# Informed search algorithms

## Chapter 4

# Outline

- ◊ Best-first search
- ◊ A$^*$ search
- ◊ Heuristics
- ◊ Hill-climbing
- ◊ Simulated annealing
- ◊ Genetic algorithms
- ◊ Local search in continuous spaces

# Review: Tree search

```
function Tree-Search( problem, fringe) returns a solution, or failure
   fringe ← Insert(Make-Node(Initial-State[problem]), fringe)
   loop do
      if fringe is empty then return failure
      node ← Remove-Front(fringe)
      if Goal-Test[problem] applied to State(node) succeeds return node
      fringe ← InsertAll(Expand(node, problem), fringe)
```

A strategy is defined by picking the *order of node expansion*

# Best-first search

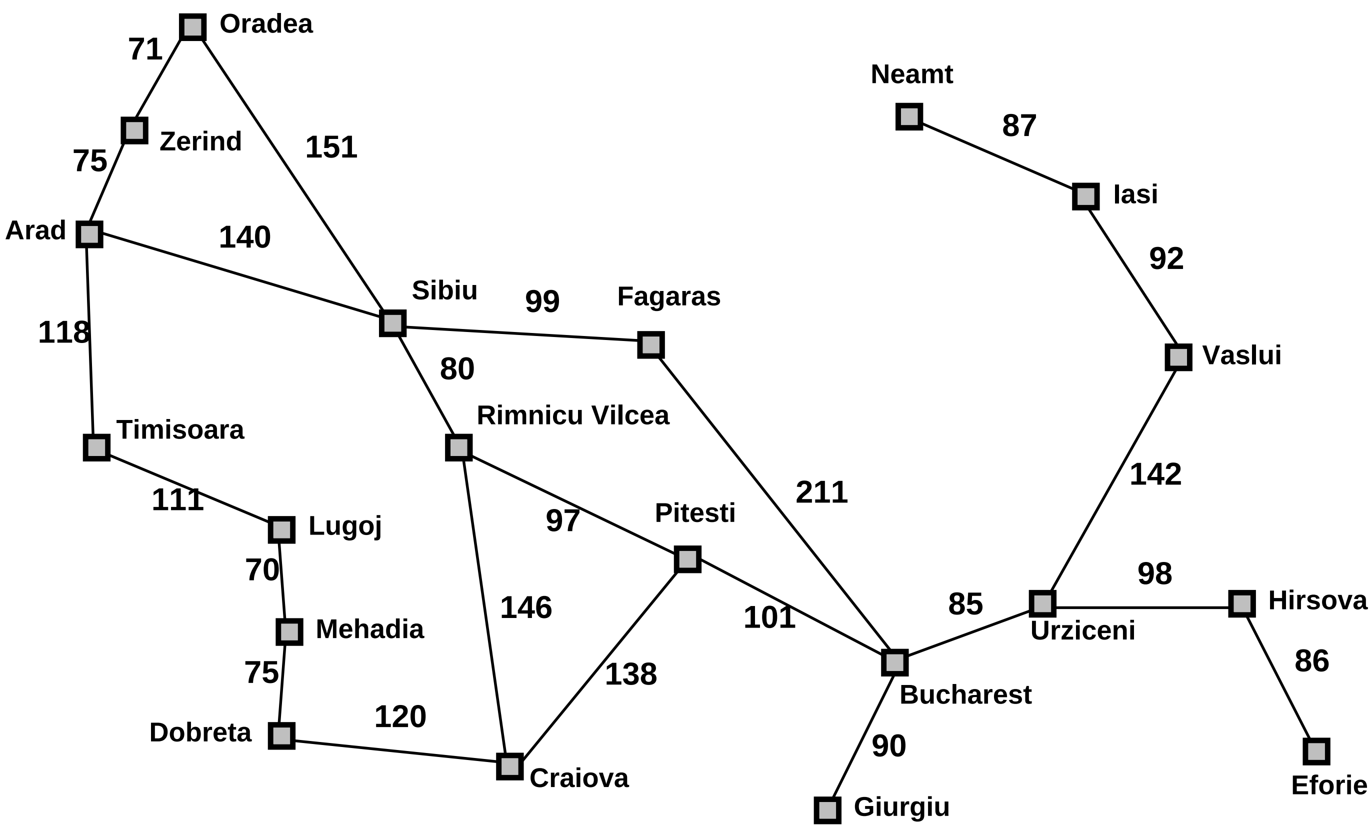
Idea: use an *evaluation function* for each node
– estimate of "desirability"

$\Rightarrow$ Expand most desirable unexpanded node

Implementation:
$fringe$ is a queue sorted in decreasing order of desirability

Special cases:
greedy search
$\text{A}^*$ search

# Romania with step costs in km




Straight−line distance
to Bucharest

| | |
|---|---|
| **Arad** | 366 |
| **Bucharest** | 0 |
| **Craiova** | 160 |
| **Dobreta** | 242 |
| **Eforie** | 161 |
| **Fagaras** | 178 |
| **Giurgiu** | 77 |
| **Hirsova** | 151 |
| **Iasi** | 226 |
| **Lugoj** | 244 |
| **Mehadia** | 241 |
| **Neamt** | 234 |
| **Oradea** | 380 |
| **Pitesti** | 98 |
| **Rimnicu Vilcea** | 193 |
| **Sibiu** | 253 |
| **Timisoara** | 329 |
| **Urziceni** | 80 |
| **Vaslui** | 199 |
| **Zerind** | 374 |

# Greedy search

Evaluation function $h(n)$ (heuristic)
$= $ estimate of cost from $n$ to the closest goal

E.g., $h_{\mathrm{SLD}}(n) =$ straight-line distance from $n$ to Bucharest

Greedy search expands the node that *appears* to be closest to goal

# Greedy search example

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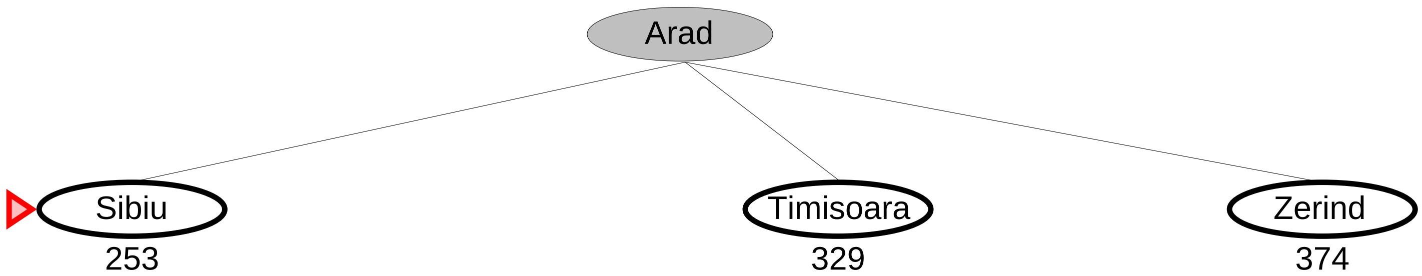

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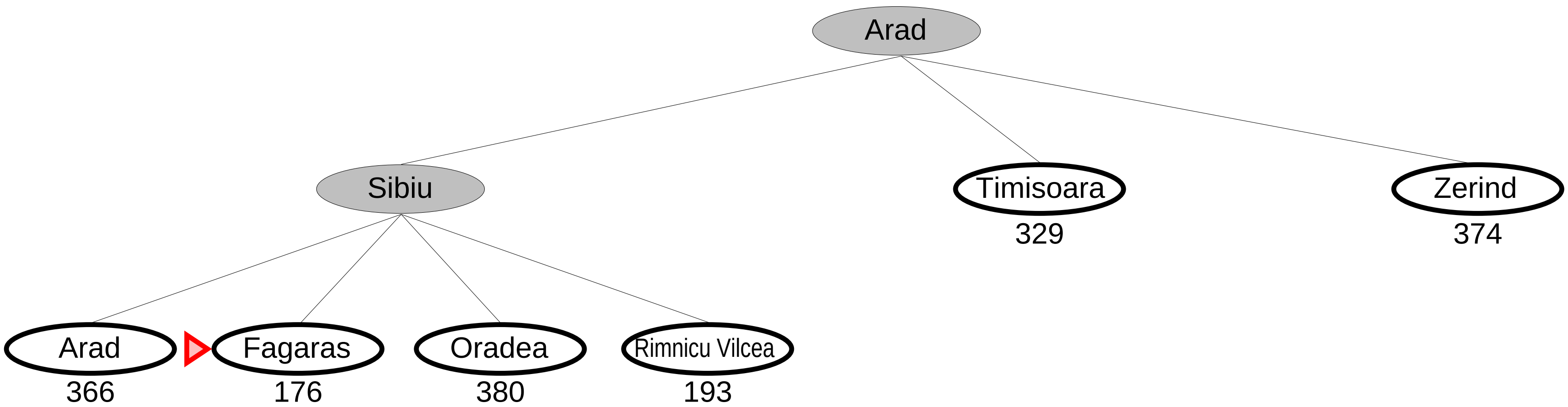

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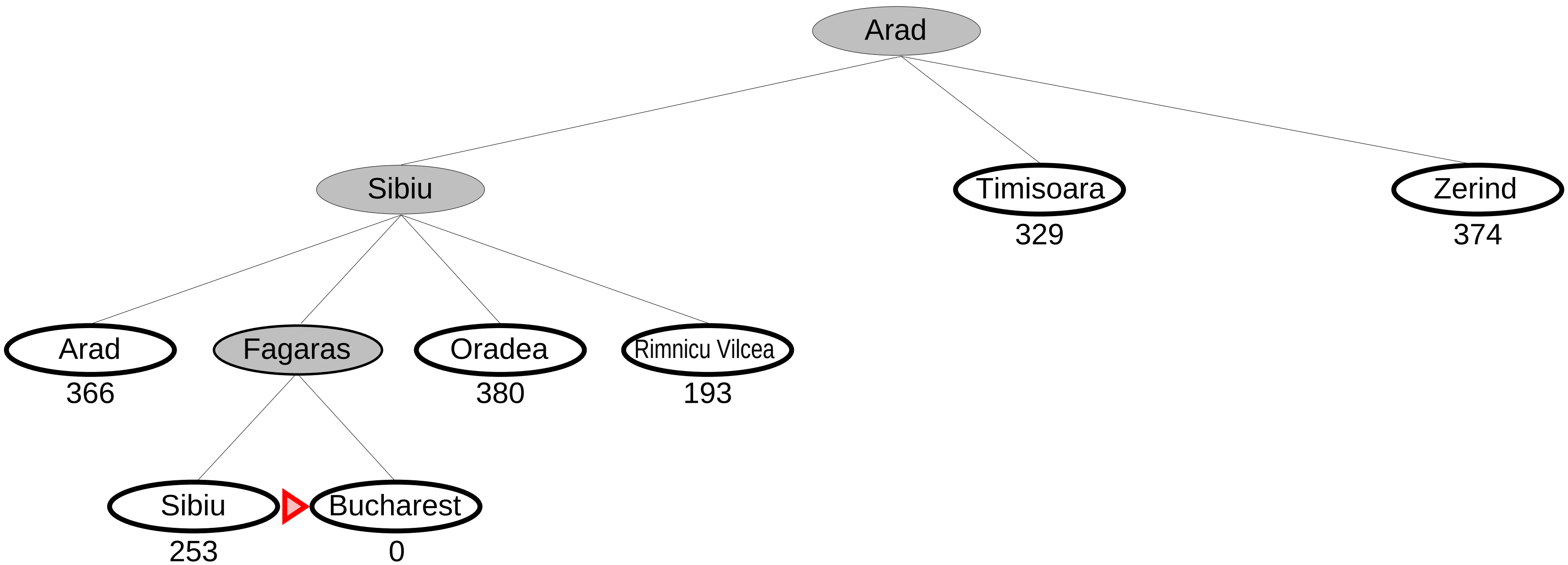

# Properties of greedy search

Complete??

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Complete?? No–can get stuck in loops, e.g., with Oradea as goal,
Iasi $\to$ Neamt $\to$ Iasi $\to$ Neamt $\to$
Complete in finite space with repeated-state checking

Time??

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Optimal?? No

# A* search

Idea: avoid expanding paths that are already expensive

Evaluation function $f(n) = g(n) + h(n)$

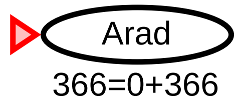
$g(n)$ = cost so far to reach $n$
$h(n)$ = estimated cost to goal from $n$
$f(n)$ = estimated total cost of path through $n$ to goal

A* search uses an *admissible* heuristic
i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the *true* cost from $n$.
(Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal $G$.)

E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance

Theorem: A* search is optimal

# A* search example



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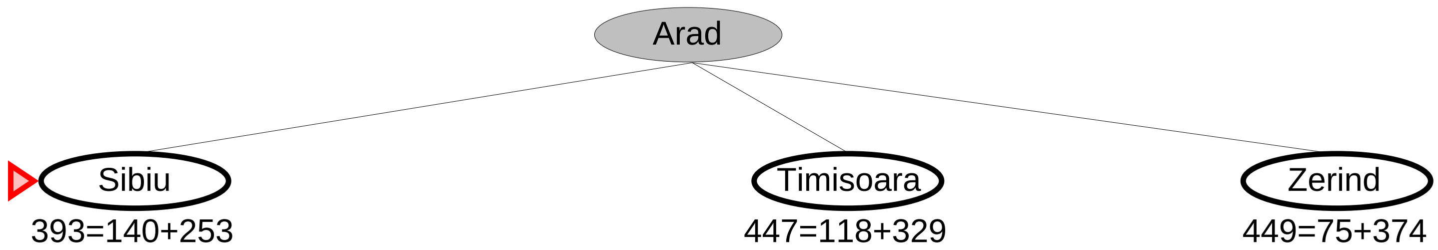

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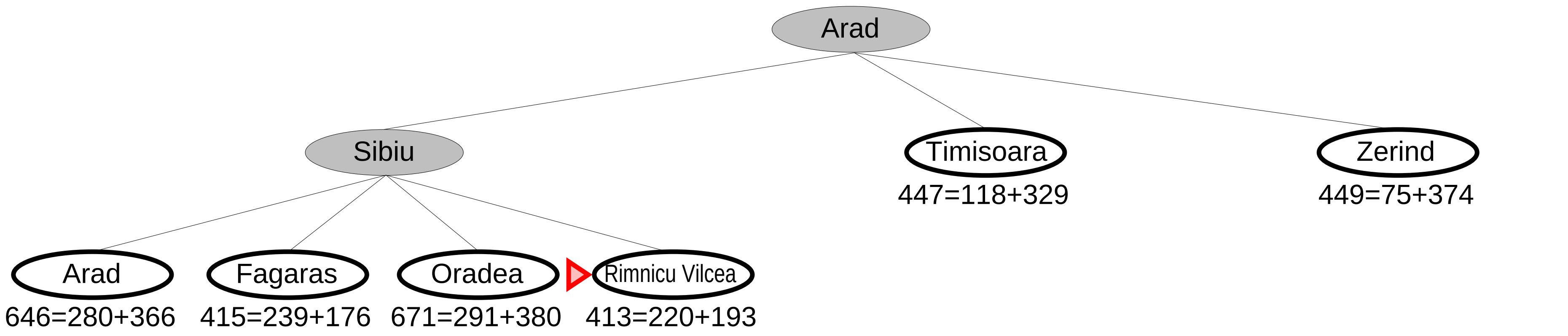

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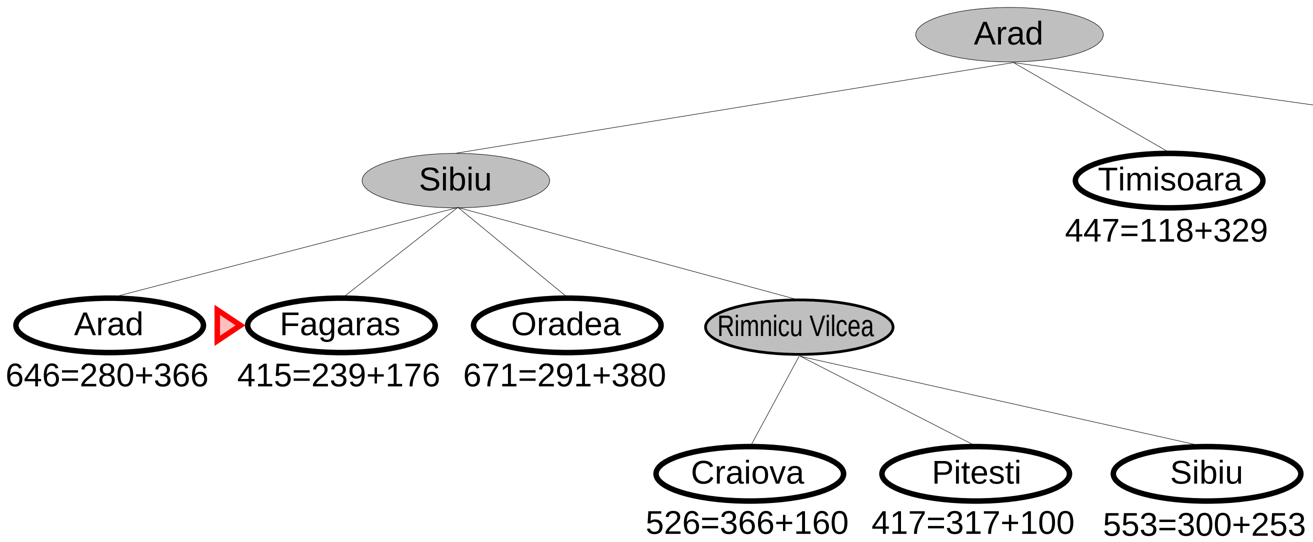

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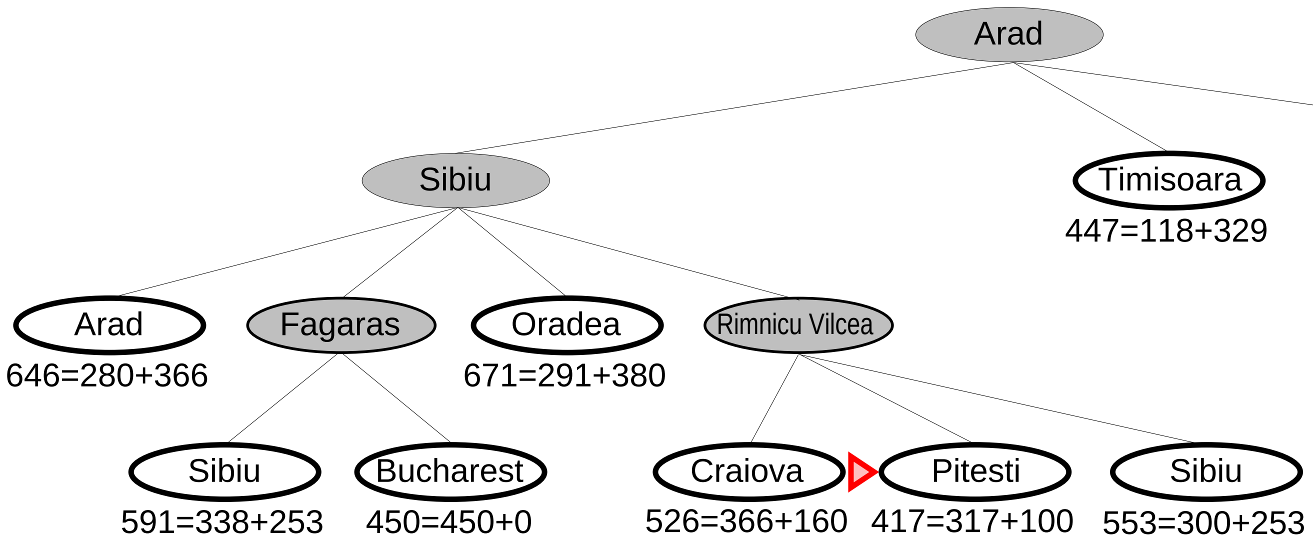

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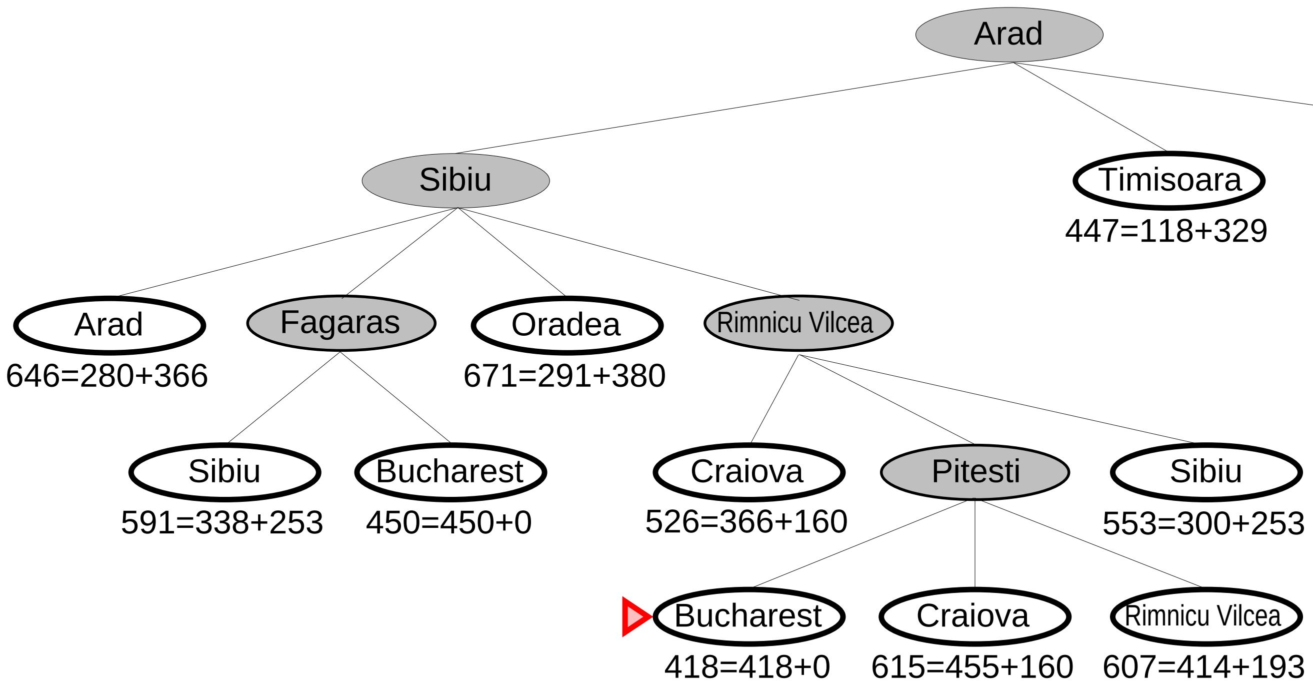
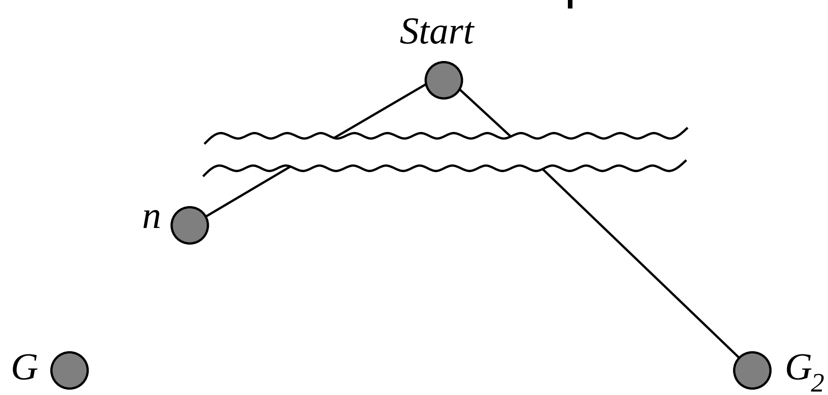


# Optimality of A$^*$ (standard proof)

Suppose some suboptimal goal $G_2$ has been generated and is in the queue.
Let $n$ be an unexpanded node on a shortest path to an optimal goal $G_1$.



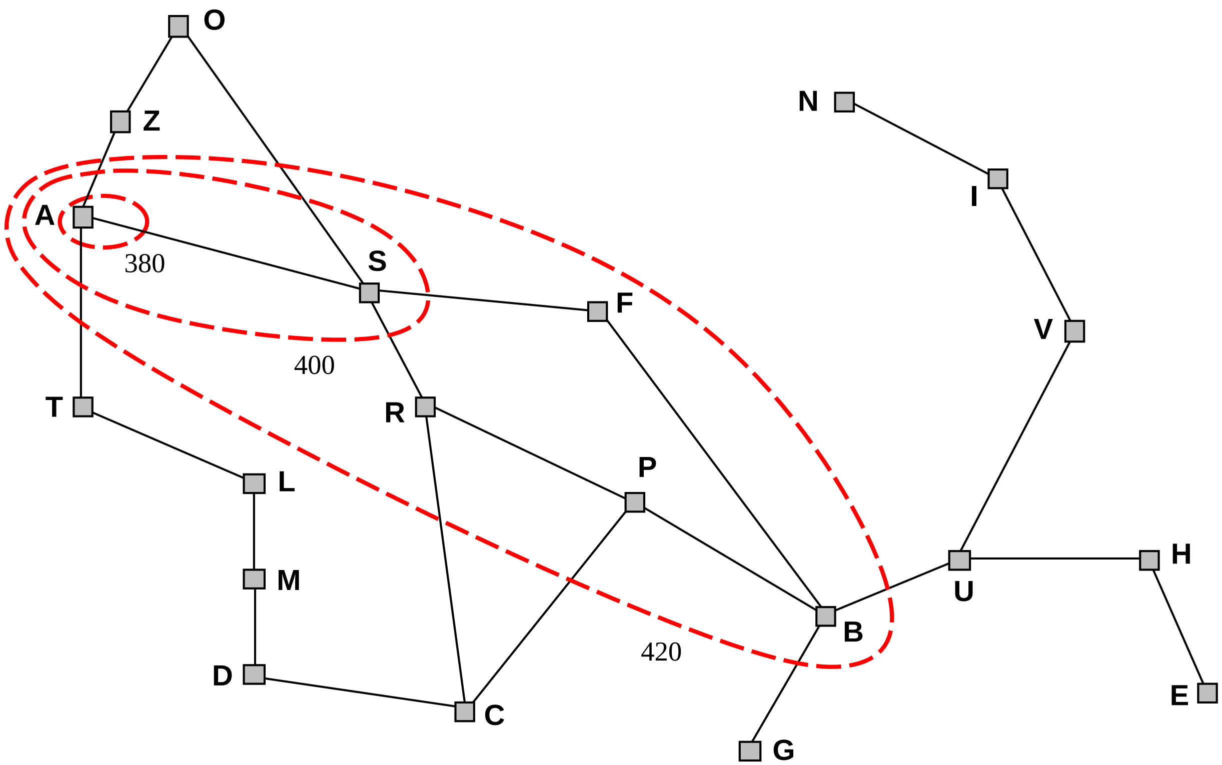

$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G_1) && \text{since } G_2 \text{ is suboptimal} \\
&\geq f(n) && \text{since } h \text{ is admissible}
\end{aligned}
$$

Since $f(G_2) > f(n)$, A$^*$ will never select $G_2$ for expansion

# Optimality of A$^*$ (more useful)

Lemma: A$^*$ expands nodes in order of increasing $f$ value$^*$

Gradually adds "$f$-contours" of nodes (cf. breadth-first adds layers)
Contour $i$ has all nodes with $f = f_i$, where $f_i < f_{i+1}$



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# Properties of A$^*$

Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time?? Exponential in [relative error in $h$ $\times$ length of soln.]

Space?? Keeps all nodes in memory

Optimal?? Yes—cannot expand $f_{i+1}$ until $f_i$ is finished

A$^*$ expands all nodes with $f(n) < C^*$
A$^*$ expands some nodes with $f(n) = C^*$
A$^*$ expands no nodes with $f(n) > C^*$

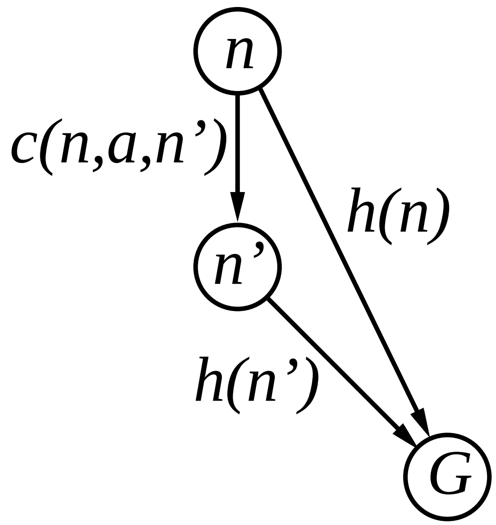
# Proof of lemma: Consistency

A heuristic is *consistent* if

$$h(n) \leq c(n,a,n') + h(n')$$

If $h$ is consistent, we have

$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n,a,n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

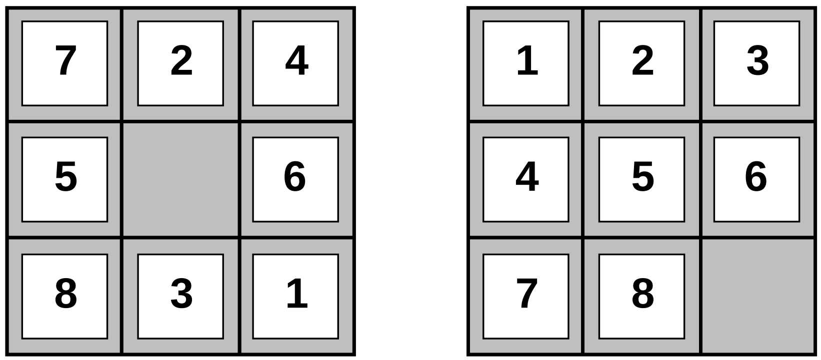



I.e., $f(n)$ is nondecreasing along any path.

# Admissible heuristics

E.g., for the 8-puzzle:

$h_1(n)$ = number of misplaced tiles
$h_2(n)$ = total Manhattan distance
(i.e., no. of squares from desired location of each tile)




**Start State** **Goal State**

$\underline{h_1(S) =}$??
$\underline{h_2(S) =}$??

# Admissible heuristics

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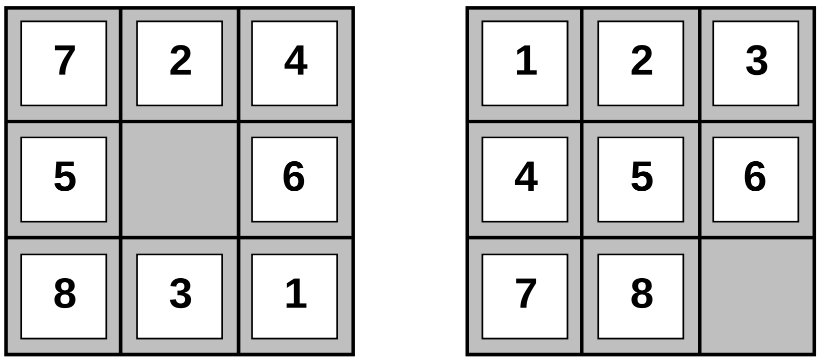



Start State Goal State

$h_1(S) =??$ 6
$h_2(S) =??$ 4+0+3+3+1+0+2+1 = 14

# Dominance

If $h_2(n) \geq h_1(n)$ for all $n$ (both admissible)
then $h_2$ *dominates* $h_1$ and is better for search

Typical search costs:

$d = 14$ IDS = 3,473,941 nodes
A$^*(h_1)$ = 539 nodes
A$^*(h_2)$ = 113 nodes
$d = 24$ IDS $\approx$ 54,000,000,000 nodes
A$^*(h_1)$ = 39,135 nodes
A$^*(h_2)$ = 1,641 nodes

# Relaxed problems

Admissible heuristics can be derived from the *exact*
solution cost of a *relaxed* version of the problem

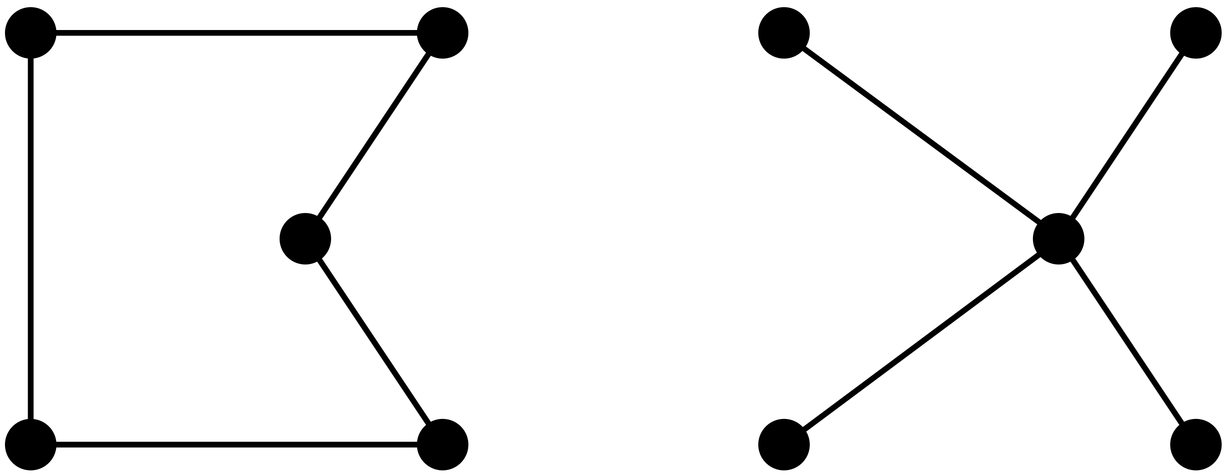
If the rules of the 8-puzzle are relaxed so that a tile can move *anywhere*, then $h_1(n)$ gives the shortest solution

If the rules are relaxed so that a tile can move to *any adjacent square*, then $h_2(n)$ gives the shortest solution

Key point: the optimal solution cost of a relaxed problem
is no greater than the optimal solution cost of the real problem

# Relaxed problems contd.

Well-known example: travelling salesperson problem (TSP)
Find the shortest tour visiting all cities exactly once



Minimum spanning tree can be computed in $O(n^2)$
and is a lower bound on the shortest (open) tour

# Iterative improvement algorithms

In many optimization problems, *path* is irrelevant;
the goal state itself is the solution

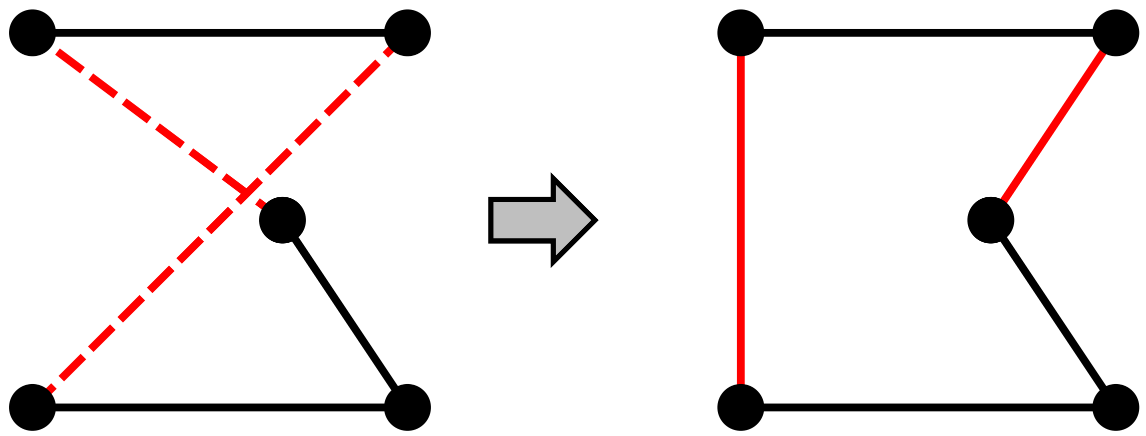
Then state space = set of "complete" configurations;
find *optimal* configuration, e.g., TSP
or, find configuration satisfying constraints, e.g., timetable

In such cases, can use *iterative improvement* algorithms;
keep a single "current" state, try to improve it

Constant space, suitable for online as well as offline search

# Example: Travelling Salesperson Problem

Start with any complete tour, perform pairwise exchanges

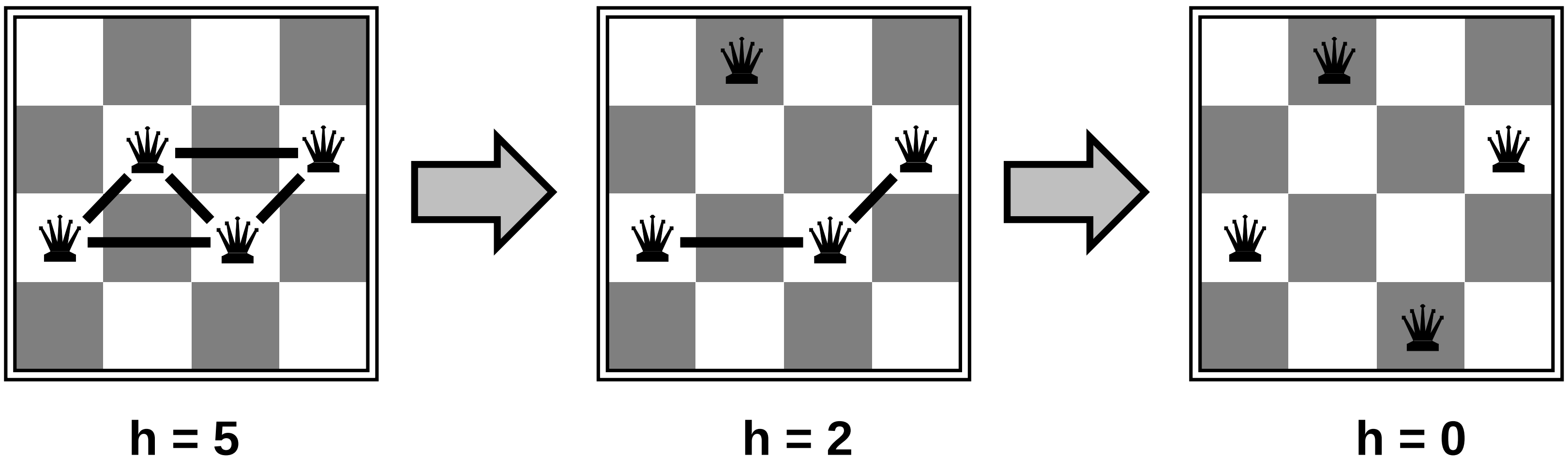


Variants of this approach get within 1% of optimal very quickly with thousands of cities

# Example: $n$-queens

Put $n$ queens on an $n \times n$ board with no two queens on the same row, column, or diagonal

Move a queen to reduce number of conflicts




Almost always solves $n$-queens problems almost instantaneously for very large $n$

# Hill-climbing (or gradient ascent/descent)

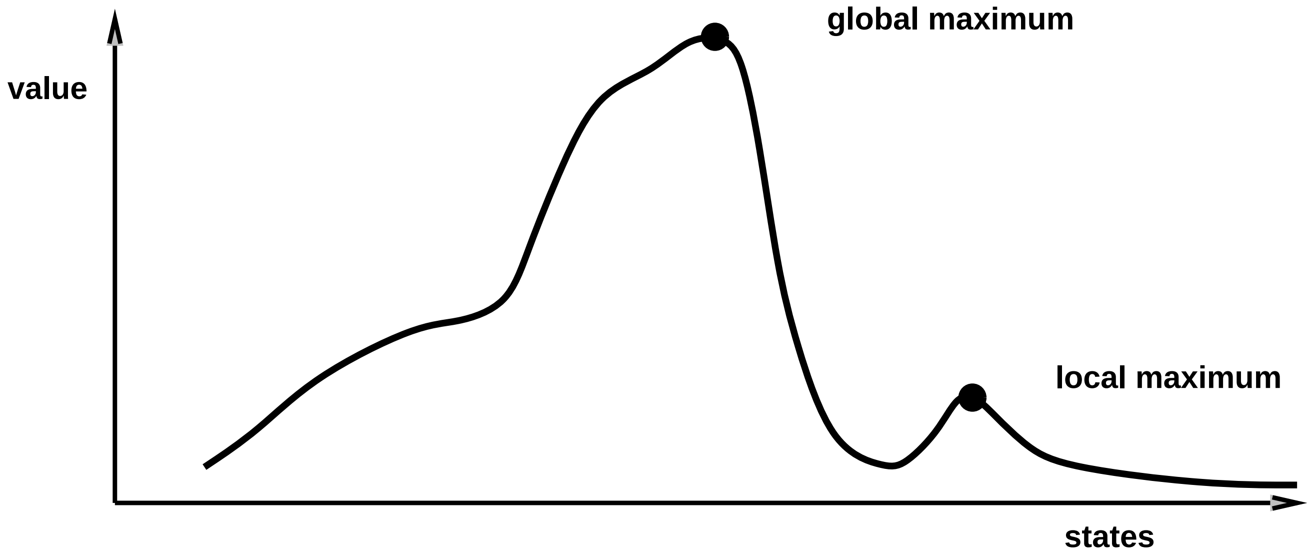
"Like climbing Everest in thick fog with amnesia"

```
function HILL-CLIMBING(problem) returns a state that is a local maximum
   inputs: problem, a problem
   local variables: current, a node
                    neighbor, a node

   current ← MAKE-NODE(INITIAL-STATE[problem])
   loop do
        neighbor ← a highest-valued successor of current
        if VALUE[neighbor] < VALUE[current] then return STATE[current]
        current ← neighbor
   end
```

# Hill-climbing contd.

Problem: depending on initial state, can get stuck on local maxima




In continuous spaces, problems w/ choosing step size, slow convergence

# Simulated annealing

Idea: escape local maxima by allowing some "bad" moves
*but gradually decrease their size and frequency*

**function** SIMULATED-ANNEALING( *problem*, *schedule*) **returns** a solution state
  **inputs**: *problem*, a problem
    *schedule*, a mapping from time to "temperature"
  **local variables**: *current*, a node
    *next*, a node
    $T$, a "temperature" controlling prob. of downward steps

  *current* ← MAKE-NODE(INITIAL-STATE[*problem*])
  **for** $t \leftarrow 1$ **to** $\infty$ **do**
    $T \leftarrow$ *schedule*[$t$]
    **if** $T = 0$ **then return** *current*
    *next* ← a randomly selected successor of *current*
    $\Delta E \leftarrow$ VALUE[*next*] – VALUE[*current*]
    **if** $\Delta E > 0$ **then** *current* ← *next*
    **else** *current* ← *next* only with probability $e^{\Delta E/T}$

# Properties of simulated annealing

At fixed "temperature" $T$, state occupation probability reaches Boltzman distribution

$$p(x) = \alpha e^{\frac{E(x)}{kT}}$$

$T$ decreased slowly enough $\Longrightarrow$ always reach best state

Is this necessarily an interesting guarantee??

Devised by Metropolis et al., 1953, for physical process modelling

Widely used in VLSI layout, airline scheduling, etc.

# Local beam search

Idea: keep $k$ states instead of 1; choose top $k$ of all their successors

Not the same as $k$ searches run in parallel!
Searches that find good states recruit other searches to join them

Problem: quite often, all $k$ states end up on same local hill

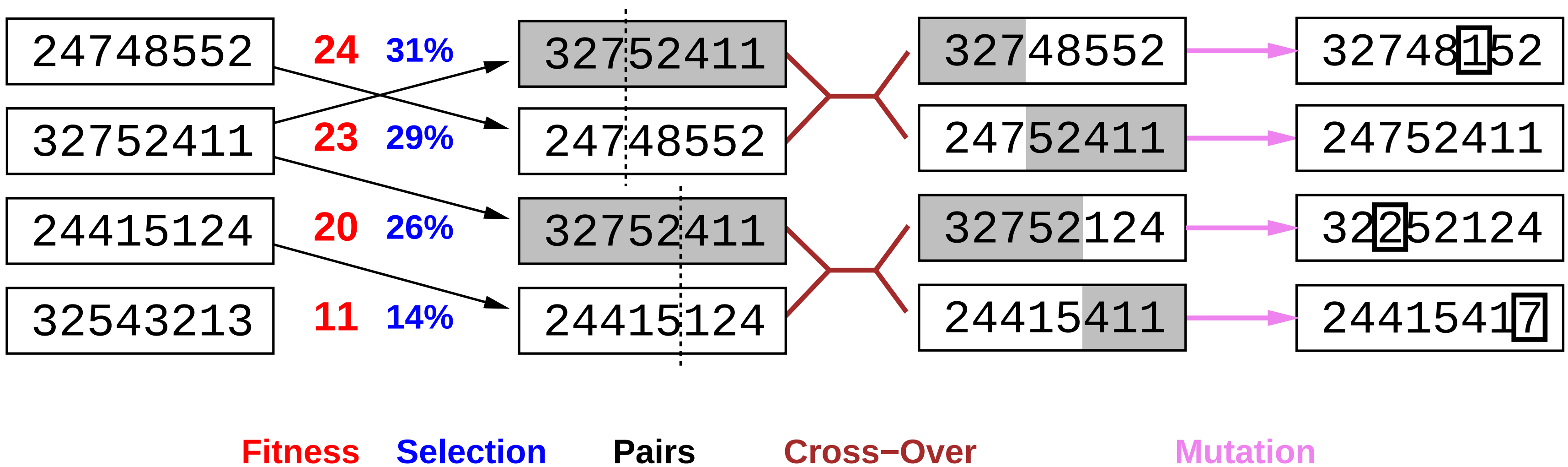
Idea: choose $k$ successors randomly, biased towards good ones

Observe the close analogy to natural selection!

# Genetic algorithms

Search and optimization methods inspired by natural selection

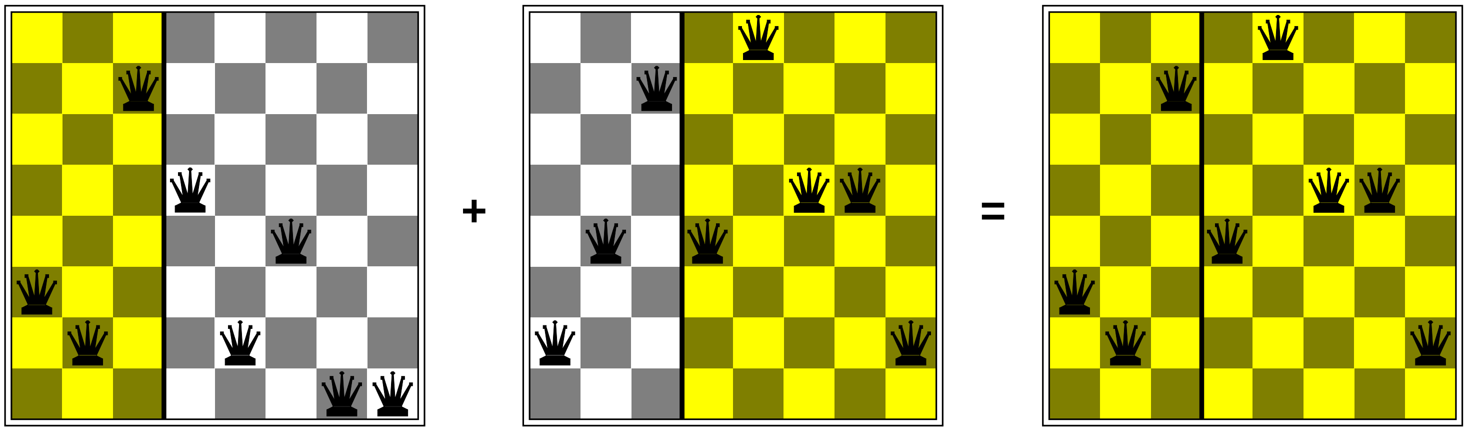
Population based stochastic search + generate successors from **pairs** of states



# Genetic algorithms contd.

GAs require states encoded as strings (GPs use trees reprsenting programs)

Crossover helps when substrings are meaningful components (decomposable problems)



GAs and GPs are examples of Evolutionary Computation methods

# Continuous state spaces

Suppose we want to site three airports in Romania:
- 6-D state space defined by $(x_1, y_2)$, $(x_2, y_2)$, $(x_3, y_3)$
- objective function $f(x_1, y_2, x_2, y_2, x_3, y_3) =$
  sum of squared distances from each city to nearest airport

Discretization methods turn continuous space into discrete space, e.g., empirical gradient considers $\pm\delta$ change in each coordinate

Gradient methods compute

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial y_3}\right)$$

to increase/reduce $f$, e.g., by $\mathbf{x} \leftarrow \mathbf{x} + \alpha \nabla f(\mathbf{x})$

Sometimes can solve for $\nabla f(\mathbf{x}) = 0$ exactly (e.g., with one city).
Newton–Raphson (1664, 1690) iterates $\mathbf{x} \leftarrow \mathbf{x} - \mathbf{H}_f^{-1}(\mathbf{x}) \nabla f(\mathbf{x})$
to solve $\nabla f(\mathbf{x}) = 0$, where $\mathbf{H}_{ij} = \partial^2 f / \partial x_i \partial x_j$