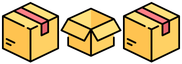
BDP 509: Applied Game Theory



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Summer Session II, 2022

Lecture Five: Sequential Games

July 20, 2022

Today's Tasks

1. Notifications
2. Review and practice question
3. Sequential games and extensive form representation
4. Nash equilibrium and strategic form representation
5. Subgame perfect equilibrium and backwards induction
6. Some canonical games:
   6.1 Entry game
   6.2 Cake sharing game
   6.3 Centipede game

Review

A mixed strategy is when our player chooses a probability distribution over their actions.

For example, our player mix by choosing a probability p for action **A**, and playing the other action with the residual probability:

$$p \circ \mathbf{A} + (1-p) \circ \mathbf{B}$$

To find the mixed strategy Nash equilibrium we find the intersection of the best response functions which tells us the best response for each mixture of their opponents.

Theorem: any player who plays a mixed strategy in a Nash equilibrium must be indifferent between the pure strategies they mix between.

Practice question

		B	
		left	*right*
A	*up*	1, 5	2, 4
	down	1, 3	0, 5

Suppose A plays *up* with probability q, and B plays *left* with probability p.

Then for A, $up \succ down$ (i.e. set $q = 1$) if:

$$\begin{aligned} p + (1-p) \times 2 &> p \\ 1 &> p \end{aligned}$$

That is, *up* is always a best response, and *down* is only a best response if $p = 1$.

Practice question

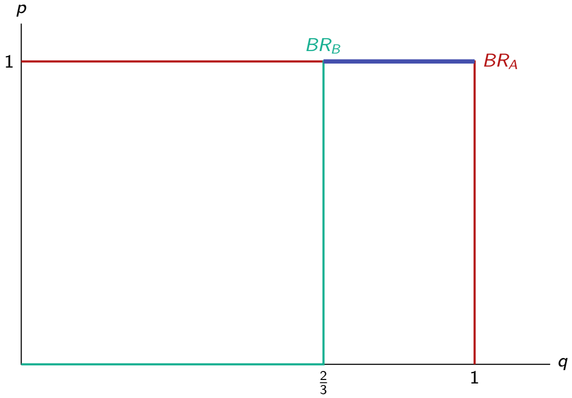
			B	
			left	*right*
A	*up*		1, 5	2, 4
	down		1, 3	0, 5

Then for B, *left* $\succ$ *right* (i.e. set $p = 1$) if:

$$q \times 5 + (1 - q) \times 3 > q \times 4 + (1 - q) \times 5$$
$$q > \frac{2}{3}$$

That is, *left* is a best response if $q \geq \frac{2}{3}$ and *right* is a best response if $q \leq \frac{2}{3}$.

Practice question



Thus, there are an infinite number of NE: $(q \circ \textit{up} + (1 - q) \circ \textit{down}$, $\textit{left})$ where $q \geq \frac{2}{3}$.

Simultaneous and sequential games

So far we have been working with simultaneous games, that is games where our players choose their actions at the same time. It wasn't assumed that these players literally choose actions at the same time, but was assumed they did so without knowledge of the other players actions e.g. in isolation.

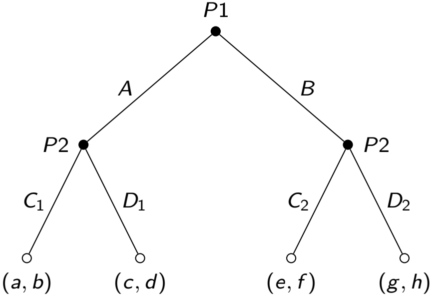
A sequential game is a game where there is some order of play and players observe what those who have went before them have played before they choose their own actions.

Examples of sequential games include:

1. Open auctions (e.g. English or Dutch)
2. Chess
3. Bargaining and veto games
4. Roll call voting

Extensive form representation

We can represent such games graphically with what is called the extensive form or a game tree:



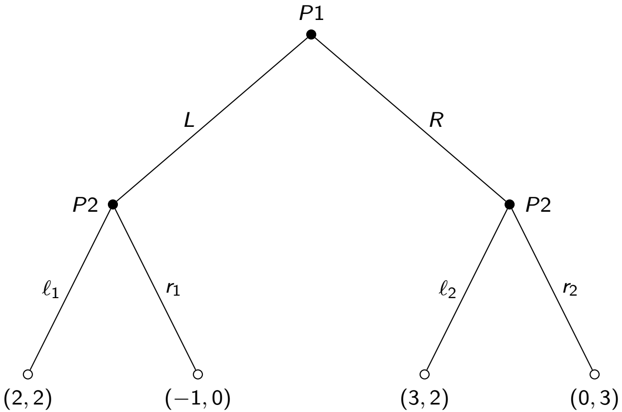
Terminology

Some terminology to be familiar with:

- The initial node is where the game begins, calls on the first player to act and happens after no actions have been played.
- A (generic) node is where a particular player is called to act and happens after a sequence of actions leading from the initial node. This is also called a history.
- Terminal nodes are the final nodes where no player is called to act and the payoff is specified and delivered.
- A path is a series of actions that lead from the initial node to a terminal node.

Recall that, as before, a strategy is a complete, contingent course of action. So in our new extensive games, a strategy for a player says what they'll do at every node they are called to act, whether or not this node is ultimately reached.

Example: threat game



Strategic form representation

We can use the tools we've already developed to study this game by translating this game into a strategic/normal form game:

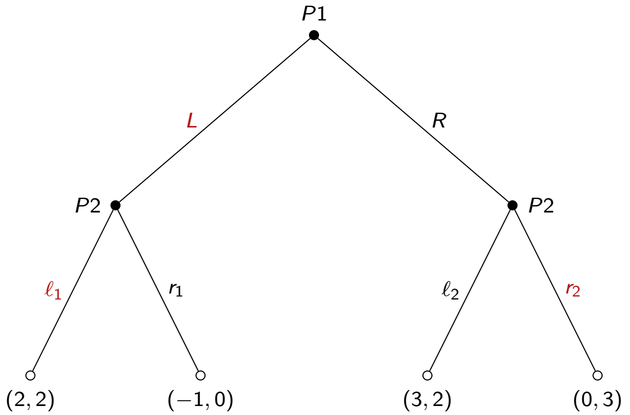
		$P1$: L	R
$P2$	ℓ_1, ℓ_2	$2, 2$	$2, 3$
	ℓ_1, r_2	$2, 2$	$3, 0$
	r_1, ℓ_2	$0, -1$	$2, 3$
	r_1, r_2	$0, -1$	$3, 0$

As such there are two (pure strategy) Nash equilibrium of this game: $(L, \ell_1 r_2)$ and $(R, r_1 r_2)$.

One of these equilibria looks suspicious ...

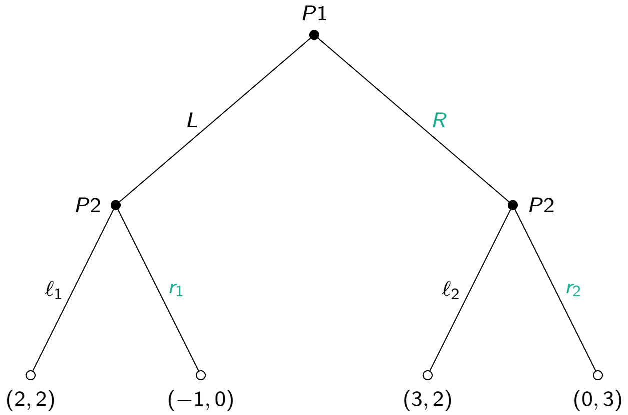
Nash equilibrium #1

Let's visualise $(L, \ell_1 r_2)$:



Nash equilibrium #2

Now consider $(R, r_1 r_2)$:



An issue . . .

In the second Nash equilibrium of the threat game, $P2$ would not have been playing a best response had the node following L been reached.This was a still a Nash equilibrium as (unilaterally) changing r_1 to ℓ_1 would not have changed $P2$'s payoff; the path would have remained at R, r_2 and $P2$ would have still received a payoff of 3.

Similarly $P1$ would not have been able to increase their payoff from switching to L. If they unilaterally did, their payoff would go from 0 to -1.

However, would $P1$ believe that r_1 would have indeed been played if they did switch to L? After all, $P2$ would be able to see that L was played and in that situation would have been best to play ℓ_1!

Nash equilibrium does not concern itself with these types of "double deviations". It only asks when all players are playing mutually best responses.

Subgame perfect Nash equilibrium

A subgame is a generic node who's terminal nodes are only reachable from this node. It can be thought of as a self contained game. In our perfect information world, all (non terminal) nodes are subgames.

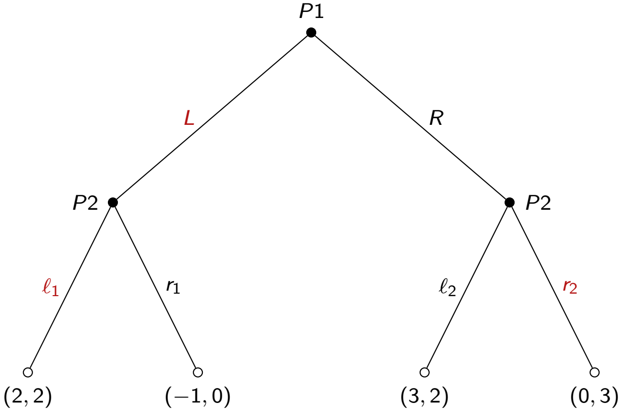
How many subgames are there in the threat game? Three: the node following L, the node following R and the initial node.

A subgame perfect Nash equilibrium is a strategy profile who's strategies are Nash equilibrium in every subgame.

Because the entire game is also a subgame, all SPNE are NE, but the converse need not be true!

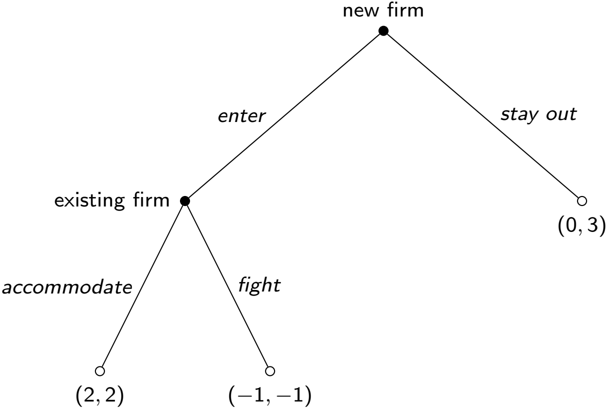
Backwards induction

We can find the subgame perfect Nash equilibrium by starting at the penultimate nodes, finding the best responses and working backwards. This process is called backwards induction:



Entry game

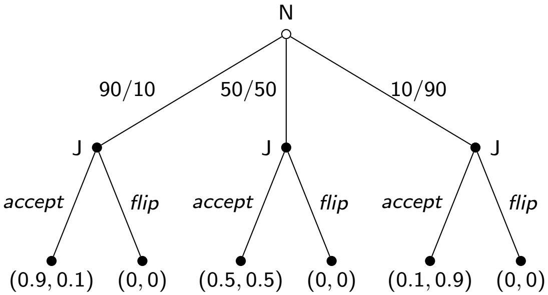
A new firm is deciding whether to move into an existing firms market:



There are two Nash equilibria: (*enter*, *accommodate*) and (*stay out*, *fight*), but only the first is a subgame perfect Nash equilibrium.

Cake sharing game

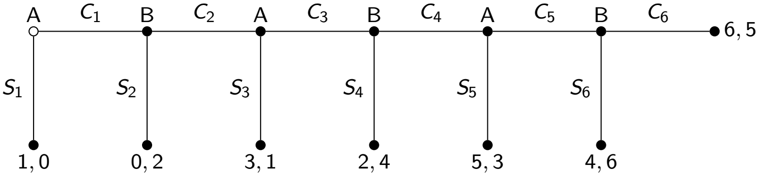
There is one large slice of cake left and two siblings, Nawaaz and Jamil, are eyeing it off. Nawaaz, the older sibling, is the only one allowed to wield the knife but Jamil has a trick up his sleeve ...



The subgame perfect Nash equilibrium is where Nawaaz splits the cake 90/10 and Jamil is forced to accept every split. But if Jamil can credibly threaten to flip the cake when he doesn't agree with a split, he can essentially choose whatever split he likes!

Centipede game

Consider the following sequential game between player A and player B:



The subgame perfect Nash equilibrium is: $(S_1 S_3 S_5, S_2 S_4 S_6)$ and the payoff is $(1, 0)$ which is Pareto dominated by 5 of the 6 other outcomes!