Math 3001 Analysis 1
Homework Set 5

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**Problem 3.** Let $(a_n)_n \in \mathbb{N}$ be a sequence of nonzero real numbers. Assume that there is a real number $r$ with $0 < r < 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r \qquad \text{for all } n \in \mathbb{N}$$

Then the series $\sum_{k=0}^{\infty} a_k$ converges.

a) Prove by induction on $n$ that $|a_n| \leq r^n|a_0|$ for all $n \in \mathbb{N}$.

b) Use (a) to show that $\lim_{n\to\infty} a_n = 0$(ration test for sequences).

a) *Proof.* Note that by given

$$|a_{n+1}| \leq r|a_n| \qquad (*)$$

To prove the claim by induction, we seek to show $P(n) := |a_n| \leq r^n|a_0|$. We first show the base case, $P(0)$:

$$\begin{aligned} |a_0| &\leq r^0|a_0| \\ &= |a_0| \end{aligned}$$

Take as the induction hypothesis $P(k) : a_k \leq r^k|a_0|$. We seek to show that $P(k)$ implies $P(k+1) : a_{k+1} \leq r^{k+1}|a_0|$.

$$\begin{aligned} r^{k+1}|a_0| &= r \cdot r^k|a_0| \\ &\geq r|a_k| && \text{(Induction hypothesis)} \\ &\geq |a_{k+1}| && \text{(By *)} \end{aligned}$$

Thus we have $P(k) \implies P(k+1)$ and by induction $P(n)$. $\square$

b) *Proof.* Note that $(r^n)_{n\in\mathbb{N}}$ is strictly decreasing for $0 < r < 1$:

$$\begin{aligned} r^n - r^{n+1} &= r^n(1-r) \\ &> 0 \end{aligned}$$

Also, $r^n > 0$. So $r^n \to 0$. Then:

$$\begin{aligned}
|a_n| &\leq r^n |a_0| \\
\lim_{n\to\infty} |a_n| &\leq \lim_{n\to\infty} r^n |a_0| \\
&= |a_0| \lim_{n\to\infty} r^n \\
&= |a_0| \cdot 0 \\
&= 0
\end{aligned}$$

Since $|a_n|$ are non-negative, $\lim_{n\to\infty} a_0 = 0$. □

**Problem 4.** Consider the following two sequences and determine whether they converge or not. In the convergent case determine their limit.

$$\text{(a) } x_n = \frac{13n^3 + 5n^2 + 3n + 8}{13 + 5n + 3n^2 + 8n^3}, \quad \text{(b) } y_n = \frac{2^n}{(n+2)!}$$

a) *Proof.*

$$\frac{13n^3 + 5n^2 + 3n + 8}{13 + 5n + 3n^2 + 8n^3} = \frac{13n^3 + 5n^2 + 3n + 8}{13 + 5n + 3n^2 + 8n^3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$
$$\frac{13 + \frac{5}{n} + \frac{3}{n^2} + \frac{8}{n^3}}{\frac{13}{n^3} + \frac{5}{n^2} + \frac{3}{n} + 8}$$

Since the limit of sums is the sum of the limits, and the limit of quotients is the quotients of the limits, we can consider each part separately. Starting with the numerator:

$$\begin{aligned}
\lim_{n\to\infty} \left(13 + \frac{5}{n} + \frac{3}{n^2} + \frac{8}{n^3}\right) &= \lim_{n\to\infty} 13 + \lim_{n\to\infty} \frac{5}{n} + \lim_{n\to\infty} \frac{3}{n^2} + \lim_{n\to\infty} \frac{8}{n^3} \\
&= 13
\end{aligned}$$

And the denominator:

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{13}{n^3} + \frac{5}{n^2} + \frac{3}{n} + 8\right) &= \lim_{n\to\infty} \frac{13}{n^3} + \lim_{n\to\infty} \frac{5}{n^2} + \lim_{n\to\infty} \frac{3}{n} + \lim_{n\to\infty} 8 \\
&= 8
\end{aligned}$$

Combining them with the quotient rule for limits, we have:

$$\lim_{n\to\infty} \frac{13n^3 + 5n^2 + 3n + 8}{13 + 5n + 3n^2 + 8n^3} = \lim_{n\to\infty} \frac{13 + \frac{5}{n} + \frac{3}{n^2} + \frac{8}{n^3}}{\frac{13}{n^3} + \frac{5}{n^2} + \frac{3}{n} + 8} = \frac{13}{8}$$

□

b) *Proof.*

$$\begin{aligned}
\left|\frac{y_{n+1}}{y_n}\right| &= \left|\frac{\frac{2^{n+1}}{((n+1)+2)!}}{\frac{2^n}{(n+2)!}}\right| \\
&= \left|\frac{2^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n}\right| \\
&= \left|\frac{2}{n+3}\right| \\
&< \frac{2}{3} \qquad \forall n \in \mathbb{N}
\end{aligned}$$

So by the ratio test for sequences, $(y_n)_{n\in\mathbb{N}}$ converges. Now choose $N \in \mathbb{N}$ such that

$$2^n \leq n! \qquad \forall n > N$$

. Specifically, take $N = 4$. Then, for all $n \geq N$

$$\begin{aligned}
\frac{2^n}{(n+2)!} &\leq \frac{n!}{(n+2)!} \\
&= \frac{1}{(n+1)(n+2)} \\
&< \frac{1}{n^2} < \frac{1}{n} \\
&\to 0
\end{aligned}$$

□

**Problem 5.** Use the comparison, root, or ratio test to determine whether the following series converge or diverge:

$$\text{a)} \sum_{n=0}^{\infty} \frac{2^n n^3}{3^n} \quad \text{b)} \sum_{n=0}^{\infty} \frac{5^n}{3^n(n^4+2)} \quad \text{c)} \sum_{n=1}^{\infty} \left(\frac{1}{5}+\frac{1}{n}\right)^n \quad \text{d)} \sum_{n=0}^{\infty} \frac{1}{n^n}$$

a) *Proof.* By ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)^3}{3^{n+1}}}{\frac{2^n n^3}{3^n}} &= \lim_{n\to\infty} \frac{2^{n+1}(n+1)^3}{3^{n+1}} \frac{3^n}{2^n n^3} \\
&= \lim_{n\to\infty} \frac{2(n^3+3n^2+3n+1)}{3n^3} \\
&= \lim_{n\to\infty} \frac{2n^3+6n^2+6n+2}{3n^3} \\
&= \lim_{n\to\infty} \frac{2n^3}{3n^3} + \lim_{n\to\infty} \frac{6n^2}{3n^3} + \lim_{n\to\infty} \frac{6n}{3n^3} + \lim_{n\to\infty} \frac{2}{3n^3} \\
&= \lim_{n\to\infty} \frac{2}{3} + \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{2}{n^2} + \lim_{n\to\infty} \frac{2}{3n^3} \\
&= \frac{2}{3} + 0 + 0 + 0 = \frac{2}{3}
\end{aligned}$$

The series converges. □

b) *Proof.* By ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{5^{n+1}}{3^{n+1}((n+1)^4+2)}}{\frac{5^n}{3^n(n^4+2)}} &= \lim_{n\to\infty} \frac{5^{n+1}}{3^{n+1}\left((n+1)^4+2\right)} \frac{3^n\left(n^4+2\right)}{5^n} \\
&= \lim_{n\to\infty} \frac{5\left(n^4+2\right)}{3\left(n^4+4n^3+6n^2+4n+1+2\right)} \\
&= \lim_{n\to\infty} \frac{5}{3} \cdot \frac{n^4+2}{n^4+4n^3+6n^2+4n+3} \\
&= \lim_{n\to\infty} \frac{5}{3} \cdot \frac{1+\frac{2}{n^4}}{1+\frac{4}{n}+\frac{6}{n^2}+\frac{4}{n^3}+\frac{3}{n^4}} \\
&> 1 \qquad\qquad \text{(By the same argument used in problem 2a)}
\end{aligned}$$

The series diverges. □

c) *Proof.* By root test:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{5}+\frac{1}{n}\right)^n\right|} &= \lim_{n\to\infty} \left(\frac{1}{5}+\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{5} + \lim_{n\to\infty} \frac{1}{n} \\
&= \frac{1}{5} < 1
\end{aligned}$$

The series converges. □

d) *Proof.* By root test:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} &= \lim_{n\to\infty} \left|\frac{1}{n}\right| \\
&= 0
\end{aligned}$$

The series converges. □