Optimal Allocation with Noisy Inspection*

Nawaaz Khalfan†

October 26, 2022

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Abstract

A principal receives an unknown reward from allocating to an agent who has private information about the reward. Prior to allocating, the principal may elicit a report from the agent and inspect them at a cost, but must do so without transfers. When the private information is noisy, the unique separating mechanism that maximizes the principal's expected return segments signals into two groups, inspects high types, allocating to them only if the inspected return is sufficiently positive, and doesn't inspect low types, compensating them with a small probability of allocation. This relates to a number of applied settings such as employer hiring strategies, public grant mechanisms and portfolio investment rules.

1 Overview

Appraising the value of an asset is an essential precursor to its exchange. Employers interview potential employees, public funds assess grant applications, venture capitalists evaluate investment opportunities. This process is often costly and information that could be used to lower, and even circumvent, these costs is often privately held.

This article considers a principal whose return from allocating to an agent is uncertain and inherent to the agent they allocate to. The principal has the ability to inspect (*interview, assess, evaluate*) the agent at a cost and learn about the true return, as well as the opportunity to receive a report from the agent of the private information they have about the return. The agent, independent of their private information, strictly prefers to be allocated to. Inspection then has two purposes: the verification of private information and the discovery of additional information.

The question I will address here is how the principal inspects and allocates in order to acquire information and maximize their expected return. Direct transfers of value between the agent

*This paper was informed by extensive comments and suggestions from Rakesh Vohra, George Mailath and Aislinn Bohren, as well as the feedback from the University of Pennsylvania's Microeconomic Theory Workshop.

†University of Pennsylvania; email: khalfan@sas.upenn.edu

and principal in this analysis are prohibited. This is to reflect the observation that payments for disclosure are rarely a feature of appraisal. For example, when assessing applicants for promotion, managers rarely pay, or receive payment, for individual submissions in this process. Furthermore, in restricting this channel, we study the direct value of inspection, which can then be uses as a benchmark for particular applications of interest.

The environment presented in this article naturally encompasses many applied settings. To fix attention, consider the following three stylized examples.

1. **Hiring**: a manager, the principal, seeks to fill an open position in their firm with a potential employee, the agent. The agent would like to be hired, receiving the benefits of employment at this firm, and has some idea of their future productivity in this position. This information, however, is less than perfect. The principal could interview the agent and discover the true productivity, but this is costly for the principal. What interview and hiring protocol should the firm enact?

2. **Funding**: a governing board, the principal, sets the rules by which it allocates a scarce, publicly owned resource, such as funding for a project and an applicant, the agent. The agent is primarily interested in being approved, valuing their own use above rival uses, and the principal wants to fund positive net value projects, perhaps weighing a combination of private and public preferences for the funding's use. How should a resource restricted board design testing and approval rules to support socially valuable projects?

3. **Investing**: an investor, the principal, sets the mechanism by which it evaluates and finances an early investment opportunity, the agent. The investor may be governed by the motivation to strengthen an existing portfolio or even personal philanthropic concerns, but is restricted in outlining these preferences publicly. The agent wishes to be financed and to continue their startup, and has the most information about the startup. This information doesn't fully determine the investor's value for the opportunity without an appraisal. How much information can the investor elicit through their evaluation and funding decisions?

The three key aspects in each of these examples is that agents have noisy private information about the principal's allocation reward, it is costly for the principal to discover the reward themselves, and the use of transfers to align incentives are limited. It is in this environment that inspection has the potential to serve two different roles in the acquisition of information. Directly, it informs the principal of the reward they will receive if they carry through with allocating to the agent, and indirectly, it disciplines the agent's disclosure of information by conditioning allocation on the outcome of an inspection. To this end, I will address the tradeoff between these sources of information: direct acquisition through *discovery*, and indirect acquisition through *screening*.

Modelling the agent's private information as a signal of *favourableness*, the unique separating mechanism that maximizes the expected reward for the principal has a simple structure. To elicit truthful reports, the principal pools two types of signals from the agent - *high* and *low*. For agents with high signals, they are always inspected and only allocated to after inspection if the reward is sufficiently positive. For agents with low signals, they are never inspected, and compensated for their report with a small probability of unconditional allocation.

This induces three types of inefficiencies for the principal: over-allocation to agents with low types, over-inspection of agents with both high and low types, and under-allocation post inspection. When the full commitment assumption is relaxed, the losses from over-allocating to low types are magnified and the other losses suppressed, to the net detriment of the principal's objective.

The questions raised in this article cannot be adequately addressed in settings where agents are perfectly informed of the principal's reward and inspection only *verifies* reports. It is these models that make up the literature on mechanism design, with costly inspection and no transfers. A seminal paper in this branch is Ben-Porath, Dekel and Lipman (2014) who analyse a model of N such agents and show that inspection is used to check the highest reported type above a threshold, providing an adequate incentive for low types not to masquerade. Inspection is never used to inform the principal of any additional information, a feature that seems to be present in each of our examples. At the other extreme, if agents had no information, or no ability to communicate with the principal, the principal faces a decision problem that resembles Weitzman (1979).[1] He shows that the principal should inspect agents sequentially, stopping when a sufficiently valuable agent is discovered. In this setting, if the agents had some private information and the ability to communicate, we might suspect that the principal could save on inspection costs by collecting this information, treating some preferentially in order to do this, and use the information to make better allocations.

I will begin by further detailing this literature in order to provide context for the contributions in this article. I will then outline a framework for analysing this environment and provide a benchmark for comparison (section 3), subsequently prove the main results for how inspection is used to acquire information (section 4), and conclude the main analysis with how the results extend to environments where the commitment assumption is relaxed (section 5).

2 Literature

Mechanisms with perfect verification:

- Green and Laffont (1986) - hard evidence

[1]More accurately, the extension in Doval (2018).

- Ben-Porath, Dekel and Lipman (2014) - perfect inspection
- Mylovanov and Zapechelnyuk (2017) - perfect inspection after allocation
- Silva (2019*b*) - perfect inspection with no commitment
- Epitropou and Vohra (2019) - perfect inspection with sequentially arriving agents

Inspection is perfect and is used by the principal for verification only.

Perfect information: Green and Laffont (1986), Ben-Porath, Dekel and Lipman (2014), Mylovanov and Zapechelnyuk (2017), Epitropou and Vohra (2019).

Mechanisms with imperfect verification:

- Pereyra and Silva (2021) is the closest, but they have object allocation where $k < n$, inspection is costless and they consider the efficient mechanism.
- Ball and Kattwinkel (2019) is also similar but the agent has a type dependent utility function, an effort choice and receives a transfer from the principal. The focus of the paper is to derive endogenous signal structures (in the family of pass/fail tests), and we consider this as an extension.

Both papers use inspection the check the agents report. There is a conceptual difference between *verifying* an agent's report and *inspecting* their inherent qualities.

Efficient mechanisms: Ball and Kattwinkel (2019), Silva (2019*a*), Siegel and Strulovici (2021), Pereyra and Silva (2021), Erlanson and Kleiner (2020).

Other: accounting, auditing and disclosure

- Jonathan Glover?
- Anil Arya?
- Tirole's Corporate Finance?
- Dai 2009 - arbitration

Agents private information determines their value.

Transfers: Townsend (1979), Border and Sobel (1987), Mookherjee and Png (1989), Alaei et al. (2020).

Limited transfers: Mylovanov and Zapechelnyuk (2017), Silva (2019*b*), Li (2021).

Judicial mechanism design:

- Silva (2019*a*)
- Siegel and Strulovici (2021)

Unexpected application, but the models are less sophisticated and tailored to the application.

Scoring rules: McCarthy (1956), Savage (1971), Gneiting and Raftery (2007).

3 Environment

A *principal* receives an unknown, real reward, R, from allocating to an *agent*. If the agent is allocated to, they receive a payoff of 1, and 0 otherwise. Prior to allocation, the principal may *inspect* the agent at a cost to their final payoff, $c > 0$, and in return learn the true value of the reward, r. The agent has their own private information about the prize, a signal s, which defines their *type*.

Here, we are considering one principal and one agent. An equivalent setup is a single principal with k objects to allocate, among $\ell \leq k$ agents, each of whom has unit demand, the same preferences for each object, and an independent signal of the principal's reward. The problem where $\ell > k$ is left for future research.

The agent has a strict preference to be allocated to, and their payoffs are normalized around this. Changing the intensity of this preference, and even making this intensity type-dependent, makes no difference to the analysis as long as we maintain the strict preference for allocation. As such, this normalization is without loss of generality. It will of course matter when interpreting particular applications and extensions, but this is left to the responsibility of the reader.

The principal has the ability to commit to a rule that determines what they do after any report from the agent and any subsequent realization of r if inspection takes place. Following the main analysis, we can explore how the environment changes when we relax this assumption, which I will outline in section 5. One can thus think of the full commitment assumption as the most interesting case study in understanding this environment.

Direct transfers of value between the principal and agent are *not* permitted. This reflects the observation that transfers are seldom used for direct disclosure of information in allocation problems. A place where we may see transfers occurring in practice is bargaining between the principal and agent once the principal decides to allocate. We can then interpret the restriction as the principal not being able to commit to altering later stage bargaining outcomes before their allocation decision. The model then claims that the principal's reward, and the agent's preference to be allocated to, reflects the expected outcome of the ensuing bargaining game.

An alternative interpretation is that this is an environment where bargaining does not occur and in practice, there are many settings where this is true. In some allocation problems, the use of money is seen as counterproductive, for example in the assignment of public housing. In others, there are commitments from one or both of the parties not to use or accept transfers, an institutional restriction that occurs in many forms of governance involving public offices. And finally, in some, the entire bargaining power over the value that can be shared may belong to just one of the parties, and so bargaining does not occur in any real sense.

Even if the restriction isn't true of an application at hand, it is still important to understand what inspection offers the principal in isolation of transfers. As such, the ensuing analysis, can be viewed through a predictive lens if a policy to restrict transfers will be adopted, or a prescriptive lens if one is being considered.

The timing and structure of the game is fixed and common knowledge.

1. The principal commits to an inspection and allocation policy; nature assigns signals according to a commonly known generating process.
2. The agent observes their signal (type), s, and submits a report to the principal.
3. The principal implements their policy conditional on the report and any subsequent inspection realizations that are generated by the policy.
4. All remaining uncertainty is resolved, and net payoffs are awarded.

The primary question to address is which inspection and allocation rule the principal should select in order to maximize their expected return, subject to the agent's incentives to report.

This report could in practice be a lengthy and complicated message, but given our objective here is to study the resulting outcomes and, at most, the total information exchanged by these messages, we will instead work with the *direct mechanism* by appealing to the revelation principle (Myerson, 1981). That is, we restrict attention to the message game where the agent directly reports their type, and require that the agent's expected return from reporting truthfully is weakly greater than that of reporting any other type.

Listing the principal's available actions conditional on report s, let:

- $x(s)$ be the inspection probability,
- $y(s)$ be the allocation probability without inspection, and

- $z(s, r)$ be the allocation probability after inspection and realizing reward r.

To fix language, I will refer to x as the *inspection rule*, y as the *pre-inspection allocation*, and z as the *post-inspection allocation*. Together, (x, y, z) constitute the principal's *mechanism*, and this mechanism is *feasible* if:

$$x(s) \in [0,1],\ y(s) \in [0,1],\ z(s,r) \in [0,1] \quad \forall\ s, r$$

Here we are endowing the principal with the ability to ration the object they're allocating. In the hiring example, this can be thought of limiting the hours the employee works, and in the funding examples, partially funding the applicant's project. If the object is indivisible, partial allocation could be thought as the outcome of a lottery over the object. It is this lottery interpretation that I'll use to fix language throughout. The decision to inspect may also be partial, in which case the lottery interpretation is natural, but could also be a further rationing.[2] What we are restricting here is that allocation can be at most one, representing the capacity constraint on the principal's allocation.

The principal chooses this mechanism to maximize their ex ante expected return. At the interim stage, after learning s, their payoff is determined by two events. They may allocate without inspecting, receiving the expected return given the signal, and this occurs with the probability that they don't inspect, $(1 - x(s))$, and that they do allocate, $y(s)$. Alternatively, they may allocate after inspecting and learning the return, r, receiving this net of the inspection cost, c, which, conditional on this r being the true reward, occurs with the probability they do inspect, $x(s)$, and they do allocate, $z(s, r)$. Let v_s refer to the principal's interim payoff given s.

$$v_s(x, y, z) \coloneqq (1 - x(s))y(s)\mathbb{E}(r|s) + x(s)(\mathbb{E}(z(s, r).r|s) - c)$$

Their ex ante expected return is their expected interim payoff, which I will refer to as the *objective*, v.

$$v(x, y, z) \coloneqq \mathbb{E}_s\left[(1 - x(s))y(s)\mathbb{E}(r|s) + x(s)(\mathbb{E}(z(s, r).r|s) - c)\right]$$

Optimizing involves maximizing the objective subject to the agent's incentive compatibility constraints. These require an agent of type s to receive as high a payoff from reporting s than any other type, $\hat{s}$. Given the agent's normalized payoffs, this is the likelihood of being allocated to by the mechanism, and occurs with the net probability of the two events outlined above. Let u_s be this payoff for type s and $u_{s,\hat{s}}$ be the payoff from reporting $\hat{s}$ more generally, so that $u_{s,s} = u_s$.

$$u_s(x, y, z) \coloneqq (1-x(s))y(s)+x(s)\mathbb{E}(z(s, r)|s) \geq (1-x(\hat{s}))y(\hat{s})+x(\hat{s})\mathbb{E}(z(\hat{s}, r)|s) \eqqcolon u_{s,\hat{s}}(x, y, z) \quad \forall\ \hat{s}$$

This *incentive compatibility* constraint for the pair $(s, \hat{s})$ is labelled $IC_{s,\hat{s}}$ for reference. Note that

[2] You would then have to claim that inspecting an agent for part of the object, is only partially costly.

the agent's type only augments their return directly through determining the distribution of r and thus the likelihood of being allocated to conditional on inspection. Also note that given their return is the net probability of being allocated to, there is no need for an individual rationality constraint as all type's receive a weakly positive return.

In total, the mathematical program that the principal solves is:

$$\begin{aligned}
\max_{(x,y,z)} \quad & \mathbb{E}_s\left[(1-x(s))y(s)\mathbb{E}(r|s) + x(s)(\mathbb{E}(z(s,r).r|s) - c)\right] \\
\text{s.t.} \quad & IC_{s,\hat{s}} : (1-x(s))y(s) + x(s)\mathbb{E}(z(s,r)|s) \geq (1-x(\hat{s}))y(\hat{s}) + x(\hat{s})\mathbb{E}(z(\hat{s},r)|s) \quad \forall\ \hat{s} \quad \forall\ s \\
& F_{s,r} : x(s) \in [0,1],\ y(s) \in [0,1],\ z(s,r) \in [0,1] \quad \forall\ r \quad \forall\ s
\end{aligned}$$

3.1 Signals

In practice, the agent's private information could be complicated and nuanced. For a job applicant, their private information includes their educational performance, feedback from old colleagues, observations about the firm's prior hiring decisions, and many other factors. Given we are primarily interested in the informational content of this object, though, we will instead collapse this into a single parameter and ask what characteristics we'd expect this parameter to have.

Let the agent's information be represented by a private signal, $s \in \{s_0, s_1, \ldots, s_N\}$. Suppose $s = s_n$ with probability $p_n \in (0,1)$ and that this is a properly defined probability: $\sum_n p_n = 1$. Denote P_n as the cumulative mass function, so that $P_n = \sum_{m \leq n} p_m$.

I have opted for a discrete formulation of signals for exposition alone. The use of discrete signals allows us to think clearly about the different incentives that are important to this problem, and in section 4.4 I extend the result to an appropriate limiting environment via Helly's selection theorem. One could instead conduct this exercise with continuous types, and it is my contention that this would provide no substantive additional information.

First, we'd like signals to informative. If $s = s_n$, suppose now the reward that the principal receives from allocating to the agent is a random variable $R|s_n \sim \Pi_n$ where Π_n is absolutely continuous and admits a density function π_n. Denote the unconditional distribution of the reward R by Π and assume that it is also absolutely continuous, admits a density function π and has support $\mathcal{R} = [\underline{r}, \overline{r}] \subseteq \mathbb{R}$. Finally, assume that each of these distributions has a finite mean.

I've chosen to introduce signals first and rewards as deriving from these signals, but the order in which nature selects rewards and signals is unimportant as long as the information about these stays the same. That is, you could think of the agent's as having an underlying reward, and receiving a signal about this reward which they then report to the principal, or as receiving the signal and when then they are inspected, or when the game concludes, a reward is generated given the signal.

Next, these signals need to have some structure in order to conduct analysis. To this end, I will make one key assumption that says that higher signals are more *favourable* than lower signals:

Assumption 1 *The signals are completely ordered by the monotone likelihood ratio property (MLRP). That is:*

$$\frac{\pi_n(r_1)}{\pi_m(r_1)} \geq \frac{\pi_n(r_0)}{\pi_m(r_0)} \quad \text{for all } r_1 > r_0 \text{ and } n > m$$

That is, higher signals generate higher rewards relatively more likely than lower signals so. In this sense, an agent with a higher signal is more favourable to the principal, and is a notion of that was outlined by Milgrom (1981) and has been widely adopted since.

Another popular ordering, and one whose characteristics we will call upon often in this analysis, is that of *first order stochastic dominance* (FOSD). A visual demonstration of the differences between MLRP and FOSD, and the reason for selecting the former, is highlighted in section 4.1.1. For now, note that MLRP is a stronger notion than FOSD:

Claim 1 *If the signals are completely ordered by the monotone likelihood ratio property, they are ordered by first order stochastic dominance. That is:*

$$\Pi_n(r) \leq \Pi_m(r) \quad \text{for all } r \text{ if } n > m$$

Proof: If a higher signal generates higher rewards relatively more likely than lower signals, then it must be true on average for rewards greater than any fixed reward, $\hat{r}$, so that $1-\Pi_n(\hat{r}) \geq 1-\Pi_m(\hat{r})$ for $n > m$. A complete proof is in the Appendix. ∎

Given this assumption, it is convenient and unambiguous to relabel the signals by their induced expected reward, so that $s_n = \mathbb{E}(r|s_n)$. As such, our information parameter now has a neat interpretation, and we will call on this interpretation where helpful. We can now state the principal's problem explicitly.

Principal's problem:
In return for a report of the agent's signal, s_n, the principal may inspect the agent, x_n, allocate to

the agent without inspecting, y_n, or allocate to the agent after inspecting and observing r, $z_{n,r}$.

$$\begin{aligned}
\max_{(x,y,z)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n(\int r z_{n,r}\pi_{n,r}\,dr - c)]p_n \\
\text{s.t.} \quad & IC_{n,m} : (1-x_n)y_n + x_n(\int z_{n,r}\pi_{n,r}\,dr) \geq (1-x_m)y_m + x_m(\int z_{m,r}\pi_{n,r}\,dr) \quad \forall\ n,m \\
& F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n
\end{aligned}$$

For ease of notation, let:

- $\psi_n(z) \coloneqq \int r z_r \pi_{n,r}\,dr - c$ as the expected net reward for the principal from inspecting given some arbitrary post-inspection allocation rule z, and
- $\phi_n(z) \coloneqq \int_r z_r \pi_{n,r}\,dr$ as the expected allocation to the agent from being inspected given some arbitrary post-inspection allocation rule z.

3.2 Symmetric information benchmark

As a benchmark, consider the problem where the principal has full information regarding the agent's signal, and whose solution we refer to as the *first best* policy.

$$\begin{aligned}
\max_{(x_n,y_n,z_n)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n(\int r z_{n,r}\pi_{n,r}\,dr - c)]p_n \\
\text{s.t.} \quad & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n
\end{aligned}$$

Given the agent has nothing additional to report - information is *symmetric* - the agent has no strategically relevant actions, and this is a straightforward problem to optimize. It outlines exactly what incentives the principal has for how they treat each agent and a basis for which to measure the losses associated with private information from the principal's perspective.

The next claim outlines the first best policy and, as with all claims, theorems and propositions in this paper, a sketch of the proof is provided in the main body and a full proof in the appendix. Further, let $\mathbb{1}\{\mathcal{Q}\}$ be the indicator function that is equal to 1 if the statement $\mathcal{Q}$ is true given the arguments, and 0 otherwise.[3]

Claim 2 *The first best policy* (x_n^*, y_n^*, z_n^*) *is given by:*

- $z_{n,r}^* = \mathbb{1}\{r \geq 0\}$,
- $y_n^* = \mathbb{1}\{\mathbb{E}(r|s_n) \geq 0\}$, *and*
- $x_n^* = \mathbb{1}\{\psi_n(z_n^*) \geq \max\{\mathbb{E}(r|s_n), 0\}\}$.

[3] The standard definition of an indicator function is $\mathbb{1}_A(x) \coloneqq 1$ if $x \in A$ and 0 if $x \notin A$. We're more interested in the set A and less in the argument x, so I'm suppressing the argument and promoting the set.

Proof: Conditional on the decision to inspect, x_n, $z_{n,r}$ selects when to allocate post-inspection and should then be maximized when $r \geq 0$ and minimized otherwise. Call this the *ideal* post-inspection allocation rule. Similarly, conditional on the decision to not inspect, $1 - x_n$, y_n selects when to allocate pre-inspection without additional information and should then be maximized when $\mathbb{E}(r|s_n) \geq 0$ and minimized otherwise. Finally, x_n selects when to inspect, and should be maximized when the expected net reward for the principal from inspecting given the ideal post-inspection allocation rule is both greater than outright allocating or outright rejecting. ■

There are only three relevant policy combinations of the first best policy:

1. *no allocation*, **N**, given by $x_n = 0, y_n = 0$,
2. *ideal inspection*, **I**, given by $x_n = 1, z_{n,r} = \mathbb{1}\{r \geq 0\}$, and
3. *full allocation*, **A**, given by $x_n = 0, y_n = 1$.

Let $\psi_n^* := \psi_n(z_{n,r}^*)$, that is, the expected return from ideal inspection. Preferences over these policies have a fixed order with respect to s_n due to the FOSD ordering of the signals.

Claim 3 *There exists some s_α and s_β, with $s_\alpha \leq s_\beta$, such that:*

- *if $s_n \leq s_\alpha$ then $0 \geq \max\{\psi_n^*, \mathbb{E}(r|s_n)\}$,*
- *if $s_n \in (s_\alpha, s_\beta)$ then $\psi_n^* > \max\{0, \mathbb{E}(r|s_n)\}$, and*
- *if $s_n \geq s_\beta$ then $\mathbb{E}(r|s_n) \geq \max\{\psi_n^*, 0\}$.*

Proof: By FOSD, the expected return from ideal inspection is increasing in the signal as the cost is fixed and the likelihood of a positive reward is increasing. As such, ψ_n^* has a single crossing with 0, after which ideal inspection is preferred to rejecting outright. Label the corresponding crossing signal as s_α if it's negative and 0 otherwise. The rate at which ψ_n^* increases is though is less than $\mathbb{E}(r|s_n)$ as the informativeness of the signal must also increase, eventually rendering inspection as little more informative than the signal itself. As such, ψ_n^* and $\mathbb{E}(r|s_n)$ also have a single crossing, after which ideal inspection is less preferred to allocating outright. Label the corresponding crossing signal be s_β if it's positive and 0 otherwise. ■

In total this says that there are three regions of interest with respect to the principal's preferences: *low* signals, who the principal would like to reject outright and not allocate to, *intermediate* signals, who the principal would like to inspect and allocate if they are shown to yield positive rewards, and *high* signals, who the principal would like to allocate to outright and save on inspection costs. As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the upper envelope of 0, $\mathbb{E}(r|s_n)$ and ψ_n^* gives us a first best objective as a function of the signal as shown in Figure 1,

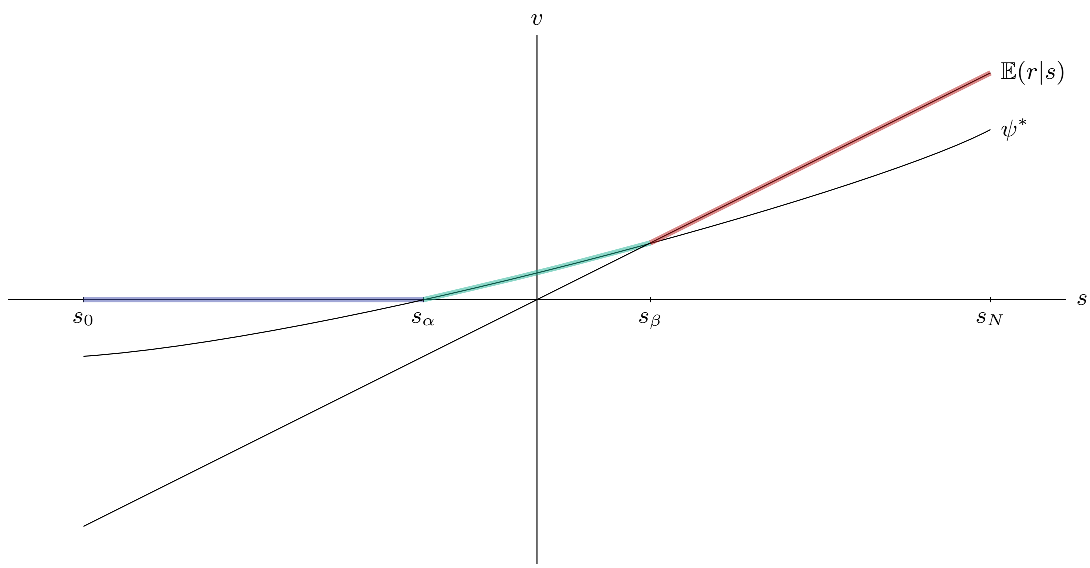

Figure 1: first best objective, v^*

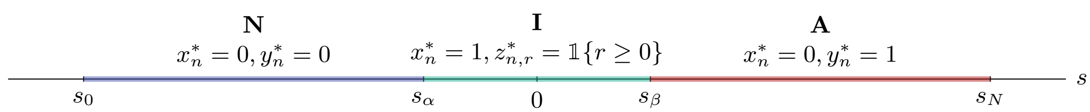

Figure 2: first best policy, (x^*, y^*, z^*)

and a first best policy as a function of the signal as shown in Figure 2, in a general setting.

If the return from inspection is high enough for all signals, then $\psi_n^* \geq \max\{0, \mathbb{E}(r|s_n)\}$ for all n and so $s_\alpha < s_0$ and $s_\beta > s_N$. For example, if $c = 0$ then it's trivially the case that inspection is always optimal. Looking forward to the core problem, the first best policy is then also incentive compatible, as all signals are treated equally. As such, we will continue under the assumption that either $s_\alpha > s_0$ or $s_\beta < s_N$.

4 Acquiring information

Now let's return to the problem of acquiring the private information from the agents. We will refer to the solution of maximizing the as the *second-best* policy. The first observation I want to draw your attendi

Relaxed problem:
Instead of solving the core problem directly, we will solve a relaxation of the problem that only

requires that $IC_{n,m}$ holds for $m = n+1$ i.e. the upward local IC (ULIC) constraints.

$$\begin{aligned}
\max_{(x_n,y_n,z_n)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n(\int r z_{n,r}\pi_{n,r}\, dr - c)]p_n \\
\text{s.t.} \quad & IC_{n,n+1} : (1-x_n)y_n + x_n(\int z_{n,r}\pi_{n,r}\, dr) \geq (1-x_{n+1})y_{n+1} + x_{n+1}(\int z_{n+1,r}\pi_{n,r}\, dr) \quad \forall\ n < N \\
& F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n
\end{aligned}$$

4.1 Threshold post-inspection rules

To derive the optimal policy for this problem, we will first show that the post-inspection rules are not only deterministic but only allocate when the realized reward is high.

Claim 4 *Optimal post-inspection rules are threshold mechanisms. That is, for each n there exists some τ_n such that:*

$$z_{n,r} = \mathbb{1}\{r \geq \tau_n\}$$

Proof: Suppose (x, y, z) is incentive compatible, optimal, and that for some n, z_n is not a threshold mechanism. Define τ_n such that:

$$\int z_{n,r}\pi_{n,r}\, dr = \int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r}\, dr$$

Given Π_n is absolutely continuous, τ_n is well-defined.

Consider a new policy which replaces z_n with this threshold post-inspection rule about τ_n. Clearly this is incentive compatible for n as it's defined such that they receive the same likelihood of allocation given they are inspected as before. That is, $IC_{n,n+1}$ continues to hold.

Now consider $IC_{n-1,n}$:

$$(1-x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r}\, dr) \geq (1-x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r}\, dr)$$

We'd like to show this continues to hold under the new policy. That is:

$$(1-x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r}\, dr) \geq (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\, dr)$$

Note that we can rewrite the right hand side of the original $IC_{n-1,n}$ by decomposing $z_{n,r}$ into the threshold rule and the residual that would reconstitute $z_{n,r}$:

$$
\begin{aligned}
&(1-x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r}\,dr) \\
&= (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr) + x_n(\int^{\tau_n} z_{n,r}\pi_{n-1,r}\,dr - \int_{\tau_n}(1-z_{n,r})\pi_{n-1,r}\,dr)
\end{aligned}
$$

By MLRP, $\pi_{n-1,r} \geq \pi_{n,r}\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}$ if $r < \tau_n$ and $\pi_{n-1,r} \leq \pi_{n,r}\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}$ if $r > \tau_n$. As such the right hand side of $IC_{n-1,n}$ must be:

$$
\begin{aligned}
&\geq (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr) + x_n\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}(\int^{\tau_n} z_{n,r}\pi_{n,r}\,dr - \int_{\tau_n}(1-z_{n,r})\pi_{n,r}\,dr) \\
&= (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr) + x_n\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r}\,dr - \int z_{n,r}\pi_{n,r}\,dr) \\
&= (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr)
\end{aligned}
$$

Where the the final equality comes from the definition of τ_n. As such, if the initial policy is incentive compatible for $n-1$ then the new policy is also incentive compatible.

Finally, the new policy must generate a higher return for the principal given we've shifted allocation weight from low values of r to high values of r evaluated under the same Π_n. Another way of showing this is that $\mathbb{1}\{r \geq \tau_n\}\Pi_n$ stochastically dominates $z_n\Pi_n$ and the principal evaluates an increasing function, r, with respect to these (censored) distributions. This implies (x, y, z) cannot have been optimal, a contradiction the proposition. ∎

The next example demonstrates this proof graphically and highlights the role of the monotone likelihood ratio property.

4.1.1 Example: FOSD or MLRP

Consider Figure 3, which gives an example of a post-inspection rule that we propose to transform into a threshold rule. The top function, z_n, is this post-inspection rule, the middle function, π_n, is the reward distribution under the signal that the principal targets with z_n, and the bottom function, π_{n-1}, is the reward distribution for a lower signal who we're covering from deviating.

We've normalized π_n to be constant and given two examples of the lower signals reward distribution. The left is an example of a distribution that is first order stochastically dominated by Π_n but does not follow the monotone likelihood ratio property. The right distribution is both first order stochastically dominated by Π_n and follows the monotone likelihood ratio property. The later can be seen by noting that the relative likelihood of rewards (relative to the higher signal) is decreasing in the reward, made obvious by a constant π_n. Both follow the right mass property given by FOSD: higher signals place a higher likelihood on generating a greater rewards then any fixed standard.

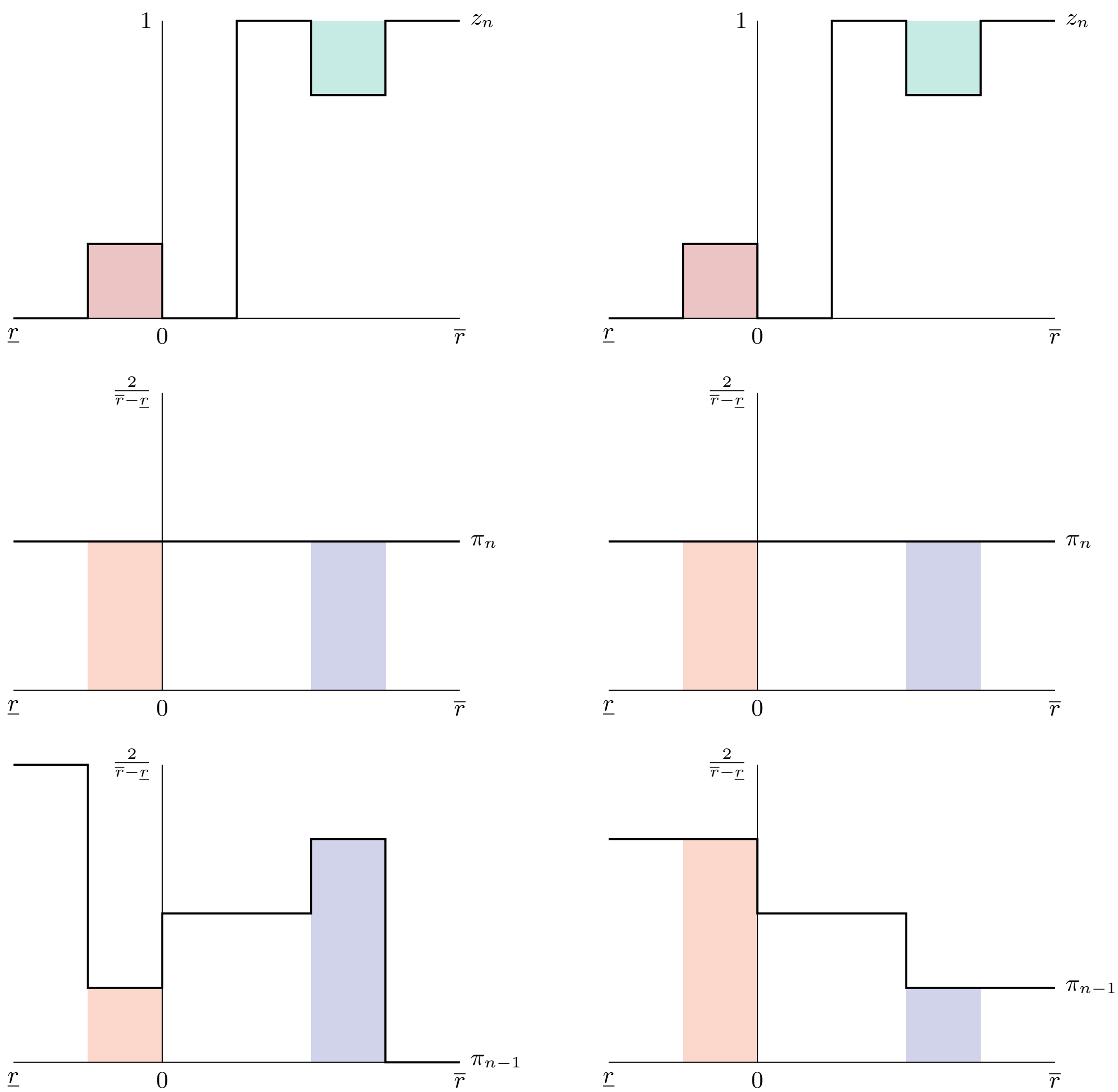

Figure 3: FOSD (left and right) is not enough for the transformation, but MLRP (right only) is.

Transforming z_n in this example is a matter of moving the red mass up to the green deficit. This is done in a way to make n indifferent between the rules, and in our example n assigns the same probability to these two events: the orange likelihood and the blue likelihood respectfully.

For this transformation to continue to satisfy $n-1$'s incentive constraint, without any additional restrictions on (x, y, z), we must guarantee that deviating to n's new policy leaves them no better off than had they deviated prior under the initial policy. FOSD is not enough to guarantee this. This is because lower signals may still put a higher relative likelihood of greater rewards than they had for smaller rewards higher signals do without violating the total right-mass property given by FOSD. In the example in the left distribution, $n-1$ puts a low orange likelihood on the event that r falls in the range where the principal is reducing the likelihood of allocation, and a large blue likelihood on the event where allocation is now being guaranteed, despite always putting lower probability on the total event that r is greater than any fixed value.

The MLRP prevents this possibility as the relative likelihood must be decreasing for higher values of r. As such, in the right distribution the orange likelihood is now greater than the blue likelihood and so on net makes $n-1$ worse off under this deviation.

4.2 Binding ULIC

We'll now show that this threshold characteristic implies that each upward local incentive compatibility constraint holds with equality in any optimal solution. For ease of language, we'll say that these constraints *bind* but do not mean to imply that they have a positive shadow price. In fact, for many local constraints in the second-best policy, incentive compatibility is free.

In the same way we denoted $\psi_n(z_n) \coloneqq \int_r r z_{n,r}\pi_{n,r}\,dr - c$ as the expected net reward to the principal from inspecting given some arbitrary post-inspection allocation rule z_n, denote $\phi_n(z_n) \coloneqq \int_r z_{n,r}\pi_{n,r}\,dr$ as the expected allocation to the agent from being inspected given some arbitrary post-inspection allocation rule z_n. Further use $\psi_n(\tau_n)$ and $\phi_n(\tau_n)$ as the respective expected values under the threshold rule defined by τ_n.

Consider the following partition of the signal space:

1. $S_0 \coloneqq \{n \mid 0 \geq \mathbb{E}(r|s_n),\ 0 \geq \psi_n(\tau_n)\}$
2. $S_\alpha \coloneqq \{n \mid 0 \geq \mathbb{E}(r|s_n),\ \psi_n(\tau_n) > 0\}$
3. $S_\beta \coloneqq \{n \mid \mathbb{E}(r|s_n) > 0,\ \psi_n(\tau_n) > \mathbb{E}(r|s_n)\}$
4. $S_1 \coloneqq \{n \mid \mathbb{E}(r|s_n) > 0,\ \mathbb{E}(r|s_n) \geq \psi_n(\tau_n)\}$

For a fixed vector of thresholds, τ, this partition outlines the ideal policy choice: no allocation if $n \in S_0$, inspect if $n \in S_\alpha \cup S_\beta$ and allocate without inspection if $n \in S_1$. If $\tau_n = 0$ for each n, this partition is as described by the first best policy and displayed in Figure 2. If $\tau_n \neq 0$ then $\psi_n(\tau_n) \leq \psi_n^*$ as the principal is either over or under allocating conditional on the realized reward. As such the S_0 and S_1 are supersets of their first best counterparts while $S_\alpha \cup S_\beta$ is a subset.

We will use this partition to show that under any optimal policy, (x, y, z), each ULIC constraint binds and forms a chain of equality conditions from 0 to $N-1$, reducing our problem to a single mathematical program. In each of the following arguments, the approach is the same: suppose a particular type of constraint doesn't bind for a solution (x, y, z) and find a feasible policy improvement, contradicting the optimality of the proposed solution.

CLAIM 5 *If* $n \in S_0$ *or* $n+1 \in S_1$*,* $IC_{n,n+1}$ *binds.*

Proof: Suppose $n \in S_0$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing y_n and x_n improves the objective as $0 \geq \mathbb{E}(r|s_n)$ and $0 \geq \psi_n(\tau_n)$, and decreases the left hand side so tightens $IC_{n,n+1}$. Further, this only relaxes $IC_{n-1,n}$.

$\therefore$ Either $u_n = 0$ ($x_n = 0$ and $y_n = 0$) for all $n \in S_0$, or $IC_{n,n+1}$ binds. Note that if $u_n = 0$ for all $n \in S_0$ then $IC_{n,n+1}$ trivially binds as $u_{n,n+1} \geq 0$.

Now consider $n+1 \in S_1$ and suppose $IC_{n,n+1}$ doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing y_{n+1} and expanding x_{n+1} improves the objective as $\mathbb{E}(r|s_{n+1}) > 0$ and $\mathbb{E}(r|s_{n+1}) \geq \psi_{n+1}(\tau_{n+1})$, and increases the right hand side so tightens $IC_{n,n+1}$. Further, this only relaxes $IC_{n+1,n+2}$.

$\therefore$ Either $u_{n+1} = 1$ ($x_{n+1} = 0$ and $y_{n+1} = 1$) for all $n+1 \in S_0$, or $IC_{n,n+1}$ binds. Note that if $u_{n+1} = 1$ for all $n+1 \in S_1$ then $IC_{n,n+1}$ trivially binds as $u_n \leq 1$. ■

A note about the edge cases: as we've assumed that either $s_\alpha > s_0$ or $s_\beta < s_N$, S_0 and S_1 cannot both be empty. As we'll show there is a chain of IC constraints linking all signals, so $u_0 = 0$ ($x_0 = 0$ and $y_0 = 0$) implies $u_n = 0$ for all n, and $u_N = 1$ ($x_N = 0$ and $y_N = 1$) implies $u_n = 1$ for all n. This is a contradiction if S_0 and S_1 are both nonempty, and if only one is nonempty, this trivially proves our claim that each IC constraint must bind. We should expect such cases to persist as there may be environments where the second best policy treats all signals equally, globally rejecting or globally accepting, as separating and inspecting may be too costly.

CLAIM 6 *If* $n, n+1 \in S_\alpha$*,* $IC_{n,n+1}$ *binds.*

Proof: Suppose $n, n+1 \in S_\alpha$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing y_n improves the objective as $\psi_n(\tau_n) \geq 0 \geq \mathbb{E}(r|s_n)$, and decreases the left hand side so tightens $IC_{n,n+1}$ and only relaxes $IC_{n-1,n}$.

Suppose $y_n = 0$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Suppose $\tau_n < 0$, then raising τ_n will improve the objective, decrease the left hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n-1,n}$. Similarly, suppose $\tau_{n+1} > 0$, then lowering τ_{n+1} will improve the objective, increase the right hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n+1,n+2}$.

Suppose $y_n = 0$, $\tau_n \geq 0$ and $\tau_{n+1} \leq 0$ and $IC_{n,n+1}$ doesn't bind. Consider expanding x_{n+1} which would improve the objective as $\psi_{n+1}(\tau_{n+1}) > 0 \geq \mathbb{E}(r|s_{n+1})$. This tightens $IC_{n,n+1}$ if:

$$0 < -y_{n+1} + \phi_n(\tau_{n+1}) \quad \Rightarrow \quad \phi_n(\tau_{n+1}) > y_{n+1}$$

Note that if this is true, then it also relaxes $IC_{n+1,n+2}$ as $\phi_{n+1}(\tau_{n+1}) \geq \phi_n(\tau_{n+1})$ by FOSD.

Suppose by contradiction $\phi_n(\tau_{n+1}) \leq y_{n+1}$, and $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = x_n\phi_n(\tau_n) &> (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ &\geq (1 - x_{n+1})\phi_n(\tau_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ &= \phi_n(\tau_{n+1}) \end{aligned}$$

But as $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$ and $\phi_n(\tau_{n+1}) \geq \phi_n(0)$, this is a contradiction. As such, expanding x_{n+1} must tighten $IC_{n,n+1}$.

Finally, suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_{n+1} = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > \phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've have already established this cannot be the case.

$\therefore IC_{n,n+1}$ binds. ■

Claim 7 *If $n, n+1 \in S_\beta$, $IC_{n,n+1}$ binds.*

Proof: Suppose $n, n+1 \in S_\beta$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then expanding y_{n+1} improves the objective as $\psi_{n+1}(\tau_{n+1}) \geq 0 \geq \mathbb{E}(r|s_{n+1})$, and increases the right hand side so tightens $IC_{n,n+1}$ and only relaxes $IC_{n+1,n+2}$.

Suppose $y_{n+1} = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Suppose $\tau_n < 0$, then raising τ_n will improve the objective, decrease the left hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n-1,n}$. Similarly, suppose $\tau_{n+1} > 0$, then lowering τ_{n+1} will improve the objective, increase the right hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n+1,n+2}$.

Suppose $y_{n+1} = 1$, $\tau_n \geq 0$ and $\tau_{n+1} \leq 0$ and $IC_{n,n+1}$ doesn't bind. Consider expanding x_n which would improve the objective as $\psi_n(\tau_n) > \mathbb{E}(r|s_n) > 0$. This tightens $IC_{n,n+1}$ if:

$$-y_n + \phi_n(\tau_n) < 0 \quad \Rightarrow \quad \phi_n(\tau_n) < y_n$$

Note that if this is true, then it also relaxes $IC_{n-1,n}$ as $\phi_{n-1}(\tau_n) \leq \phi_n(\tau_n)$ by FOSD.

Suppose by contradiction $\phi_n(\tau_n) \geq y_n$, and $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ (1 - x_n)\phi_n(\tau_n) + x_n\phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ \phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \end{aligned}$$

But, $\phi_n(\tau_n) \leq \phi_n(0)$ and $(1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$, a contradiction. As such, expanding x_n tightens $IC_{n,n+1}$.

Finally, suppose $y_{n+1} = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_n = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = \phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established this cannot be the case.

$\therefore IC_{n,n+1}$ binds. ∎

So far, we've shown constraints within the sets are binding, as well as those leading from S_0 or those leading to S_1. Now we're left to check constraints across the inspection sets.

Claim 8 *If $n \in S_\alpha$ and $n + 1 \in S_\beta$, $IC_{n,n+1}$ binds.*

Proof: Suppose $n \in S_\alpha$, $n+1 \in S_\beta$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as before, reducing y_n, expanding y_{n+1}, raising τ_n if $\tau_n < 0$ and lowering τ_{n+1} if $\tau_{n+1} > 0$ all improve the objective, tighten $IC_{n,n+1}$, relax $IC_{n-1,n}$ and $IC_{n+1,n+2}$.

Suppose, $y_n = 0$, $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

This cannot be the case as $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$ and $(1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$.

$\therefore IC_{n,n+1}$ binds. ■

We already know $S_0 < S_1$ and $S_\alpha < S_\beta$ as the value of $\mathbb{E}(r|s_n)$ is given by the signal structure and not the policy.[4] So the only two types of constraints to check are: $n \in S_\alpha$, $n+1 \in S_0$ and $n \in S_1$, $n+1 \in S_\beta$.

Claim 9 *If $n \in S_\alpha$ and $n+1 \in S_0$, $IC_{n,n+1}$ binds.*

Proof: Suppose $n \in S_\alpha$, $n+1 \in S_0$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as in the proof of Claim 6, reducing y_n, raising τ_n if $\tau_n < 0$ and lowering τ_{n+1} if $\tau_{n+1} > 0$ all improve the objective, tighten $IC_{n,n+1}$ and relax adjacent IC constraints.

Suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Also as in the proof of Claim 6, it must be that $\phi_n(\tau_{n+1}) > y_{n+1}$. Suppose by contradiction $\phi_n(\tau_{n+1}) \leq y_{n+1}$, and $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = x_n\phi_n(\tau_n) &> (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ &\geq (1-x_{n+1})\phi_n(\tau_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ &= \phi_n(\tau_{n+1}) \end{aligned}$$

[4] Here, $<$ refers to the *below* set relation defined by: set A is below set B, $A < B$, if $\forall a \in A$ and $\forall b \in B$, $a < b$.

But as $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$ and $\phi_n(\tau_{n+1}) \geq \phi_n(0)$, this is a contradiction.

Now consider reducing the probability that $n+1$ is allocated to without inspection, in favour of inspection using the threshold assigned to n. In particular, conditional on not inspecting $n+1$, instead of allocating with probability y_{n+1}, allocate with probability λy_{n+1} and inspect using the threshold τ_n with probability $(1-\lambda)$, for some $\lambda \in (0,1)$.

This improves the objective as $0 \geq \mathbb{E}(r|s_{n+1})$ and $\psi_{n+1}(\tau_n) > 0$. The second fact here comes from the observation that despite $0 \geq \psi_{n+1}(\tau_{n+1})$, $\psi_n(\tau_n) > 0$ implies $\psi_{n+1}(\tau_n) > 0$. To see this, take a fixed threshold τ and rearrange:

$$\begin{aligned} \psi_n(\tau) &> 0 \\ \int \mathbb{1}\{r \geq \tau\} r\pi_{n,r}\, dr - c &> 0 \\ \int \mathbb{1}\{r \geq \tau\} r\pi_{n,r}\, dr &> c \end{aligned}$$

As $\mathbb{1}\{r \geq \tau\}r$ is an increasing function of r when $\tau > 0$, the left hand side is increasing in n as ensured by FOSD.

This tightens $IC_{n,n+1}$ as $\phi_n(\tau_n) > y_{n+1}$.[5] To see this, observe that:

$$\phi_n(\tau_n) > x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) > y_{n+1}$$

Which follows the condition that $IC_{n,n+1}$ didn't bind and our claim that $\phi_n(\tau_{n+1}) > y_{n+1}$.

And this change only relaxes $IC_{n+1,n+2}$ ensured by FOSD and the same rationale:

$$\phi_{n+1}(\tau_n) > x_n\phi_{n+1}(\tau_n) \geq x_n\phi_n(\tau_n) > y_{n+1}$$

Finally, suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_{n+1} = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > \phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've have already established this cannot be the case.

$\therefore IC_{n,n+1}$ binds. ■

Claim 10 *If $n \in S_1$ and $n+1 \in S_\beta$, $IC_{n,n+1}$ binds.*

[5] And does not contradict for small enough λ.

Proof: Suppose $n \in S_1$, $n+1 \in S_\beta$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as in the proof of Claim 7, expanding y_{n+1}, raising τ_n if $\tau_n < 0$ and lowering τ_{n+1} if $\tau_{n+1} > 0$ all improve the objective, tighten $IC_{n,n+1}$ and relax adjacent IC constraints.

Suppose $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, and $IC_{n,n+1}$ doesn't bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Also as in the proof of Claim 7, it must be that $y_n > \phi_n(\tau_n)$. Suppose by contradiction $\phi_n(\tau_n) \geq y_n$, and $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned}
u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) &> (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\
(1-x_n)\phi_n(\tau_n) + x_n\phi_n(\tau_n) &> (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\
\phi_n(\tau_n) &> (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1})
\end{aligned}$$

But, $\phi_n(\tau_n) \leq \phi_n(0)$ and $(1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$, a contradiction.

Now consider reducing the probability that n is allocated to without inspection, in favour of inspection using the threshold assigned to $n+1$. In particular, conditional on not inspecting n, instead of allocating with probability y_n, allocate with probability λy_n and inspect using the threshold τ_{n+1} with probability $(1-\lambda)$, for some $\lambda \in (0,1)$.

This improves the objective as $\psi_n(\tau_{n+1}) \geq \mathbb{E}(r|s_n)$. This fact comes from the observation that despite $\mathbb{E}(r|s_n) \geq \psi_n(\tau_n)$, $\psi_{n+1}(\tau_{n+1}) > \mathbb{E}(r|s_{n+1})$ implies $\psi_n(\tau_n) > \mathbb{E}(r|s_n)$. To see this, take a fixed threshold τ and rearrange:

$$\begin{aligned}
\psi_{n+1}(\tau_{n+1}) &> \mathbb{E}(r|s_{n+1}) \\
\int \mathbb{1}\{r \geq \tau\} r\pi_{n+1,r}\, dr - c &> \int r\pi_{n+1,r}\, dr \\
-c &> \int \mathbb{1}\{r < \tau\} r\pi_{n+1,r}\, dr
\end{aligned}$$

As $\mathbb{1}\{r < \tau\}r$ is an increasing function of r when $\tau < 0$, the right hand side is increasing in n as ensured by FOSD.

This tightens $IC_{n,n+1}$ as $y_n > \phi_n(\tau_{n+1})$.[6] To see this, observe that:

$$y_n > (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) > \phi_n(\tau_{n+1})$$

Which follows from our claim that $y_n > \phi_n(\tau_n)$ and the condition that $IC_{n,n+1}$ didn't bind.

And this change only relaxes $IC_{n-1,n}$ ensured the same rationale and FOSD:

$$y_n > \phi_n(\tau_{n+1}) > \phi_{n-1}(\tau_{n+1})$$

Finally, suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_n = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = \phi_n(\tau_n) > (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've have already established this cannot be the case.

$\therefore IC_{n,n+1}$ binds. ■

Then, by claims 5 through 10, $IC_{n,n+1}$ must bind for all n.

4.3 The linear program and its solution

With the results from 4 and 4.2, we can rewrite our problem as:

$$\begin{aligned}
\max_{(x_n,y_n,\tau_n)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n\psi_n(\tau_n)]p_n \\
\text{s.t.} \quad & IC_{n,n+1} : (1-x_n)y_n + x_n\phi_n(\tau_n) = (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \quad \forall\ n < N \\
& F : 0 \leq x_n, y_n, \tau_n \leq 1 \quad \forall\ r \quad \forall\ n
\end{aligned}$$

With this, we can conclude that the optimal inspection rule, x, is an extreme solution and is monotonic in n.

Claim 11 *Optimal inspection rules are threshold mechanisms. That is, there exists an n_0 such that $x_n = \mathbb{1}\{n \geq n_0\}$.*

Proof: First observe that for each n we can represent $(1-x_n)y_n$ recursively using the binding

[6] And does not contradict for small enough λ.

$IC_{n,n+1}$ constraints:

$$
\begin{aligned}
(1-x_n)y_n &= (1-x_{n+1})y_{n+1} && + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= (1-x_{n+2})y_{n+2} && + x_{n+2}\phi_{n+1}(\tau_{n+2}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) \\
& && + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= (1-x_{n+3})y_{n+3} && + x_{n+3}\phi_{n+2}(\tau_{n+3}) - x_{n+2}\phi_{n+2}(\tau_{n+2}) \\
& && + x_{n+2}\phi_{n+1}(\tau_{n+2}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) \\
& && + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= \cdots \\
&= (1-x_N)y_N && + \sum_{m=n}^{N-1}[x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]
\end{aligned}
$$

Which can also be written as the following arrangement:

$$(1-x_n)y_n = (1-x_N)y_N + x_N\phi_{N-1}(\tau_N) + \sum_{m=n+1}^{N-1} x_m[\phi_{m-1}(\tau_m) - \phi_m(\tau_m)] - x_n\phi_n(\tau_n)$$

Note that this also restricts the choice of τ. For example, suppose for some pair $n_0 < n_1$, $x_{n_0} = 1$, $x_{n_1} = 1$, and $x_m = 0$ for $n_0 < m < n_l$. Then by the binding constraint τ_{n_0} and τ_{n_1} must satisfy:

$$\phi_{n_0}(\tau_{n_0}) = \phi_{n_1-1}(\tau_{n_1})$$

This says that if $n_0 = n_1 - 1$ then $\tau_{n_0} = \tau_{n_1}$, and if $n_0 < n_1 - 1$, then $\tau_{n_0} > \tau_{n_1}$ and uniquely detirmined. We will return to this example after the substition.

Substituting the arrangement into the objective function:

$$
\begin{aligned}
&\sum_n [(1-x_n)y_n \mathbb{E}(r|s_n) + x_n \psi_n(\tau_n)]p_n \\
&= \sum_n [\langle (1-x_N)y_N + \sum_{m=n}^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\rangle \mathbb{E}(r|s_n) + x_n\psi_n(\tau_n)]p_n \\
&= (1-x_N)y_N \sum_n \mathbb{E}(r|s_n)p_n + \sum_n \sum_{m=n}^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\mathbb{E}(r|s_n)p_n + \sum_n x_n\psi_n(\tau_n)p_n \\
&= (1-x_N)y_N \sum_n \mathbb{E}(r|s_n)p_n + \sum_m^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)] \sum_n^m \mathbb{E}(r|s_n)p_n + \sum_n x_n\psi_n(\tau_n)p_n \\
&= (1-x_N)y_N \mathbb{E}(r) + \sum_m^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\mathbb{E}(r|s \le s_m)P_m + \sum_n x_n\psi_n(\tau_n)p_n
\end{aligned}
$$

This shows us that we that for a fixed y_N and τ the objective is linear in x_n, whose only restriction is that $x_n \in [0,1]$:

$$
\begin{aligned}
\max_{(x_n,\tau_n),y_N} (1-x_N)y_N\mathbb{E}(r) \quad &+ \quad x_N[\phi_{N-1}(\tau_N)\mathbb{E}(r|s \le s_{N-1})P_{N-1} + \psi_N(\tau_N)p_N] \\
&+ \sum_{n=1}^{N-1} \quad x_n[\phi_{n-1}(\tau_n)\mathbb{E}(r|s \le s_{n-1})P_{n-1} - \phi_n(\tau_n)\mathbb{E}(r|s \le s_n)P_n + \psi_n(\tau_n)p_n] \\
&+ \quad x_0[-\phi_0(\tau_0)\mathbb{E}(r|s_0)p_0 + \psi_0(\tau_0)p_0]
\end{aligned}
$$

Let a_n be the coefficient on x_n in the objective, and observe that a_n is only a function of τ_n. We can immediately conclude:

- $x_n = \mathbb{1}\{a_n(\tau_n) \ge 0\}$, and
- $y_N = \mathbb{1}\{\mathbb{E}(r) \ge 0\}$.

This means the restrictions on τ in the previous example are the only relevant restriction to our problem. Returning to the case when $n_0 < n_1 - 1$ ...

$\therefore x_n = \mathbb{1}\{n \ge n_0\}$ for some $n_0 \in \{0, \dots, N, N+1\}$. ■

Label the pooling mechanism and separating mechanism ...

4.4 Limiting solution

Consider the continuous extension ... Helly's selection theorem ...

Claim 12 *The second best policy $(x_s^\star, y_s^\star, z_s^\star)$ is given by:*

- $x_s^\star = \mathbb{1}\{s \ge s_\gamma\}$,

- $y^\star_s = \phi^*_\gamma$, *and*
- $z^\star_{s,r} = \mathbb{1}\{r \geq \tau^\star\}$,

where γ *is the solution to* $\phi_\gamma(\tau^\star)\mathbb{E}(r|s_\gamma) = \psi_\gamma(\tau^\star)$ *and* $\tau^\star$ *is the solution to:*

$$\tau = \left[\frac{\pi_{s_\gamma,\tau} P_{s_\gamma}}{\int_{s_\gamma}^{s_N} \pi_{s,\tau} p_s \, ds}\right] (-\mathbb{E}(r|s \leq s_\gamma))$$

Proof: Finally, we're left to check that the solution to the relaxed problem indeed satisfies the omitted global IC constraints ... ■

As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the second best objective as a function of the signal as shown in Figure 4, and a second best policy as a function of the signal as shown in Figure 5.

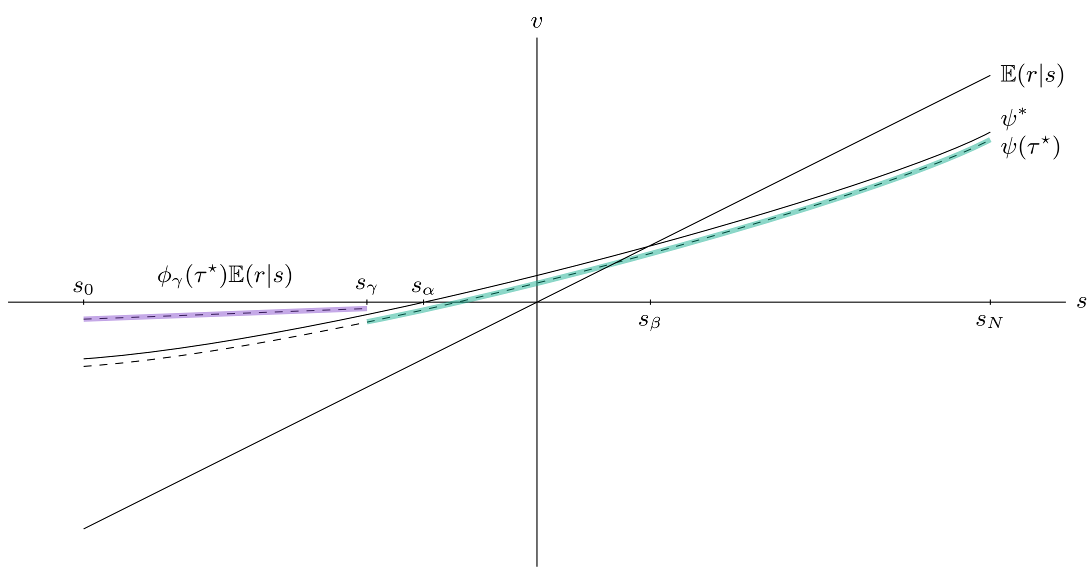

Figure 4: second-best objective, $v^\star$

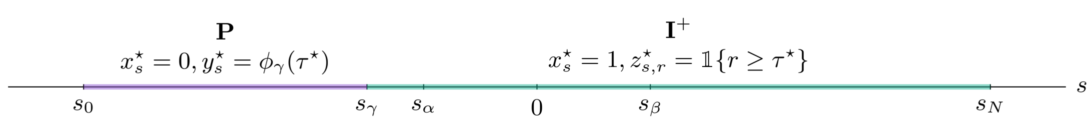

Figure 5: second-best policy, $(x^\star, y^\star, z^\star)$

As we can see, there are four types of losses for the principal introduced by the agents private information:

- *Under-allocation post inspection*: agents who are inspected, $s \in (s_\gamma, s_N]$, and who generate a marginally positive reward are now rejected in order to gather pre-inspection information about lower types.

- *Over-inspection at the top*: agents who's private information would be enough to guarantee allocation, $s \in (s_\beta, s_N]$, are now inspected in order to gather pre-inspection information about lower types.
- *Over-inspection at the bottom*: agents who's private information would be enough to marginally reject without inspection, $s \in (s_\gamma, s_\alpha)$, are now inspected as it's cheaper to do so than accepting them at a higher rate.
- *Over-allocation at the bottom*: agents who's private information would be enough to reject without inspection, $s \in [s_0, s_\gamma]$, are now allocated to with positive probability.

When these losses are less than those associated with the pooling policies, separation is optimal for the principal.

5 Commitment

There are three natural relaxations of the full commitment assumption in our model:

1. **partial commitment**, or *pre-inspection commitment*: the principal can commit to an inspection rule but cannot commit to a post-inspection allocation rule,
2. **limited commitment**, or *pre-assessment commitment*: the principal cannot commit to either inspection rule or post-inspection allocation rule, but can commit to a pre-inspection allocation, and
3. **no commitment**: the principal has no commitment at all, that is they can't even commit to a pre-inspection allocation.

Note that in the no commitment case the principal can only choose between the pooling equilibria and reports can convey no information. We know what this looks like, so let's now turn to the first two relaxations.

Need to show claim 2 works with τ=0. Straightforward as it's the same as first best, so only relies on the first set of claims which don't rely on changing tau.

5.1 Partial commitment

Claim 13 *The second-best policy $(x^\star, y^\star, z^\star)$ under partial commitment is given by:*

- $x_s^\star = \mathbb{1}\{s \geq s_\delta\}$,
- $y_s^\star = \phi_\delta^*$, *and*

- $z^{\star}_{s,r} = z^{*}_{s,r} = \mathbb{1}\{r \geq 0\}$,

where δ is the solution to $\phi^*_\delta \mathbb{E}(r|s_\delta) = \psi^*_\delta$.

As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the second-best objective as a function of the signal as shown in Figure 6, and a second-best policy as a function of the signal as shown in Figure 7.

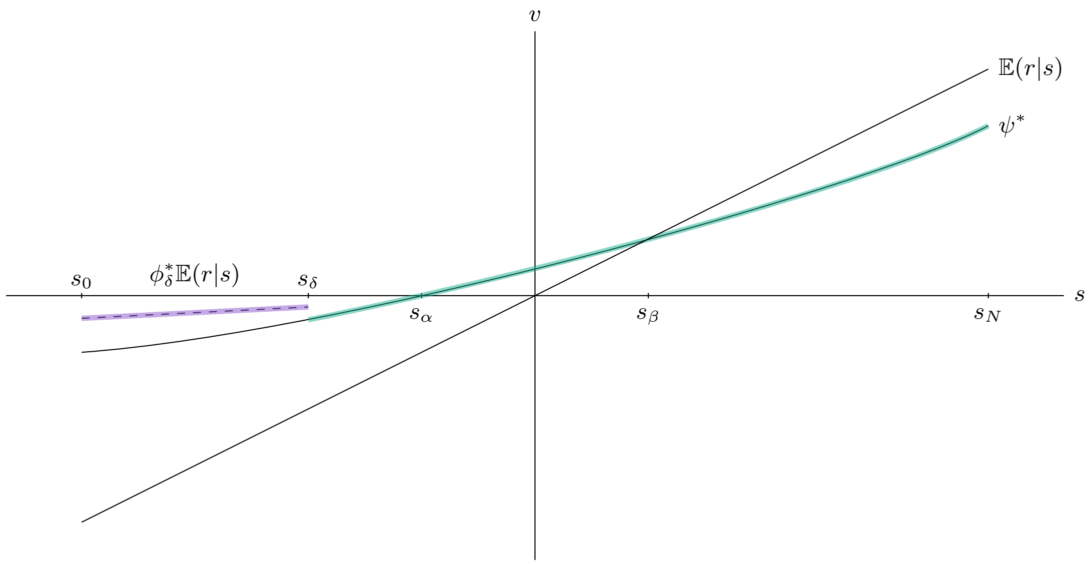

Figure 6: second best objective under partial commitment, $v^\star$

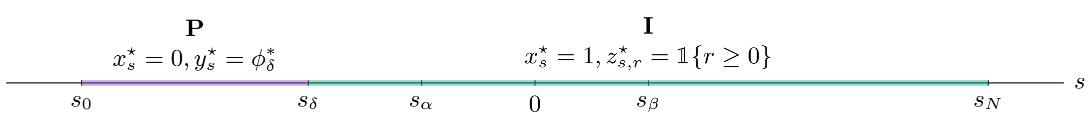

Figure 7: second best policy under partial commitment, $(x^\star, y^\star, z^\star)$

5.2 Limited commitment

Claim 14 *The second best policy $(x^\star, y^\star, z^\star)$ under limited commitment is given by:*

- $x^\star_s = \mathbb{1}\{s \geq s_\alpha\}$,
- $y^\star_s = \phi^*_\alpha$, *and*
- $z^\star_{s,r} = z^*_{s,r} = \mathbb{1}\{r \geq 0\}$,

where α is the solution to $\psi^*_\alpha = 0$.

As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the second best objective as a function of the signal as shown in Figure 8, and a second best policy as a function of the signal as shown in Figure 9.

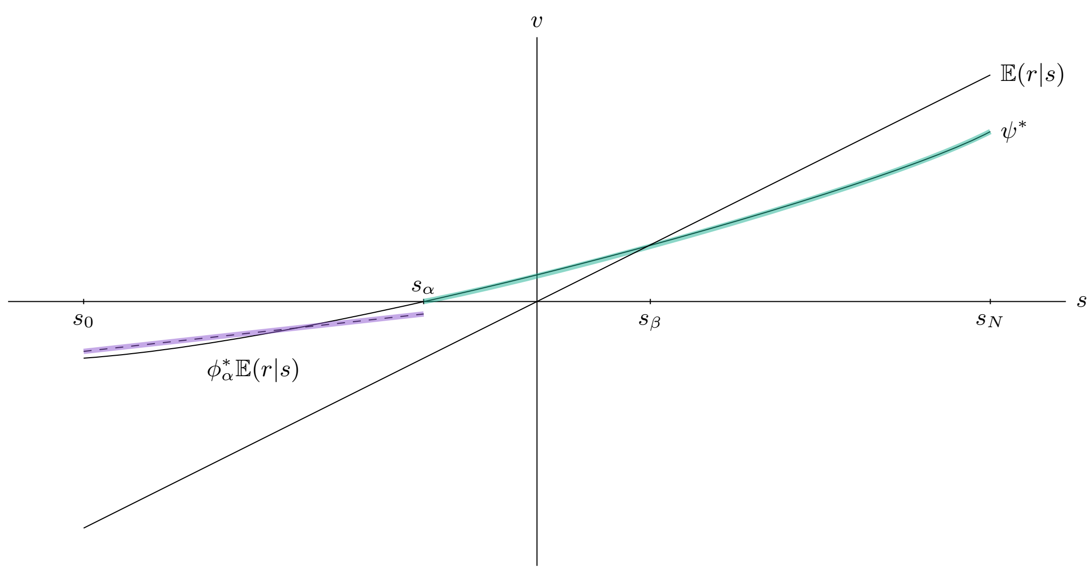

Figure 8: second best objective under limited commitment, $v^\star$

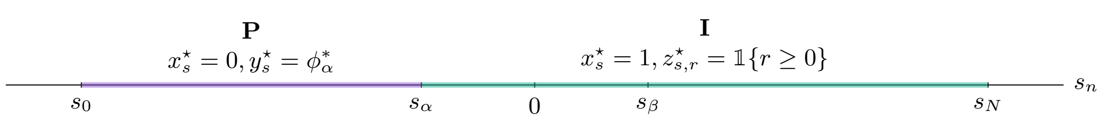

Figure 9: second-best policy under limited commitment, $(x^\star, y^\star, z^\star)$

6 Summary

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A Environment

One might consider two assumptions as being important restrictions on the space of signals for an economic problem such as this.

A1. The signals are ordered by **first order stochastic dominance (FOSD)**: $\Pi_n(r) \leq \Pi_m(r)$ for all r if $n > m$.

A2. The signals bear the **monotone likelihood ratio property (MLRP)**: $\pi_n(r_1)/\pi_m(r_1) \geq \pi_n(r_0)/\pi_m(r_0)$ for all $r_1 > r_0$ and $n > m$.

Claim 1 *A2 implies A1.*

Proof: Rearranging the condition in A2 for some $r_1 > r_0$ and $n > m$:

$$\pi_n(r_1)\pi_m(r_0) \geq \pi_n(r_0)\pi_m(r_1)$$

We can integrate this expression up to r_1 with respect to r_0 to get:

$$\int^{r_1} \pi_n(r_1)\pi_m(r_0)dr_0 \geq \int^{r_1} \pi_n(r_0)\pi_m(r_1)dr_0$$
$$\pi_n(r_1)\Pi_m(r_1) \geq \Pi_n(r_1)\pi_m(r_1)$$
$$\frac{\pi_n(r_1)}{\pi_m(r_1)} \geq \frac{\Pi_n(r_1)}{\Pi_m(r_1)}$$

Similarly, we can also integrate the original expression down to r_0 with respect to r_1 to get:

$$\int_{r_0} \pi_n(r_1)\pi_m(r_0)dr_1 \geq \int_{r_0} \pi_n(r_0)\pi_m(r_1)dr_1$$
$$(1 - \Pi_n(r_0))\pi_m(r_0) \geq \pi_n(r_0)(1 - \Pi_m(r_0))$$
$$\frac{1 - \Pi_n(r_0)}{1 - \Pi_m(r_0)} \geq \frac{\pi_n(r_0)}{\pi_m(r_0)}$$

Combining and rearranging these last two expressions for any particular $r = r_0 = r_1$ gives us A1:

$$\Pi_m(r) \geq \Pi_n(r)$$

■

A.1 First best benchmark

If the solution to the problem,

$$\begin{aligned} \max_{(x_n, y_n, z_n)} \quad & \sum_n [(1-x_n) y_n \mathbb{E}(r|s_n) + x_n (\int r z_{n,r} \pi_{n,r} \, dr - c)] p_n \\ \text{s.t.} \quad & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\, r \quad \forall\, n \end{aligned}$$

is the *first best* policy, then:

CLAIM 2 *The first best policy (x_n^*, y_n^*, z_n^*) is given by:*

- $z_{n,r}^* = \mathbb{1}(r \geq 0)$,
- $y_n^* = \mathbb{1}(\mathbb{E}(r|s_n) \geq 0)$, *and*
- $x_n^* = \mathbb{1}(\psi_n(z_n^*) \geq \max\{\mathbb{E}(r|s_n), 0\})$.

Proof: If $z_{n,r} < 1$ for some $r > 0$ then increasing $z_{n,r}$ weakly increases the objective function, and if $z_{n,r} > 0$ for some $r < 0$ then decreasing $z_{n,r}$ weakly increases the objective function. Similarly if $y_n < 1$ for some $\mathbb{E}(r|s_n) > 0$ then increasing y_n weakly increases the objective function, and if $y_n > 0$ for some $\mathbb{E}(r|s_n) < 0$ then decreasing y_n weakly increases the objective function. Then, the only weakly unimprovable policies are $z_{n,r} = \mathbb{1}\{r \geq 0\}$ and $y_n = \mathbb{1}\{\mathbb{E}(r|s_n) \geq 0\}$. Given this, the objective is linear in x_n and so the maximum is obtained by selecting the larger coefficient: setting $x_n = 1$ when $\psi_n(z_n^*) \geq \max\{\mathbb{E}(r|s_n), 0\}$, and 0 otherwise. ■

And there are only three relevant policy combinations of this policy with a fixed order with respect to s_n:

CLAIM 3 *There exists some s_α and s_β, with $s_\alpha \leq s_\beta$, such that:*

- *if $s_n \leq s_\alpha$ then $0 \geq \max\{\psi_n^*, \mathbb{E}(r|s_n)\}$,*
- *if $s_n \in (s_\alpha, s_\beta)$ then $\psi_n^* > \max\{0, \mathbb{E}(r|s_n)\}$, and*
- *if $s_n \geq s_\beta$ then $\mathbb{E}(r|s_n) \geq \max\{\psi_n^*, 0\}$.*

Proof: We will prove this claim by constructing thresholds $\tilde{s}_\alpha$ and $\tilde{s}_\beta$ and adjusting them to match the succinct claim.

By FOSD, ψ_n^* is increasing in n, as c is fixed and the likelihood that $r > 0$ is increasing. As such there exists a $\tilde{s}_\alpha$ such that $\psi_n^* > 0$ if $s_n > \tilde{s}_\alpha$. Note that trivially we can set $\tilde{s}_\alpha$ as any value less than s_0 if $\psi_0^* > 0$ and any value greater than s_N if $\psi_N^* < 0$.

Additionally, there exists a $\tilde{s}_\beta$ such that $\psi_n^* < \mathbb{E}(r|s_n)$ if $s_n > \tilde{s}_\beta$. To see this, observe that the following are equivalent:

$$\begin{aligned} \psi_n^* &< \mathbb{E}(r|s_n) \\ \int \mathbb{1}\{r \geq 0\} r \pi_{n,r}\, dr - c &< \int r \pi_{n,r}\, dr \\ -c &< \int \mathbb{1}\{r < 0\} r \pi_{n,r}\, dr \end{aligned}$$

and as $\mathbb{1}\{r < 0\} \cdot r$ is increasing in r, the right hand side is increasing in n by FOSD. As before we can set $\tilde{s}_\beta$ as any value less than s_0 if $\psi_0^* < \mathbb{E}(r|s_0)$ and any value greater than s_N if $\psi_N^* > \mathbb{E}(r|s_N)$.

Finally by definition $s_n = \mathbb{E}(r|s_n)$ and so $\mathbb{E}(r|s_n) > 0$ when $s_n > 0$. This means that $\tilde{s}_\alpha$ is only policy relevant when less than 0 and $\tilde{s}_\beta$ when greater than zero. As such, define $s_\alpha = \min\{\tilde{s}_\alpha, 0\}$ and $s_\beta = \max\{\tilde{s}_\beta, 0\}$. ■

B Acquiring information

C Commitment

D Signal choice

E Application costs