

# Probabilistic Graphical Models Case study: Bayesian retweet modeling

Khaled Fouda

March 2021

## Abstract

Here we build a probabilistic model over the number of retweets and tweeter's reaction time (predictors) in terms of the distributions over available data. We begin with a brief background over the required theory. In section two we explore our data and assign the prior distributions. In section three we compute the posterior distributions of our parameters. Finally, we go over the implementation and output analysis.

## Contents

<b>1</b>	<b>Background</b>	<b>3</b>
1.1	Bayesian Probabilities . . . . .	3
1.1.1	Conjugate Priors . . . . .	4
1.1.2	Noninformative prior distributions . . . . .	5
1.1.3	References and extra readings . . . . .	5

1.2	Stationary Markov Chains . . . . .	5
1.3	MCMC Sampling . . . . .	6
1.3.1	Gibbs Sampling . . . . .	6
1.3.2	The Metropolis-Hastings Algorithm . . . . .	7
1.3.3	Convergence . . . . .	8
1.3.4	References and extra readings . . . . .	9
<b>2</b>	<b>Model likelihood</b>	<b>10</b>
<b>3</b>	<b>Priors and Posteriors</b>	<b>14</b>
3.1	Conjugacy . . . . .	20
3.1.1	Normal-IG for Normal likelihood . . . . .	20
3.1.2	Gamma for-Inverse Gamma . . . . .	22
3.2	Joint Posterior . . . . .	24
3.3	Conditional Posteriors . . . . .	24
3.3.1	$\alpha^x, \tau^x$ . . . . .	25
3.3.2	$\alpha, \sigma_\Delta^2$ . . . . .	26
3.3.3	$a_t, b_t$ . . . . .	27
3.3.4	$b_j^x$ . . . . .	28
3.3.5	$\sigma_b^2$ . . . . .	28
3.3.6	$\mu_j^x, \beta$ . . . . .	29
<b>4</b>	<b>Implementation</b>	<b>30</b>
<b>5</b>	<b>References</b>	<b>30</b>

# 1 Background

## 1.1 Bayesian Probabilities

In Bayesian statistics of one response variable we assign two distribution. The first  $p(y|\theta)$  where we have only a subset of  $y$  and we need to model it so we can predict future values of it. We choose this distribution based on either the shape of the frequency ( looks like normal or exponential) or based on the nature of the data ( bernoulli for binary data, human reaction time follows an inverse gamma distribution).

The second distribution we assign is the prior  $p(\theta)$  which can either represent our belief of how the data is distributed (ex: expect the parameter to be normally distributed with mean 3 and sd 1) or it can have no information at all -noninformative- and the model has to learn it all by itself.

What we are interested in is two other distributions. The first is the posterior

$$p(\theta|y) = \frac{p(\theta, y)}{\sum_{\theta} p(\theta, y)} \propto p(\theta, y) = p(\theta) p(y|\theta)$$

In order to find an accurate distribution of  $\theta$  we will sample this posterior distribution. However the denominator is quite difficult to compute so we will depend on the algorithm that lets us sample the posterior from  $p(\theta) p(y|\theta)$ . We will denote the latter as the likelihood from now on.

We are also interested in predicting future values of  $y$  and hence we need a marginal distribution free of any other parameters and hence

$$p(y) = \frac{p(\theta, y)}{p(\theta|y)}$$

### 1.1.1 Conjugate Priors

Given the distribution of the likelihood, if the the prior and posterior has the same distribution with different parameters then we say that the prior is a natural conjugate prior.

This is very useful because if we know that the posterior has a common distribution then we can easily sample from it.

Moreover, if the likelihood belong to the exponential family then there exist a natural conjugate prior associated with it. Below is a table with some examples of likelihood functions and their conjugate priors that we will use to define our priors later. Note that these priors are not unique.

The likelihood of an exponential family has the following representation

$$p(y|\theta) = \prod_{i=1}^n p(y_i|\theta) = \prod f(y_i) g(\theta)^n \exp(\phi(\theta) \sum u(y_i)) \propto g(\theta)^n \exp(\phi(\theta) \sum u(y_i))$$

So if we choose our prior as

$$p(\theta) \propto g(\theta)^m \exp(v \phi(\theta))$$

Then the posterior can be represented as

$$p(\theta|y) \propto g(\theta)^{n+m} \exp(v \phi(\theta) \sum u(y_i))$$

Note that  $\sum u(y_i)$  is a sufficient statistic for  $\theta$ .

Likelihood	Model parameters	Prior/Posterior	prior param	posterior param
Binomial	p(probability)	Beta	$\alpha, \beta$	$\alpha + \sum y_i, \beta + nN - \sum x_i$
Normal	$\mu, \sigma^2$	Normal, Inverse Gamma	$\mu_p, \sigma_b^2, \alpha, \beta$	* check reference
Inverse Gamma	$\alpha, \beta$	Normal, Gamma	$\mu, \sigma^2, \alpha_g, \beta_g$	* check reference

### 1.1.2 Noninformative prior distributions

When prior distributions have no population basis, they can be difficult to construct, and there has long been a desire for prior distributions that can be guaranteed to play a minimal role in the posterior distribution. The rationale for using noninformative prior distributions is to let the data speak for themselves, so that inferences are unaffected by information external to the current data.

One way of choosing the noninformative priors is by Jeffreys' in-variance principle which states that the noninformative prior is proportional to fisher's information.:

$$p(\theta) \propto I(Y) = -E \frac{d^2}{d\theta^2} \log(p(y|\theta))$$

For example, if our likelihood has a binomial distribution  $y \sim \text{Bin}(n, \theta)$  then we can compute fisher's information to be  $\theta^{-.5} (1 - \theta)^{-.5}$  and we can conclude that the prior  $\theta \sim \text{Beta}(.5, .5)$

### 1.1.3 References and extra readings

1. **A Compendium of Conjugate Priors** Deriving conjugate priors for many common distributions.
2. **Bayesian Data Analysis, by Andrew Gelman** Chapter 2: Single-parameter models.

## 1.2 Stationary Markov Chains

A stationary MC doesn't change over time. That is  $p(x_{i+1}|x_i) = p(x_{k+1}|x_k)$

**Definition** A stationary distribution of a Markov chain is a distribution  $\pi$  such that :

$$\pi(y) = \sum_x p(y|x)\pi(x)$$

**Definition** A Markov chain is said to be reversible if there exists a probability distribution  $\pi$  such that

$$p(x|y)\pi(y) = p(y|x)\pi(x)$$

**Result** If a MC is reversible then it is stationary since,

$$\sum_x p(x|y)\pi(y) = \sum_x p(y|x)\pi(x) = \pi(y)$$

## 1.3 MCMC Sampling

Simulating a Markov chain until stationarity is achieved then simulation this stationary Markov chain to estimate  $\theta$ .

### 1.3.1 Gibbs Sampling

Now assume we have two parameters  $\theta$ ,  $\alpha$  with joint posterior distribution  $p(\theta, \alpha|y)$  and that this distribution is known distribution but some complicated expression. However, there is a big chance - if we model smart - that the conditional posteriors  $p(\theta|\alpha, y)$  and  $p(\alpha|\theta, y)$  are very easy to sample from.

Hence the algorithm is as follows:

1. Initialize  $\theta$  and  $\alpha$  to some values.
2. For each time step  $t$ :
  - Sample  $\theta_t \sim p(\theta|\alpha, y)$
  - Sample  $\alpha_t \sim p(\alpha|\theta, y)$
- The sampling order can be sequential like we did above or be random.

- This can be generalized to the case of  $p$  parameters.
- If the conditional posteriors are hard to sample from then we cannot use this method.

How to compute the conditional posteriors? Luckily we don't need to sum or integrate. Because we are dealing with proportionality not exact distribution we can safely assume that the conditional posterior of  $\theta$  is all the elements of the joint posterior that contains  $\theta$ . All other parameters as well as  $y$  are considered constant and can be discarded.

### 1.3.2 The Metropolis-Hastings Algorithm

Gibbs sampling is a special case of the MH algorithm. Now assume that the conditional posteriors are also complicated.

We need to define a transition function  $Q(\theta^{new}|\theta^{old})$  Then at each step  $t$ :

1. Sample  $\theta^{new} \sim Q(\theta^{new}|\theta^{old})$
2. Accept the new sample with probability  $\frac{p(\theta^{new})}{p(\theta^{old})} \frac{Q(\theta^{old}|\theta^{new})}{Q(\theta^{new}|\theta^{old})}$
3. otherwise, set  $\theta^{new} = \theta^{old}$

Remember that our goal is to reach a stationary MC process. For a reversible process and stationary process the acceptance probability above is 1.

Moreover, In the basic Metropolis algorithm, we assume that the proposal distribution is symmetric, that is  $Q(\theta^{new}|\theta^{old}) = Q(\theta^{old}|\theta^{new})$  and the acceptance probability is reduced to  $p = \frac{p(\theta^{new})}{p(\theta^{old})}$

- We didn't use the posterior distribution, we used the prior density and our proposed sampler.

- A common choice for the sampler is one that provided random small steps.
  - For example, if our prior is  $\theta \sim Normal()$  we can use  $Q(\theta^{new}|\theta^{old}) = N(\theta^{old}, \epsilon)$  where  $\epsilon$  is chosen to be very small to guarantee having small steps.
- If we have a mix of easy-to-sample conditional posteriors and hard-to-sample ones, we can use Gibbs for the easy ones and M-H for the hard ones inside the sample loop. This approach is called Metropolis-within-Gibbs.
- As with Gibbs sampling, subsequent samples are highly correlated. We can solve this by counting only every  $k^{th}$  accepted sample to reduce the dependency.
- How do we know we have reached stationarity?

### 1.3.3 Convergence

After running an MCMC we need to verify that we have indeed reached a stationary process.

**Gelman-Rubin Approach** aims to diagnose convergence. Lets assume we run  $m/2$  Markov chains each with  $4n$  iterations and is randomly initialized. Discard the first half and split the resulting  $2n$  samples into two sub-chains. now we can say we have  $m$  markov chains with  $n$  samples each.

The portion we discarded constitutes the burn-in period where the chains are assumed to be in their transient phase. We now hope that the  $m \times n$  samples are from the desired stationary distributions. To verify that we are going to compare the between-chain variance with the within-chain variance.



Let  $\theta_{i,j}$  be the sample  $i$  from the chain  $j$ . The between- and within-chain variances,  $B$  and  $W$ , computed as:

$$B = \frac{n}{m-1} \sum_{j=1}^m (\bar{\theta}_{.j} - \bar{\theta}_{..})^2$$

$$W = \frac{1}{m} \sum_{j=1}^m s_j^2 \text{ where } s_j^2 = \frac{1}{n-1} \sum_{i=1}^n (\theta_{ij} - \bar{\theta}_{.j})^2$$

Notice The similarities between them and the  $MS_{ERROR}$  and  $MS_{EFFECT}$  in the ANOVA effect model.

We can now estimate  $Var(\theta|y)$  as a weighted average of  $W$  and  $B$  with :

$$\hat{Var}(\theta|y) = \frac{n-1}{n}W + \frac{1}{n}B$$

This estimated is unbiased if we reach stationarity. Moreover,  $W$  is an underestimate of  $Var(\theta|y)$  but it could approach it when  $n \rightarrow \infty$ . We can therefore monitor convergence through

$$\hat{R} = \sqrt{\frac{\hat{Var}(\theta|y)}{W}}$$

Values of  $\hat{R} \leq 1.1$  are acceptable but the closer to 1 the better. We then monitor  $\hat{R}$  for each of the parameter we estimate.

### 1.3.4 References and extra readings

- **MCMC and bayesian Modeling by Martin Haugh** These lecture notes provide an introduction to Bayesian modeling and MCMC algorithms including the Metropolis-Hastings and Gibbs Sampling algorithms.

- **Pattern recognition and machine learning by C.Bishop** Chapter 11: Sampling Method.

## 2 Model likelihood

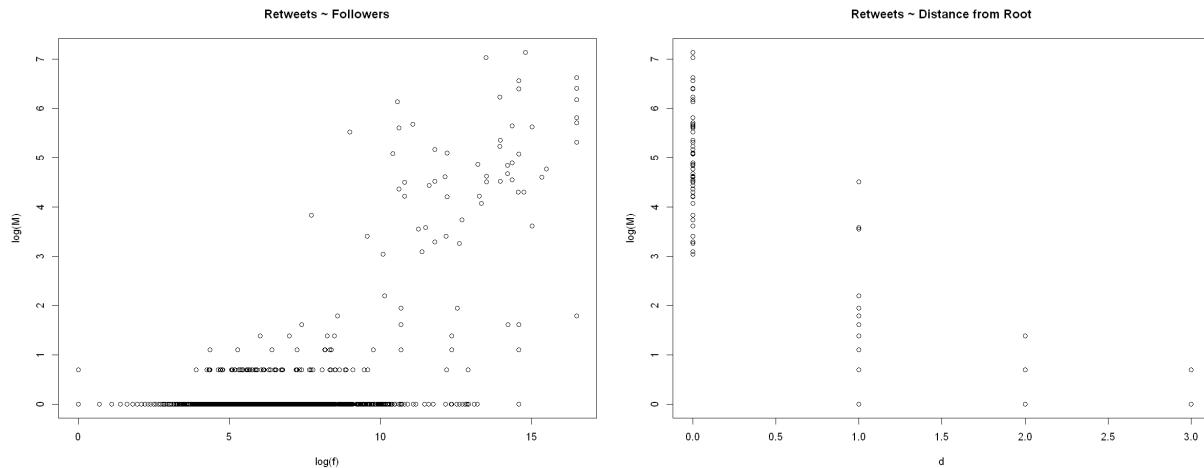
Our data consists of 52 root tweets and recursively all the retweets to them. We have in total 12248 data rows. The relevant elements of each row are:

- **case:** 1 to 52 to keep track to which root the retweet belongs.
- **FollowerCount:** The number of followers for the re-tweeter. Denote it  $f_j^x$  where  $x$  is the case number and  $j$  is the re-tweeter's order in that case.
- **DistanceFromRoot:** = 0 if it's the root (ie: the tweet's owner). = 1 if the user retweeted it directly from the root.  $\zeta$  1 if it was retweeted from a previous retweet. Denote it  $d_j^x$
- **TimeDiff:** The time in minutes between the tweet's origin and the user's retweet of it. I took  $\log(x+1)$  to get rid of the outliers. Denote it  $S_j^x$
- **Retweets:** The number of retweets received for that user  $j$ . For the root user, this is the total number of retweets he received for the original tweet  $M_0^x$ . For the re-tweeters, it's the number of retweets they received on their retweet (skewed near 0). Denote it  $M_j^x$

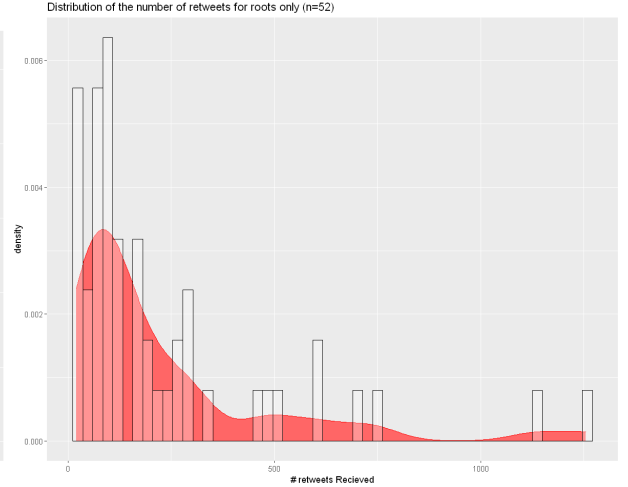
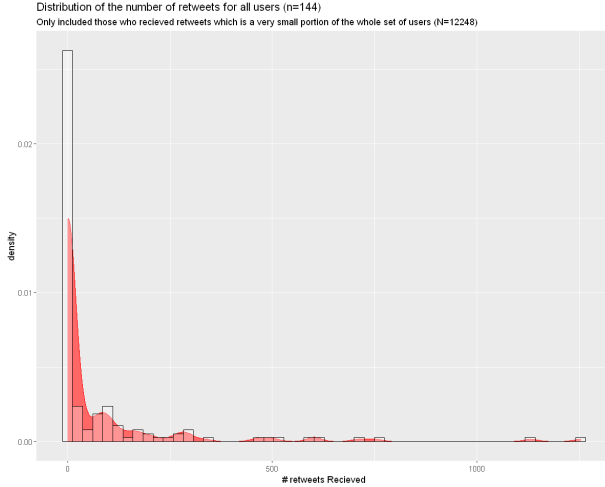
We are interested in modeling the total number of retweets the root user receives  $M_0^x$  and the possibility of being able to predict it before observing the full data.

We assume the the number of retweets  $M_j^x$  to be related to be related to the number of followers  $f_j^x$  and the distance from the root  $d_j^x$ . By fitting an ANOVA model we confirm our assumption since the p-values for the model  $\log(M_j^x) \sim \log(f_j^x) + d_j^x$  are all very small.

The plots below illustrate the linear relations. The more follower a user get, the more retweets they receive and the further from the root the less likely they will receive retweets.

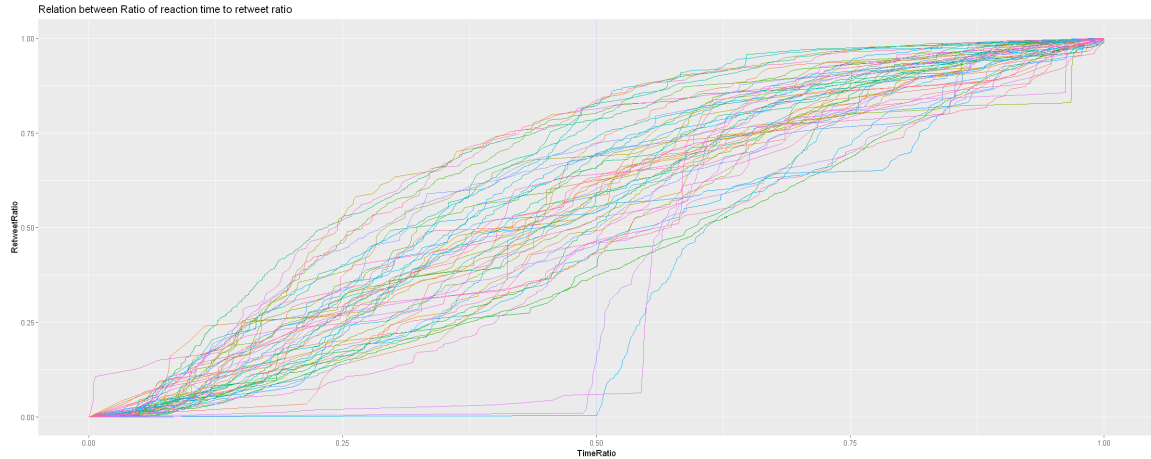


Below is the distribution of  $M_j^x$ . For the first plot, all users except those who didn't receive any retweets are shown. In the second, only root users are considered. We notice that the data is highly skewed.



We will assume that the followers are all equally likely to retweet. If we have a number  $b_j^x$  representing the probability that any of the followers  $f_j^x$  will retweet then we can model  $M_j^x \sim \text{Binomial}(f_j^x, b_j^x)$ . We may also assume that  $b_j^x$  is related to both  $f_j^x$  and  $d_j^x$ . More on modeling them in the next section.

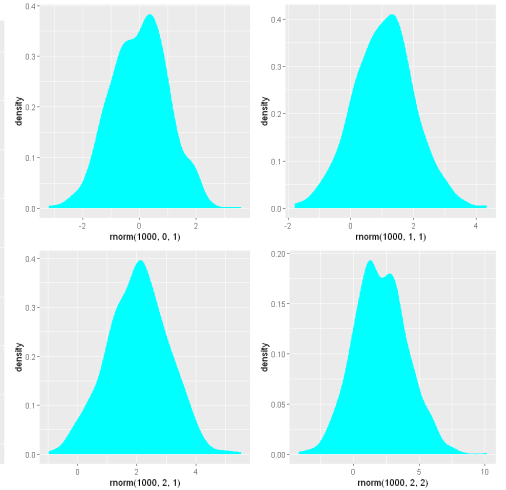
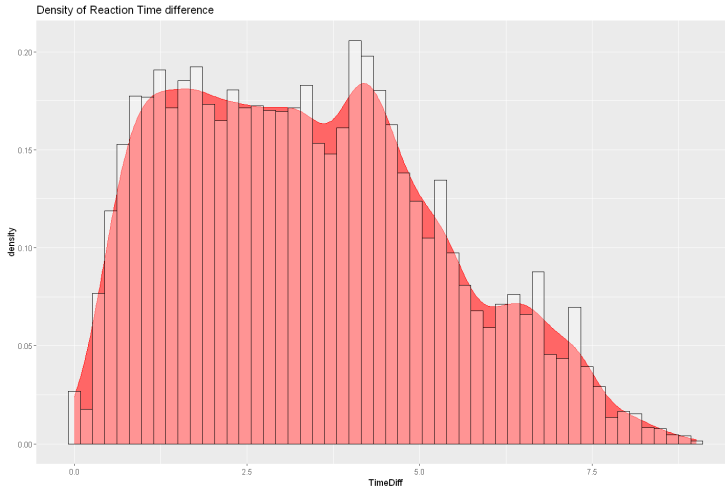
Now we consider a another variable of interest. The reaction time. To understand how it affects the total number of retweets, lets analyze the following plot where we plot the time-ratio and retweet-ratio for every root user (in colors) to notice any change in growth of retweets overtime.



We can see that the growth in retweets is logistic where in the first portion of time the growth is exponential but after time it slows down and eventually flatten out.

That shows an interesting relation between the two variable and therefore we will be interested in modeling the reaction time as well.

Below on the left we have the distribution of the reaction time ( after taking the log) and on right we have the density of several normal distributions. From this we can assume that the reaction time (in log) has a normal distribution.



### 3 Priors and Posteriors

To summarize our likelihood, we have:

$$P(M_j^x | f_j^x, b_j^x) = \text{Binomial}(M_j^x | f_j^x, b_j^x) = \binom{f_j^x}{m_j^x} (b_j^x)^{m_j^x} (1 - b_j^x)^{f_j^x - m_j^x}$$

$$P(S_j^x | \alpha^x, \tau^x) = \text{Normal}(S_j^x | \alpha^x, \tau^x) \propto (\tau^x)^{-n/2} \exp\left\{-\frac{(s_j^x - \alpha^x)^2}{2\tau^x}\right\}$$

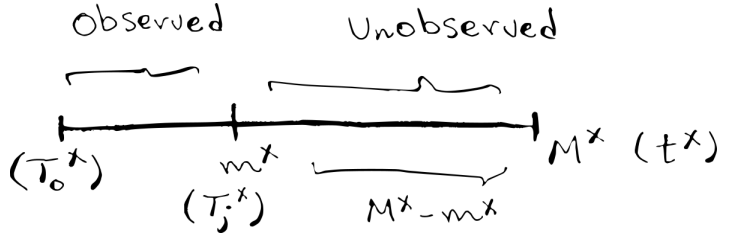
Note here  $\tau^x$  is the variance for user  $x$  I omitted the square to avoid confusion.

Now we write the joint likelihood equation. We first consider the training case where all the retweets are available. Let  $S^x = \cup_j S_j^x$  and  $M^x = \cup_j M_j^x$  ie, the retweets for user  $x$ .

$$P(S^x, M^x | \alpha^x, \tau^x, b^x, m^x) = \prod_{j=0}^{m^x} \text{Norm}(s_j^x | \alpha^x, \tau^x) \text{Bin}(M_j^x | b_j^x, f_j^x)$$

Now assume tat the data is partially observed. Let  $m^x$  be the observed retweets and  $M^x$  be the eventual total number of retweets. Therefore we have  $M^x - m^x$  unobserved retweets and if the last retweet observed is at time  $T_j^x$  and  $t^x$  is the time we observe the last tweet, then the reaction times between  $T_j^x$  and  $t^x$  are unobserved.

Recall that the reaction time in our data is  $S_j^x = \log(t_j^x - T_0^x + 1)$  where  $T_0^x$  is the time of the first tweet and the added 1 is to avoid having a log of 0.



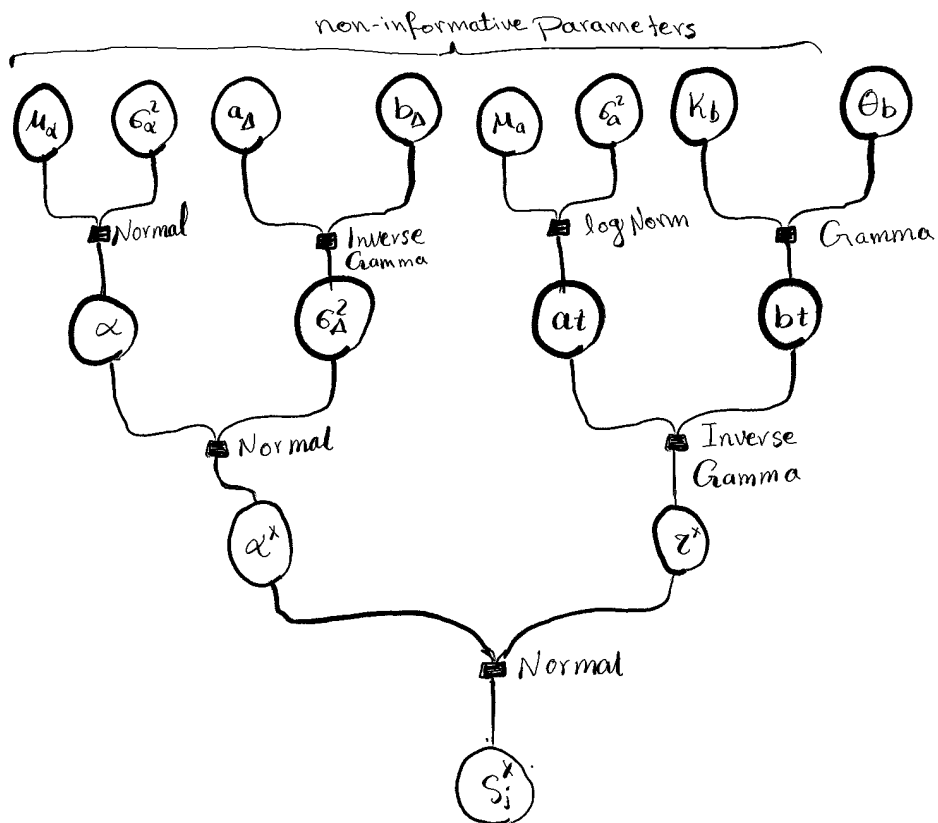
Here we asume that the data is ordered. ie,  $m_2^x$  is the second retweet received at time  $t_2^x > t_1^x$ . We can then use order statistic properties to compute the joint likelihood when the data is partially observed.

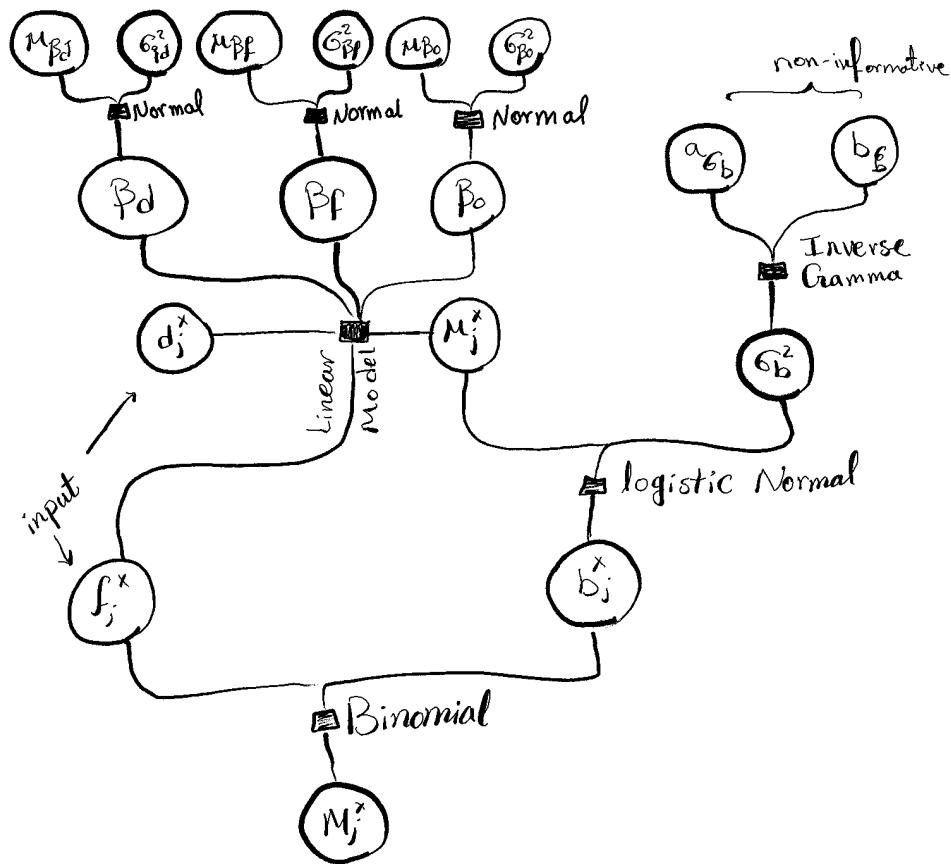
$$P(S_{t^x}^x, m_{t^x}^x | \alpha^x, \tau^x, m^x, M^x) = \prod_{j=0}^{m^x} \binom{M_j^x}{m_j^x} f(s_j^x | \alpha^x, \tau^x) (1 - F(t^x - T_j^x | \alpha^x, \tau^x))^{M_j^x - m_j^x}$$

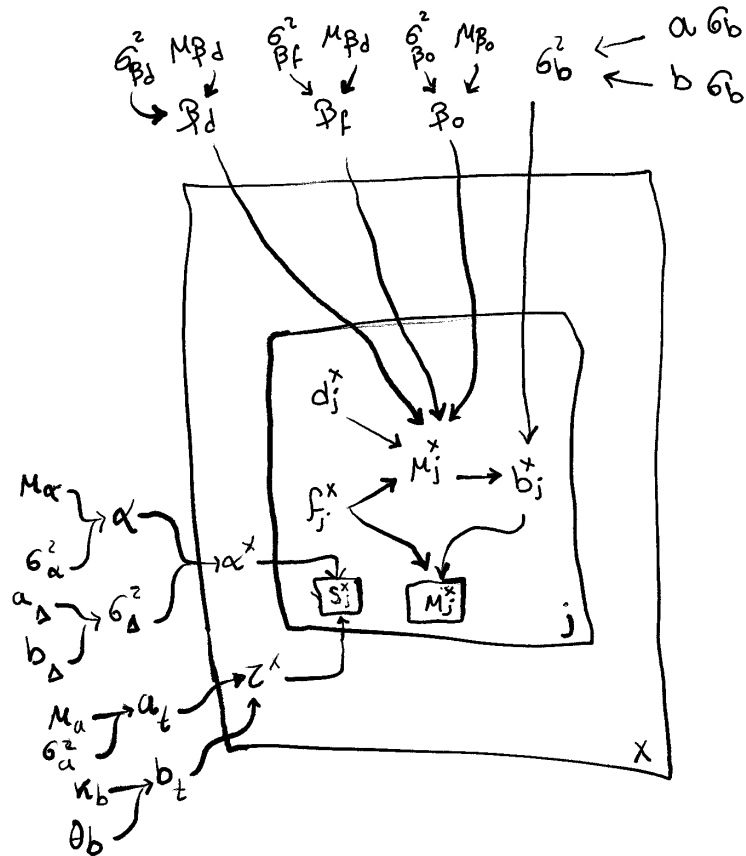
The combination term is there because we have  $M_j^x - m_j^x$  possible unobserved retweets.

The chosen model and priors are illustrated below. We will analyze then in details for the rest of this section.









## 3.1 Conjugacy

We first need to justify some of our priors. We assume that the Normal-IG is a conjugate prior for the normal likelihood and that Gamma is conjugate prior for the IG.

### 3.1.1 Normal-IG for Normal likelihood

We begin to prove the first assumption. Assume that the likelihood  $D \sim \text{Normal}(\mu, \sigma^2)$  and consider the case when  $\sigma^2$  is known but  $\mu$  isn't. (we will treat each parameter individually while fixing the other. Since we are using Gibbs sampling where we sample from each parameter while fixing the others, this should not cause problems.).

$$p(D|\mu) \sim \text{Normal}(\mu, \sigma^2) \propto \exp\left\{\frac{-1}{2\sigma^2} \sum_x (x_i - \mu)^2\right\}$$

We claim that the prior of the following shape is conjugate

$$p(\mu) \propto \exp\left\{\frac{-1}{2\sigma_o^2}(\mu - \mu_o)^2\right\} \sim \text{Normal}(\mu|\mu_o, \sigma_o^2)$$

**proof:** we now show that the posterior is normally distributed.

$$\begin{aligned} p(\mu|D) &\propto p(\mu)p(D|\mu) \propto \exp\left\{\frac{-1}{2\sigma_o^2}(\mu - \mu_o)^2 + \frac{-1}{2\sigma^2} \sum_x (x_i - \mu)^2\right\} \\ &= \exp\left\{\frac{-1}{2\sigma^2} \sum_x (x_i^2 - 2x_i\mu + \mu^2) - \frac{1}{2\sigma_o^2}(\mu^2 - 2\mu\mu_o + \mu_o^2)\right\} \end{aligned}$$

$$\exp\left\{\frac{-\mu^2}{2}\left[\frac{1}{\sigma_o^2} + \frac{n}{\sigma^2}\right] + \mu\left[\frac{\mu_o}{\sigma_o^2} + \frac{\sum x_i}{\sigma^2}\right] - \left[\frac{\mu_o^2}{2\sigma_o^2} + \frac{\sum x_i^2}{2\sigma^2}\right]\right\}$$

We want the expression above to be similar to:

$$\exp\left\{\frac{-1}{2\sigma_n^2}(\mu^2 - 2\mu\mu_n + \mu_n^2)\right\}$$

For some parameters  $\sigma_n^2$  and  $\mu_n$ . We now solve the two equations for the parameters.

$$\frac{-\mu^2}{2\sigma_n^2} = \frac{-\mu^2}{2}\left[\frac{1}{\sigma_o^2} + \frac{n}{\sigma^2}\right]$$

Solving this we get for  $\sigma_n^2$  we get

$$\sigma_n^2 = \frac{\sigma^2\sigma_o^2}{n\sigma_o^2 + \sigma^2} = (n + \sigma^2\sigma_o^{-2})^{-1}\sigma^2 \quad (1)$$

And by equating the terms for  $\mu$  we get:

$$\frac{\mu\mu_n}{\sigma_n^2} = \mu\left[\frac{\mu_o}{\sigma_o^2} + \frac{\sum x_i}{\sigma^2}\right]$$

then,

$$\mu_n = (n + \sigma^2\sigma_o^{-2})^{-1}\sum x_i \quad (2)$$

And hence the posterior  $p(\mu|D) \sim Normal(\mu|\mu_n, \sigma_n^2)$

Moreover, The predictive posterior also has a normal distribution,

$$p(x|D) = \int p(x|\mu)p(\mu|D) d\mu = \int N(x|\mu, \sigma^2) N(\mu|\mu_n, \sigma^2) d\mu = Normal(x|\mu_n, \sigma_n^2 + \sigma^2)$$

But for this presentation we will focus on the posterior model more than the predictive posterior.

We now fix  $\mu$  and show that  $\sigma^2$  has an IG conjugate. Let  $\lambda = \sigma^2$  to avoid confusion about the square. We claim that  $\lambda \sim IG(\lambda|\alpha, \beta)$  is conjugate.

**proof:**

$$p(\lambda) \propto \lambda^{-\alpha-1} \exp(-\beta/\lambda)$$

Then the posterior is,

$$\begin{aligned} p(\lambda|D) &\propto p(\lambda)p(D|\lambda) \propto \lambda^{-\alpha-1} \exp(-\beta/\lambda) \lambda^{-1} \exp\left(\frac{-1}{\lambda} \sum (x_i - \mu)^2\right) \\ &= \lambda^{-(\alpha+n/2)-1} \exp\left(\frac{-1}{\lambda} \left(\frac{1}{2} \sum (x_i - \mu)^2 + \beta\right)\right) \end{aligned}$$

From this we see that  $\lambda \sim IG(\alpha_n, \beta_n)$  where,

$$\alpha_n = \alpha + n/2 \text{ and } \beta_n = \frac{1}{2} \sum (x_i - \mu)^2 + \beta \quad (3)$$


---

### 3.1.2 Gamma for-Inverse Gamma

Now assume we have an Inverse-Gamma likelihood of the shape

$$p(D|\alpha, \beta) = \prod IG(x_i|\alpha, \beta) \propto \left(\prod x_i\right)^{-\alpha-1} e^{-\beta \sum \frac{1}{x_i}} \beta^\alpha \left(\frac{1}{\Gamma(\alpha)}\right)^n$$

Assume first that  $\alpha$  is fixed. We claim that  $\beta \sim \text{Gamma}(\beta|\alpha_o|\beta_o)$  is a conjugate prior.

**Proof:** The posterior can be calculated as,

$$p(\beta|D) \propto p(D|\beta)p(\beta) \propto e^{-\beta \sum \frac{1}{x_i}} \beta^\alpha \beta^{\alpha_o-1} e^{-\beta\beta_o}$$

$$= \beta^{n\alpha + \alpha_o} \exp\{-\beta(\beta_o + \sum \frac{1}{x_i})\}$$

Hence we can see that the resulting posterior is

$$\text{Gamma}(\beta \mid n\alpha + \alpha_o, \beta_o + \sum \frac{1}{x_i}) \quad (4)$$

Now we fix  $\beta$  and try to find a conjugate prior for  $\alpha$ . By expressing the likelihood above as an exponential family - by taking  $e^{log}$  - we get,

$$\begin{aligned} p(D|\alpha) &= e^{(-\alpha-1)\log(\prod x_i)} \beta^{n\alpha} e^{n(\log(1)-\log(\Gamma(\alpha)))} \\ &\propto e^{(-\alpha-1)\sum \log(x_i)} (\beta^\alpha e^{-\log(\Gamma(\alpha))})^n \\ &= e^{\phi(\alpha)\sum u_i(x_i)} [g(\alpha)]^n \end{aligned}$$

A conjugate prior will have the shape of,

$$\propto e^{\phi(\alpha)v} [g(\alpha)]^m$$

For some  $v$  and  $n$ . However, there is no obvious conjugate so we will use an alternative method (Metropolis-Hastings) to sample  $\alpha$ .

Nevertheless, we still need to assign a prior distribution for alpha. Since  $\sum \log(x_i)$  is a sufficient statistic for  $\alpha$ , it's reasonable to assume a log-normal prior.

---

### 3.2 Joint Posterior

Going back to our diagram. The joint posterior is the product of the likelihoods and priors which is the same as taking the product of the CPD in the graph. In other words, the product of the densities of each node given its parents. Here we are considering the full model where all the data is observed. If we considered the partially observed data then we need to use the joint distribution  $p(S_{t^x}^x, m_{t^x}^x)$  we computed above.

Let  $\Phi = (\alpha, \sigma_\Delta^2, \bigcup_x \alpha^x, a_t, b_t, \bigcup_x \tau^x, \beta_o, \beta_f, \beta_d, \sigma_b^2, \bigcup_x \bigcup_j b_j^x)$  be the list of all the model parameters.

$$p(\Phi|M, S) = p(\alpha)p(\sigma_\Delta^2) \prod_x p(\alpha^x|\alpha, \sigma_\Delta^2)p(a_t)p(b_t) \prod_x p(\tau^x|a_t, b_t) \prod_x \prod_j p(S_j^x|\alpha^x, \tau^x) \\ * p(\beta_o, \beta_f, \beta_d)p(\sigma_b^2) \prod_x \prod_j p(b_j^x|\mu_j^x, \sigma_b^2) \prod_x \prod_j p(M_j^x|f_j^x, b_j^x)$$


---

### 3.3 Conditional Posteriors

It's not easy to sample directly from the joint posterior distribution above. However, using MCMC sampling we focus at one parameter a time and consider everything else to be constant.

For this section we will derive the conditional posterior for each parameter using the two conjugates we proved above and using either the diagram or the joint posterior.

Using the joint posterior above, for parameter  $\zeta$  we only consider the densities that has  $\zeta$  in its parameterization.



### 3.3.1 $\alpha^x, \tau^x$

Using the concept above, we have the posterior to be,

$$p(\alpha^x | \bigcup_j S_j^x) \propto p(\alpha^x | \alpha, \sigma_\Delta^2) \prod_j p(S_j^x | \alpha^x, \tau^x) = \text{Prior X likelihood}$$

$$p(\tau^x | \bigcup_j S_j^x) \propto p(\tau^x | a_t, b_t) \prod_j p(S_j^x | \alpha^x, \tau^x) = \text{Prior X likelihood}$$

And since we defined the likelihood to be Gaussian, we define the prior/posterior of  $\alpha^x$  and  $\tau^x$  be Gaussian and Inverse-Gamma as shown earlier. We will use

$$p(\alpha^x | \bigcup_j S_j^x) \propto N(\alpha^x | \alpha, \sigma_\Delta^2) \prod_j N(S_j^x | \alpha^x, \tau^x)$$

$$p(\tau^x | \bigcup_j S_j^x) \propto IG(\tau^x | a_t, b_t) \prod_j N(S_j^x | \alpha^x, \tau^x)$$

And using the parameterization we derived in equations (1)(2)(3) we have that the posterior of  $\alpha^x$  is Gaussian with parameters:

$$\mu' = (M^x + \tau^x \sigma_\Delta^{-2})^{-1} \sum_j s_j^x$$

$$\sigma^{2'} = (M^x + \tau^x \sigma_\Delta^{-2})^{-1} \tau^x$$

and  $\tau^x$  is an IG with parameters:

$$\alpha' = a_t + \frac{M^x}{2}$$

$$\beta' = b_t + .5 \sum_{j=1}^{M^x} (s_j^x - \alpha^x)^2$$

Where  $M^x$  is the total number of retweets for root tweet  $x$ .

### 3.3.2 $\alpha, \sigma_\Delta^2$

Again, we do the same exact thing we did.

$$p(\alpha|\alpha^x) \propto p(\alpha|\mu_\alpha, \sigma_\alpha^2) \prod_x p(\alpha^x|\alpha, \sigma_\Delta^2)$$

$$p(\sigma_\Delta^2|\alpha^x) \propto p(\sigma_\Delta^2|a_\Delta, b_\Delta) \prod_x p(\alpha^x|\alpha, \sigma_\Delta^2)$$

Here we have the likelihood to be Gaussian as well. We choose the priors/posteriors to be Normal-IG where the paramters of the priors are noninformative.  $N(0, 100^2)$  and  $IG(.5, .5)$ . Hence,

$$p(\alpha|\alpha^x) \propto N(\alpha|\mu_\alpha, \sigma_\alpha^2) \prod_x N(\alpha^x|\alpha, \sigma_\Delta^2)$$

$$p(\sigma_\Delta^2|\alpha^x) \propto IG(\sigma_\Delta^2|a_\Delta, b_\Delta) \prod_x N(\alpha^x|\alpha, \sigma_\Delta^2)$$

And using the parameterization we derived in equations (1)(2)(3) we have that the posterior of  $\alpha$  to be Normal with parameters:

$$\mu' = (X + \sigma_\Delta^2 \sigma_\alpha^{-2})^{-1} \sum_x \alpha^x$$

$$\sigma^{2'} = (X + \sigma_\Delta^2 \sigma_\alpha^{-2})^{-1} \sigma_\Delta^2$$

and  $\sigma_\Delta^2$  is an IG with parameters:

$$\alpha' = a_\Delta + \frac{X}{2}$$

$$\beta' = b_\Delta + .5 \sum_{x=1}^X (\alpha^x - \alpha)^2$$

Where  $X$  is the number of root tweets.

### 3.3.3 $a_t, b_t$

As shown in 3.1.2, For an IG likelihood, Gamma is a conjugate prior for the rate parameter. Hence,

$$p(b_t|\tau^x) \propto p(b_t|k_b, \theta_b) \prod_x p(\tau^x|a_t, b_t) \propto G(b_t|k_b, \theta_b) \prod_x IG(\tau^x|a_t, b_t)$$

We choose the prior's parameters to be noninformative.  $k_b = .5, \theta_b = 1/500$ . The posterior can be evaluated using formula (4).

$$p(b_t|\tau^x) \propto Gamma(Xa_t + k_b, \theta_b + \sum_x \frac{1}{\tau^x})$$

As for  $a_t$  we will give it a noninformative log-normal(0, 10<sup>2</sup>) prior and a normal step-wise transition function,

$$Q(a_{t+1}|a_t) \sim Normal(a_t, .2)$$

Where the small variance is to guarantee making only small steps. We can then use M-H method to sample  $a_t$ .

### 3.3.4 $b_j^x$

$b_j^x$  represents the probability that a follower  $f_j^x$  will retweet the tweet  $x$ . We give it a logistic Normal prior and transition function. We take the logit to guarantee that it's bounded between 0 and 1. Moreover, its mean is a Bayesian linear regression model of  $f_j^x$  and  $d_j^x$ . More on that later. We will use M-H to sample  $b_j^x$ .

### 3.3.5 $\sigma_b^2$

Since the likelihood is logit-Normal, if we transformed the  $x$ s by  $\text{logit}(x)$  we get the a usual Normal likelihood and we can give  $\sigma_b^2$  and conjugate prior/posterior Inverse-Gamma().

The prior's parameters are noninformative :  $p(\sigma_b^2 | a_{\sigma_b}, b_{\sigma_b}) \sim IG(.5, .5)$ .

The posteriors parameters are calculated using formula (3) to be:

$$p(\sigma_b^2 | \bigcup_x \bigcup_j b_j^x) \sim IG(a', b')$$

where,

$$a' = a_{\sigma_b} + \frac{N}{2}$$

$$b' = b_{\sigma_b} + .5 \sum_x \sum_j (\text{logit}(b_j^x) - \mu_j^x)^2$$

Where  $N$  is the total number of retweets in the dataset. ie.  $\sum_x \sum_j (1)$

### 3.3.6 $\mu_j^x, \beta$

We assumed that the mean of  $b_j^x$  to be related to the number of followers  $f_j^x$  and the distance from the root  $d_j^x$ .

Fitting a linear model would be a reasonable choice.

Let  $X = \{1, \log(f_j^x + 1), \log(d_j^x + 1)\}$  and  $\beta = \{\beta_o, \beta_f, \beta_d\}$  Then the likelihood can be expressed as

$$\text{logit}(b_j^x) \sim \text{Normal}(\beta^T X, \sigma_b^2)$$

Let us assume a noninformative Multivariate-Gaussian prior on  $\beta$  with parameters:

$$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and covariance matrix } S = \begin{pmatrix} 100^2 & 0 & 0 \\ 0 & 100^2 & 0 \\ 0 & 0 & 100^2 \end{pmatrix}$$

The log posterior distribution is then,

$$\begin{aligned} \log p(\beta | \bigcup_x \bigcup_j \text{logit}(b_j^x)) &\propto \log p(\beta | \mu, S) + \sum_x \sum_j \log p(\text{logit}(b_j^x) | \beta) \\ &\propto -\frac{1}{2} \beta^T S^{-1} \beta - \frac{1}{2\sigma_b^2} \sum_x \sum_j (\text{logit}(b_j^x) - \beta^T X) \end{aligned}$$

Assume  $X$  has the shape (3,N),  $\beta$  has the shape (3,1) and  $L = \bigcup_x \bigcup_j \text{logit}(b_j^x)$  has the shape (N,1) Then the equation above can be vectorized to,

$$= -\frac{1}{2} [\beta^T S^{-1} \beta + \frac{1}{\sigma_b^2} \{\beta^T X^T X \beta - 2L^T X \beta + L^T L\}]$$

Matching that with the density of a multivariate Gaussian with Covariance matrix  $\Sigma$  and mean vector  $\mu$

$$-\frac{1}{2}(\beta - \mu)^T \Sigma^{-1}(\beta - \mu)$$

We get the following parameters for the posterior multivariate Gaussian of  $\beta$

$$\text{Precision Matrix: } \Sigma^{-1} = \sigma_b^{-2} X^T X + S^{-1}$$

$$\mu = \sigma_b^{-2} \Sigma X^T L$$

---

## 4 Implementation

## 5 References

### References

- [1] pennsylvania state university *Applied Multivariate Statistical Analysis*.
- [2] University at Buffalo *Probabilistic Graphical Models*.
- [3] Daphne Koller *Probabilistic Graphical Models*.