Optimal Allocation with Noisy Inspection

Nawaaz Khalfan

January 20, 2023

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A core economic activity

- employers **interview** potential employees
- public funds **assess** grant applications
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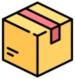

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2. verification or *screening*

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How should the principal design the inspection and allocation mechanism to maximize their ex ante expected return?

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1. **Hiring**: a firm seeks to fill an open position in their operation with a potential employee.
2. **Grant assignment**: a public fund is tasked with assessing a grant application.
3. **Impact investment**: a venture capitalist sets the mechanism by which it reviews and invests in startups.

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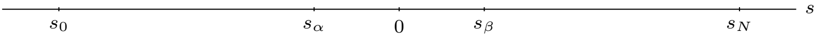

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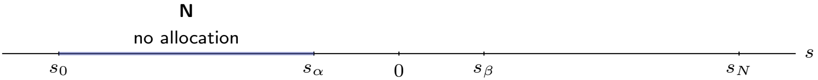

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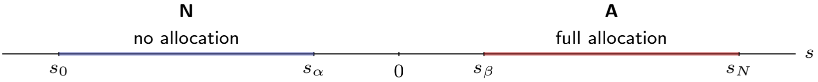

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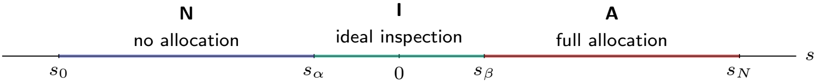

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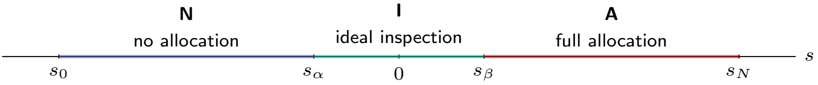

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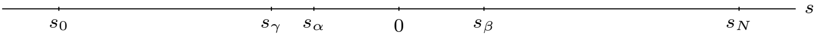

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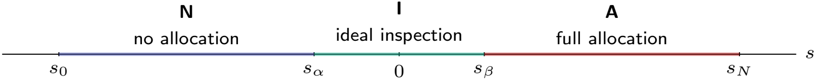

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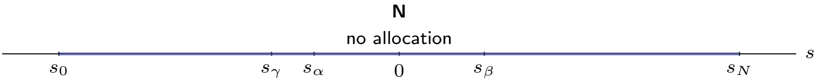

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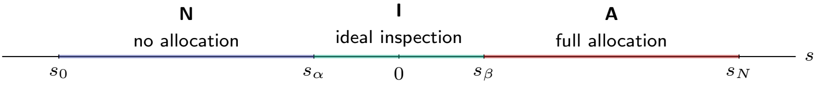

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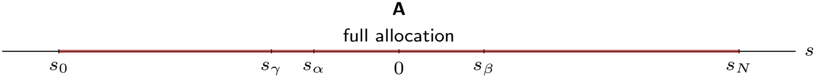

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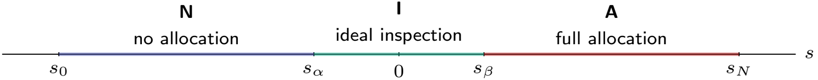

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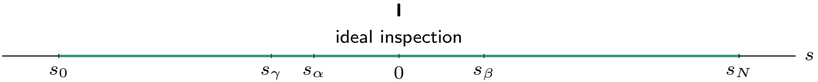

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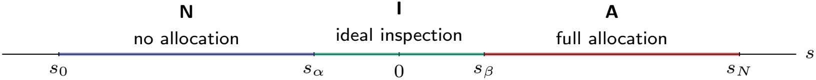

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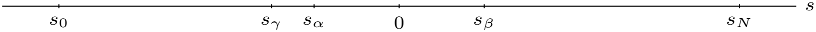

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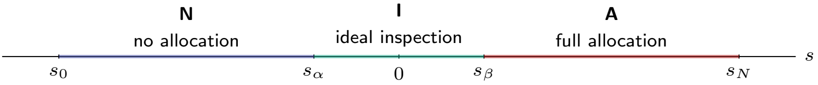

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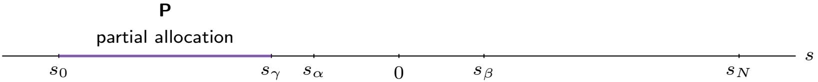

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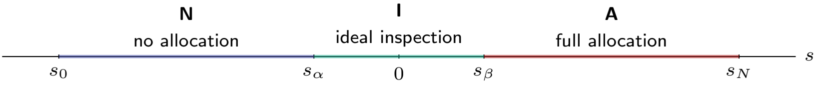

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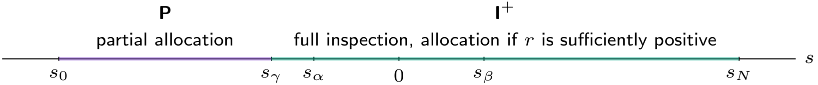

Losses

Three types of losses from private information:

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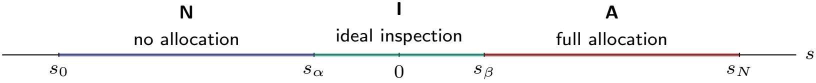

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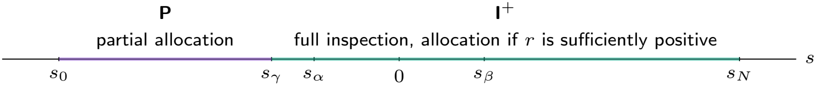

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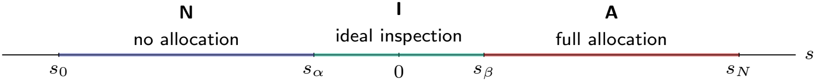

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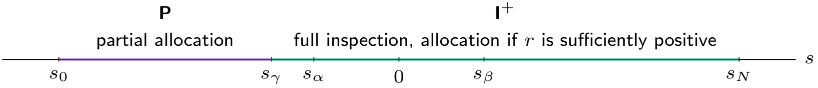

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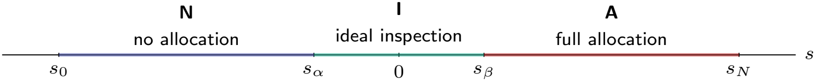

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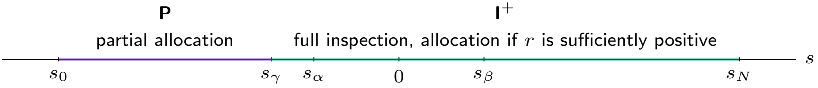

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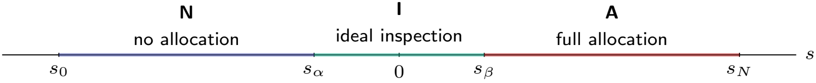

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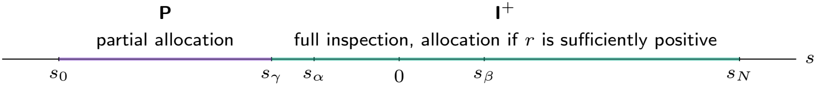

Literature

Perfect information: Green and Laffont (1986), Ben-Porath, Dekel and Lipman (2014), Mylovanov and Zapechelnyuk (2017), Epitropou and Vohra (2019).

Transfers: Townsend (1979), Border and Sobel (1987), Mookherjee and Png (1989), Alaei et al. (2020).

Limited transfers: Mylovanov and Zapechelnyuk (2017), Silva (2019*b*), Li (2021).

Efficient mechanisms: Ball and Kattwinkel (2019), Silva (2019*a*), Siegel and Strulovici (2021), Pereyra and Silva (2021), Erlanson and Kleiner (2020).

Scoring rules: McCarthy (1956), Savage (1971), Gneiting and Raftery (2007).

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Direct transfers of value between the principal and agent are prohibited.

Signals

Suppose $s \in \{s_0, s_1, \ldots, s_N\}$, where $s = s_n$ with probability $p_n \in (0, 1)$, $\sum_n p_n = 1$, and P_n is the cmf.

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Suppose that the signals are ordered by the monotone likelihood ratio property, **MLRP**.

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Note that MLRP $\Rightarrow$ FOSD.

It's without loss to relabel the signals by their induced expected reward, so that $s_n = \mathbb{E}(r|s_n)$.

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2. The agent observes their signal and submits a report to the principal.
3. The principal implements the mechanism conditional on the report and any reward realizations.
4. All remaining uncertainty is resolved, and rewards are distributed.

Mechanism

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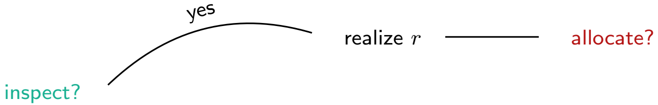

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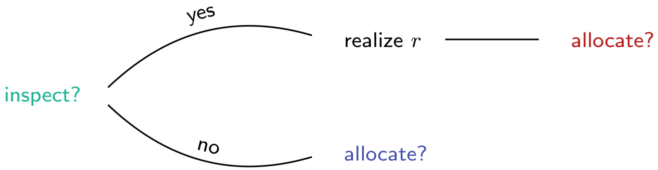

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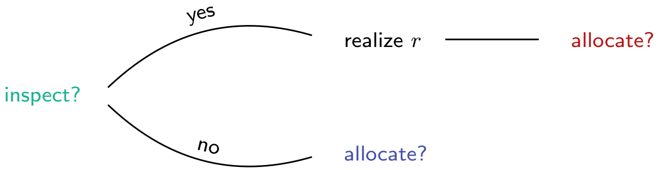


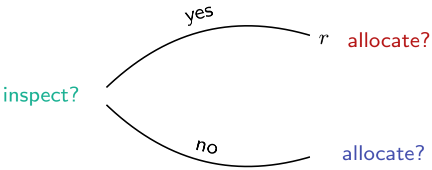
Then, a **mechanism** specifies for each type s,

- x_s: an inspection rule,
- y_s: a pre-inspection allocation, and
- $z_{s,r}$: a post-inspection allocation for each r.

These are potentially probabilistic choices, so are bounded between 0 and 1.

Rewards

Principal's objective:



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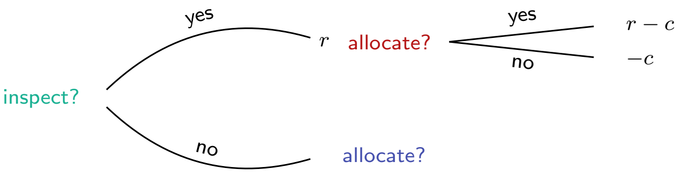

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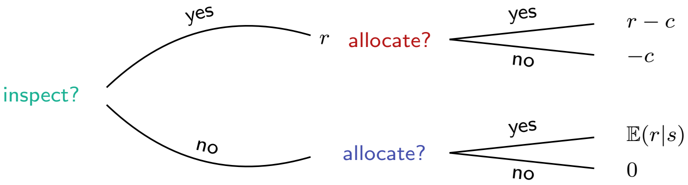

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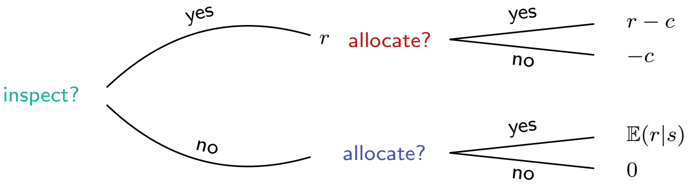

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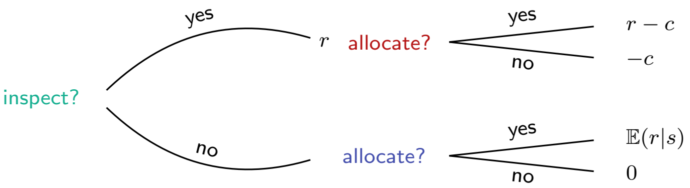

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An **optimal allocation** is a mechanism that maximizes the ex ante expected objective subject to *incentive compatibility* (IC) for each type s:

$$u(s|s) \geq u(\hat{s}|s) \quad \forall \hat{s}$$

Optimal allocation

The principal's problem:

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$$(1 - x_s)$$

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$$(1 - x_s) y_s$$

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$$(1 - x_s)y_s\mathbb{E}(r|s)$$

Optimal allocation

The principal's problem:

$$(1 - x_s) y_s \mathbb{E}(r|s) + x_s$$

Optimal allocation

The principal's problem:

$$(1 - x_s)y_s\mathbb{E}(r|s) + x_s(\mathbb{E}(z_{s,r}.r|s) - c)$$

Optimal allocation

The principal's problem:

$$\max_{(x,y,z)} \quad \mathbb{E}_s \left[(1 - x_s) y_s \mathbb{E}(r|s) + x_s (\mathbb{E}(z_{s,r}.r|s) - c) \right]$$

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$$\text{s.t.} \quad IC_{s,\hat{s}} : (1-x_s)y_s + x_s\mathbb{E}(z_{s,r}|s) \geq (1-x_{\hat{s}})y_{\hat{s}} + x_{\hat{s}}\mathbb{E}(z_{\hat{s},r}|s) \quad \forall\ s, \hat{s}$$

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$$F_{s,r} : x_s \in [0,1],\ y_s \in [0,1],\ z_{s,r} \in [0,1] \quad \forall\ s,r$$

Optimal allocation

The principal's problem:

$$\max_{(x,y,z)} \quad \sum_n [(1-x_n) y_n \mathbb{E}(r|s_n) + x_n \psi_n(z_n)] p_n$$

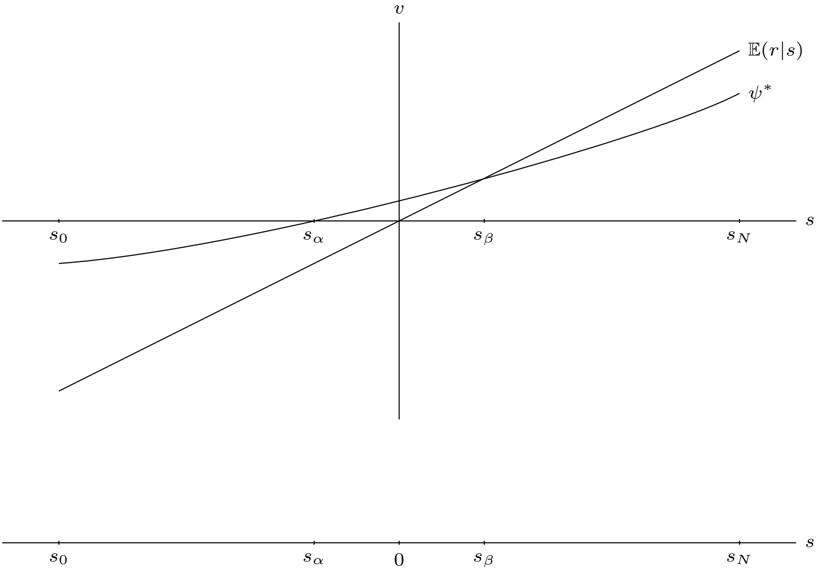
$$\text{s.t.} \quad IC_{n,m} : (1-x_n) y_n + x_n \mathbb{E}(z_{n,r}|n) \geq (1-x_m) y_m + x_m \mathbb{E}(z_{m,r}|n) \quad \forall\ n, m$$

$$F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n$$

where:

- $\psi_n(z_n) \coloneqq \mathbb{E}(z_{s,r}.r|s) - c = \int r z_{n,r} \pi_{n,r}\, dr - c$, is the expected reward from inspecting n with post-inspection allocation rule z_n.

First best policy, $*$



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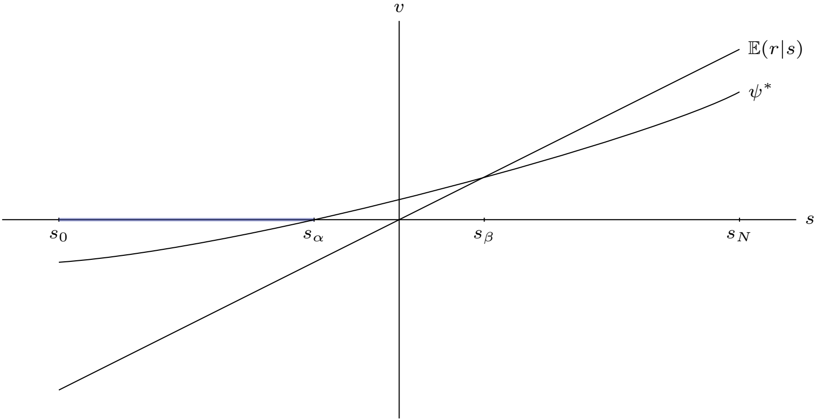

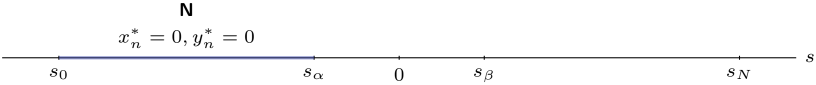

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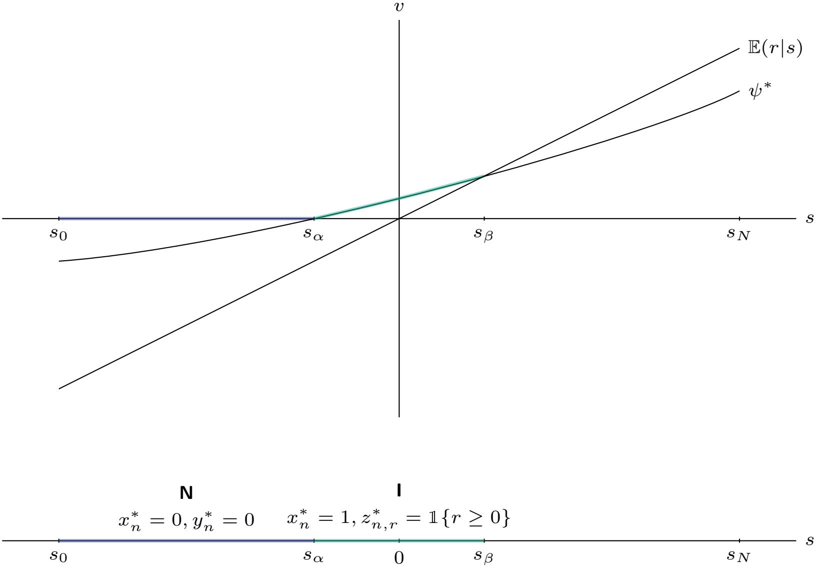

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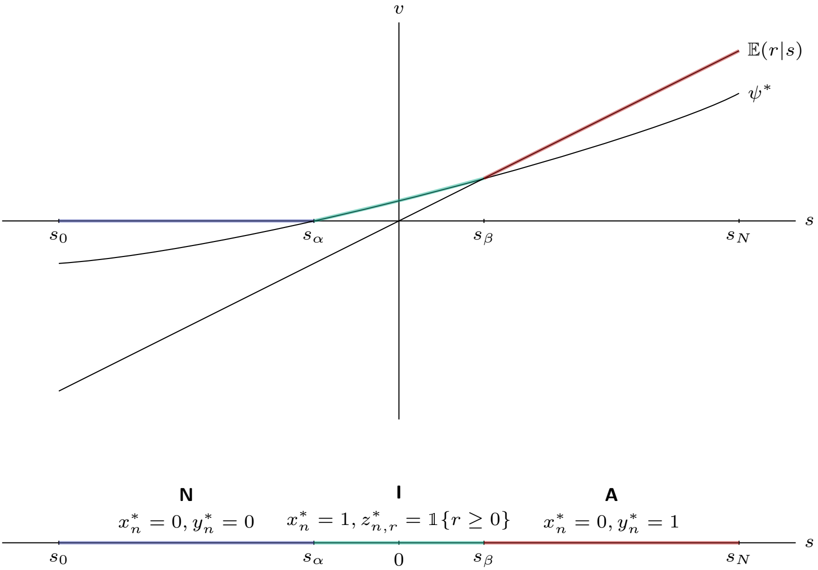

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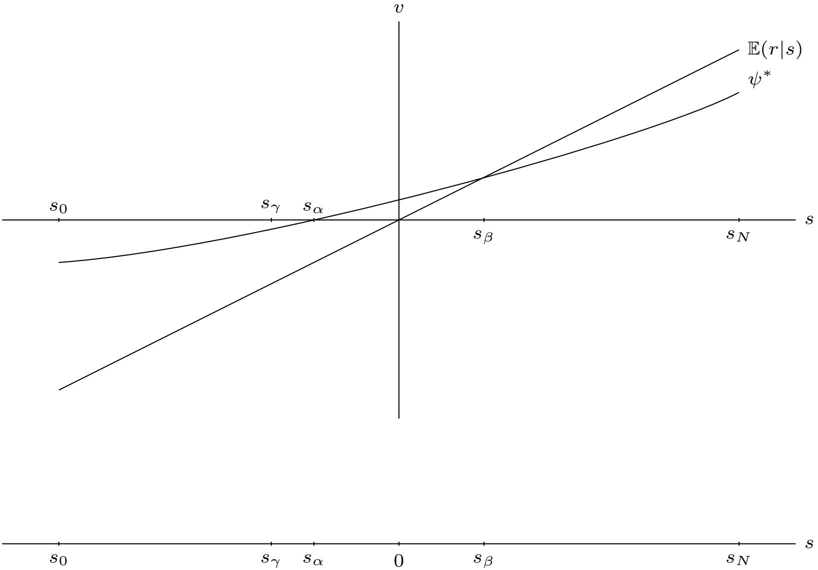

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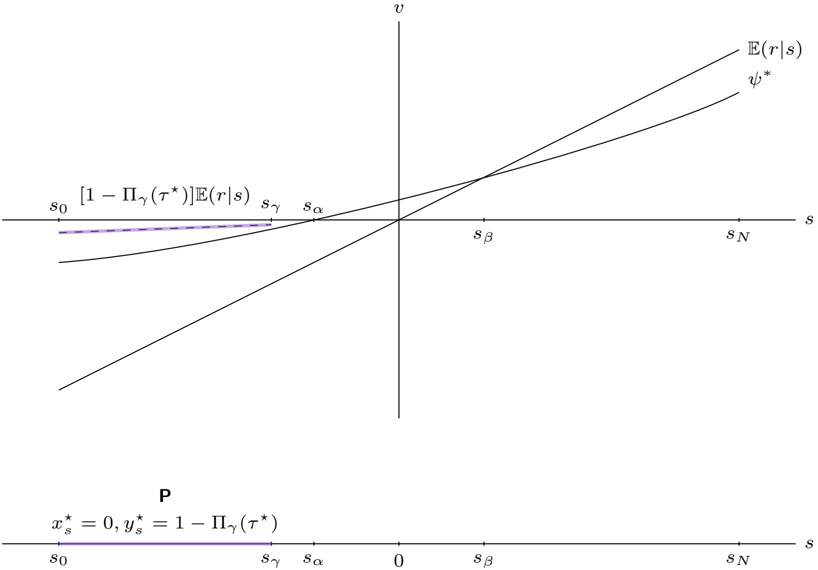

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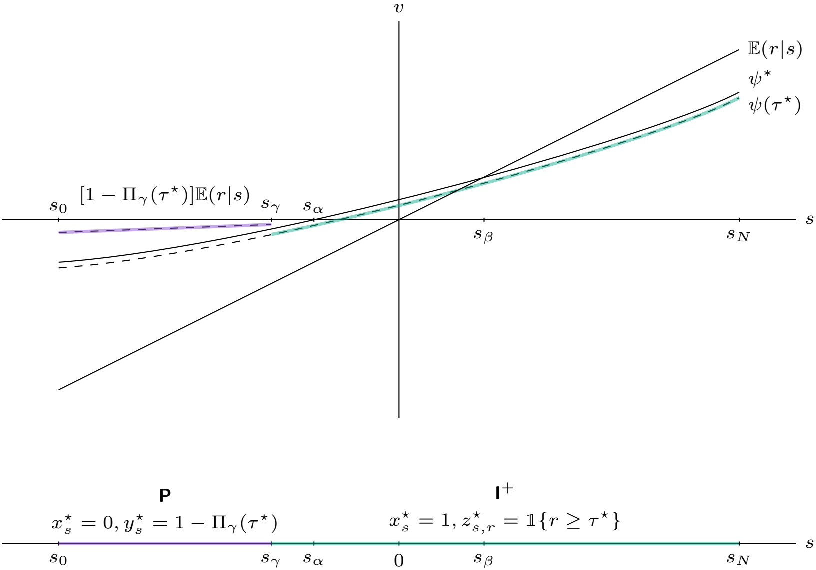

A solution recipe

Consider a **relaxation** of the principal's problem that only requires the upward local IC constraints to be satisfied. That is:

$$IC_{n,n+1} : (1-x_n)y_n + x_n\mathbb{E}(z_{n,r}|n) \geq (1-x_{n+1})y_{n+1} + x_{n+1}\mathbb{E}(z_{n+1,r}|n) \quad \forall\, n < N$$

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$\Rightarrow$ Optimal post-inspection thresholds are constant: $\tau_n = \tau\ \forall n$.

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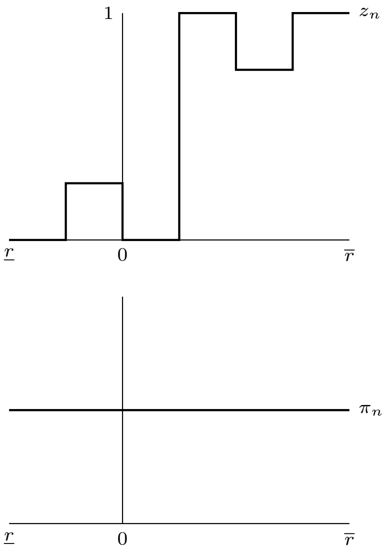
Idea: For each n find the τ_n such that:

$$\int z_{n,r}\pi_{n,r}\, dr = \int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r}\, dr$$

This transformation will always improve the objective, maintain the expected payoff for n, and weakly reduce the expected deviation payoff for $n-1$.

The transformation



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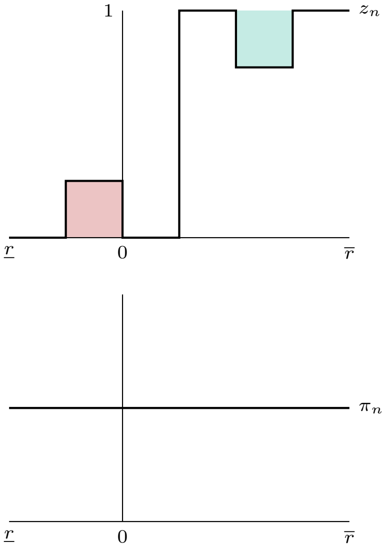

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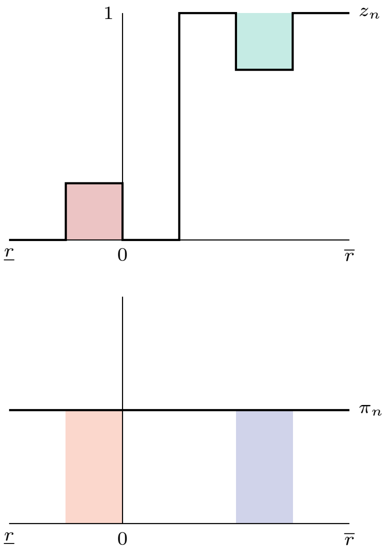
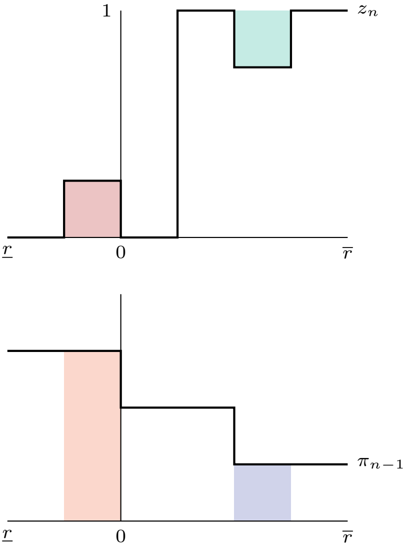
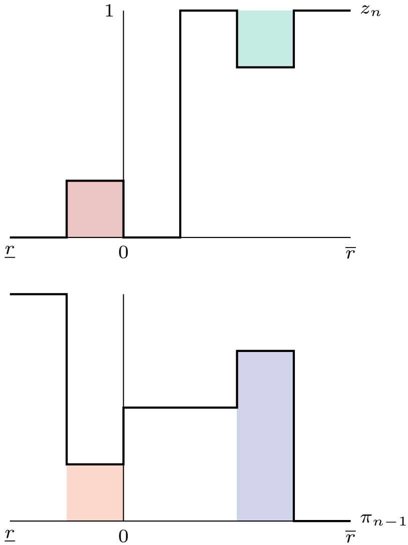


MLRP



FOSD



Threshold tests

Post-inspection allocations are then determined by a simple threshold test.

For the agent:

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For convenience, let's denote:

$$\overline{\Pi}_n(\tau) \coloneqq 1 - \Pi_n(\tau)$$

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Claim 2: Each upward local incentive compatibility constraint binds. That is, for each $n < N$:

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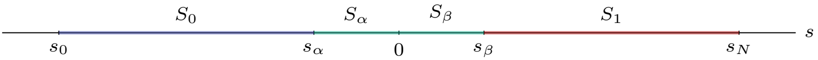
Note, that if $\tau_n = 0$ for each n, this corresponds with our first best policy:



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&= \cdots \\
&= (1 - x_N)y_N \qquad + \sum_{m=n}^{N-1}\left[x_{m+1}\overline{\Pi}_m(\tau_{m+1}) - x_m\overline{\Pi}_m(\tau_m)\right]
\end{aligned}$$

A linear objective

Our value function becomes:

$$
\begin{aligned}
v = & \quad (1 - x_N) y_N \mathbb{E}(r) \\
& + \quad x_N [\overline{\Pi}_{N-1}(\tau_N) \mathbb{E}(r|s \leq s_{N-1}) P_{N-1} + \psi_N(\tau_N) p_N] \\
& + \sum_{n=1}^{N-1} \quad x_n [\overline{\Pi}_{n-1}(\tau_n) \mathbb{E}(r|s \leq s_{n-1}) P_{n-1} - \overline{\Pi}_n(\tau_n) \mathbb{E}(r|s \leq s_n) P_n + \psi_n(\tau_n) p_n] \\
& + \quad x_0 [-\overline{\Pi}_0(\tau_0) \mathbb{E}(r|s_0) p_0 + \psi_0(\tau_0) p_0]
\end{aligned}
$$

This is a linear function in x_n. Similar to the proof of claim 2, we can then use variation arguments to prove that, $x_n \in \{0, 1\}$.

For example, our constraint directly implies that for any consecutive signals such that $x_n = x_{n+1} = 1$, then $\tau_n = \tau_{n+1}$, as $\overline{\Pi}_n(\tau_{n+1}) = \overline{\Pi}_n(\tau_n)$.

Optimal separating policy

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- the value of those signals above γ, that we inspect with threshold τ.

$$\max_{\gamma,\tau} \; Pr(r > \tau | s_\gamma)\mathbb{E}(r | s \leq s_\gamma) \cdot Pr(s \leq s_\gamma) + v(\mathbf{I}(\tau) | s > s_\gamma) \cdot Pr(s > s_\gamma)$$

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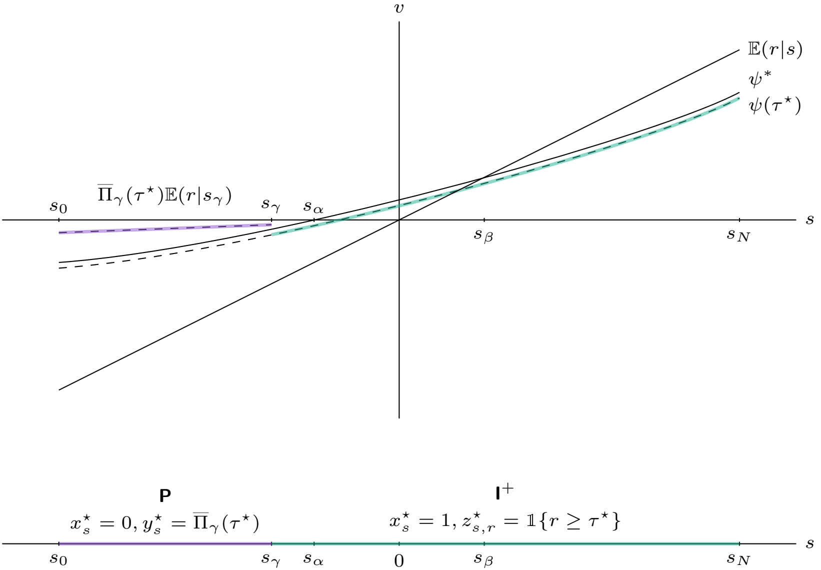
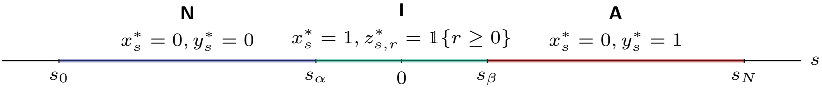
This satisfies the **global** incentive compatibility constraints for all γ and τ, and thus must be a solution to the original problem.

Second best solution

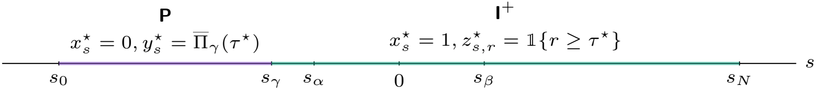


Losses

Public information benchmark:



Optimal (separating) mechanism:



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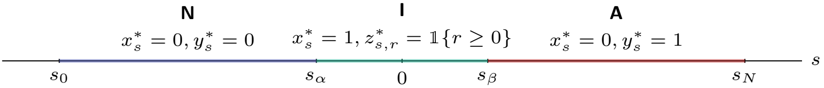

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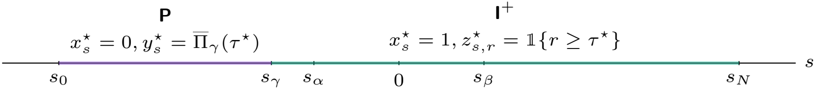

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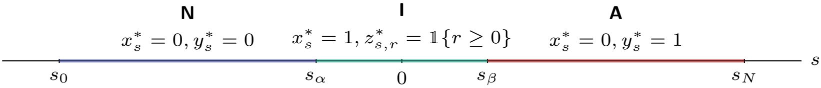

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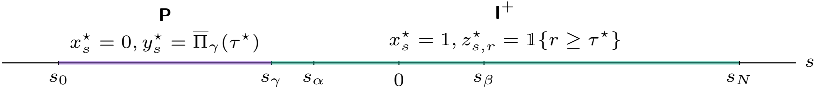

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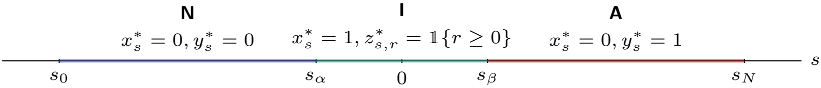

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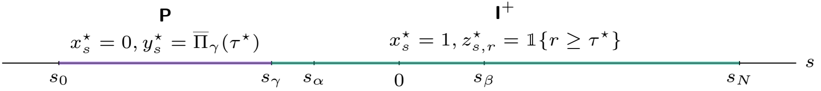

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3. under-allocation post-inspection: for $s \in [s_\gamma, s_N]$ and $r \in [0, \tau^\star]$, $z_{s,r}^\star = 0$.

Noisy inspection

Optimal inspection balances *discovery* and *verification*.

When agents have **noisy private information**, the principal:

- over-inspects high and low types,
- under-allocates to agents who are inspected, and
- over-allocates to agents who are not inspected.

Weakening commitment magnifies the losses from over-allocating to agents who aren't inspected.

For separating to be optimal, signals need to be sufficiently accurate, costs sufficiently small and information sufficiently valuable.

Outstanding questions?

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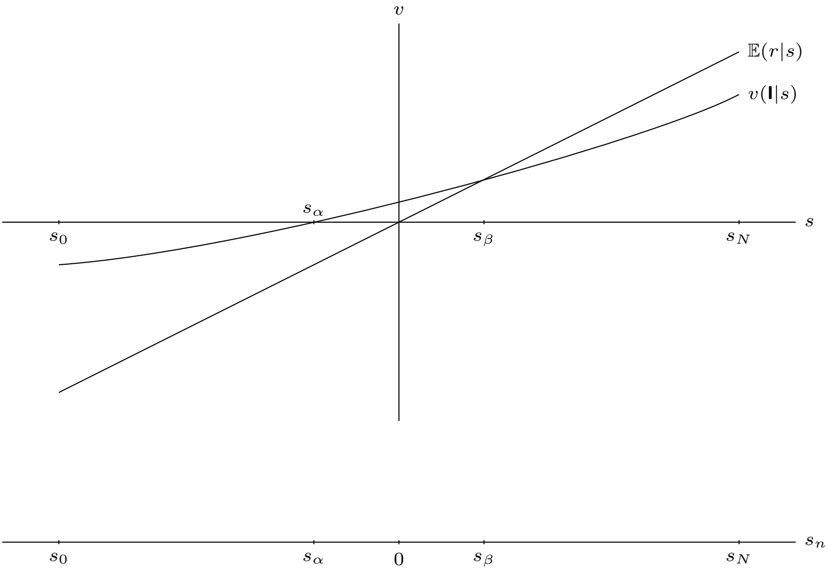
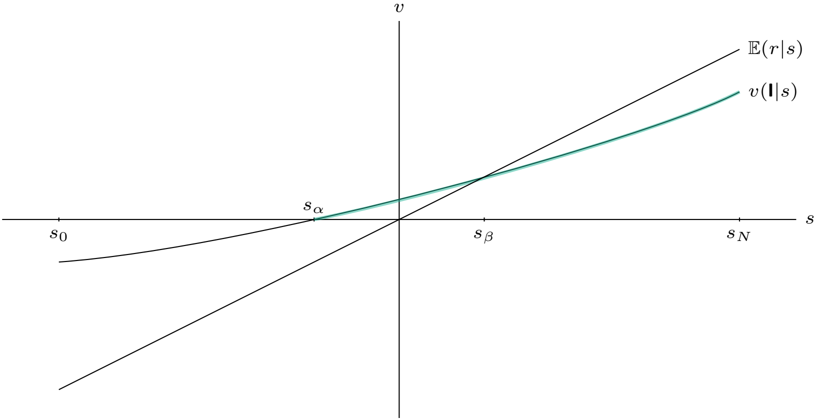
3. **no commitment**: the principal cannot commit to allocations or an inspection rule.

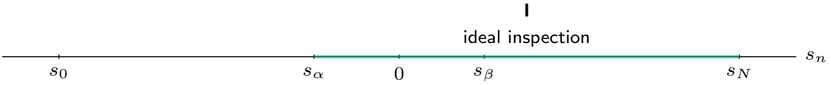
For **no commitment**, the principal can only choose between the pooling mechanisms and reports convey no information. We know what this looks like, so let's turn to the first two relaxations.

Pre-assessment commitment

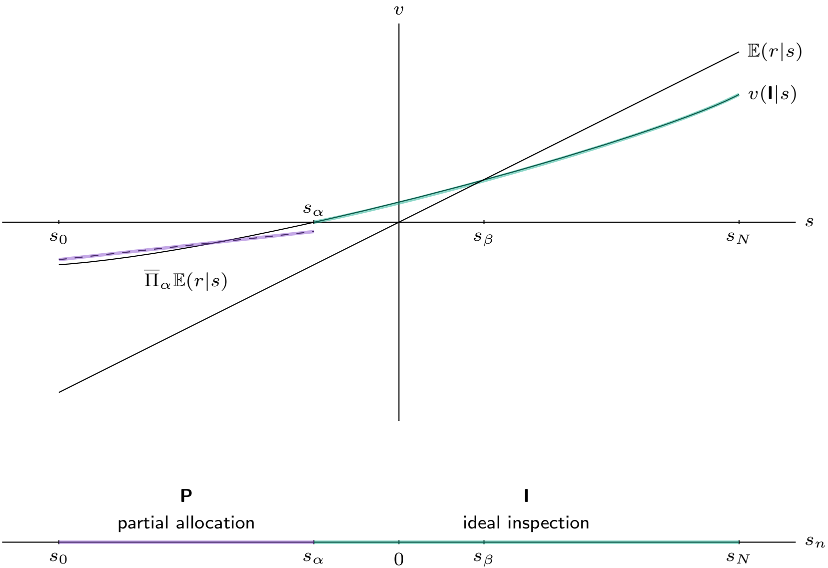


Pre-assessment commitment

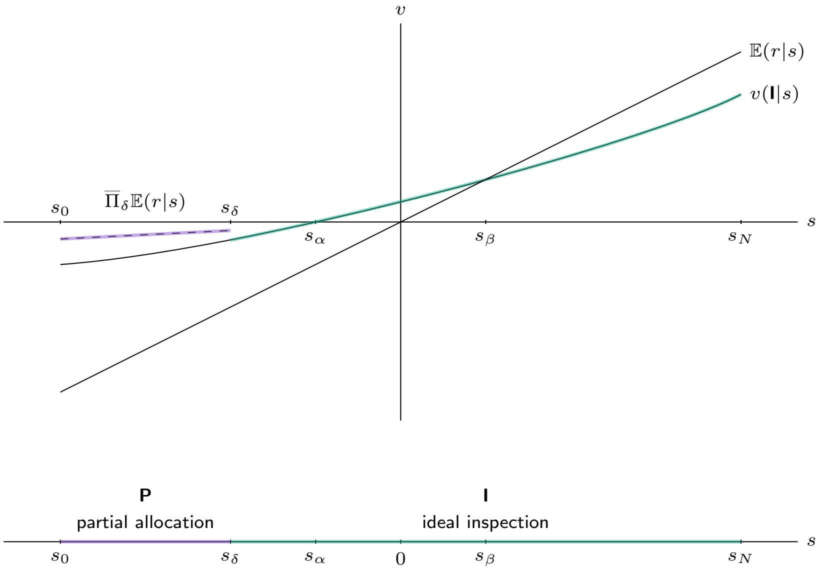




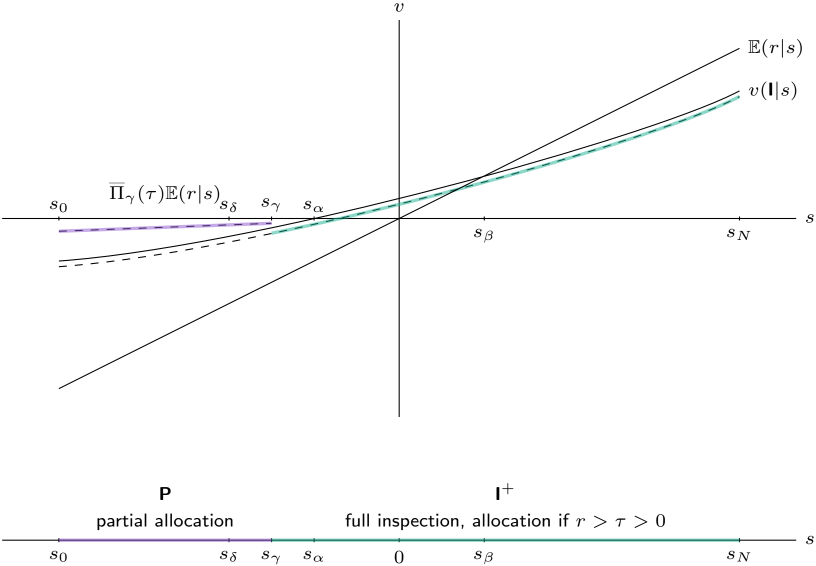
Pre-assessment commitment



Pre-inspection commitment



Full commitment



Gaussian environment

Suppose the prior over rewards is given by: $r \sim N(\mu, 1)$, and the agent receives a signal of this reward, $\hat{s} = r + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$.

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$$s = \frac{\sigma^2}{\sigma^2 + 1}\left[\mu + \frac{\hat{s}}{\sigma^2}\right] \quad \text{and} \quad \hat{\sigma}^2 = \frac{\sigma^2}{\sigma^2 + 1}$$

The induced distribution of signals, P, is then given by: $s \sim N(\mu, \frac{1}{\sigma^2+1})$.

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The induced distribution of signals, P, is then given by: $s \sim N(\mu, \frac{1}{\sigma^2+1})$.

The environment is by a triple:

- μ, the ex-ante expected reward of allocating to an agent,
- $\alpha := 1/\sigma^2$, the precision of the agent's signal of the reward, and
- c, the inspection cost to the principal.

Pooling equilibria

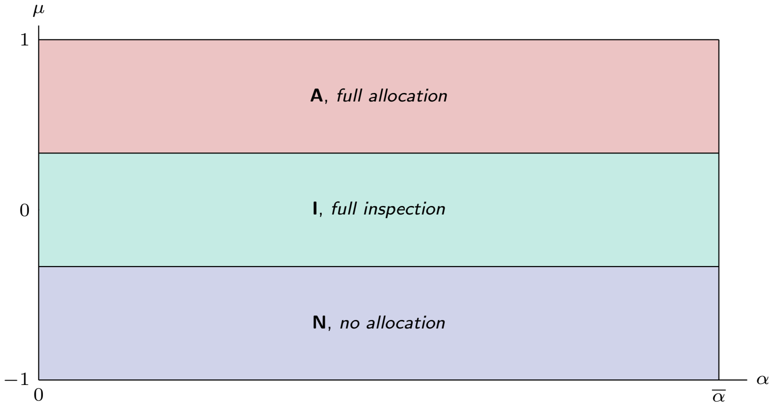

Figure: third-best policy as a function of precision, α, and prior mean, μ

Comparative statics

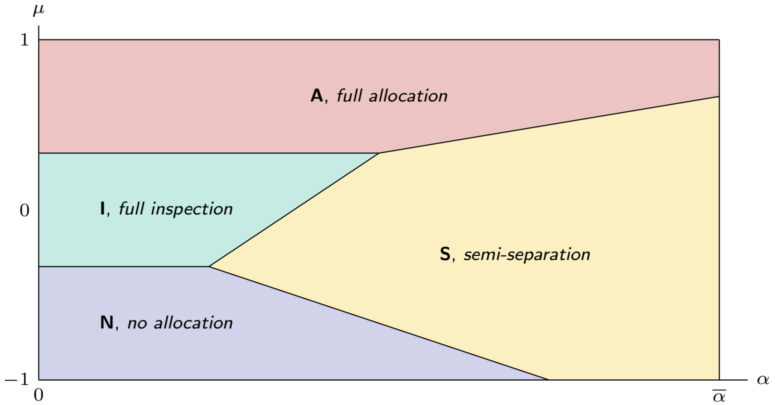

Figure: second-best policy as a function of precision, α, and prior mean, μ