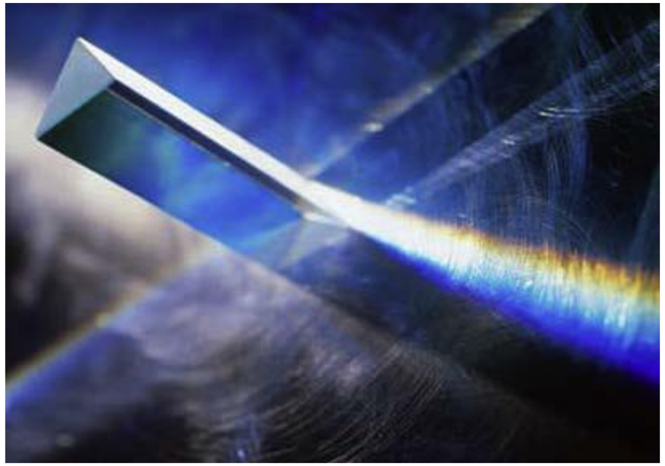
Chapter 4

Real and Complex Fourier Series

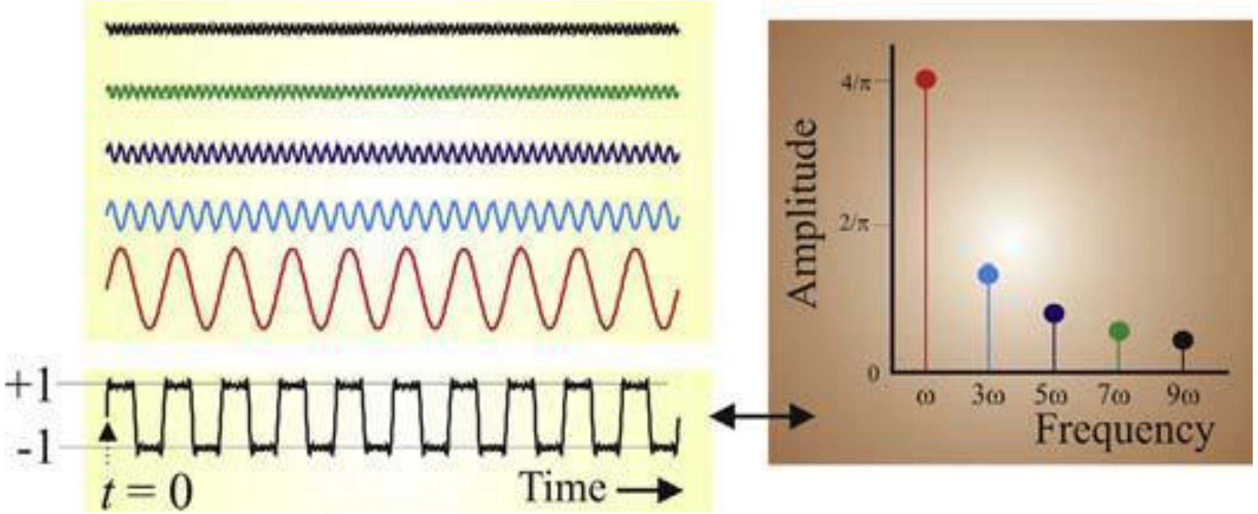
INTRODUCTION

The Fourier series as a technique to represent arbitrary periodic functions as a **summation of sine and cosine waves**. Also, the decomposition of signals into underlying frequency components is familiar to most; examples are the color spectrum obtained from decomposing white light with a **prism** or **decomposing sound into pure tone components**.

This example illustrates the basis of spectral analysis: a time domain signal (i.e., the [almost] square wave) can be decomposed into five sine waves, each with a different frequency and amplitude. The graph depicting these frequency and amplitude values in the figure is a frequency domain representation of the (almost) square wave in the time domain.



A prism performs spectral decomposition of white light in bands with different wavelengths that are perceived by us as different colors.



The sum of five sine waves approximates a square wave with amplitude 1 (bottom trace). The amplitude of the sine waves decreases with frequency. The spectral content of the square wave is shown in a graph of amplitude versus frequency (right).

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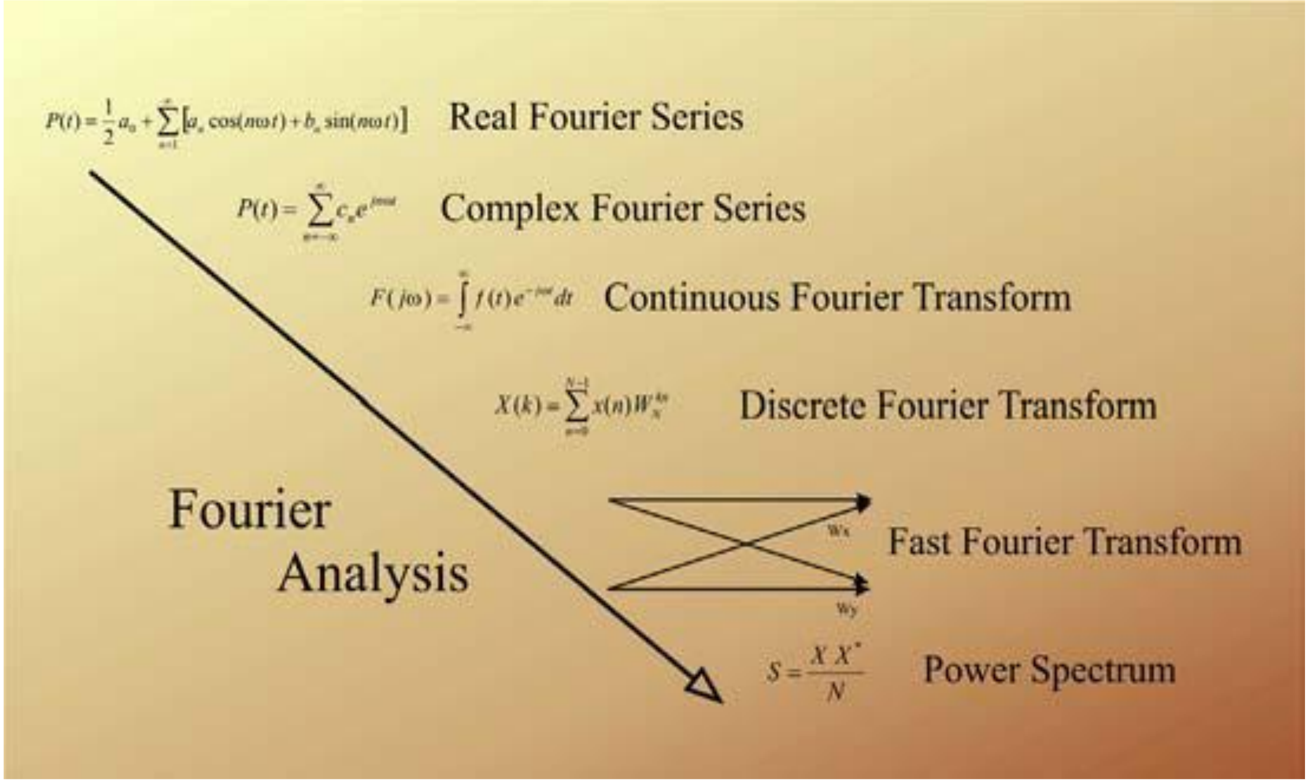
different types of Fourier analysis



The relationship between different types of Fourier analysis. The real and complex Fourier series can represent a function as the sum of waves as shown in the pervious slide.The continuous and discrete versions of the Fourier transform provide the basis for examining real-world signals in the frequency domain. The computational effort to obtain a Fourier transform is significantly reduced by using the fast Fourier transform (FFT) algorithm. The FFT result can subsequently be applied to compute spectral properties such as a power spectrum describing the power of the signal’s different frequency components.

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THE FOURIER SERIES

- The Fourier series provides a basis for analysis of signals in the frequency domain. In this section we show that a function f(t) (such as the [almost] square wave in pervious slide) with period T

  [i.e., f(t) = f(t + T)],frequency f = 1/T, and angular frequency ω defined as ω= 2πf can be represented by a series P(t):

$$P(t) = \frac{1}{2}a_0 + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \cdots + b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \cdots$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

- with the first term 1/2a0 representing the **DC (direct current) component**, and the remaining sine and cosine waves weighted by the **an and bn** coefficients represent **the AC (alternating current)** components of the signal.

Minimization of the Difference Between P(t) and f(t)

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The difference is considered the error of the approximation: i.e., the error E that is made by the approximation is [P(t) f(t)], which can be minimized by reducing E2 over a full period T of the time series:

$$E^2 = \int_t^{t+T} [P(t) - f(t)]^2 dt \qquad \partial E^2/\partial a_n = 0 \text{ and } \partial E^2/\partial b_n = 0$$

$$\partial\left[\int_t^{t+T} [P(t) - f(t)]^2 dt\right] \bigg/ \partial a_n \rightarrow \int_t^{t+T} \partial\left\{[P(t) - f(t)]^2 dt\right\} \bigg/ \partial a_n$$

$$2\int_T \left[(P(t) - f(t))\frac{\partial(P(t) - f(t))}{\partial a_n}\right] dt = 0 \rightarrow \partial(P(t) - f(t))/\partial a_n = \partial P(t)/\partial a_n$$

$$\partial\left[\int_t^{t+T} [P(t) - f(t)]^2 dt\right] \bigg/ \partial b_n \rightarrow \int_t^{t+T} \partial\left\{[P(t) - f(t)]^2 dt\right\} \bigg/ \partial b_n$$

$$2\int_T \left[(P(t) - f(t))\frac{\partial(P(t) - f(t))}{\partial b_n}\right] dt = 0 \rightarrow \partial(P(t) - f(t))/\partial b_n = \partial P(t)/\partial b_n$$

TABLE 5.1 Evaluation of $\partial P(t)/\partial a_n$ for Different Values of n

Index	Derivative
$n = 0$	$\partial P(t)/\partial a_0 = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial a_0 = 1/2$
$n = 1$	$\partial P(t)/\partial a_1 = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial a_1 = \cos(\omega t)$
$n = 2$	$\partial P(t)/\partial a_2 = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial a_2 = \cos(2\omega t)$
n	$\partial P(t)/\partial a_n = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial a_n = \cos(n\omega t)$

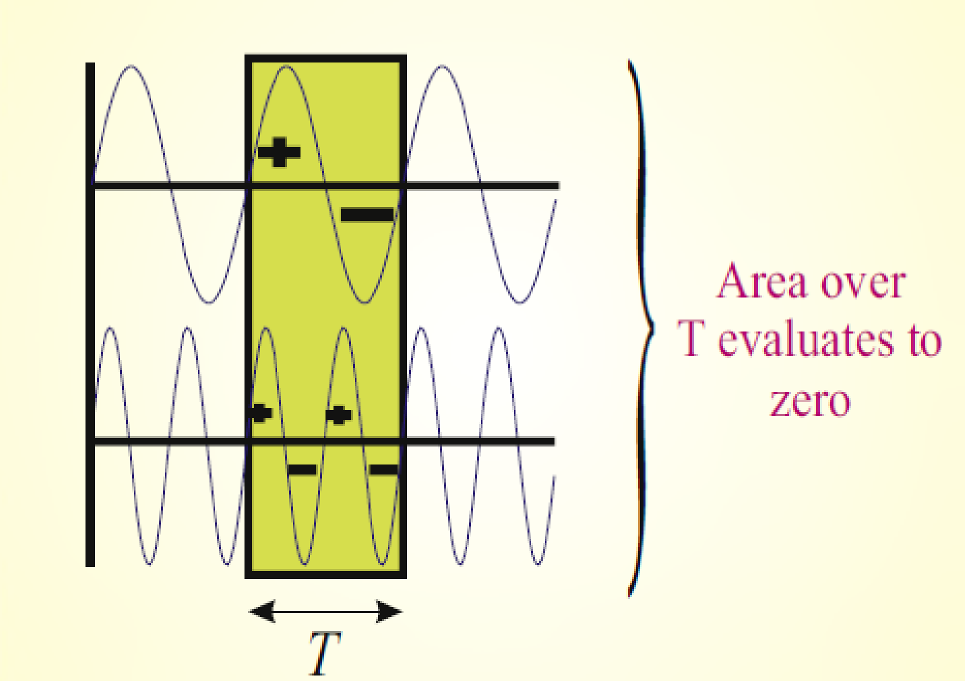
TABLE 5.2 Evaluation of $\partial P(t)/\partial b_n$ for Different Values of n

Index	Derivative
$n = 0$	Index does not exist for b coefficient
$n = 1$	$\partial P(t)/\partial b_1 = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial b_1 = \sin(\omega t)$
$n = 2$	$\partial P(t)/\partial b_2 = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial b_2 = \sin(2\omega t)$
n	$\partial P(t)/\partial b_n = \partial\left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right] \Big/ \partial b_n = \sin(n\omega t)$

Table 5.1, 5.2, Signal processing for Neuroscientist, second edition by Wim van Drongelen

Coefficients a0, an, bn

- In the following sections we use the obtained results to derive expressions for the coefficients an and bn. To simplify matters, we will frequently rely on two helpful properties: the fact that **(1)** the integral of a cosine or sine wave over one or more periods evaluates to zero, and **(2)** the orthogonal characteristics of the integrals at hand.



$$\int_T \cos(N\omega t) = 0 \text{ and } \int_T \sin(N\omega t) = 0$$

Figure A5.1-1, Signal processing for Neuroscientist, second edition by Wim van Drongelen

$$\int_T \cos(n\omega t)\cos(m\omega t)dt = \begin{cases} T/2 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}$$

$$\int_T \sin(n\omega t)\sin(m\omega t)dt = \begin{cases} T/2 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}$$

$$\int_T \sin(n\omega t)\cos(m\omega t)dt = 0 \text{ for all } m \text{ and } n$$

Coefficients a0

Returning to the an coefficients: for n = 0 we found that the derivative associated with minimization evaluates to ½. Substitution of this result into $2\int_T \left[(P(t)-f(t))\frac{\partial(P(t)-f(t))}{\partial a_n}\right]dt = 0$ gives us an expression for a0:

$$2\int_T (P(t)-f(t))\tfrac{1}{2}dt = \int_T P(t)dt - \int_T f(t)dt = 0 \rightarrow \int_T f(t)dt = \int_T P(t)dt$$

$$\int_T f(t)dt = \int_T \left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]dt$$

$$= \int_T \frac{1}{2}a_0 dt + \int_T a_1\cos(\omega t)dt + \int_T a_2\cos(2\omega t)dt\ldots + \int_T b_1\sin(\omega t)dt + \ldots$$

$$\int_T f(t)\,dt = \frac{1}{2}a_0\int_T dt = \frac{1}{2}a_0 T \rightarrow \boxed{a_0 = \frac{2}{T}\int_T f(t)\,dt}$$

Coefficients a1 and an

For n = 1 we obtained cos(ωt) for the partial derivative (Table 5.1); substituting this result into

$$2\int_T [(P(t) - f(t))\cos(\omega t)]dt = 2\int_T P(t)\cos(\omega t)dt - 2\int_T f(t)\cos(\omega t)dt = 0$$

$$\rightarrow \int_T f(t)\cos(\omega t)dt = \int_T P(t)\cos(\omega t)dt$$

Filling in the terms for the Fourier series P(t)

$$\int_T f(t)\cos(\omega t)dt = \int_T \left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots\right.$$

$$\left. + b_1\sin(\omega t) + \ldots\right]\cos(\omega t)dt$$

$$= \underbrace{\int_T \frac{1}{2}a_0\cos(\omega t)dt}_{0} + \int_T a_1(\cos(\omega t))^2 dt + \underbrace{\int_T a_2\cos(2\omega t)\cos(\omega t)dt}_{0}\ldots$$

$$+ \underbrace{\int_T b_1\sin(\omega t)\cos(\omega t)dt}_{0} + \ldots$$

Coefficients a1, an …

- Therefore, all the terms in equation evaluate to zero, excep $\int_T f(t)\cos(\omega t)dt = \int_T a_1(\cos(\omega t))^2 dt$ all owing us to simplify equation as follow:

$$\int_T f(t)\cos(\omega t)dt = \int_T a_1(\cos(\omega t))^2 dt \rightarrow \int_T f(t)\cos(\omega t)dt = \int_T a_1(\cos(\omega t))^2 dt = \int_T \frac{1}{2}a_1[1+\cos(2\omega t)]dt$$

$$\boxed{\cos(A)\cos(A) = ½[\cos(0) + \cos(2A)]}$$

$$\rightarrow \frac{1}{2}a_1\left[\int_T dt + \int_T \cos(2\omega t)dt\right] = \frac{1}{2}[t]_0^T + 0 = \frac{T}{2}a_1$$

$$a_1 = \frac{2}{T}\int_T f(t)\cos(\omega t)dt$$

The above procedure can be applied to find the other coefficients an. The integrals of the products cos(n ω t) × cos(mωt) in the series all evaluate to zero with the exception of those in which m = n .
The property that products of functions are zero unless they have the same coefficient is characteristic of **orthogonal functions** .
The integral of the products cos(n ω t) × sin(m ω t) all evaluate to zero also. This leads to the general formula for an:

$$\boxed{a_n = \frac{2}{T}\int_T f(t)\cos(n\omega t)dt}$$

Coefficients b1, bn

- For n = 1 we obtained sin(ωt) for the partial derivative (Table 5.2); substituting this result into

$$2\int_T [(P(t) - f(t))\sin(\omega t)]dt = 2\int_T P(t)\sin(\omega t)dt - 2\int_T f(t)\sin(\omega t)dt = 0$$

Filling in the terms for the Fourier series P(t)

$$\int_T f(t)\sin(\omega t)dt = \int_T P(t)\sin(\omega t)dt$$

$$\int_T f(t)\sin(\omega t)dt = \int_T \left[\frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t)\ldots + b_1\sin(\omega t) + \ldots\right]\sin(\omega t)dt$$

$$= \cancelto{0}{\int_T \frac{1}{2}a_0\sin(\omega t)dt} + \cancelto{0}{\int_T a_1\cos(\omega t)\sin(\omega t)dt} + \cancelto{0}{\int_T a_2\cos(2\omega t)\sin(\omega t)dt}\ldots + \int_T b_1(\sin(\omega t))^2 dt + \ldots$$

Coefficients b1, bn …

$$\int_T f(t)\sin(\omega t)dt = \int_T b_1(\sin(\omega t))^2 dt \rightarrow \int_T b_1(\sin(\omega t))^2 dt = \int_T \frac{1}{2}b_1[1-\cos(2\omega t)]dt = \frac{1}{2}b_1\left[\int_T dt - \int_T \cos(2\omega t)dt\right] = \frac{T}{2}b_1$$

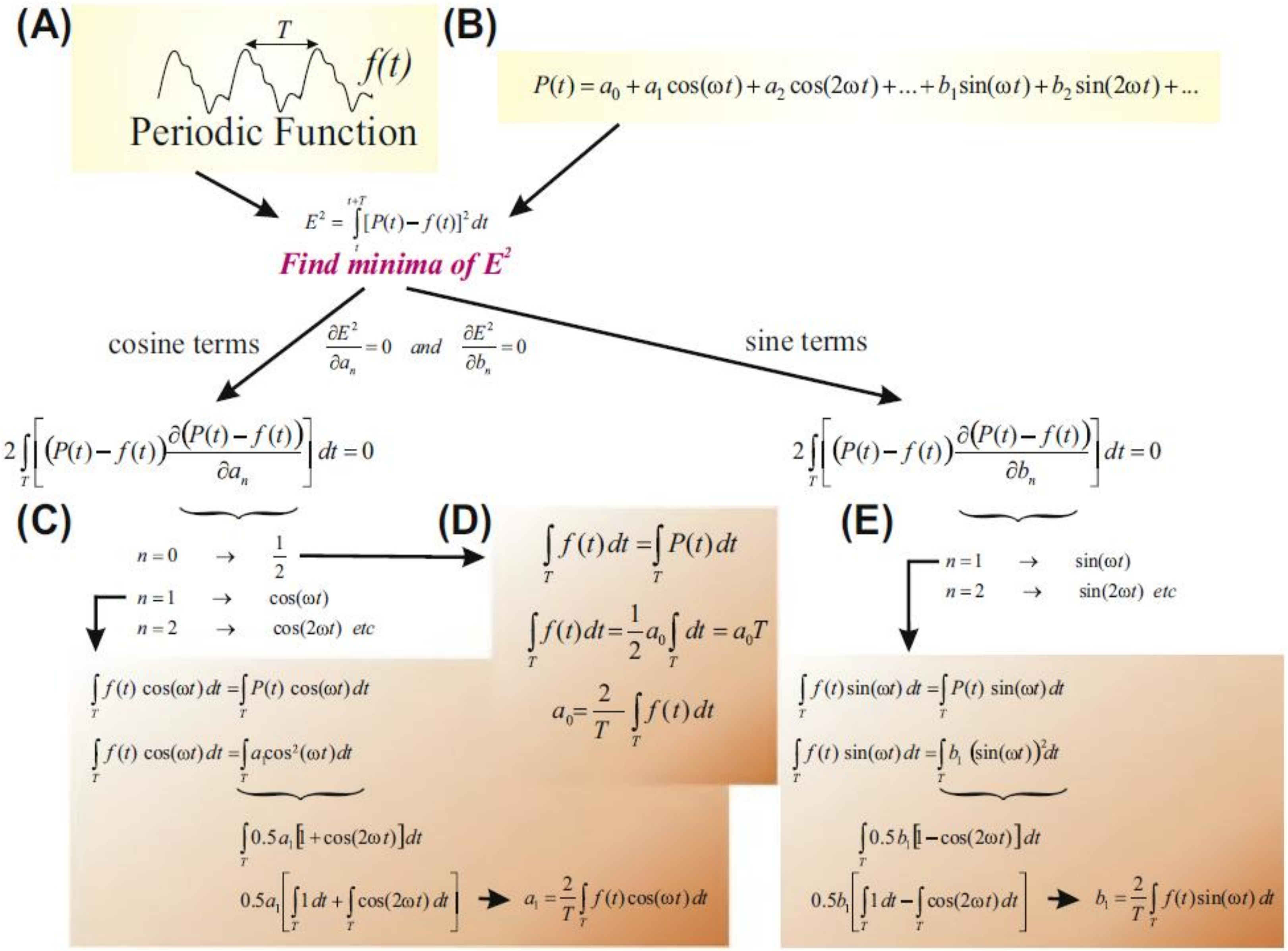
$$\boxed{\sin(A)\sin(A) = ½[\cos(0) - \cos(2A)]}$$

$$b_1 = \frac{2}{T}\int f(t)\sin(\omega t)dt$$

And finally, applying the same procedure to solve for **bn**:

$$\boxed{b_n = \frac{2}{T}\int_T f(t)\sin(n\omega t)dt}$$

Summary



$$\boxed{a_0 = \frac{2}{T}\int_T f(t)\,dt}$$

$$\boxed{a_n = \frac{2}{T}\int_T f(t)\cos(n\omega t)dt}$$

$$\boxed{b_n = \frac{2}{T}\int_T f(t)\sin(n\omega t)dt}$$

Overview of the real Fourier series representation of f(t), a periodic function (A). (B) The real Fourier series P(t). (C) and (D) Determination of coefficients a0 and a1 in P(t). (E) The same as (C) for the b1 coefficient (note that there is no b0). Determination of an and bn coefficients is similar to the procedure for a1 and b1.

Figure A5.1-1, Signal processing for Neuroscientist, second edition by Wim van Drongelen

THE COMPLEX FOURIER SERIES

The Fourier series of a periodic function is frequently presented in the complex form.
The notation for the complex Fourier series is:

$$a_0 = \frac{2}{T} \int_T f(t)\, dt$$

$$a_n = \frac{2}{T} \int_T f(t) \cos(n\omega t) dt$$

REAL FOURIER SERIES

$$P(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$b_n = \frac{2}{T} \int_T f(t) \sin(n\omega t) dt$$

Complex FOURIER SERIES

$$P(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$c_n = \frac{1}{T} \int_T f(t) e^{-jn\omega t} dt$$

$\int_T \cdots$ indicates that the integral must be evaluated over a full period T, where it is not important what the starting point is.

e.g., $-T/2 \rightarrow T/2$ or $0 \rightarrow T$

$$e^{jx} = \cos(x) + j \sin(x)$$

Euler's relation

The Real and Complex Fourier series notations are equivalent

Figure A5.1-1, Signal processing for Neuroscientist, second edition by Wim van Drongelen

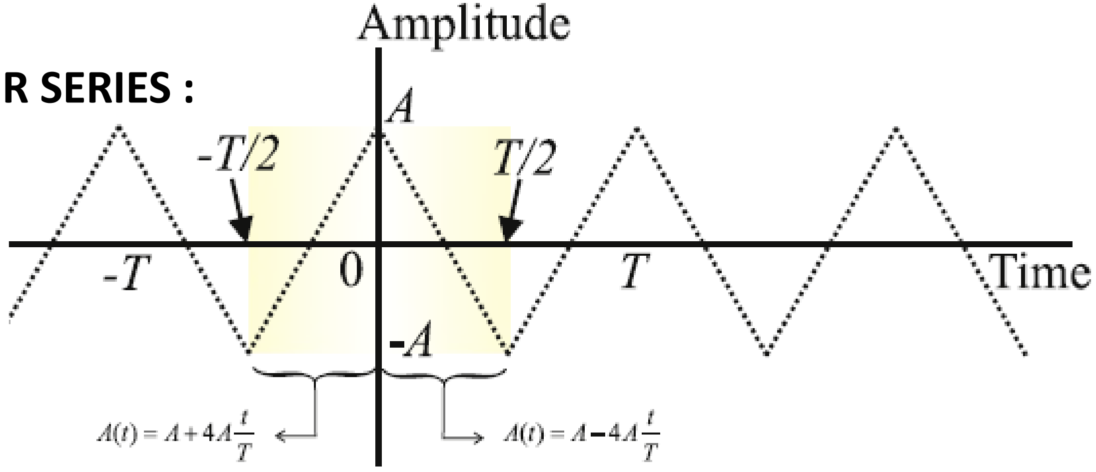
Example

Finding Real FOURIER SERIES :

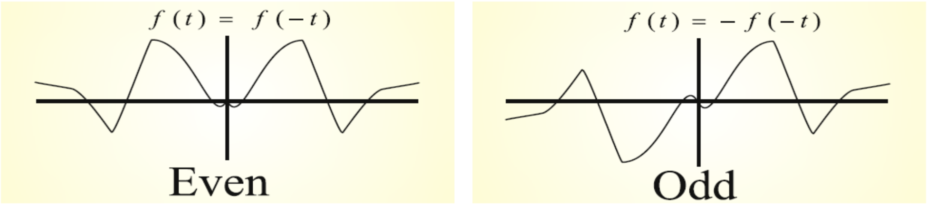
$$a_0 = \frac{2}{T}\int_T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_T f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_T f(t)\sin(n\omega t)\,dt$$



$b_n = 0$ $\quad$ $a_n = 0$



$$f_{even} = (f(t) + f(-t))/2. \qquad f_{odd} = (f(t) - f(-t))/2$$

$$f(t) = f_{even} + f_{odd} = \frac{f(t)+f(-t)}{2} + \frac{f(t)-f(-t)}{2} = \frac{2f(t)}{2} = f(t)$$

DC component is absent, therefore a0=0.
The time series is even ,i.e., the sinusoidal odd components in the Fourier series are absent, therefore bn = 0.

From these observations we conclude that we only have to calculate the an coefficients to obtain the representation of the real Fourier series. We can calculate these coefficients by integration over a full period from T/2 to T/2. In order to avoid trying to integrate over the discontinuity at t = 0, we can break the function up into two components.

3. $A(t) = A + 4At/T$ for $-T/2 \le t \le 0$, and
4. $A(t) = A - 4At/T$ for $0 \le t \le T/2$

Figure A5.2, Figure 5.5 Signal processing for Neuroscientist, second edition by Wim van Drongelen

$$\boxed{a_n = \frac{2}{T}\int_T f(t)\cos(n\omega t)dt}$$

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$$a_n = \frac{2}{T}\int_{-T/2}^{0}\left(A + 4A\frac{t}{T}\right)\cos(n\omega t)dt + \frac{2}{T}\int_{0}^{T/2}\left(A - 4A\frac{t}{T}\right)\cos(n\omega t)dt$$

$$a_n = \underbrace{\frac{2A}{T}\int_{-T/2}^{0}\cos(n\omega t)dt + \frac{2A}{T}\int_{0}^{T/2}\cos(n\omega t)dt}_{I} \quad \underbrace{+\frac{8A}{T^2}\int_{-T/2}^{0} t\cos(n\omega t)dt - \frac{8A}{T^2}\int_{0}^{T/2} t\cos(n\omega t)dt}_{II}$$

$$\underbrace{\frac{2A}{T}\frac{1}{n\omega}[\sin(n\omega t)]_{-T/2}^{0} + \frac{2A}{T}\frac{1}{n\omega}[\sin(n\omega t)]_{0}^{T/2}}_{I} = 0$$

$$\int t\cos(n\omega t)\,dt = \frac{1}{n^2\omega^2}\int \underbrace{(n\omega t)}_{u}\ \underbrace{\cos(n\omega t)\,d(n\omega t)}_{dv}$$

with: $\begin{cases} u = n\omega t \quad \rightarrow \quad du = d(n\omega t) \\ dv = \cos(n\omega t)\,d(n\omega t) \rightarrow v = \sin(n\omega t) \end{cases}$

$t \rightarrow n\omega t$ $\rightarrow$ $$\int \underbrace{(n\omega t)}_{u}\ \underbrace{\cos(n\omega t)\,d(n\omega t)}_{dv} = \underbrace{(n\omega t)\sin(n\omega t)}_{uv} - \int \underbrace{\sin(n\omega t)}_{v}\ \underbrace{d(n\omega t)}_{du} = (n\omega t)\sin(n\omega t) + \cos(n\omega t) + C$$

$$a_n = \frac{2A}{n^2\pi^2}\left(\left[\underbrace{\cos(0)}_{1} - \cos\underbrace{(-n\pi)}_{\substack{-1 \text{ for } n=odd \\ 1 \text{ for } n=even}}\right] - \left[\underbrace{\cos(n\pi)}_{\substack{-1 \text{ for } n=odd \\ 1 \text{ for } n=even}} - \underbrace{\cos(0)}_{1}\right]\right)$$

$$a_n = \frac{8A}{(n\pi^2)} \qquad \text{if n is odd}$$

$$a_n = 0 \qquad \text{if n is even}$$

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