Math 4001 Analysis 2
Homework Set 3

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Problem Apos. 12.19. Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ be defined by the equation $\mathbf{f}(t) = (\cos t, \sin t)$. Then $\mathbf{f}'(t)(u) = u(-\sin t, \cos t)$ for every real u. Then Mean-Value formula

$$\mathbf{f}(y) - \mathbf{f}(x) = \mathbf{f}'(z)(y - x)$$

cannot hold when $x = 0, y = 2\pi$, since the left member is zero and the right member is a vector of length 2π. Nevertheless, Theorem 12.9 states that for every vector $\mathbf{a}$ in $\mathbb{R}^2$ there is a z in the inverval $(0, 2\pi)$ such that

$$\mathbf{a} \cdot \{\mathbf{f}(y) - \mathbf{f}(x)\} = \mathbf{a} \cdot \{\mathbf{f}'(z)(y - x)\}.$$

Determine z in terms of $\mathbf{a}$ when $x = 0$ and $y = 2\pi$.

Proof.

$$\begin{aligned}
\mathbf{a}\left(\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\right) &= \mathbf{a} \cdot 2\pi \begin{bmatrix}-\sin z\\ \cos z\end{bmatrix} \\
0 &= 2\pi\left(a_2 \cos z - a_1 \sin z\right) \\
a_1 \sin z &= a_2 \cos z \\
\frac{\sin z}{\cos z} &= \frac{a_2}{a_1} \\
z &= \arctan\left(\frac{a_2}{a_1}\right)
\end{aligned}$$

□

Problem Apos. 12.21. State and prove a generalization of the result in Exercise 12.20 for a real-valued function differentiable on an n-ball $B(x)$.

Proof. Let $g(t)$ be defined such that $g(0) = f(\mathbf{x}), g(1) = f(\mathbf{y})$. Explicitly:

$$g(t) = f[ty_1 + (1-t)x_1, y_2, \cdots, y_n] + f[x_1, ty_2 + (1-t)x_2, y_3, \cdots, y_n] + \cdots + f[x_1, x_2, \cdots, ty_n + (1-t)x_n]$$

An attempt to express this more succinctly gives

$$g(t) = \sum_{i}^{n} f_i$$

where f_k is defined such that it's k-th argument is $ty_k + (1-t)x_k$ and all arguments before are x_i, and all arguments after are y_i.

Then,

$$\begin{aligned}
g(1) &= f[(1)y_1 + (1-1)x_1, y_2, \cdots, y_n] + f[x_1, (1)y_2 + (1-1)x_2, y_3, \cdots, y_n] + \cdots \\
&\quad + f[x_1, x_2, \cdots, (1)y_n + (1-1)x_n] \\
&= f(y_1, y_2, \cdots, y_n) + f(x_1, y_2, y_3, \cdots, y_n) + \cdots + f(x_1, x_2, \cdots, y_n) \\
g(0) &= f[(0)y_1 + (1-0)x_1, y_2, \cdots, y_n] + f[x_1, (0)y_2 + (1-0)x_2, y_3, \cdots, y_n] + \cdots \\
&\quad + f[x_1, x_2, \cdots, (0)y_n + (1-0)x_n] \\
&= f(x_1, y_2, \cdots, y_n) + f(x_1, x_2, y_3, \cdots, y_n) + \cdots + f(x_1, x_2, \cdots, x_n)
\end{aligned}$$

and we have

$$\begin{aligned}
g(1) - g(0) = f(y_1, y_2, \cdots, y_n) &+ f(x_1, y_2, y_3, \cdots, y_n) + \cdots + f(x_1, x_2, x_3 \cdots, y_n) \\
&- f(x_1, y_2, y_3, \cdots, y_n) - f(x_1, x_2, y_3, \cdots, y_n) - \cdots - f(x_1, x_2, \cdots, x_n) \\
= f(\mathbf{y}) - f(\mathbf{x})
\end{aligned}$$

We then compute the derivatives, noting that for each f, we have n partial derivatives, each with respect to each argument of the function f. This means that $D_i f_k = D_i f[x_1, x_2, \cdots ty_i + (1-t)x_i](y_i - x_i)$ when $i = k$, and 0 when $i \neq k$.

To illustrate, consider the process applied to just the first summand of g:

$$\begin{aligned}
f'[ty_1 + (1-t)x_1, y_2, \cdots, y_n] &= D_1 f[ty_1 + (1-t)x_1, y_2, \cdots, y_n](y_1 - x_1) + D_2 f[ty_1 + (1-t)x_1, y_2, \cdots, y_n](\\
&\quad + D_n f[ty_1 + (1-t)x_1, y_2, \cdots, y_n](0) \\
&= D_1 f[ty_1 + (1-t)x_1, y_2, \cdots, y_n](y_1 - x_1)
\end{aligned}$$

If we let $z_k(t) = ty_k + (1-t)x_k$, we can write it more simply, and the second summand becomes $D_2 f[x_1, z_2, y_3, \cdots, y_n](y_2 - x_2)$. The third: $D_3 f[x_1, x_2, z_3, y_4 \cdots, y_n](y_3 - x_3)$. And so on. In general the k-th summand of $g'(t)$ is given by

$$D_k f_k \cdot (y_k - x_k)$$

Then $g'(t)$ simplifies to

$$g'(t) = D_1 f[z_1, y_2, \cdots, y_n](y_1 - x_1) + D_2 f[x_1, z_2, y_3, \cdots, y_n](y_2 - x_2) + \cdots + D_n f[x_1, \cdots, z_n](y_n - x_n)$$

By the Mean Value Theorem, with $z_k = z_k(t_0) = t_0 y_k + (1-t_0)x_k$

$$\begin{aligned}
g(1) - g(0) &= g'(t_0)(1-0) \\
f(\mathbf{y} - \mathbf{x}) &= D_1 f[z_1, y_2, \cdots, y_n](y_1 - x_1) + D_2 f[x_1, z_2, \cdots, y_n](y_2 - x_2) + \cdots + D_n f[x_1, \cdots, z_n](y_n - x_n)
\end{aligned}$$

$\square$

Problem Apos. 13.2. Let $\mathbf{f} = (f_1, f_2, f_3)$ be the vector-valued function defined (for every point (x_1, x_2, x_2) in $\mathbb{R}^3$ for which $x_1 + x_2 + x_3 \neq -1$) as follows:

$$f_k(x_1, x_2, x_3) = \frac{x_k}{1 + x_1 + x_2 + x_3} k = 1, 2, 3$$

Show that $J_{\mathbf{f}}(\mathbf{x}) = (1 + x_1 + x_2 + x_3)^{-4}$. Show that $\mathbf{f}$ is one-to-one and compute $\mathbf{f}^{-1}$ explicitly.

Proof. The Jacobian determinant $J_{\mathbf{f}}(x)$ is given by

$$\begin{aligned}
J_{\mathbf{f}}(x) &= \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix} \\
&= \frac{\partial f_1}{\partial x_1}\left(\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_2}\right) - \frac{\partial f_1}{\partial x_2}\left(\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\frac{\partial f_2}{\partial x_3}\right) + \frac{\partial f_1}{\partial x_3}\left(\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_1}\right) \\
&= \frac{1 + x_2 + x_3}{(1 + x_1 + x_2 + x_3)^2}\left(\frac{(1 + x_1 + x_3)(1 + x_1 + x_2)}{(1 + x_1 + x_2 + x_3)^4} - \frac{(-x_2)(-x_3)}{(1 + x_1 + x_2 + x_3)^4}\right) \\
&\quad - \frac{-x_1}{(1 + x_1 + x_2 + x_3)^2}\left(\frac{(-x_2)(1 + x_1 + x_2)}{(1 + x_1 + x_2 + x_3)^4} - \frac{(-x_3)(-x_2)}{(1 + x_1 + x_2 + x_3)^4}\right) \\
&\quad + \frac{-x_1}{(1 + x_1 + x_2 + x_3)^2}\left(\frac{(-x_2)(-x_3)}{(1 + x_1 + x_2 + x_3)^4} - \frac{(1 + x_1 + x_3)(-x_3)}{(1 + x_1 + x_2 + x_3)^4}\right) \\
&= \frac{1 + x_2 + x_3}{(1 + x_1 + x_2 + x_3)^2}\left(\frac{1 + 2x_1 + x_2 + x_1^2 + x_1x_2 + x_3 + x_1x_3 + x_2x_3 - x_2x_3}{(1 + x_1 + x_2 + x_3)^4}\right) \\
&\quad - \frac{-x_1}{(1 + x_1 + x_2 + x_3)^2}\left(\frac{-x_2 - x_1x_2 - x_2^2 - x_2x_3}{(1 + x_1 + x_2 + x_3)^4}\right) \\
&\quad + \frac{-x_1}{(1 + x_1 + x_2 + x_3)^2}\left(\frac{x_2x_3 + x_3 + x_1x_3 + x_3^2}{(1 + x_1 + x_2 + x_3)^4}\right) \\
&= \frac{1}{(1 + x_1 + x_2 + x_3)^6}\left(1 + x_1^2 + x_2^2 + x_3^2 + 2x_1 + 2x_2 + 2x_3 + 2x_1x_2 + 2x_3x_2 + 2x_1x_3\right) \\
&= \frac{1}{(1 + x_1 + x_2 + x_3)^6}(1 + x_1 + x_2 + x_3)^2 \\
&= \frac{1}{(1 + x_1 + x_2 + x_3)^4} \\
&= (1 + x_1 + x_2 + x_3)^{-4}
\end{aligned}$$

Since $J_{\mathbf{f}}(x) \neq 0$ and continuous for any x_1, x_2, x_3 for which the function is defined (specifically $x_1 + x_2 + x_3 \neq -1$), the function is one-to-one by the Inverse Function Theorem.

Finally, $\mathbf{f}^{-1}(x_1, x_2, x_3)$ is given by

$$\mathbf{f}^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1(1 + x_1 + x_2 + x_3) \\ x_2(1 + x_1 + x_2 + x_3) \\ x_3(1 + x_1 + x_2 + x_3) \end{bmatrix}$$

since

$$\mathbf{f}^{-1}(\mathbf{f}(x)) = \mathbf{f}^{-1}\left(\begin{bmatrix} \frac{x_1}{1+x_1+x_2+x_3} \\ \frac{x_2}{1+x_1+x_2+x_3} \\ \frac{x_2}{1+x_1+x_2+x_3} \end{bmatrix}\right) = \begin{bmatrix} \frac{x_1}{1+x_1+x_2+x_3}(1+x_1+x_2+x_3) \\ \frac{x_2}{1+x_1+x_2+x_3}(1+x_1+x_2+x_3) \\ \frac{x_2}{1+x_1+x_2+x_3}(1+x_1+x_2+x_3) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}$$

□

Problem Apos. 13.5. a) State conditions on f and g which will ensure that the equations $x = f(u, v), y = g(u, v)$ can be solved for u and v in a neighborhood of (x_0, y_0). If the solutions are $u = F(x, y), v = G(x, y)$, and if $J = \partial(f, g)/\partial(u, v)$, show that

$$\frac{\partial F}{\partial x} = \frac{1}{J}\frac{\partial g}{\partial v}, \frac{\partial F}{\partial y} = -\frac{1}{J}\frac{\partial f}{\partial v}, \frac{\partial G}{\partial x} = -\frac{1}{J}\frac{\partial g}{\partial u}, \frac{\partial G}{\partial y} = \frac{1}{J}\frac{\partial f}{\partial u}$$

b) Compute J and the partial derivatives of F and G at $(x_0, y_0) = (1, 1)$ when $f(u, v) = u^2 - v^2, g(u, v) = 2uv$.

Proof. a) The conditions on f and g that would ensure invertability are that the partial derivatives of each function be continuous in a neighborhood around (x_0, y_0). Also, the Jacobian determinant should not be 0, which gives us

$$J_{\mathbf{f}}(x) = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u} \neq 0$$

This leads to an interesting result, whereby the ratio of partials cannot be equivalent

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} \neq \frac{\frac{\partial g}{\partial u}}{\frac{\partial g}{\partial v}}$$

To show $\frac{\partial F}{\partial x} = \frac{1}{J}\frac{\partial g}{\partial v}$, first we calculate J

$$J = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}$$

Note also the following by chain rule:

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

Since $u = F(x, y)$, we can also express the above expression in the following way:

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial f}{\partial u}\frac{\partial F}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial G}{\partial x}$$

Similarly, we have

$$\frac{\partial y}{\partial x} = 0 = \frac{\partial y}{\partial u}\frac{\partial F}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial G}{\partial x}$$

Note that each line is equivalent to the result of multiplying the vector $\begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial x} \end{bmatrix}$ with the Jacobian Matrix. So we have a system of equations we may use to solve for the given vector via Cramer's Rule:

$$\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial F}{\partial x} = \frac{\begin{vmatrix} 1 & \frac{\partial f}{\partial v} \\ 0 & \frac{\partial g}{\partial v} \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial g}{\partial v}$$

$$\frac{\partial G}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & 1 \\ \frac{\partial g}{\partial u} & 0 \end{vmatrix}}{J} = -\frac{1}{J}\frac{\partial g}{\partial u}$$

Similarly, we can solve for the other two components:

$$\frac{\partial y}{\partial y} = 1 = \frac{\partial g}{\partial u}\frac{\partial F}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial G}{\partial y}$$

$$\frac{\partial x}{\partial y} = 0 = \frac{\partial f}{\partial u}\frac{\partial F}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial G}{\partial y}$$

$$\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial F}{\partial y} = \frac{\begin{vmatrix} 0 & \frac{\partial f}{\partial v} \\ 1 & \frac{\partial g}{\partial v} \end{vmatrix}}{J} = -\frac{1}{J}\frac{\partial f}{\partial v}$$

$$\frac{\partial G}{\partial y} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & 0 \\ \frac{\partial g}{\partial u} & 1 \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial f}{\partial u}$$

b) Given that $f(u,v) = u^2 - v^2, g(u,v) = 2uv$, we have

$$J = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

We obtain the partial derivatives at (1,1) as

$$\frac{\partial F}{\partial x} = \frac{1}{J}\frac{\partial g}{\partial v} = \frac{1}{4u^2+4v^2}2u = \frac{u}{2u^2+2v^2}$$

$$\frac{\partial F}{\partial y} = -\frac{1}{J}\frac{\partial f}{\partial v} = -\frac{1}{4u^2+4v^2}(-2v) = \frac{v}{2u^2+2v^2}$$

$$\frac{\partial G}{\partial x} = -\frac{1}{J}\frac{\partial g}{\partial u} = \frac{1}{4u^2+4v^2}2v = -\frac{v}{2u^2+2v^2}$$

$$\frac{\partial G}{\partial y} = \frac{1}{J}\frac{\partial f}{\partial u} = \frac{1}{4u^2+4v^2}2u = \frac{u}{2u^2+2v^2}$$

At $(x_0, y_0) = (1, 1)$, we have $u^2 - v^2 = 1, 2uv = 1$, so solving for u and v we get

$$u = \pm\sqrt{1 \pm \sqrt{\frac{1 \pm \sqrt{2}}{2}}}$$

$$v = \pm\sqrt{\frac{-1 \pm \sqrt{2}}{2}}$$

Plugging these into the partials gives

$$\frac{\partial F}{\partial x} = \frac{\pm\sqrt{1 \pm \sqrt{\frac{1\pm\sqrt{2}}{2}}}}{1 \pm 2\sqrt{\frac{1\pm\sqrt{2}}{2}} \pm \sqrt{2}}$$

$$\frac{\partial F}{\partial y} = \frac{\pm\sqrt{\frac{-1\pm\sqrt{2}}{2}}}{1 \pm 2\sqrt{\frac{1\pm\sqrt{2}}{2}} \pm \sqrt{2}}$$

$$\frac{\partial G}{\partial x} = -\frac{\pm\sqrt{\frac{-1\pm\sqrt{2}}{2}}}{1 \pm 2\sqrt{\frac{1\pm\sqrt{2}}{2}} \pm \sqrt{2}}$$

$$\frac{\partial G}{\partial y} = \frac{\pm\sqrt{1 \pm \sqrt{\frac{1\pm\sqrt{2}}{2}}}}{1 \pm 2\sqrt{\frac{1\pm\sqrt{2}}{2}} \pm \sqrt{2}}$$

$\square$

Problem D1. Given is the function $F(x, y) = x^2 + xy + y^2 - 3$. Show that $y = f(x)$ in a neighborhood of the point $(1, 1)$. Then determine $\partial f/\partial x$ at this point.

Proof. Let $\Omega \subset \mathbb{R}^2$. By the IFT, if $F(x, y) = 0$ for some $(x, y) \in \Omega$ and the partial derivatives $\frac{\partial F}{\partial x}$ are continuous in a neighborhood around that point, then there exists some $f(x)$ such that $f(x) = y$ and $F(x, f(x)) = 0$. So we seek to show that $F(x, y) = 0$ at $(1, 1)$ and that J is not 0 at (1,1).

$$F(1, 1) = (1)^2 + 1 \cdot 1 + (1)^2 - 3 = 3 - 3 = 0$$

$$\frac{\partial F}{\partial y}(1, 1) = x + 2y|_{(1,1)} = 1 + 2 = 3 \neq 0$$

Thus, by the Implicit Function Theorem, there exists a function $f(x)$ such that $y = f(x)$ in a neighborhood around $(1, 1)$. For a larger domain, we can note that, as long as both x and

y are not both 0, the derivative is nonzero. There are some additional restrictions, noting that

$$y = \sqrt{3 - x^2 - xy}$$

So that $x^2 + xy > 3$ to avoid an imaginary result for y.

$$\frac{\partial f}{\partial x}(1,1) = \frac{\frac{-\partial F}{\partial x}}{\frac{\partial F}{\partial y}}(1,1) = \left.\frac{2x + y}{x + 2y}\right|_{(1,1)} = \frac{2+1}{1+2} = 1$$

$\square$