Problem 1)

a. $\frac{1}{1+e^{a}} = 1 - \frac{1}{1+e^{-a}} \Rightarrow LCM \, (e^{a}+1)(e^{-a}+1)$

$\Rightarrow \frac{(e^{a}+1)(e^{-a}+1)}{1+e^{a}} = (e^{a}+1)(e^{-a}+1) - (e^{a}+1)$

$\Rightarrow (e^{a})^{-1} + 1 = (e^{a}+1)((e^{a})^{-1}+1) - (e^{a}+1)$

$e^{a} = y$

$\Rightarrow (y)^{-1} + 1 = (y+1)((y)^{-1}+1) - (y+1)$

i.e $y = 1$

$\Rightarrow$

$(1)^{-1} + 1 = 2 \cdot (1^{-1} + 1) - 2$

$2 = 4 - 2$

$2 = 2$

r.a.a

(b) $\frac{d}{da}\sigma(a) = (1-\sigma(a))\sigma(a)$

since $\sigma(-a) = 1-\sigma(a)$

$\Rightarrow \frac{d\sigma(a)}{da} = \frac{d}{da}\frac{1}{1+e^{-1}} \Rightarrow \frac{d}{da}(1+e^{-1})^{-1}$

$\Rightarrow -(1+e^{-a})^{-2}\cdot(-e^{-a}) \Rightarrow \frac{e^{-a}}{(1+e^{-a})^2}$

$\Rightarrow \frac{1}{1+e^{-a}}\cdot\frac{e^{-a}}{1+e^{-a}} \Rightarrow \sigma(a)\cdot\sigma(-a)$

$\Rightarrow (1-\sigma(a))\cdot\sigma(a)$

(c) $\sigma^{-1}(b) = \log\frac{b}{1-b}$

Problem 2:

$$\log h_\theta(x^i) = \log \frac{1}{1+e^{-\theta^T x^i}}$$
$$= -\log\left(1+e^{-\theta^T x^i}\right)$$

$$\log(1-h_\theta(x^i)) = \log\left(1 - \frac{1}{1+e^{-\theta^T x^i}}\right)$$
$$= \log\left(e^{-\theta^T x^i}\right) - \log\left(1+e^{-\theta^T x^i}\right)$$
$$= -\theta^T x^i - \log\left(1+e^{-\theta^T x^i}\right)$$

Insert these two terms in $J(\theta)$:

$$J(\theta) = -\frac{1}{m}\sum_{i=1}^{m}\left[-y^i\left(\log\left(1+e^{-\theta^T x^i}\right)\right) + (1-y^i)\cdot\left(-\theta^T x^i - \log\left(1+e^{-\theta^T x^i}\right)\right)\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[y^i\theta^T x^i - \theta^T x^i - \log\left(1+e^{-\theta^T x^i}\right)\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\Big[\underbrace{y^i\theta^T x^i}_{a} - \underbrace{\log\left(1+e^{\theta^T x^i}\right)}_{b}\Big]$$

Partial derivatives of a and b to θ:

$$a = y_i x^i$$
$$b = \frac{x^i e^{\theta^T x^i}}{1+e^{\theta^T x^i}}$$
$$= x^i h_\theta(x^i)$$

$$\frac{\partial}{\partial\theta_j}J(\theta) = -\frac{1}{m}\sum_{i=1}^{m}\left[y^i x^i - x^i h_\theta(x^i)\right]$$

$$= -\frac{1}{m}\sum\left[x^i\left(h_\theta(x^i) - y^i\right)\right]$$