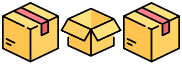
BDP 509: Applied Game Theory



Instructor: Nawaaz Khalfan

Summer Session II, 2022

Lecture Four: Mixed Strategies

July 18, 2022

Today's Tasks

1. Notifications
2. Review and utility over money
3. Mixed strategies
4. Matching pennies and mixed strategy Nash equilibria
5. Revisiting our canonical games
   5.1 Prisoners dilemma
   5.2 Stag hunt
   5.3 Final example

Review

A lottery is a consequence that results in a range of different outcomes with a known probability e.g. $\mathbf{L}$ could give a probability p of $\mathbf{X}$ occurring and a probability $(1-p)$ of $\mathbf{Y}$ occurring, and we can represent that by writing:

$$\mathbf{L} = p \circ \mathbf{X} + (1-p) \circ \mathbf{Y}$$

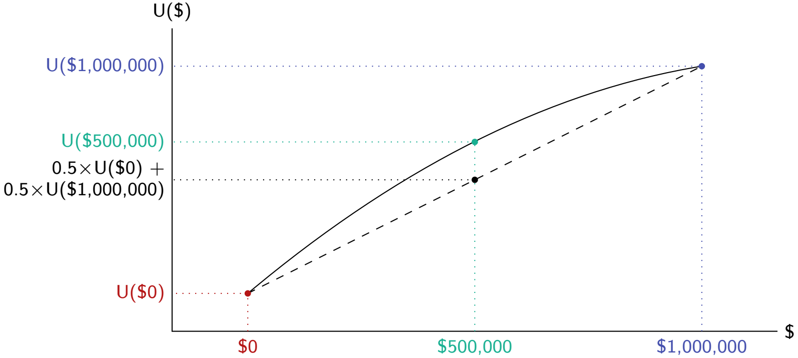
The expected utility hypothesis says that we can represent a players preference over lotteries with a utility function that allows us to compute payoffs as the expected value over the consequences it involves:

$$\mathrm{U}(\mathbf{L}) = \mathbb{E}\mathrm{U}(\mathbf{L}) \coloneqq p \times \mathrm{U}(\mathbf{X}) + (1-p) \times \mathrm{U}(\mathbf{Y})$$

Von Neumann and **Morgenstern**, 1947, proved that preferences this is without loss as long as the preferences satisfy four characteristics, namely: completeness, transitivity, continuity and independence.

Utility over money

We use utility functions over monetary payoffs that display diminishing marginal utility to represent players with risk aversion:



As such, risk aversion doesn't pose a problem the expected utility hypothesis but the Allais paradox does!

Mixed strategy

Suppose a player has the choice between playing action **A** and action **B**.

A pure strategy is choosing one action to play.

A mixed strategy is choosing a probability distribution over the players available actions.

For example, our player could choose to play **A** with 90% probability and **B** with 10% probability. We can represent this strategy with the following notation:

$$0.9 \circ \mathbf{A} + 0.1 \circ \mathbf{B}$$

More generally, any mixed strategy for our player can be thought of as choosing a single probability p for action **A**, and playing the other action with the residual probability:

$$p \circ \mathbf{A} + (1 - p) \circ \mathbf{B}$$

How many variables would you need to describe a mixed strategies if the player had 3 actions to choose from? What about N actions?

Matching pennies

Let's revisit the matching pennies game:

		US Goalie	
		left	*right*
Aus. Striker	*left*	$-1, 1$	$1, -1$
	right	$1, -1$	$-1, 1$

We now want to let the players play mixed strategies and see if we can find the Nash equilibrium.

Recall that a Nash equilibrium is a strategy profile where all players are playing a best response to each-others strategies. So if a NE involves a player choosing a mixed strategy, it must be a best response for this player!

Best response to *p*

		US Goalie	
		left	*right*
Aus. Striker	*left*	$-1, 1$	$1, -1$
	right	$1, -1$	$-1, 1$

Suppose the US Goalie plays left with probability p and right with probability $(1 - p)$, then what is the best response for the Australian Striker?

If the Aus. striker plays left, then their expected payoff is:

$$p \times (-1) + (1 - p) \times 1 = 1 - 2p$$

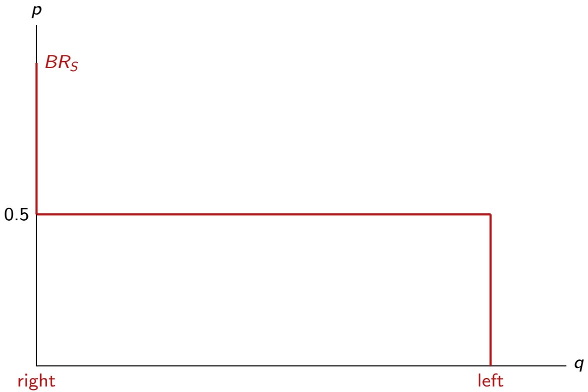
If the Aus. striker plays right, then their expected payoff is:

$$p \times 1 + (1 - p) \times (-1) = 2p - 1$$

As such, left is a best response to p if:

$$1 - 2p \geq 2p - 1 \Leftrightarrow p \leq 0.5$$

Graphical representation



When to mix?

Note that it was only a best response to mix if the Aus. striker was indifferent between left and right.

Theorem: any player who plays a mixed strategy in a Nash equilibrium must be indifferent between the pure strategies they mix between.

Best response to q

			US Goalie	
			left	*right*
Aus. Striker	*left*		$-1, 1$	$1, -1$
	right		$1, -1$	$-1, 1$

Suppose the Aus. Striker plays left with probability q and right with probability $(1 - q)$, then what is the best response for the US Goalie?

If the US goalie plays left, then their expected payoff is:

$$q \times 1 + (1 - q) \times (-1) = 2q - 1$$

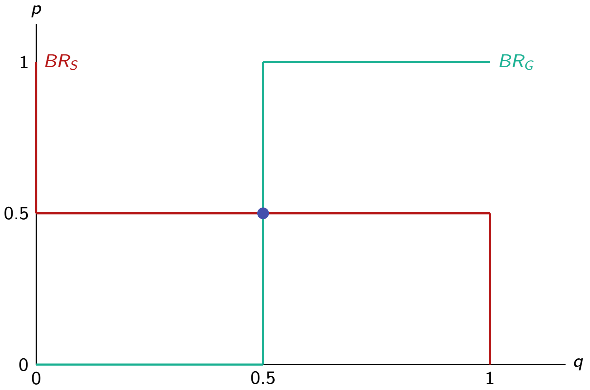
If the US goalie plays right, then their expected payoff is:

$$q \times (-1) + (1 - p) \times 1 = 1 - 2q$$

As such, left is a best response to q if:

$$2q - 1 \geq 1 - 2q \Leftrightarrow q \geq 0.5$$

Mixed strategy NE



Thus, our mixed strategy Nash equilibrium is when $p = 0.5$ and $q = 0.5$, or more exactly:

$$NE = (0.5 \circ \text{left} + 0.5 \circ \text{right}, 0.5 \circ \text{left} + 0.5 \circ \text{right})$$

Expected payoff?

What is the expected payoff under this MSNE for each of our players?

For an arbitrary mixed outcome defined by p and q, the expected payoff for either of our players is:

$$\begin{aligned}\mathbb{E}U(p,q) = {} & p.q.U(\mathit{left},\mathit{left}) + p.(1-q).U(\mathit{left},\mathit{right}) \\ & + (1-p).q.U(\mathit{right},\mathit{left}) + (1-p).(1-q).U(\mathit{right},\mathit{right})\end{aligned}$$

However, remember our theorem that says any player who mixed must be indifferent between their pure strategies! So we can make this calculation simpler by just computing:

$$\mathbb{E}U(\mathit{left},q) = q.U(\mathit{left},\mathit{left}) + (1-q).U(\mathit{left},\mathit{right})$$

For our MSNE, that means the US goalie's expected payoff is:

$$\mathbb{E}U(\mathit{left},q) = 0.5 \times 1 + 0.5 \times (-1) = 0$$

Prisoners dilemma

		Bob	
		deny	*confess*
Alice	*deny*	$-1, -1$	$-3, 0$
	confess	$0, -3$	$-2, -2$

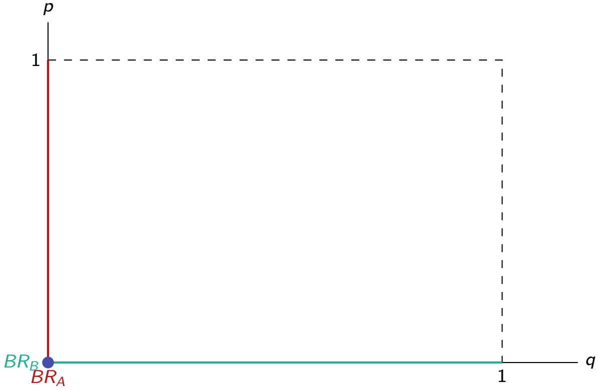
Suppose Bob plays *deny* with probability p, and Alice plays *deny* with probability q.

Then for Alice *deny* $\succ$ *confess* (i.e. set $q = 1$) if:

$$p \times (-1) + (1-p)(-3) > p \times 0 + (1-p) \times (-2)$$
$$0 > 1 \ \#$$

deny is never a best response so always set $q = 0$! (And the same for Bob, $p = 0$ is always a best response.)

Prisoners dilemma



Thus, the only Nash equilibrium is (*deny*, *deny*).

Stag hunt

		Hunter 2	
		stag	*hare*
Hunter 1	*stag*	$4,4$	$1,3$
	hare	$3,1$	$2,2$

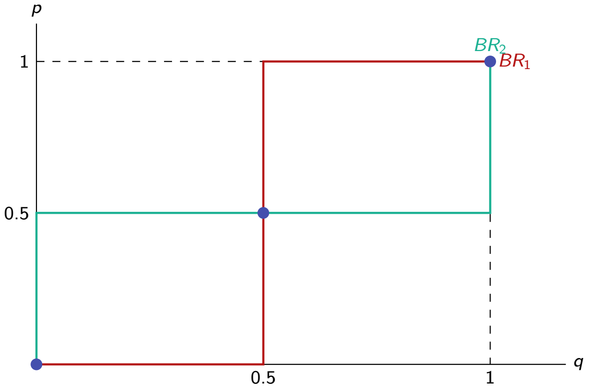
Suppose Hunter 2 plays *stag* with probability p, and Hunter 1 plays *stag* with probability q.

Then for Hunter 1, *stag* $\succ$ *hare* (i.e. set $q = 1$) if:

$$p \times 4 + (1 - p) > p \times 3 + (1 - p) \times 2$$
$$p > 0.5$$

That is, *stag* is a best response if the probability that the other hunter plays *stag* is high enough (and vise versa, and the equivalent for Hunter 2).

Stag hunt



Thus, there are three NE: (*stag*, *stag*), (*hare*, *hare*) and, $(0.5 \circ \mathit{stag} + 0.5 \circ \mathit{hare}, 0.5 \circ \mathit{stag} + 0.5 \circ \mathit{hare})$.

Final example

		B	
		left	*right*
A	*up*	$1, 5$	$2, 4$
	down	$1, 3$	$0, 5$

Suppose A plays *up* with probability q, and B plays *stag* with probability p.

Then for A, *up* $\succ$ *down* (i.e. set $q = 1$) if:

$$\begin{aligned} p + (1 - p) \times 2 &> p \\ 1 &> p \end{aligned}$$

That is, *up* is always a best response, and *down* is only a best response if $p = 1$.

Final example

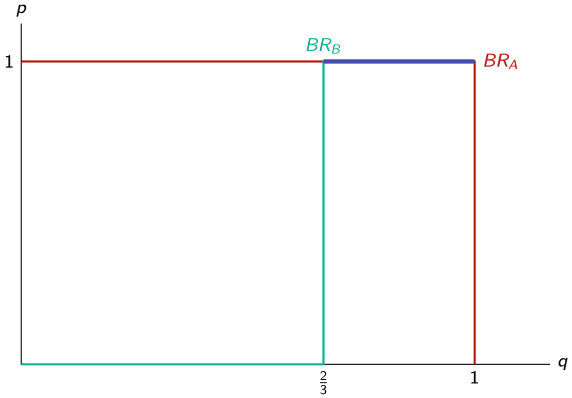
		B	
		left	*right*
A	*up*	$1, 5$	$2, 4$
	down	$1, 3$	$0, 5$

Then for B, *left* $\succ$ *right* (i.e. set $p = 1$) if:

$$q \times 5 + (1 - q) \times 3 > q \times 4 + (1 - q) \times 5$$
$$q > \frac{2}{3}$$

That is, *left* is a best response if $q > \frac{2}{3}$ and *right* is a best response if $q < \frac{2}{3}$.

Final example



Thus, there are an infinite number of NE: $(q \circ up + (1-q) \circ down, left)$ where $q \geq \frac{2}{3}$.