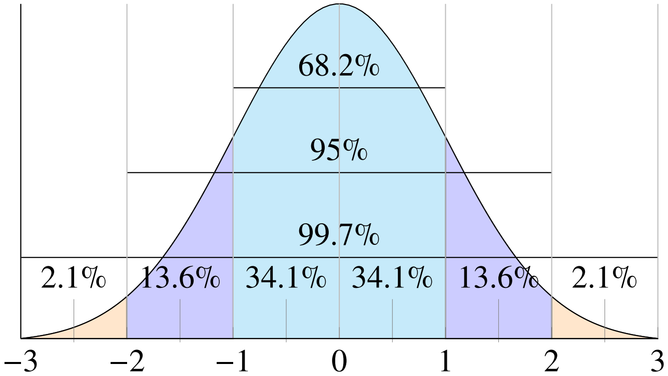
*and we denote this as* $X \sim \mathcal{N}(0,1)$.

We then state the important theorem:

**Théorème**

*Let $X$ be a random variable following the normal distribution with parameters $m$ and $\sigma$, and let $Y = \dfrac{X - m}{\sigma}$. Then we have*

$$X \sim \mathcal{N}(m, \sigma) \Longleftrightarrow Y \sim \mathcal{N}(0, 1).$$


68.2%
95%
99.7%
2.1%
13.6%
34.1%
34.1%
13.6%
2.1%
−3
−2
−1
0
1
2
3

# Expectation and Variance.

Let $X \sim \mathcal{N}(0, 1)$. Then $E(X) = 0$ and $V(X) = 1$.

# Cumulative Distribution Function

Let $X \sim \mathcal{N}(0, 1)$. The cumulative distribution function of $X$, denoted by $\Pi(x) = P(X \leq x)$, cannot be computed using standard integration methods. These probabilities are obtained through approximation and are available in tables based on the value of $x$.

# Table for computing $\Pi(x) = P(X \leq x)$.

| | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

# Properties of $\Pi(x)$.

Let $X \sim \mathcal{N}(0, 1)$, and let $\Pi(x)$ be its cumulative distribution function. Then:

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2. For all $x \in \mathbb{R}^+$, $P(|X| \leq x) = 2\Pi(x) - 1$ and $P(|X| \geq x) = 2(1 - \Pi(x))$.

# How to use the $\Pi(x)$ Table.

Let $X \sim \mathcal{N}(0,1)$. We want to calculate $P(X \leq 1.15), P(X \leq 0.10), P(X \leq -2.53), P(|X| \leq 1)$. Using the table of $\Pi(x)$ and its properties, we have: $P(X \leq 1.15)$ :

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| | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |

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$$P(X \leq 1.15) = \Pi(1.15) = 0.8749 \approx 0.875.$$

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|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |

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|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |

$$P(X \leq -2.53) = 1 - \Pi(2.53) \approx 0.0057.$$

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|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |

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$$P(|X| \leq 1) = P(-1 \leq X \leq 1) = 2\Pi(1) - 1 \approx 0.683.$$

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We obtain a smaller interval containing $\alpha$, which is $\Pi^{-1}(A) \leq \alpha \leq \Pi^{-1}(B)$.
So, we can approximate the value of $\alpha$ based on its position within the interval.

# Example

We want to find $\alpha$ for which we have $P(|X| \leq \alpha) = 0.05$.
Since $P(|X| \leq \alpha) = 2\Pi(\alpha) - 1 = 0.05$, we have
$\Pi(\alpha) = \frac{1}{2}(0.05 + 1) = 0.525$.

# Example

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Since $P(|X| \leq \alpha) = 2\Pi(\alpha) - 1 = 0.05$, we have
$\Pi(\alpha) = \frac{1}{2}(0.05 + 1) = 0.525$.
In the table, the smallest interval containing this value is $[0.5239; 0.5279]$,

| | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |

which corresponds to the interval of values for $x$: $[0.06; 0.07]$.

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which corresponds to the interval of values for $x$: $[0.06; 0.07]$.
Estimating the position of the value 0.525 within the interval $[0.5239; 0.5279]$ to within $10^{-3}$, we obtain the corresponding value of $\alpha$ within the interval $[0.06; 0.07]$, namely, $\alpha = 0.0625$.

## Calculating the Probability of an Interval of a Normal Random Variable with Parameters $m$ and $\sigma$

Let $X$ be a random variable that follows the normal distribution with parameters $m$ and $\sigma$.
The probability of an interval $P(a \leq X \leq b)$ can be calculated using the cumulative distribution function of the standard normal distribution $\Pi(x)$ by defining $Y = \frac{X-m}{\sigma}$. We then have:

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The probability of an interval $P(a \leq X \leq b)$ can be calculated using the cumulative distribution function of the standard normal distribution $\Pi(x)$ by defining $Y = \frac{X-m}{\sigma}$. We then have:

$$P(a \leq X \leq b) = P\left(\frac{a-m}{\sigma} \leq Y \leq \frac{b-m}{\sigma}\right) = \Pi\left(\frac{b-m}{\sigma}\right) - \Pi\left(\frac{a-m}{\sigma}\right).$$

# Example

Let $X \sim \mathcal{N}(-8, 5)$. We want to calculate the probability $P(-9.5 \leq X \leq -7)$. To do this, we set $Y = \frac{X+8}{5}$, and we have

$$\begin{aligned} P(-9.5 \leq X \leq -7) &= P\left(\frac{-9+8}{5} \leq Y \leq \frac{-7+8}{5}\right) \\ &= \Pi(0.2) - \Pi(-0.3) = \Pi(0.2) - [1 - \Pi(0.3)] \\ &\approx 0.198. \end{aligned}$$