Chapter 6

Fourier Transform Applications

Amplitude spectrum and phase

A related approach to displaying the results of spectral analysis is to show the amplitude spectrum AS ,the square root of the power spectrum. If one wants the amplitude in the spectrum to correspond with the amplitude of sinusoidal signals in the time domain, one must normalize by 2/N:

$$AS = \frac{2}{N}\sqrt{XX^*}$$

A third commonly used presentation is the phase spectrum. This depicts the phase versus frequency. with I(X) and R(X) denoting the imaginary and real parts of X, respectively. Unlike in the power and amplitude spectra no normalization is required for the phase.

$$\varphi = \arctan I(X)/R(X)$$

relationships between the epoch length and sample rate with precision and range of spectrum.

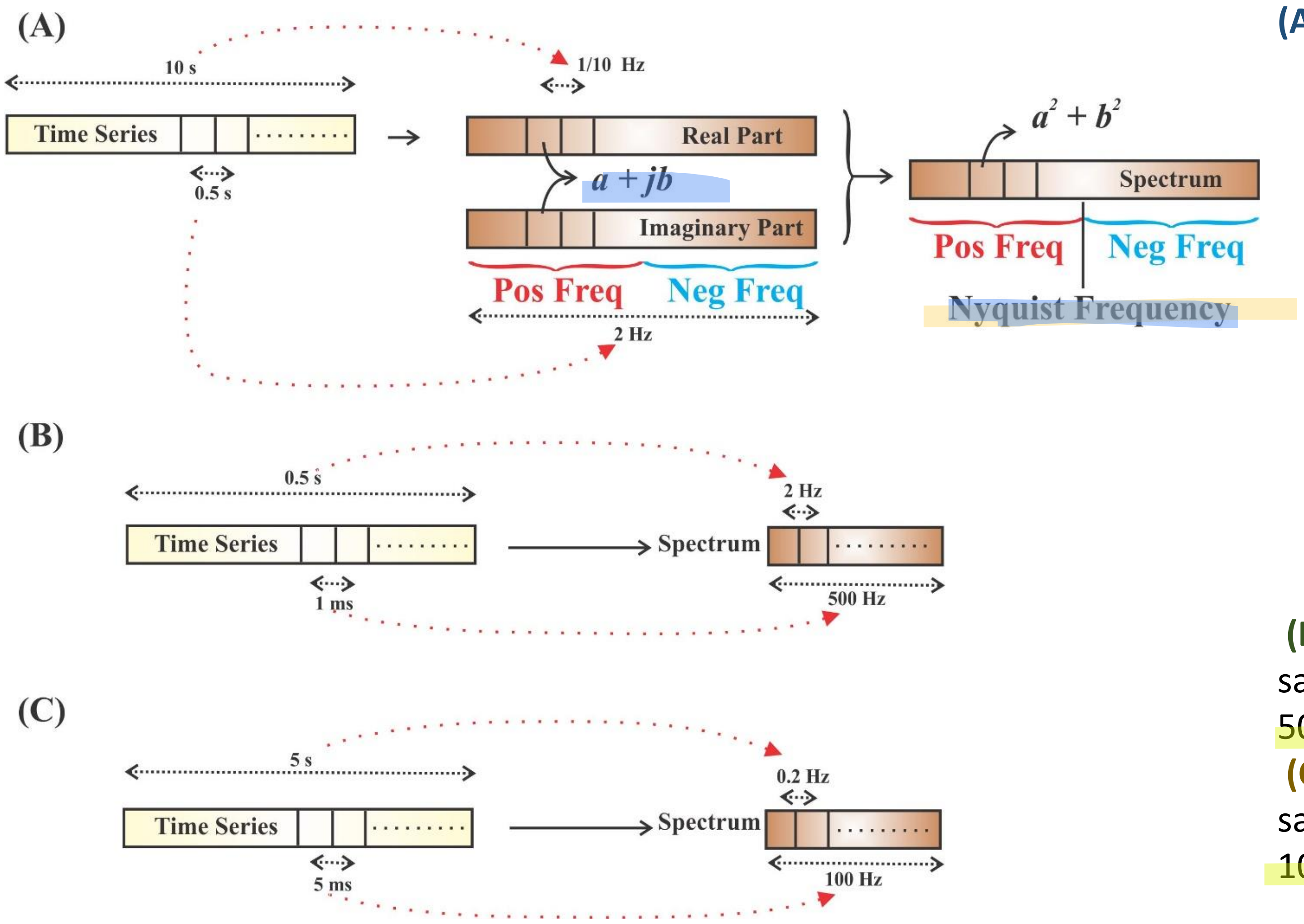
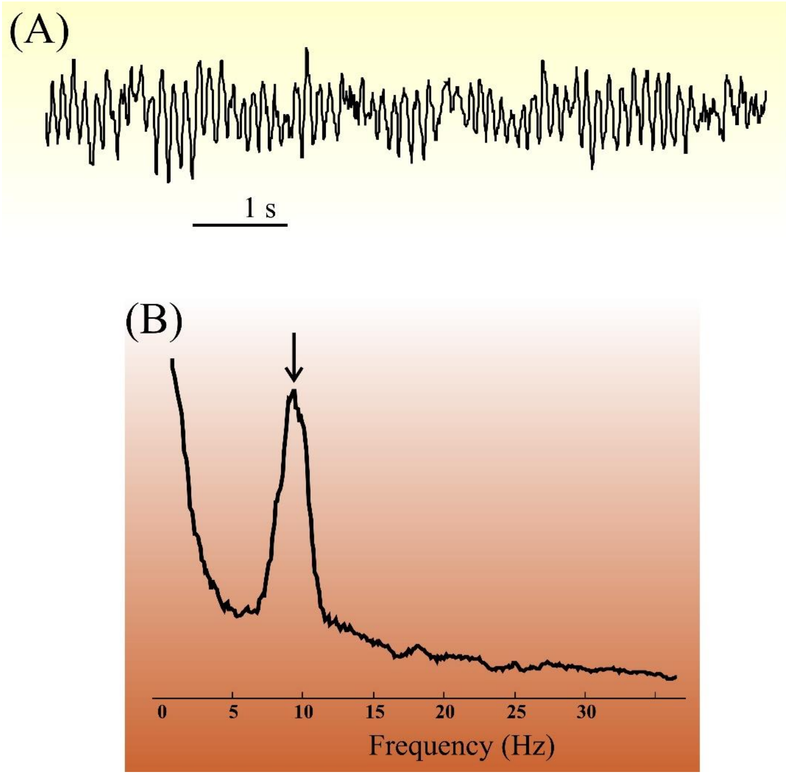


Figure 7.1, Signal processing for Neuroscientist, second edition by Wim van Drongelen

(A) An example of a time series sampled at an interval of 0.5 s for 10 s. The discrete Fourier transform consists of even (real) and odd (imaginary) parts in which the range (2 Hz) and resolution (1/10 Hz) are directly related to the time series. The spectrum resulting from these real and imaginary coefficients is even. The frequency scale can be represented as a full circle where $0 - \pi$ can be considered positive frequencies and $\pi - 2\pi$ as negative ones. Because the power spectrum is even, the part reflecting the negative frequencies is identical to the part containing positive frequencies, and therefore it is common practice to depict only the first half of the spectrum (up to the Nyquist frequency).

(B) An example of a 0.5-s epoch sampled at 1 kHz (1 ms sample interval) resulting in 1/0.5 =2 Hz precision and 1000/2 = 500 Hz range.

(C) An example of a 5-s epoch sampled at 200 Hz (5 ms sample interval) resulting in 1/5 = 0.2 Hz precision and 200/2 = 100 Hz range

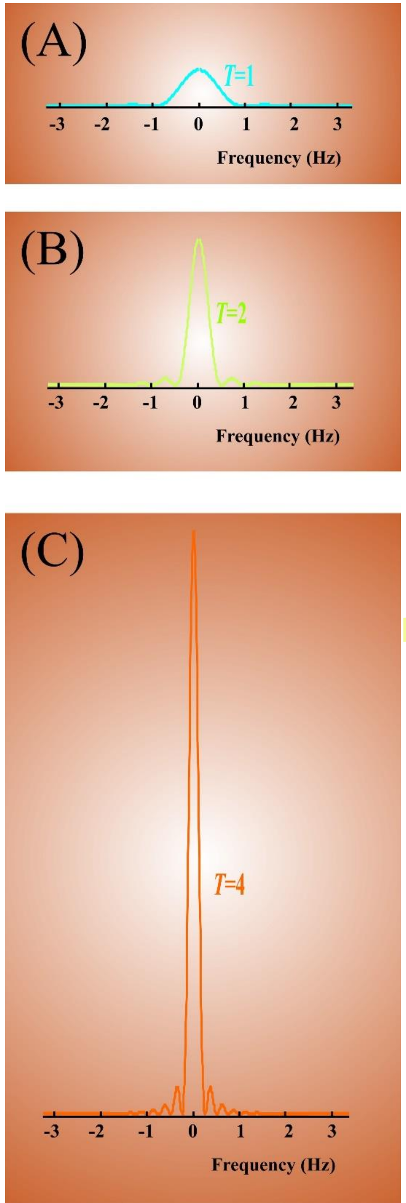


An example of spectral analysis of an electroencephalography trace recorded from position O2 shown in (A). The trace includes strong oscillation in the **alpha band**. Accordingly, the power spectrum in (B) shows the clear presence of a component slightly below 10 Hz (arrow) representing this alpha rhythm. For clarity, the spectrum in (B) was smoothed in the frequency domain using a rectangular 1.5-Hz window

Figure 7.3, Signal processing for Neuroscientist, second edition by Wim van Drongelen

In this examples the spectrum is determined from a finite epoch of data. Because we evaluate the signal over a limited time interval consisting of N samples, we are implicitly multiplying the theoretically infinite input signal of the FFT (or DFT) algorithm with a rectangular function (i.e., a rectangular data window). As we will see in the Fourier transform section, this multiplication in the time domain corresponds with a convolution in the frequency domain. Because the DFT/FFT is determined by default over a limited epoch of a time series, we can analyze the effect of such a limitation using the continuous Fourier transform (CFT).

The power spectrum of rectangular windows



It can be seen that these spectra have ripples in the frequency domain that correspond to the inverse of their window duration, i.e., for T = 4 the ripples are at 0.25Hz, for T = 2 the ripples are at 0.5 Hz, and for T = 1 the ripples are at 1 Hz. This ripple effect causes the discrete spectrum of a pure wave such as cos(2πft) or sin(2πft) to show energy adjacent to the main peak at frequency.

Transform of a cosine wave truncated by a finite epoch length

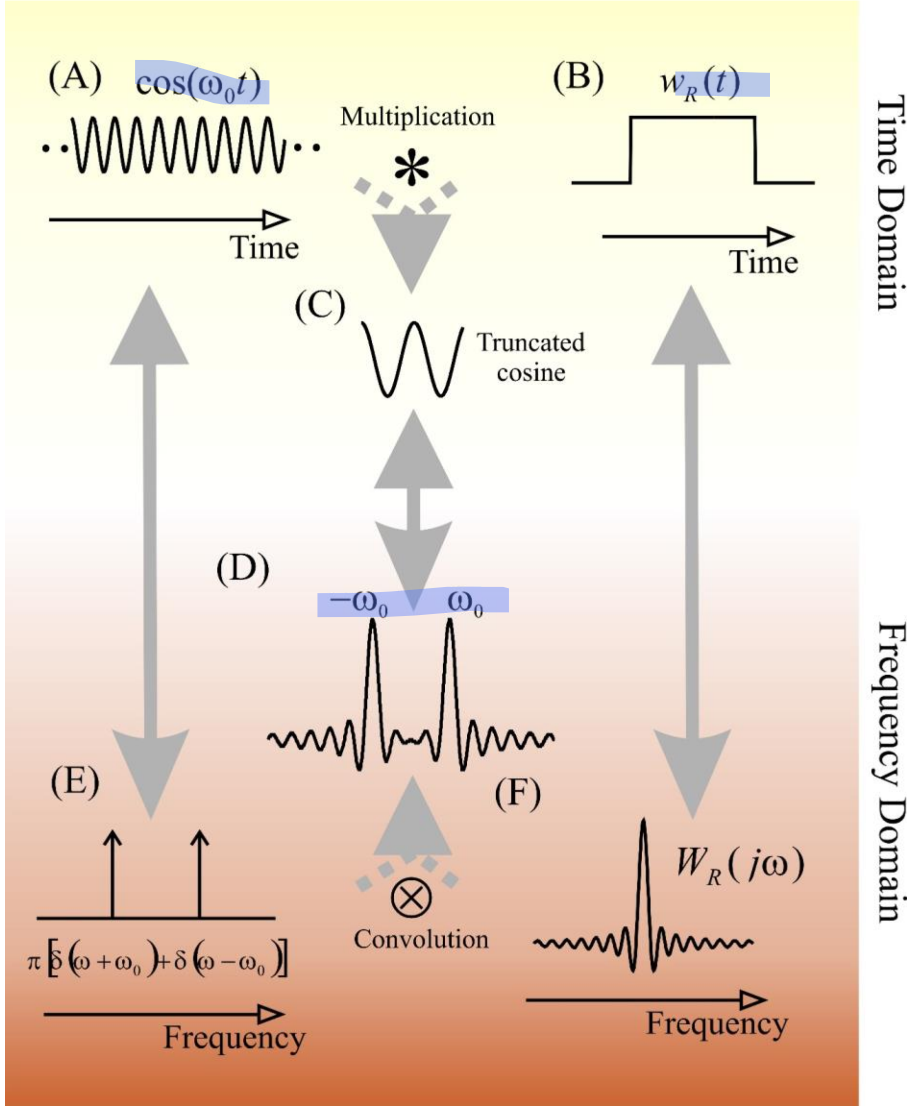


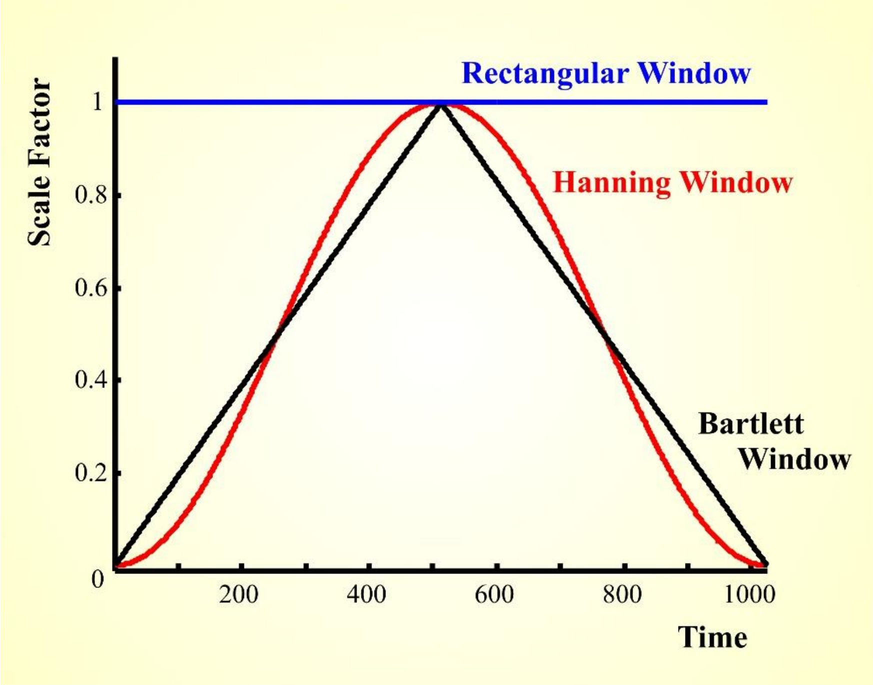
Figure 7.4, Figure 7.5 Signal processing for Neuroscientist, second edition by Wim van Drongelen

A theoretically infinite cosine wave **(A)** multiplied by a rectangular window **(B)** generates a truncated wave **(C)**. The Fourier transform of the cosine and the window in the frequency domain are shown in **(E)** and **(F)**, respectively. The transform of the truncated cosine is the convolution of its components, shown in **(D)**.

Commonly used Data Windows

TABLE 7.2 Overview of Commonly Used Data Window Functions

Data Window	Equation Window $w(t)$ for Epoch Size $-T \to T$
Bartlett (triangular, Fejér)	$w(t) = \begin{cases} 1 - \frac{\lvert t \rvert}{T} & \lvert t \rvert \le T \\ \text{else } 0 \end{cases}$
Hamming	$w(t) = \begin{cases} 0.54 + 0.46 \cos\left[\frac{\pi t}{T}\right] & \lvert t \rvert \le T \\ \text{else } 0 \end{cases}$
Hann (von Hann, hanning)	$w(t) = \begin{cases} 0.5 + 0.5 \cos\left[\frac{\pi t}{T}\right] & \lvert t \rvert \le T \\ \text{else } 0 \end{cases}$
Rectangular	$w(t) = \begin{cases} 1 & \lvert t \rvert \le T \\ \text{else } 0 \end{cases}$



Examples of three window functions of 1024 points

Table 7.2, Figure 7.6 Signal processing for Neuroscientist, second edition by Wim van Drongelen

Spectral Analysis of Physiological Signals

Spectral analysis of signals composed of pure sine waves is theoretically straightforward. In physiological signals interpretation of spectra requires caution because these time series are rarely stationary and usually contain both **nonperiodic** and **periodic** components. Even when the DC component is removed, the spectra from physiological data may contain **low-frequency components** due to slow nonperiodic activity (e.g., trends) or periodic activity with a periodicity beyond the analysis window. Similarly, the high-frequency components may be contaminated by high-frequency nonperiodic processes (e.g., sudden events). Furthermore, the periodic activity in physiological signals is usually far from purely sinusoidal, leading to spectral components **(so-called harmonics)** at higher frequencies.

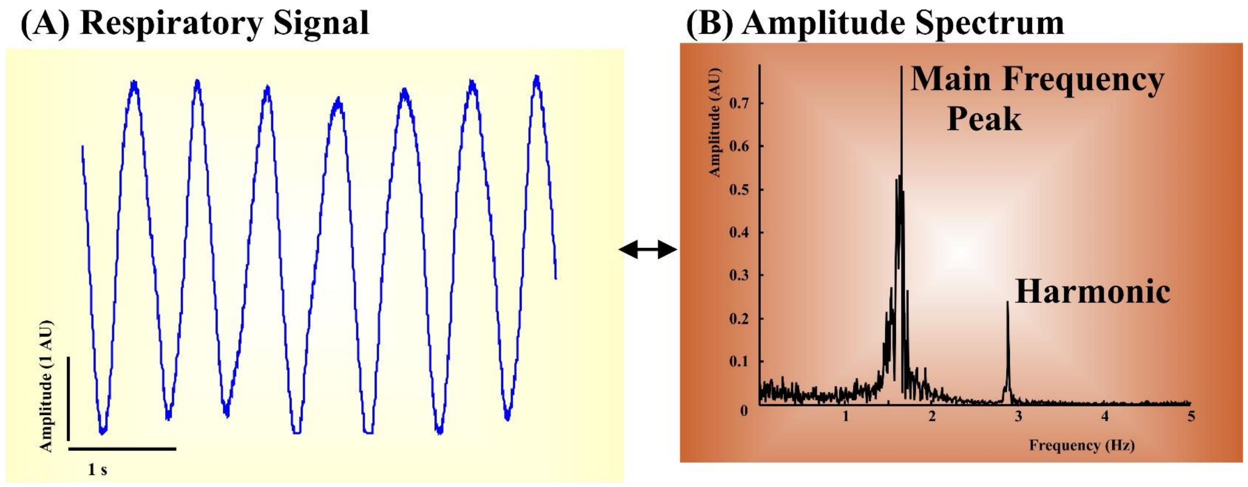


Figure 7.7 Signal processing for Neuroscientist, second edition by Wim van Drongelen

Frequency analysis of a respiratory signal from a human neonate. An epoch of the time domain signal is shown in **(A)** and the amplitude spectrum in **(B)**. Clearly the main peak 1.5 Hz shows the respiratory frequency, whereas the peak close to 3 Hz is a harmonic due to the imperfect sinusoidal signal.