

# Optimal Allocation with Noisy Inspection<sup>\*</sup>

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## Abstract

A principal receives an unknown reward from allocating to an agent who has private information about the reward. Prior to allocating, the principal may elicit a report from the agent and inspect them at a cost, but must do so without transfers. When the private information is noisy, the mechanism that maximizes the principal's expected return segments signals into two groups, inspects high types, allocating to them only if the inspected return is sufficiently positive, and doesn't inspect low types, compensating them with a small probability of allocation. This relates to a number of applied settings such as employer hiring strategies, public grant mechanisms, and portfolio investment rules.

## 1 Overview

Appraising the value of an asset is an essential precursor to its exchange. Employers interview potential employees, public funds assess grant applications, venture capitalists evaluate investment opportunities. This process is often costly, and information that could be used to lower, and even circumvent, these costs is often privately held.

This paper considers a principal whose return from allocating to (*employing, selecting, supporting*) an agent is uncertain and inherent to the agent they allocate to. The principal has the ability to inspect (*interview, assess, evaluate*) the agent at a cost and learn about the true return, as well as the opportunity to receive a report from the agent of the private information they have about it. The agent, independent of their private information, strictly prefers to be allocated to. This paper derives the optimal use of inspection to gather information and govern allocation when the information is imperfect, inspection is costly and transfers are restricted.

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The environment presented here encompasses many important settings. Consider the following three examples.

1. **Hiring:** a firm, the principal, seeks to fill an open position in their operation with a potential employee, the agent. The agent would like to be hired and aware of their own characteristics, has an estimate of their future productivity in this position. The principal can ask for this estimate, and also interview the agent themselves, discovering a better forecast of their productivity. The interview, however, is costly for the principal. What interview and hiring protocol should the firm enact?
2. **Funding:** a governing board, the principal, sets the rules by which it allocates a scarce, publicly owned resource, such as funding for an applicant's project, the agent. The agent is primarily interested in being approved, valuing their own use above rival uses, and knows the most about the project's characteristics and likelihood of succeeding. The principal wants to fund positive net value projects, perhaps weighing a combination of private and public preferences for the funding's use. How should a governing board design testing and approval rules to ultimately support socially valuable projects?
3. **Investing:** an investor, the principal, determines the way it evaluates and finances an early investment opportunity, the agent. The investor may be governed by the motivation to strengthen an existing portfolio or even personal philanthropic concerns, but is restricted in outlining these preferences publicly. The agent wishes to be financed and to continue their startup, and has the most information about the startup. This information doesn't fully determine the investor's value for the opportunity without an appraisal. How much information can the investor elicit through their evaluation and funding decisions?

These examples share three important features. Agents have noisy private information about the principal's allocation reward, it is costly for the principal to discover the reward themselves, and the use of transfers to align incentives are limited. It is in this environment that inspection has the potential to serve two different roles. Directly, it informs the principal of the reward they will receive if they carry through with allocating to the agent, and indirectly, it disciplines the agent's disclosure of information by conditioning allocation on the outcome of an inspection. This paper addresses the tradeoff between these two sources of information: direct acquisition through *discovery*, and indirect acquisition through *screening*.

Modelling the agent's private information as a signal of *favourableness*, the mechanism that maximizes the expected return for the principal has a simple structure. To elicit truthful reports, the principal segments the signals into two groups - *high* and *low*. Agents with high signals are always inspected and only allocated post-inspection if the discovered reward is sufficiently valuable. Agents with low signals are never inspected, but are compensated for their report with a small

probability of unconditional allocation.

This segmentation is the only optimal mechanism that does not entirely *pool* the signals. That is, inspection and allocation at most partially-separates signals by screening high from low, and beyond this, treats signals in these groups equally. Inspection in this mechanism is deterministic - high signals are inspected, low signals are not - and the conditional allocation is inefficient - some agents who are revealed as giving the principal a positive return are rejected. In addition to under-allocating post-inspection, the principal over-allocates to agents with low types, and over-inspects both some high and low types.

Alternatively, the principal may find it optimal to pool all signals, treating the agents non-preferentially. They do this by either never inspecting, and allocating to or rejecting all agents, or always inspecting, and allocating only when the realized reward is positive. The choice between one of these pooling mechanisms and the partially-separating mechanism depends on the prior over the rewards, the signal accuracy of the agents, and the cost of inspection.

In practice, the over-inspecting of low types and the under-allocating post-inspection may require strong commitment from the principal. This paper also derives the optimal mechanism when commitment is relaxed and demonstrates that, to the net detriment of the principal’s objective, the losses from over-allocating to low types will be magnified and the losses from over-inspecting and under-allocating post-inspection will be suppressed.

This paper is related to a branch of the mechanism design literature devoted to costly inspection with no transfers. In a seminal paper, [Ben-Porath, Dekel and Lipman \(2014\)](#) analyse a model of many, perfectly informed agents and show that inspection is used to check the highest reported type above a threshold, providing an adequate incentive for low types not to masquerade. Inspection is never used to inform the principal of additional information, a feature that is present in each of our examples. This paper shows that imperfect information recovers this quality, and demonstrates how to model and analyse this feature.

The next section discusses where this paper fits within the broader literature on verification in mechanism design. Subsequently, the environment and a benchmark for comparison is detailed (section 3), followed by a proof of the main results on how inspection is used to acquire information (section 4). Finally, the main analysis is concluded with how the results extend to environments where commitment is relaxed (section 5) and a demonstration of when this mechanism prevails over non-preferential, or *pooling*, mechanisms (section 6).

## 2 Related literature

Verification in mechanism design has received widespread interest, notably studied in [Green and Laffont \(1986\)](#). As mentioned, [Ben-Porath, Dekel and Lipman \(2014\)](#) consider perfectly informed agents and overcome the difficulty of studying a general mechanism design problem without transfers to characterize an optimal policy. The concern about applying this result, however, is that inspection is only used for verification, and this paper broadens the scope of inspection by modelling imperfectly informed agents. The extensions of [Mylovanov and Zapechelnnyuk \(2017\)](#), [Epitropou and Vohra \(2019\)](#) and [Erlanson and Kleiner \(2020\)](#) explore alternative timing and actions, though in the single agent environment presented here, they are primarily the same.

There is a growing literature concerning mechanisms with imperfect verification. [Pereyra and Silva \(2021\)](#) is the closest in their treatment, but model scarce allocation, costless and imperfect inspection technologies, and primarily focus on efficient mechanisms. [Ball and Kattwinkel \(2019\)](#) explore private information and verification, but consider effort choice and transfers. An extension of the results provided in this paper can be applied to noisy verification, but here the focus is on imperfectly informed agents.

There is a conceptual difference between *verifying* an agent’s report and *inspecting* their inherent qualities, and this is distinguished here. The literature on scoring rules considers this distinction and, as in the seminal work of [McCarthy \(1956\)](#) and [Savage \(1971\)](#), outlines mechanisms that elicit truthful reports of noisy signals.<sup>1</sup> This does not account for the incentives to collect this information, however, nor the cost in implementing the scoring rule. The work done here can then be seen as an extension of scoring rules to allocation environments without transfers.

An application of this environment concerns judicial mechanism design. [Silva \(2019a\)](#), and [Siegel and Strulovici \(2021\)](#) explore these topics with similar ideas of noisy inspection, but their models are less general, tailored to the application and vary along the dimensions of costs, transfers, and incentives.

Settings with evidence, disclosure, and audits have a long tradition in mechanism design, from [Townsend \(1979\)](#), to [Border and Sobel \(1987\)](#) and [Mookherjee and Png \(1989\)](#). These primarily study optimal mechanisms with transfers. The contribution of [Alaei et al. \(2020\)](#) unites many of their features, with deferred inspection and payments. Recent work has also looked at environments with limited transfers such as [Mylovanov and Zapechelnnyuk \(2017\)](#), [Silva \(2019b\)](#), [Li \(2021\)](#), and [Patel and Urgan \(2022\)](#). These papers, and their solutions, can then be directly compared to the optimal solution presented here to understand what qualitatively changes when transfers are prohibited.

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<sup>1</sup>For an extensive treatment, see [Gneiting and Raftery \(2007\)](#).

### 3 Environment

A *principal* receives an unknown, real return,  $R$ , from allocating to an *agent*. If the agent is allocated to, they receive a payoff of 1, and 0 otherwise. Prior to allocation, the principal may *inspect* the agent at a cost to their final payoff,  $c > 0$ , and in return learn the true value of the reward,  $r$ . The agent has their own private information about the prize, a signal  $s$ , which defines their *type*.

Here, we are considering one principal and one agent. An equivalent setup is a single principal with  $k$  objects to allocate, among  $\ell \leq k$  agents, each of whom has unit demand, the same preferences for each object, and an independent signal of the principal's reward. The problem where  $\ell > k$  is left for future research.

The agent has a strict preference to be allocated to, and their payoffs are normalized around this. Changing the intensity of this preference, and even making this intensity type-dependent, makes no difference to the analysis so long as we maintain the strict preference for allocation. As such, this normalization is without loss of generality. It will of course matter when interpreting particular applications and extensions, but this is left to the responsibility of the reader.

The principal has the ability to commit to a rule that determines what they do after any report and any subsequent realization of the reward. Following the main analysis, we will explore how the environment changes when we relax this ability, as outlined in section 5. One can then think of the full commitment setting as the most informative case study in understanding this environment.

Direct transfers of value between the principal and agent are *not* permitted. This reflects the observation that transfers are seldom used for direct disclosure of information in practice. A place where we may see transfers occurring is bargaining between the principal and agent once the principal decides to allocate. We can then interpret this restriction as the principal not being able to commit to altering later stage bargaining outcomes prior to making their allocation decision. The model then claims that the principal's reward, and the agent's preference to be allocated to, reflects the expected outcome of this ensuing bargaining game.

An alternative interpretation is that this is an environment where bargaining does not occur, and there are many settings where this is true. In some, such as the assignment of public housing, the use of money is seen as counterproductive. In others, such as the appointment to public office, there are commitments from one or both of the parties not to use or accept transfers. And finally, the entire bargaining power over the value that can be shared may belong to just one of the parties, and so bargaining does not occur in any real sense.

Even if the restriction isn't true of an application, it is still important to understand what in-

spection offers the principal in isolation of transfers. This may be for predictive and prescriptive analysis, or building an understanding of the ways in which we can gather information in different settings.

The timing and structure of the game is fixed and common knowledge.

1. The principal commits to an inspection and allocation rule; nature assigns signals according to a commonly known generating process.
2. The agent observes their signal,  $s$ , and submits a report to the principal.
3. The principal implements their rule conditional on the report and any subsequent inspection realizations that are generated by the rule.
4. All remaining uncertainty is resolved, and net payoffs are awarded.

The primary question to address is which inspection and allocation rule the principal should select in order to maximize their expected return, subject to the agent's incentives to report.

This report in practice could be a lengthy and complicated message. Given our objective here is to study the resulting outcomes and, at most, the total information exchanged by this message, we will instead work with the *direct mechanism* by appealing to the revelation principle (Myerson, 1981). That is, we restrict attention to the message game where the agent directly reports their type, and require that the agent's expected return from reporting truthfully is weakly greater than that of reporting any other type.

Listing the principal's available actions conditional on report  $s$ , let:

- $x(s)$  be the inspection probability,
- $y(s)$  be the allocation probability without inspection, and
- $z(s, r)$  be the allocation probability after inspection and realizing reward  $r$ .

To fix language, let's refer to  $x$  as the *inspection rule*,  $y$  as the *pre-inspection allocation*, and  $z$  as the *post-inspection allocation*. Together,  $(x, y, z)$  constitute the principal's *mechanism*, and this mechanism is *feasible* if:

$$x(s) \in [0, 1], y(s) \in [0, 1], z(s, r) \in [0, 1] \quad \forall s, r$$

Further, let a *policy* be a particular mechanism that is proposed to maximize some program.

Here we are endowing the principal with the ability to ration the object they're allocating. In the hiring example, this can be thought of as limiting the hours the employee works, and in the

funding examples, partially funding the applicant's project. Partial allocation can also be thought of as the outcome of a lottery over the object. The decision to inspect may also be partial, in which case the lottery interpretation is natural.<sup>2</sup> It is the lottery interpretation for both choices that we'll refer to throughout the paper. What we are restricting here is that allocation can be at most one, representing the capacity constraint on the principal's allocation.

The principal chooses this mechanism to maximize their ex ante expected return. At the interim stage, after learning  $s$ , their payoff is determined by two events. They may allocate without inspecting, receiving the expected return given the signal, and this occurs with the probability that they don't inspect,  $(1 - x(s))$ , and that they do allocate,  $y(s)$ . Alternatively, they may allocate after inspecting and learning the return,  $r$ , receiving this net of the inspection cost,  $c$ , which, conditional on  $r$  being the true reward, occurs with the probability they do inspect,  $x(s)$ , and they do allocate,  $z(s, r)$ . Let  $v_s$  refer to the principal's interim payoff given  $s$ .

$$v_s(x, y, z) := (1 - x(s))y(s)\mathbb{E}(r \mid s) + x(s)(\mathbb{E}(z(s, r) \cdot r \mid s) - c)$$

Their ex ante expected return is their expected interim payoff, which we can refer to as the *objective*,  $v$ .

$$v(x, y, z) := \mathbb{E}_s [(1 - x(s))y(s)\mathbb{E}(r \mid s) + x(s)(\mathbb{E}(z(s, r) \cdot r \mid s) - c)]$$

Optimizing involves maximizing the objective subject to the agent's incentives to report their private information truthfully. That is, an agent of type  $s$  must receive as high a payoff from reporting  $s$  than any other type,  $\hat{s}$ . Given the normalization, the agent's payoff is the likelihood of being allocated to by the mechanism, and occurs with the net probability of the two events outlined above. Let  $u_s$  be the payoff for type  $s$  and  $u_{s, \hat{s}}$  be the payoff from  $s$  reporting  $\hat{s}$ , so that  $u_{s, s} = u_s$ .

$$u_s(x, y, z) := (1 - x(s))y(s) + x(s)\mathbb{E}(z(s, r) \mid s) \geq (1 - x(\hat{s}))y(\hat{s}) + x(\hat{s})\mathbb{E}(z(\hat{s}, r) \mid s) =: u_{s, \hat{s}}(x, y, z) \quad \forall \hat{s}$$

This *incentive compatibility* constraint for the pair  $(s, \hat{s})$  is labelled  $IC_{s, \hat{s}}$  for reference. Note that the agent's type only augments their return directly through determining the distribution of  $r$  and thus the likelihood of being allocated to conditional on inspection. Also given their return is the net probability of being allocated to, there is no need for an individual rationality constraint as all type's receive a weakly positive return.

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<sup>2</sup>This could also be a further rationing, however we would then have to claim that inspecting an agent for part of the object is only partially costly.

In total, the mathematical program that the principal solves is:

$$\begin{aligned}
& \max_{(x,y,z)} \quad \mathbb{E}_s [(1 - x(s))y(s)\mathbb{E}(r \mid s) + x(s)(\mathbb{E}(z(s, r) \cdot r \mid s) - c)] \\
& \text{s.t.} \quad IC_{s,\hat{s}} : (1 - x(s))y(s) + x(s)\mathbb{E}(z(s, r) \mid s) \geq (1 - x(\hat{s}))y(\hat{s}) + x(\hat{s})\mathbb{E}(z(\hat{s}, r) \mid s) \quad \forall \hat{s} \quad \forall s \\
& \quad F_{s,r} : x(s) \in [0, 1], \quad y(s) \in [0, 1], \quad z(s, r) \in [0, 1] \quad \forall r \quad \forall s
\end{aligned}$$

### 3.1 Signals

In practice, the agent's private information could be complicated and nuanced. For a job applicant, this may include educational performance, feedback from colleagues, career ambitions, and observations about the firm's hiring decisions. As we are primarily interested in the informational content of this information, let us collapse this into a single parameter and ask what characteristics we'd like it to have.

Let the agent's information be represented by a private signal,  $s \in \{s_0, s_1, \dots, s_N\}$  and suppose  $s = s_n$  with probability  $p_n \in (0, 1)$ , so that  $\sum_n p_n = 1$ . Denote  $P_n$  as the cumulative mass function, so that  $P_n = \sum_{m \leq n} p_m$ .

This paper adopts a discrete formulation of signals for exposition alone. The use of discrete signals allows us to think clearly about the different incentives that are important to the problem, and with proposition 1, the main result is extended to an appropriate limiting environment via Helly's selection theorem. One could instead conduct this exercise entirely with continuous types, and it would not provide any substantive additional insights.

Signals are informative of the principal's allocation reward. If  $s = s_n$ , let the reward that the principal receives from allocating to the agent is a random variable  $R \mid s_n \sim \Pi_n$  where  $\Pi_n$  is absolutely continuous and admits a density function  $\pi_n$ . Denote the unconditional distribution of the reward  $R$  by  $\Pi$  and assume that it is also absolutely continuous, admits a density function  $\pi$  and has support  $\mathcal{R} = [\underline{r}, \bar{r}] \subseteq \mathbb{R}$ . Finally, assume that each of these distributions has a finite mean.

Here, signals have been introduced first and rewards second, but the order in which nature selects rewards and signals is unimportant as long as the information the players have about these at each stage remains the same. That is, you could think of the agent as having an underlying reward, and receiving a signal about this reward which they then report to the principal, or as receiving as signal and when then they are inspected, or when the game concludes, a reward is generated given the signal.

The signals have an order conducive to analysis and reflective that higher signals are more *favourable* than lower signals. Specifically, we will impose that the signals are completely or-



dered by the *monotone likelihood ratio property* (MLRP). That is:

$$\frac{\pi_n(r_1)}{\pi_m(r_1)} \geq \frac{\pi_n(r_0)}{\pi_m(r_0)} \quad \text{for all } r_1 > r_0 \text{ and } n > m$$

That is, higher signals generate higher rewards relatively more likely than lower signals. In this sense, an agent with a higher signal is more favourable to the principal, and is a notion outlined by [Milgrom \(1981\)](#) and has been widely adopted since. This is a reasonable characteristic we'd expect of the types of private information we're studying. While the information may be noisy and difficult to communicate, we'd expect agents to be broadly aware of and agree on which information the principal values more highly than others.

Another popular ordering is that of *first order stochastic dominance* (FOSD). A visual demonstration of the differences between MLRP and FOSD, and the reason for selecting the former, is highlighted in section 4.1.1. For now, note that MLRP is a stronger notion than FOSD.

**CLAIM 1** *If the signals are completely ordered by the monotone likelihood ratio property, they are ordered by first order stochastic dominance. That is:*

$$\Pi_n(r) \leq \Pi_m(r) \quad \text{for all } r \text{ and } n > m$$

**Proof:** If a higher signal generates higher rewards relatively more likely than lower signals, then it must be true on average for rewards greater than any fixed reward,  $\hat{r}$ , and the reverse must be true for rewards lower than the fixed reward. Then,  $1 - \Pi_n(\hat{r}) \geq 1 - \Pi_m(\hat{r})$  for  $n > m$ . A complete proof is provided in the Appendix. ■

Given MLRP, it is convenient and unambiguous to relabel the signals by their induced expected reward, so that  $s_n = \mathbb{E}(r \mid s_n)$ . As such, our information parameter now has a neat interpretation, and we will call on this interpretation where helpful. We can now state the principal's problem more explicitly.

### Principal's problem:

In return for a report of the agent's signal,  $s_n$ , the principal may inspect the agent,  $x_n$ , allocate to the agent without inspecting,  $y_n$ , or allocate to the agent after inspecting and observing  $r$ ,  $z_{n,r}$ .

$$\begin{aligned} \max_{(x,y,z)} \quad & \sum_n [(1 - x_n)y_n \mathbb{E}(r \mid s_n) + x_n(\int r z_{n,r} \pi_{n,r} dr - c)] p_n \\ \text{s.t.} \quad & IC_{n,m} : (1 - x_n)y_n + x_n(\int z_{n,r} \pi_{n,r} dr) \geq (1 - x_m)y_m + x_m(\int z_{m,r} \pi_{m,r} dr) \quad \forall n, m \\ & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall r \quad \forall n \end{aligned}$$

For ease of notation, let:

- $\psi_n(z) := \int r z_r \pi_{n,r} dr - c$  as the expected net reward for the principal from inspecting given an arbitrary post-inspection allocation rule  $z$ , and
- $\phi_n(z) := \int z_r \pi_{n,r} dr$  as the expected allocation to the agent from being inspected given an arbitrary post-inspection allocation rule  $z$ .

### 3.2 Symmetric information

As a benchmark, consider a problem where the principal has full access to the agent's private information, whose solution we refer to as the *first-best* policy.

$$\begin{aligned} \max_{(x,y,z)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r \mid s_n) + x_n \psi_n(z_n)] p_n \\ \text{s.t.} \quad & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall r \quad \forall n \end{aligned}$$

Given the agent has nothing additional to report - information is *symmetric* - the agent has no strategically relevant actions. This problem is then straightforward to optimize and outlines exactly what preference the principal has for treating each type of agent. This provides a basis for which to measure the losses associated with the dispersion of information from the principal's perspective.

The next claim outlines the first-best policy and, as with all claims, theorems and propositions in this paper, a sketch of the proof is provided in the main body and a full proof in the appendix. Let  $\mathbb{1}\{Q\}$  be the indicator function that is equal to 1 if the statement  $Q$  is true given the arguments, and 0 otherwise.<sup>3</sup>

CLAIM 2 *The first-best policy  $(x_n^*, y_n^*, z_n^*)$  is given by:*

- $z_{n,r}^* = \mathbb{1}\{r \geq 0\}$ ,
- $y_n^* = \mathbb{1}\{\mathbb{E}(r \mid s_n) \geq 0\}$ , and
- $x_n^* = \mathbb{1}\{\psi_n(z_n^*) \geq \max\{\mathbb{E}(r \mid s_n), 0\}\}$ .

**Proof:** Conditional on the decision to inspect,  $x_n, z_{n,r}$  selects when to allocate post-inspection and should then be maximized when  $r \geq 0$  and minimized otherwise. Call this the *ideal* post-inspection allocation rule. Similarly, conditional on the decision not to inspect,  $1 - x_n, y_n$  selects when to allocate pre-inspection without additional information and should then be maximized when  $\mathbb{E}(r \mid s_n) \geq 0$  and minimized otherwise. Finally,  $x_n$  selects when to inspect, and should be maximized when the expected net reward for the principal from inspecting given the ideal post-inspection allocation rule is both greater than outright allocating or outright rejecting. ■

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<sup>3</sup>The standard definition of an indicator function is  $\mathbb{1}_A(x) := 1$  if  $x \in A$  and 0 if  $x \notin A$ . We're more interested in the set  $A$  and less in the argument  $x$ , so suppress the argument and promote the set.

There are only three relevant policy combinations of the first-best policy.

1. *no allocation*,  $\mathbf{N}$ , given by  $x_n = 0, y_n = 0$ ,
2. *ideal inspection*,  $\mathbf{I}$ , given by  $x_n = 1, z_{n,r} = \mathbb{1}\{r \geq 0\}$ , and
3. *full allocation*,  $\mathbf{A}$ , given by  $x_n = 0, y_n = 1$ .

Let  $\psi_n^* := \psi_n(z_{n,r}^*)$  be the expected return from ideal inspection. Preferences over these policies have a fixed order with respect to  $s_n$  due to the FOSD ordering of the signals.

**CLAIM 3** *There exists some  $s_\alpha$  and  $s_\beta$ , with  $s_\alpha \leq s_\beta$ , such that:*

- *if  $s_n \leq s_\alpha$  then  $0 \geq \max\{\psi_n^*, \mathbb{E}(r \mid s_n)\}$ ,*
- *if  $s_n \in (s_\alpha, s_\beta)$  then  $\psi_n^* > \max\{0, \mathbb{E}(r \mid s_n)\}$ , and*
- *if  $s_n \geq s_\beta$  then  $\mathbb{E}(r \mid s_n) \geq \max\{\psi_n^*, 0\}$ .*

**Proof:** By FOSD, the expected return from ideal inspection is increasing in the signal as the cost is fixed and the likelihood of a positive reward is increasing. As such,  $\psi_n^*$  has a single crossing with 0, after which ideal inspection is preferred to no allocation. Label the corresponding crossing signal as  $s_\alpha$  if it's negative and 0 otherwise. The rate at which  $\psi_n^*$  increases is less than  $\mathbb{E}(r \mid s_n)$  increases as the informativeness of the signal must also increase, eventually rendering inspection as little more informative than the signal itself. As such,  $\psi_n^*$  and  $\mathbb{E}(r \mid s_n)$  also have a single crossing, after which ideal inspection is less preferred to full allocation. Label the corresponding crossing signal as  $s_\beta$  if it's positive and 0 otherwise. ■

This says that there are three regions of interest with respect to the principal's preferences: *low* signals, who the principal would like to reject outright and not allocate to, *intermediate* signals, who the principal would like to inspect and allocate if they are shown to yield positive returns, and *high* signals, who the principal would like to allocate to outright and save on inspection costs. As such, if  $s_\alpha \in (s_0, 0)$  and  $s_\beta \in (0, s_N)$  then tracing out the upper envelope of  $\psi_n^*$ ,  $\mathbb{E}(r \mid s_n)$  and 0 gives us the first-best objective as a function of the signal, as shown in Figure 1, and the corresponding first-best policy as a function of the signal, as shown in Figure 2.

If the return from inspection is high enough for all signals, then  $\psi_n^* \geq \max\{0, \mathbb{E}(r \mid s_n)\}$  for all  $n$  and so  $s_\alpha < s_0$  and  $s_\beta > s_N$ . For example, if  $c = 0$  then it's trivially the case that inspection is always optimal. Looking forward to the problem where information is privately held, the first-best policy is then incentive compatible too, as all signals are treated equally. As such, we will continue under the assumption that either  $s_\alpha > s_0$  or  $s_\beta < s_N$ .

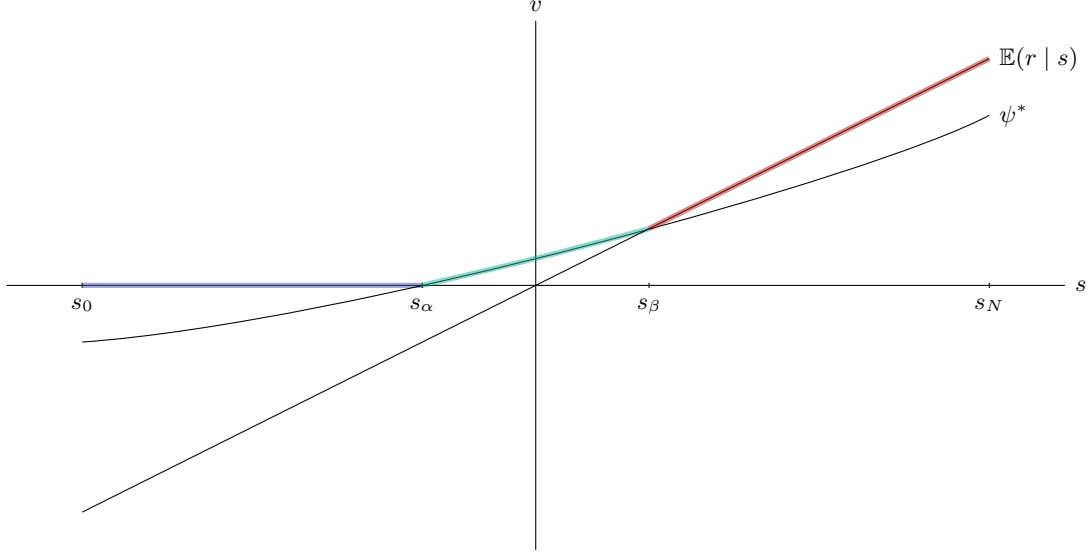


Figure 1: first-best objective,  $v^*$

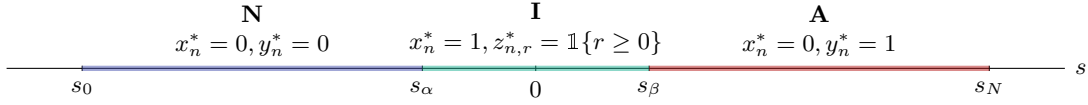


Figure 2: first-best policy,  $(x^*, y^*, z^*)$

## 4 Acquiring information

Now let's return to the problem where the signal is privately held. We will refer to the mechanism that optimizes this problem as the *second-best* policy, in contrast to the symmetric information benchmark. Instead of solving the principal's problem directly, we will solve a relaxation that only requires  $IC_{n,m}$  to hold for  $m = n + 1$ . These constraints are referred to as the *upward local incentive compatibility* (ULIC) constraints.

**Relaxed problem:**

$$\begin{aligned}
 & \max_{(x,y,z)} \sum_n [(1 - x_n)y_n \mathbb{E}(r \mid s_n) + x_n \psi_n(z_n)] p_n \\
 & \text{s.t. } IC_{n,n+1} : (1 - x_n)y_n + x_n \psi_n(z_n) \geq (1 - x_{n+1})y_{n+1} + x_{n+1} \psi_n(z_{n+1}) \quad \forall n < N \\
 & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall r \quad \forall n
 \end{aligned}$$

As we will discover, it is these constraints that matter. The first-best policy tells us that the principal would like to preferentially treat higher type agents, given the favourableness of their information. So we should expect that, under an optimal policy, agent's will primarily have an incentive to lie upwards - claim they have a higher signal in order to receive that preferential

treatment - rather than lie downwards. Further, the agent with the strongest incentive to falsely claim they have a particular type is the agent whose signal is the closest to that type. This is because the policy is designed for each type to report truthfully, so it will also be attractive to those that have a similar distribution of rewards.

If the solution to this problem also satisfies the omitted constraints, then it must be an optimal solution to the principal's problem. Proceeding with this relaxation shows us that post-inspection allocations are threshold rules, each of these constraints bind, and the inspection rule itself is a threshold rule. This pins down a second-best policy which indeed satisfies the omitted constraints.

#### 4.1 Threshold post-inspection allocation

To derive the optimal policy for this problem, we will first show that the post-inspection allocations are not only deterministic but only allocate when the realized reward is high.

**CLAIM 4** *Optimal post-inspection allocations are threshold rules. That is, for each  $n$  there exists some  $\tau_n$  such that:*

$$z_{n,r} = \mathbb{1}\{r \geq \tau_n\}$$

**Proof:** Suppose  $(x, y, z)$  is incentive compatible, optimal, and that for some  $n$ ,  $z_n$  is not a threshold rule. Define  $\tau_n$  such that:

$$\int z_{n,r} \pi_{n,r} dr = \int \mathbb{1}\{r \geq \tau_n\} \pi_{n,r} dr$$

Consider a new policy which replaces  $z_n$  with this threshold rule about  $\tau_n$ . Clearly this is incentive compatible for  $n$  as it's defined such that they receive the same likelihood of allocation given they are inspected as before. That is,  $IC_{n,n+1}$  continues to hold.

If  $n > 0$ , consider  $IC_{n-1,n}$ :

$$(1 - x_{n-1})y_{n-1} + x_{n-1} \left( \int z_{n-1,r} \pi_{n-1,r} dr \right) \geq (1 - x_n)y_n + x_n \left( \int z_{n,r} \pi_{n-1,r} dr \right)$$

This must continue to hold as the transformation of  $z_n$  shifts allocation weight away from low rewards and towards high rewards such that  $s_n$  is fully compensated. As  $s_n$  and  $s_{n-1}$  are ordered by MLRP, this compensation is not enough for  $s_{n-1}$  to also remain indifferent, and thus the transformation is weakly dominated by the original. As such, the transformation reduces the right-hand side of  $IC_{n-1,n}$  which implies that if the initial policy is incentive compatible for  $s_{n-1}$ , then the new policy is too.

Finally, the new policy must generate a higher return for the principal, given we've shifted allocation weight from low values of  $r$  to high values of  $r$  evaluated under the same distribution,  $\Pi_n$ .

This implies  $(x, y, z)$  cannot have been optimal, a contradiction of the proposition. ■

This says that we can restrict our attention to post-inspection allocations that are *threshold rules*: allocate post-inspection if and only if the reward exceeds some threshold. The next example demonstrates the proof graphically and highlights the role of MLRP over FOSD.

#### 4.1.1 Example: FOSD or MLRP

Consider Figure 3. The top function,  $z_n$  is an example of a post-inspection allocation that we would like to transform into a threshold rule. The middle function,  $\pi_n$ , is the reward distribution under the signal that the principal targets with  $z_n$  and is normalized to a uniform distribution. Finally, the bottom function,  $\pi_{n-1}$ , is the reward distribution of a lower signal that we’re protecting from deviating upwards.

There are two examples of  $\pi_{n-1}$ : the left, a distribution that is first order stochastically dominated by  $\pi_n$  but does not follow the monotone likelihood ratio property, and the right, a distribution that is both first order stochastically dominated by  $\pi_n$  and follows the monotone likelihood ratio property. Both follow the right mass property given by FOSD: better signals place a greater likelihood on generating rewards that are higher than any fixed threshold. But only the right follows MLRP: the relative likelihood of rewards (relative to the higher signal’s likelihood,  $\pi_n$ ) is decreasing in the reward.

Transforming  $z_n$  involves shifting the **red mass** up to the **green deficit**. This is done in a way to make  $s_n$  indifferent between the rules, and in our example  $s_n$  assigns the same probability to these two events: the **orange likelihood** and the **blue likelihood** respectfully.

For this transformation to continue to satisfy  $s_{n-1}$ ’s incentive constraint, without any additional restrictions on  $(x, y, z)$ , we must guarantee that deviating to  $s_n$ ’s new policy leaves them no better off than had they deviated prior under the initial policy. FOSD is not enough to guarantee this because lower signals may still put a higher relative likelihood of greater rewards than they had for smaller rewards without violating the total right-mass property given by FOSD. In the left distribution,  $s_{n-1}$  puts a low (**orange**) **likelihood** on the event that  $r$  falls in the range where the principal is reducing the likelihood of allocation, and a high (**blue**) **likelihood** on the event where allocation is now being guaranteed.

MLRP prevents this, as the relative likelihood must be decreasing for higher values of  $r$ . As such, in the right distribution the **orange likelihood** is greater than the **blue likelihood**, and so on net makes  $s_{n-1}$  worse off under the deviation.

The takeaway from this is that FOSD is not enough to guarantee, at this stage, that post-

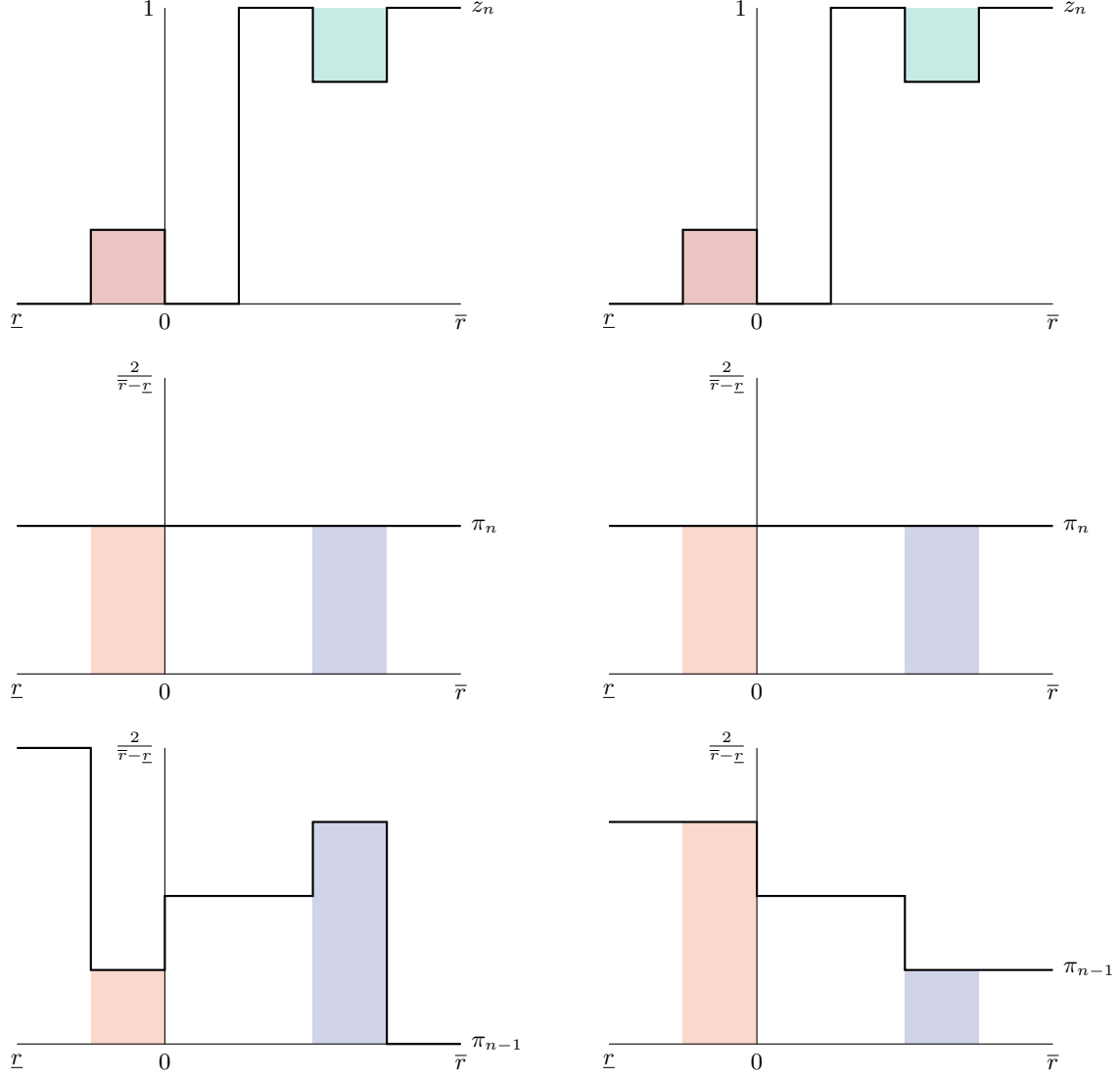


Figure 3: FOSD (left and right) is not enough for the transformation, but MLRP (right only) is.

inspection allocations must be threshold rules. Instead, some regulation on the shape of the distribution must be made, and in this paper that's the MLRP.

Claim 4 provides us with some additional structure to the functions  $\psi$  and  $\phi$ , which help to proceed with determining the optimal policy. In a slight abuse of notation, we will now refer to post-inspection allocation,  $z_n$ , by the threshold,  $\tau_n$ , that defines it, so that  $\psi_n(\tau_n)$  and  $\phi_n(\tau_n)$ , for example, represent the respective expected values under the threshold rule defined by  $\tau_n$ . Note that  $\phi_n(\tau_n)$  is now just the right tail mass above  $\tau_n$ ,  $1 - \Pi_n(\tau_n)$ .

## 4.2 Binding incentive compatibility

With this characteristic, we can see through variation arguments that each upward local incentive compatibility constraint must hold with equality in any optimal solution. For ease of language, we'll say that these constraints *bind* but do not mean to imply that they have a positive shadow price. Treating all types equally will, in some environments, be optimal and then for some types incentive compatibility is free due to feasibility.

**CLAIM 5** *In any optimal mechanism, every upward local incentive compatibility constraint binds. That is, for all  $n < N$ ,  $IC_{n,n+1}$  binds:*

$$(1 - x_n)y_n + x_n\phi_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1})$$

**Proof:** Take a feasible, incentive compatible policy,  $(x, y, z)$ , and suppose that under this policy, a particular constraint does not bind. To show this cannot be the optimal policy, we need to find a feasible alternative that is feasible, incentive compatible and give's the principal a higher expected return. As such, we need to know exactly which policy changes improve the objective.

Clearly, raising  $\tau_n$  if  $\tau_n < 0$ , or lowering  $\tau_n$  if  $\tau_n > 0$ , would constitute such an improvement for any  $s_n$ , as this involves allocating to less negative reward agents and allocating to more positive reward agents. Changing  $x_n$  and  $y_n$ , however, depends on the agent's signal.

For a fixed vector of thresholds,  $\tau$ , consider the following partition of the signal space:

1.  $S_0 := \{n \mid 0 \geq \mathbb{E}(r \mid s_n), 0 \geq \psi_n(\tau_n)\}$
2.  $S_\alpha := \{n \mid 0 \geq \mathbb{E}(r \mid s_n), \psi_n(\tau_n) > 0\}$
3.  $S_\beta := \{n \mid \mathbb{E}(r \mid s_n) > 0, \psi_n(\tau_n) > \mathbb{E}(r \mid s_n)\}$
4.  $S_1 := \{n \mid \mathbb{E}(r \mid s_n) > 0, \mathbb{E}(r \mid s_n) \geq \psi_n(\tau_n)\}$

This outlines the ideal policy choice given  $\tau$ : no allocation if  $n \in S_0$ , inspect if  $n \in S_\alpha \cup S_\beta$ , and allocate without inspection if  $n \in S_1$ . If  $\tau_n = 0$  for each  $n$ , this partition is as described by the first-best policy and displayed in Figure 2. If  $\tau_n \neq 0$  then  $\psi_n(\tau_n) \leq \psi_n^*$  as the principal is either over or under allocating conditional on the realized reward. As such, the  $S_0$  and  $S_1$  are supersets of their first-best counterparts, while  $S_\alpha \cup S_\beta$  is a subset.

Claim 5.1 in the appendix shows that  $IC_{n,n+1}$  must bind if  $n \in S_0$  or  $n + 1 \in S_1$ . If  $n \in S_0$ , then reducing  $y_n$  and  $x_n$  improves the objective as the principal would rather not allocate, and this tightens  $IC_{n,n+1}$  while only relaxing  $IC_{n-1,n}$ . Then, either  $IC_{n,n+1}$  binds, or  $u_n = 0$  ( $x_n = 0$  and  $y_n = 0$ ) and  $IC_{n,n+1}$  must trivially bind as  $u_{n,n+1} \geq 0$ . Similarly, if  $n + 1 \in S_1$  then expanding  $y_{n+1}$  and reducing  $x_{n+1}$  improves the objective as the principal would rather allocate outright,



and tightens  $IC_{n,n+1}$  while only relaxing  $IC_{n+1,n+2}$ . Then, either  $u_n < 1$  but  $IC_{n,n+1}$  binds, or  $u_n = 1$  ( $x_n = 0$  and  $y_n = 1$ ) and  $IC_{n,n+1}$  must trivially bind as  $u_n \leq 1$ .

Claims 5.2, 5.3 and 5.4 in the appendix show that  $IC_{n,n+1}$  must bind if  $n, n+1 \in S_\alpha \cup S_\beta$ . Similar to the previous argument, reducing  $y_n$  if  $n \in S_\alpha$ , or expanding  $y_{n+1}$  if  $n+1 \in S_\beta$ , must improve the objective as the principal would rather not allocate than allocate unconditionally if  $n \in S_\alpha$ , and vice versa if  $n+1 \in S_\beta$ . This tightens  $IC_{n,n+1}$  while only relaxing the adjacent constraints. Further, if  $\tau_n < 0$ , or  $\tau_{n+1} > 0$ , we can improve the objective by raising  $\tau_n$ , or lowering  $\tau_{n+1}$ , as this represents rejecting more negative rewards for  $\tau_n$  and accepting more positive rewards for  $\tau_{n+1}$ . This tightens  $IC_{n,n+1}$  and relaxes the adjacent constraints as the allocation probability is decreasing in  $\tau$ . Finally, expanding  $x_{n+1}$  if  $n \in S_\alpha$ , or expanding  $x_n$  if  $n \in S_\beta$ , must increase the objective as these are types the principal wants to inspect, and we can show that given  $\tau_n \geq 0 \geq \tau_{n+1}$  by the previous argument, this tightens  $IC_{n,n+1}$  and relaxes adjacent constraints. By exhausting these adjustments, it cannot be that  $IC_{n,n+1}$  holds but does not bind.

Now, we already know  $S_0 < S_1$  and  $S_\alpha < S_\beta$  as the value of  $\mathbb{E}(r \mid s_n)$  is given by the signal structure and not the policy.<sup>4</sup> Then, the only two types of constraints that haven't been checked are:  $n \in S_\alpha$  and  $n+1 \in S_0$ , and  $n \in S_1$  and  $n+1 \in S_\beta$ .

Claims 5.5 and 5.6 in the appendix show that  $IC_{n,n+1}$  must bind if either  $n \in S_\alpha$  and  $n+1 \in S_0$ , or  $n \in S_1$  and  $n+1 \in S_\beta$ . As in claims 5.2 and 5.3, reducing  $y_n$  if  $n \in S_\alpha$ , expanding  $y_{n+1}$  if  $n+1 \in S_\beta$ , raising  $\tau_n$  if  $\tau_n < 0$ , and lowering  $\tau_{n+1}$  if  $\tau_{n+1} > 0$ , must all improve the objective, tighten  $IC_{n,n+1}$  and relax adjacent constraints. Unlike in the previous claims, however, expanding  $x_{n+1}$  if  $n+1 \in S_0$ , and expanding  $x_n$  if  $n \in S_1$ , would make the principal worse off. Instead, we can expand inspection in a way that improves the principal's return by replicating the inspection threshold  $\tau_n$  if  $n \in S_\alpha$ , or  $\tau_{n+1}$  if  $n+1 \in S_\beta$ . As inspection of these agents using these thresholds is preferable to the principal, it must also be preferable for types  $n+1 \in S_0$  and  $n \in S_1$  respectfully. As before, this will tighten  $IC_{n,n+1}$  and relax adjacent constraints. Finally, by exhausting these adjustments, it cannot be that  $IC_{n,n+1}$  holds but does not bind.

Then, by claims 5.1 through 5.6,  $IC_{n,n+1}$  must bind for all  $n$ . ■

This says that in any optimal mechanism, for any  $n$ , the expected allocation to  $s_n$  cannot strictly exceed their allocation had they reported  $s_{n+1}$  instead. If it does, the principal could do better by varying the mechanism to tighten this constraint. This then implies the  $IC$  constraints form a chain of equality conditions from  $u_0$  to  $u_N$ , reducing our problem to a simple mathematical program.

---

<sup>4</sup>Here,  $<$  refers to the *below* set relation defined by: set  $A$  is below set  $B$ ,  $A < B$ , if  $\forall a \in A$  and  $\forall b \in B$ ,  $a < b$ .

### 4.3 Threshold inspection rules

Given claims 4 and 5, we can rewrite our problem as:

$$\begin{aligned} \max_{(x,y,\tau)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r \mid s_n) + x_n\psi_n(\tau_n)]p_n \\ \text{s.t.} \quad & IC_{n,n+1} : (1-x_n)y_n + x_n\phi_n(\tau_n) = (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \quad \forall n < N \\ & F : 0 \leq x_n, y_n, \tau_n \leq 1 \quad \forall r \quad \forall n \end{aligned}$$

With this, we can conclude that the optimal inspection rule,  $x$ , is also a threshold rule and that the inspected agents face the same post-inspection allocation.

**CLAIM 6** *Optimal inspection rules are threshold mechanisms, and the post-inspection allocation is identical for all agents. That is, there exists some  $\nu$  and  $\tau$  such that  $x_n = \mathbb{1}\{n > \nu\}$  and  $\tau_n = \tau$  for all  $n$ .*

**Proof:** First observe that:

$$\begin{aligned} u_n &= u_{n,n+1} \\ u_n &= (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \\ u_n &= (1-x_{n+1})y_{n+1} + x_{n+1}\phi_{n+1}(\tau_{n+1}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) + x_n\phi_n(\tau_{n+1}) \\ u_n &= u_{n+1} - x_{n+1}[\phi_{n+1}(\tau_{n+1}) - \phi_n(\tau_{n+1})] \end{aligned}$$

As  $\phi_{n+1}(\tau_{n+1}) > \phi_n(\tau_{n+1})$  by FOSD, this says that the likelihood of allocation is increasing in  $s_n$  and at a rate determined by the inspection rule,  $x$ , and the post-inspection allocation threshold,  $\tau$ .

Note that these two are jointly determined. That is, a choice of  $x$  restricts the choice of  $\tau$ . For example, suppose for some pair  $n_0 < n_1$ ,  $x_{n_0} = 1$ ,  $x_{n_1} = 1$ , and  $x_m = 0$  for  $n_0 < m < n_1$ . Then by the binding constraint  $\tau_{n_0}$  and  $\tau_{n_1}$  must satisfy:

$$\phi_{n_0}(\tau_{n_0}) = \phi_{n_1-1}(\tau_{n_1})$$

If  $n_0 = n_1 - 1$  then  $\tau_{n_0} = \tau_{n_1}$ , and if  $n_0 < n_1 - 1$ , then  $\tau_{n_0} > \tau_{n_1}$  and uniquely determined.

Substituting the arrangement into the objective function we find that, for a fixed  $y_N$  and  $\tau$  the objective is linear in  $x_n$ , whose only restriction is that  $x_n \in [0, 1]$ :

$$\begin{aligned} \max_{(x_n, \tau_n), y_N} \quad & (1-x_N)y_N\mathbb{E}(r) + x_N[\phi_{N-1}(\tau_N)\mathbb{E}(r \mid s \leq s_{N-1})P_{N-1} + \psi_N(\tau_N)p_N] \\ & + \sum_{n=1}^{N-1} x_n[\phi_{n-1}(\tau_n)\mathbb{E}(r \mid s \leq s_{n-1})P_{n-1} - \phi_n(\tau_n)\mathbb{E}(r \mid s \leq s_n)P_n + \psi_n(\tau_n)p_n] \\ & + x_0[-\phi_0(\tau_0)\mathbb{E}(r \mid s_0)p_0 + \psi_0(\tau_0)p_0] \end{aligned}$$

Let  $a_n$  be the coefficient on  $x_n$  in the objective, and observe that  $a_n$  is only a function of  $\tau_n$ . We can immediately conclude:

- $x_n = \mathbb{1}\{a_n(\tau_n) \geq 0\}$ , and
- $y_N = \mathbb{1}\{\mathbb{E}(r) \geq 0\}$ .

This means the restrictions on  $\tau$  in the previous example are the only relevant restriction to our problem, and as such  $x_n = \mathbb{1}\{n > \nu\}$  for some  $\nu \in \{-1, 0, \dots, N\}$  and  $\tau_n = \tau$  for all  $n$ . ■

#### 4.4 Optimal mechanism

Following directly from claim 6, we can now state the main theorem. This, and subsequent results, concern only the non-trivial optimal mechanism, with the pooling mechanisms compared in section 6.

**Theorem 1** *The second-best policy  $(x^*, y^*, z^*)$  is given by:*

- $x_n^* = \mathbb{1}\{s \geq s_{n_0}\}$ ,
- $y_n^* = \phi_{\nu^*}(\tau^*)$ , and
- $z_{n,r}^* = \mathbb{1}\{r \geq \tau^*\}$ ,

where  $\nu^*$  and  $\tau^*$  are the solution to:

$$\max_{\nu, \tau} \sum_{n=\nu+1}^N \psi_n(\tau) p_n + \phi_\nu(\tau) \mathbb{E}(r \mid s \leq s_\nu) P_\nu$$

**Proof:** As demonstrated by claim 6, this policy solves the relaxed problem. As such, we are only left to check that this solution satisfies the omitted IC constraints. Note that, for any  $\nu$  and  $\tau$ ,  $u_n = \phi_\nu(\tau)$  for all  $n \leq \nu$ , and  $u_n = \phi_n(\tau)$  for all  $n > \nu$ . As the marginal signal,  $\nu$ , is indifferent between the two treatments, by FOSD, all  $n < \nu$  must strictly prefer the pre-inspection allocation, and all  $n > \nu$  must strictly prefer inspection. As such, the global IC constraints are satisfied. ■

This gives us a simple maximization problem to solve for our two cut-offs with respect to the endowed grid of signals, and can be solved using a linear search algorithm. To get more intuition on what determines these cut-offs, however, consider a continuous extension of this problem. If we take the limit of the environment to a continuous grid, we find the following optimal mechanism.

**Proposition 1** *The second-best policy  $(x^*, y^*, z^*)$  is given by:*

- $x_s^* = \mathbb{1}\{s \geq s_\gamma\}$ ,
- $y_s^* = \phi_{\gamma^*}(\tau^*)$ , and

- $z_{s,r}^* = \mathbb{1}\{r \geq \tau^*\}$ ,

where  $\gamma^*$  and  $\tau^*$  are the solution to:

$$\phi_\gamma(\tau)\mathbb{E}(r | s_\gamma) = \psi_\gamma(\tau) \quad \text{and} \quad \tau = \left[ \frac{\pi_{s_\gamma, \tau} P_{s_\gamma}}{\int_{s_\gamma}^{s_N} \pi_{s, \tau} p_s ds} \right] (-\mathbb{E}(r | s \leq s_\gamma))$$

**Proof:** By Helly's selection theorem, a uniformly bounded sequence of monotone real functions admits a convergent subsequence. Given the policy of theorem is defined by bounded, monotone real functions, the limiting policy is also an optima of the continuous problem. Finally, the conditions on  $\gamma$  and  $\tau$  are derived using standard calculus arguments. ■

As such, if  $s_\alpha \in (s_0, 0)$  and  $s_\beta \in (0, s_N)$  then tracing out the second-best objective as a function of the signal as shown in Figure 4, and a second policy as a function of the signal as shown in Figure 5.

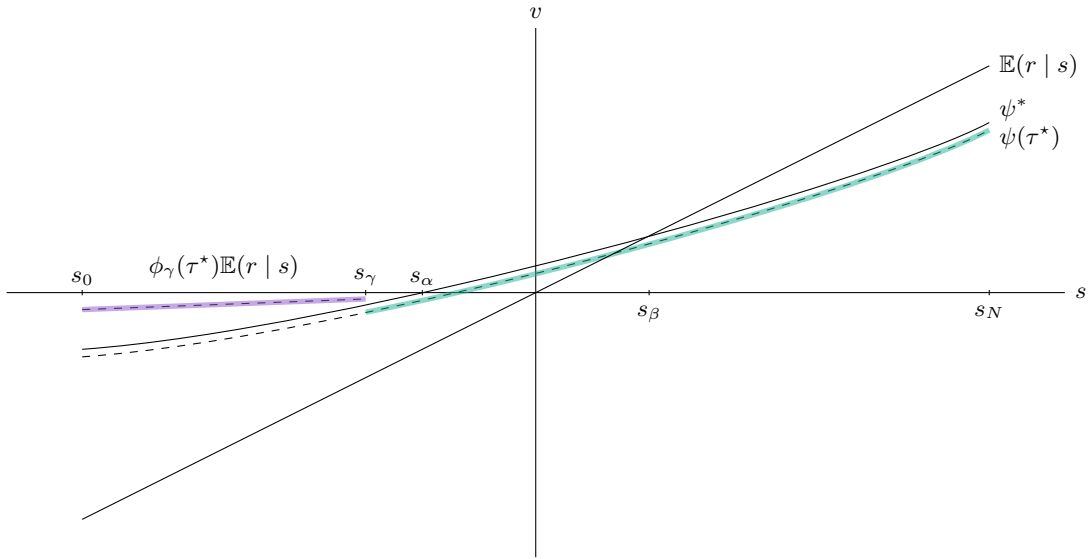


Figure 4: second-best objective,  $v^*$

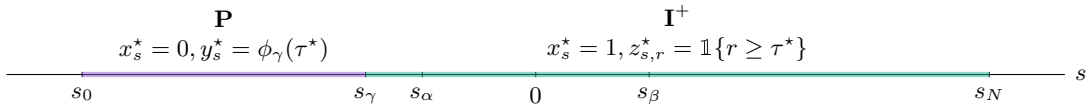


Figure 5: second-best policy,  $(x^*, y^*, z^*)$

Comparing the first-best and second-best objectives, we can characterize the losses for the principal associated with private information in terms of the agent's signal.

**Proposition 2** *The first-best objective exceeds the second-best objective for all signals. That is,*

- for  $s \in [s_0, s_\gamma]$ ,  $v_s^* = 0 \geq \phi_\gamma(\tau^*) = v_s^*$ ,
- for  $s \in (s_\gamma, s_\beta)$ ,  $v_s^* = \psi^* \geq \psi(\tau^*) = v_s^*$ , and
- for  $s \in [s_\beta, s_N]$ ,  $v_s^* = \mathbb{E}(r \mid s) \geq \psi(\tau^*) = v_s^*$ .

This says, that are four types of losses for the principal introduced by the agent's private information:

- *Over-allocation at the bottom*: agents who's private information would be sufficient to reject without inspection,  $s \in [s_0, s_\gamma]$ , are allocated to with positive probability to elicit truthful reports.
- *Over-inspection at the bottom*: agents who's private information would be marginally sufficient to reject without inspection,  $s \in (s_\gamma, s_\alpha)$ , are inspected to reduce pre-inspection allocation to lower types.
- *Over-inspection at the top*: agents who's private information would be sufficient to guarantee allocation,  $s \in (s_\beta, s_N]$ , are inspected in order to elicit truthful reports of lower agents.
- *Under-allocation post inspection*: agents who are inspected,  $s \in (s_\gamma, s_N]$ , and who generate a marginally positive reward,  $r \in [0, \tau^*)$ , are rejected in order to reduce pre-inspection allocation to lower types.

When these losses are less than those associated with the pooling policies, this partial separation is optimal for the principal.

## 5 Relaxing commitment

There are three natural relaxations to full commitment in this environment.

1. **pre-inspection commitment**, or *partial commitment*: the principal can commit to a pre-inspection allocation,  $y$ , and an inspection rule,  $x$ , but cannot commit to a post-inspection allocation,  $z$ ,
2. **pre-assessment commitment**, or *limited commitment*: the principal can commit to a pre-inspection allocation,  $y$ , but cannot commit to an inspection rule,  $x$ , or a post-inspection allocation,  $z$ , and
3. **no commitment**: the principal has no commitment at all, that is they cannot commit to any policy,  $(x, y, z)$ .

Under no commitment, the principal can only choose between the pooling equilibria, as reports convey no information. This is referred to as the *third-best* policy - the policy that optimizes the principal's objective when they can't elicit, or don't have access to, any additional information. We know what this looks like and will return to this in section 6. For now, let's consider the first two relaxations.

## 5.1 Pre-inspection commitment

**Proposition 3** *The second-best policy  $(x^*, y^*, z^*)$  under partial commitment is given by:*

- $x_s^* = \mathbb{1}\{s \geq s_\delta\}$ ,
- $y_s^* = \phi_\delta^*$ , and
- $z_{s,r}^* = z_{s,r}^* = \mathbb{1}\{r \geq 0\}$ ,

where  $\delta$  is the solution to  $\phi_\delta^* \mathbb{E}(r \mid s_\delta) = \psi_\delta^*$ .

As such, if  $s_\alpha \in (s_0, 0)$  and  $s_\beta \in (0, s_N)$  then tracing out the second-best objective as a function of the signal as shown in Figure 6, and a second-best policy as a function of the signal as shown in Figure 7.

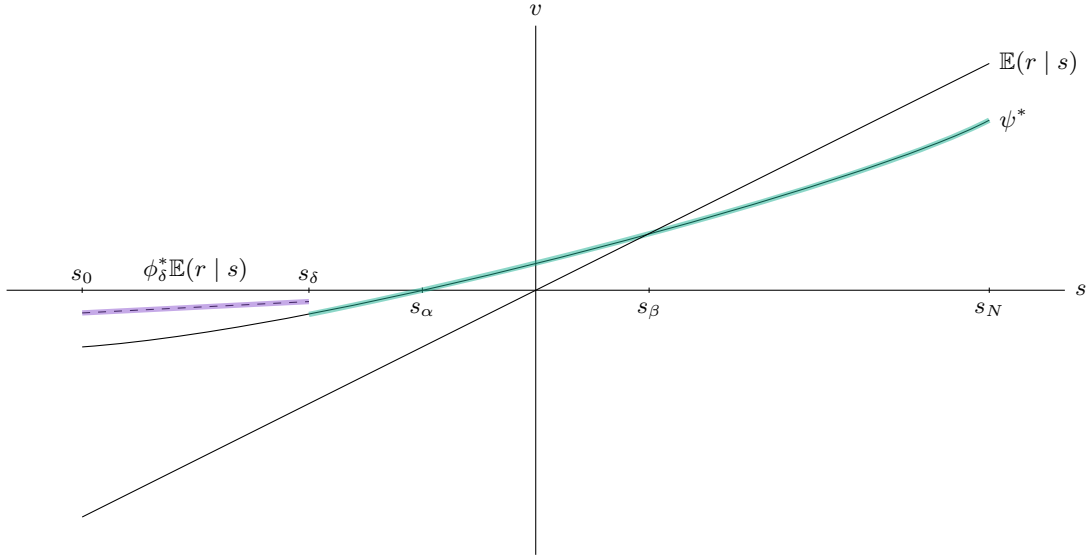


Figure 6: second-best objective under partial commitment,  $v^*$

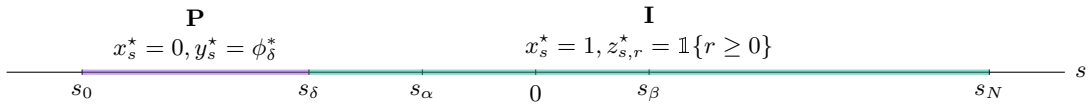


Figure 7: second-best policy under partial commitment,  $(x^*, y^*, z^*)$

## 5.2 Pre-assessment commitment

**Proposition 4** *The second-best policy  $(x^*, y^*, z^*)$  under limited commitment is given by:*

- $x_s^* = \mathbb{1}\{s \geq s_\alpha\}$ ,

- $y_s^* = \phi_\alpha^*$ , and
- $z_{s,r}^* = z_{s,r}^* = \mathbb{1}\{r \geq 0\}$ ,

where  $\alpha$  is the solution to  $\psi_\alpha^* = 0$ .

As such, if  $s_\alpha \in (s_0, 0)$  and  $s_\beta \in (0, s_N)$  then tracing out the second-best objective as a function of the signal as shown in Figure 8, and a second-best policy as a function of the signal as shown in Figure 9.

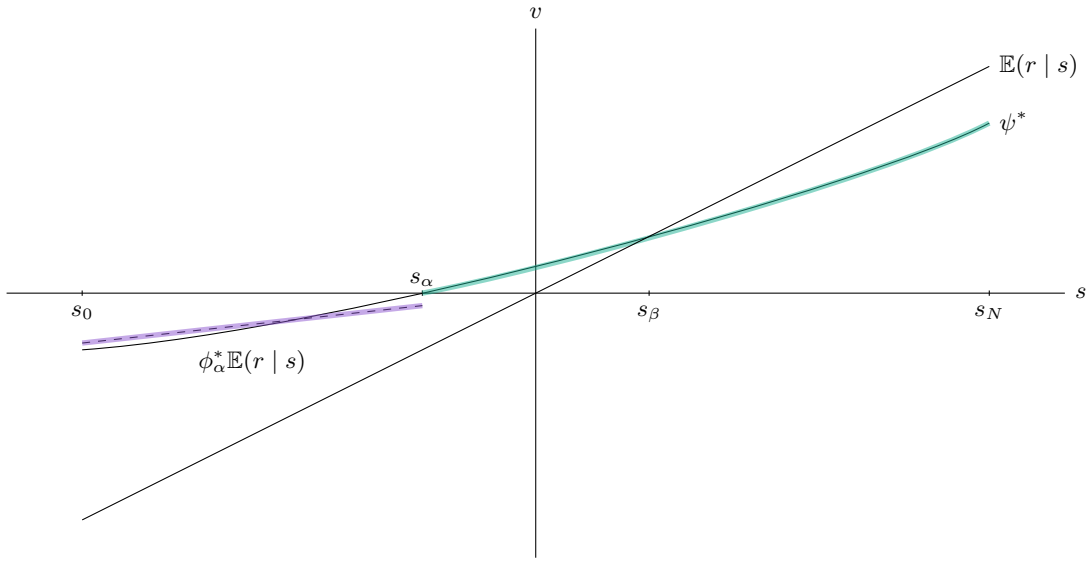


Figure 8: second-best objective under limited commitment,  $v^*$

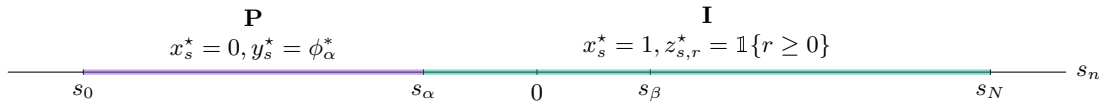


Figure 9: second-best policy under limited commitment,  $(x^*, y^*, z^*)$

## 6 Comparative statics

Our attention so far has been directed at the non-trivial optimal mechanism. We may then want to know under what conditions this mechanism is optimal against mechanisms that treat all agents non-differentially - *pooling* mechanisms. We may also want to know how the partially separating mechanism itself change under changes to the environment, an exercise known as *comparative statics*.

Let us consider the *Gaussian* environment, which provides enough structure on the setting that we can conduct this exercise in full. As demonstrated in proposition 1, extending these results to a continuous signal space poses no issue so long as the prior and posterior distributions are consistent. The normal distribution in particular satisfies MLRP if the variance is fixed across signals, and so lends itself to providing a clean example for particular comparative statics.

Suppose the prior over the rewards,  $\Pi$ , is given by:  $r \sim N(\mu, 1)$ , and the agent receives a signal of their reward determined by:  $\hat{s} = r + \varepsilon$ , where  $\varepsilon \sim N(0, \sigma^2)$ . This implies the distribution of signals,  $\hat{P}$ , is given by:  $\hat{s} \sim N(\mu, \sigma^2 + 1)$ . Together this generates a posterior distribution of rewards,  $\Pi_{\hat{s}}$ , that's given by:  $r \mid \hat{s} \sim N(s, \hat{\sigma}^2)$  where:

$$s = \frac{\sigma^2}{\sigma^2 + 1} \left[ \mu + \frac{\hat{s}}{\sigma^2} \right] \quad \text{and} \quad \hat{\sigma}^2 = \frac{\sigma^2}{\sigma^2 + 1}$$

It is without loss to relabel the signal  $\hat{s}$  by its induced expected value  $s$ , which defines our induced distribution of signals,  $P$ , given by:  $s \sim N(\mu, \frac{1}{\sigma^2 + 1})$ . Finally, define the precision,  $\alpha$ , of the signal as  $\alpha := 1/\sigma^2$ .

Then, the environment is defined by the triple  $(\mu, \alpha, c)$  where:

- $\mu$  is the ex-ante expected reward of allocating to an agent,
- $\alpha$  is the precision of the agent's signal of the reward, and
- $c$  is the inspection cost to the principal.

For each combination of  $(\mu, \alpha, c)$ , it's straightforward to evaluate the four competing mechanisms the principal entertains.

- **N**, *no allocation*, given by  $x_s = 0, y_s = 0$  for all  $s$ ,
- **I**, *ideal inspection*, given by  $x_s = 1, z_{s,r} = \mathbb{1}\{r \geq 0\}$  for all  $s$ ,
- **A**, *full allocation*, given by  $x_s = 0, y_s = 1$  for all  $s$ , and
- **S**, *semi-separation*, given by proposition 1.

As a baseline, consider the problem where the principal can only use their prior information, whose solution is referred to as the *third-best policy*. Then, the principal has to select, and is the same problem as the no commitment relaxation. For a fixed, reasonable  $c$ , this policy is plotted in figure 10 as a function of  $\alpha$  and  $\mu$ .



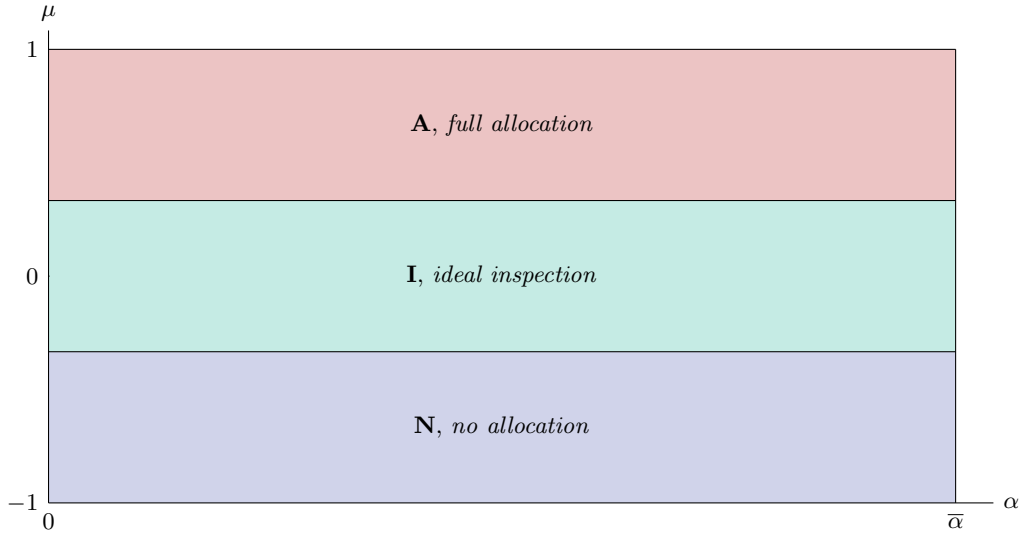


Figure 10: third-best policy as a function of precision,  $\alpha$ , and prior mean,  $\mu$

In the third-best, the principal cannot condition on the signal, so a change in the signal precision does not change the optimal policy. Changing the prior mean, however, changes the expected return from allocating and the expected return from inspecting.

Now, we can compare this directly with our second-best policy, plotted in figure 11 as a function of  $\alpha$  and  $\mu$ .

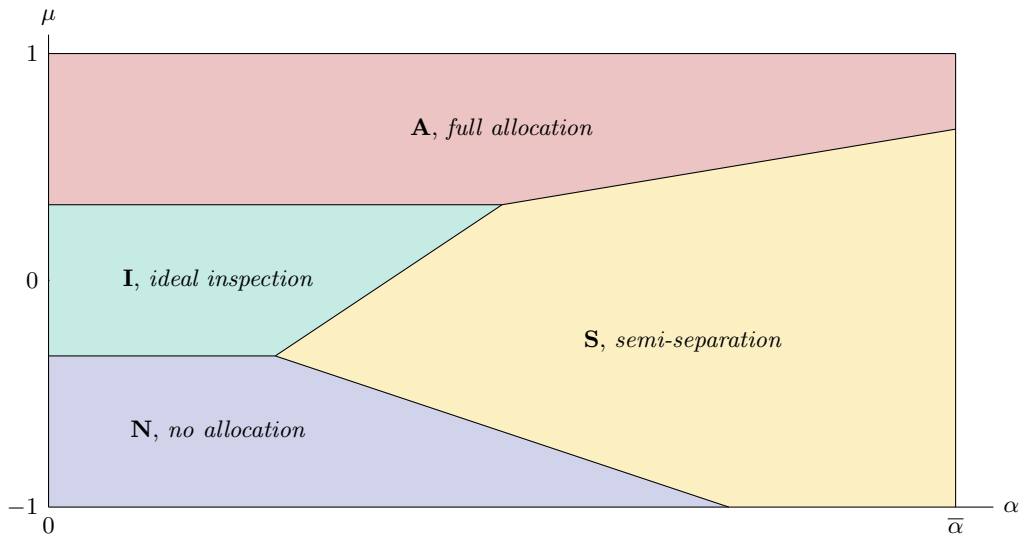


Figure 11: second-best policy as a function of precision,  $\alpha$ , and prior mean,  $\mu$

## 7 Summary

In this paper, we’ve seen that when information is noisy and privately held, costly efforts to acquire this information involve both verification and discovery. This concerns activities such as employer interviews, project assessments and investment evaluation. In these settings, information is useful for an allocation decision that the holders of the private information value, and necessarily, decisions regarding inspection and allocation account for these incentives.

To maximize their expected return, the principal segments signals into two groups, *high* and *low*. Agents with high signals are always inspected and only allocated post-inspection if the reward is sufficiently valuable. Agents with low signals are never inspected, compensated for their report with a small probability of unconditional allocation. This demonstrates that inspection has two purposes in the absence of transfers: the verification of private information by separating signals, and the discovery of additional information by conditioning allocation decisions upon the result of an inspection.

We’ve discovered the conditions under which this mechanism is optimal, and shown that if these conditions are not met, the principal can only select from non-preferential, or *pooling*, mechanisms: either universally inspecting, unconditionally allocating or blanket rejecting. We’ve also seen how this extends to many reasonable relaxations of the principal’s ability to commit, and explored the cost that this comes at.

The framework provided here can be extended to many additional problems for economic analysis. Immediately, analogous results with noise on the inspection process can be derived, showing that the principal employs a similar mechanism to discipline reports in an environment with noisy verification. One could also extend the analysis to various meta-games. For example, including a prior selection of signal precision, or allowing the principal to impose an application cost on the agents, requires no additional modelling assumptions and is a straightforward application of this paper’s contributions.

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## A Environment

One might consider two orderings on the signal space that capture the important features of private information for an economic setting such as this.

A1. The signals are ordered by **first order stochastic dominance (FOSD)**:  $\Pi_n(r) \leq \Pi_m(r)$  for all  $r$  if  $n > m$ .

A2. The signals bear the **monotone likelihood ratio property (MLRP)**:  $\pi_n(r_1)/\pi_m(r_1) \geq \pi_n(r_0)/\pi_m(r_0)$  for all  $r_1 > r_0$  and  $n > m$ .

CLAIM 1 *A2 implies A1.*

**Proof:** Rearranging the condition in A2 for some  $r_1 > r_0$  and  $n > m$ :

$$\pi_n(r_1)\pi_m(r_0) \geq \pi_n(r_0)\pi_m(r_1)$$

We can integrate this expression up to  $r_1$  with respect to  $r_0$  to get:

$$\begin{aligned} \int_{r_0}^{r_1} \pi_n(r_1)\pi_m(r_0)dr_0 &\geq \int_{r_0}^{r_1} \pi_n(r_0)\pi_m(r_1)dr_0 \\ \pi_n(r_1)\Pi_m(r_1) &\geq \Pi_n(r_1)\pi_m(r_1) \\ \frac{\pi_n(r_1)}{\pi_m(r_1)} &\geq \frac{\Pi_n(r_1)}{\Pi_m(r_1)} \end{aligned}$$

Similarly, we can also integrate the original expression down to  $r_0$  with respect to  $r_1$  to get:

$$\begin{aligned} \int_{r_0} \pi_n(r_1)\pi_m(r_0)dr_1 &\geq \int_{r_0} \pi_n(r_0)\pi_m(r_1)dr_1 \\ (1 - \Pi_n(r_0))\pi_m(r_0) &\geq \pi_n(r_0)(1 - \Pi_m(r_0)) \\ \frac{1 - \Pi_n(r_0)}{1 - \Pi_m(r_0)} &\geq \frac{\pi_n(r_0)}{\pi_m(r_0)} \end{aligned}$$

Combining and rearranging these last two expressions for any particular  $r = r_0 = r_1$  gives us A1:

$$\Pi_m(r) \geq \Pi_n(r)$$

■

## A.1 Symmetric information

If the solution to the problem,

$$\begin{aligned} \max_{(x_n, y_n, z_n)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r \mid s_n) + x_n (\int r z_{n,r} \pi_{n,r} dr - c)] p_n \\ \text{s.t.} \quad & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall r \quad \forall n \end{aligned}$$

is the *first-best* policy, then:

CLAIM 2 *The first-best policy  $(x_n^*, y_n^*, z_n^*)$  is given by:*

- $z_{n,r}^* = \mathbb{1}\{r \geq 0\}$ ,
- $y_n^* = \mathbb{1}\{\mathbb{E}(r \mid s_n) \geq 0\}$ , and
- $x_n^* = \mathbb{1}\{\psi_n(z_n^*) \geq \max\{\mathbb{E}(r \mid s_n), 0\}\}$ .

**Proof:** If  $z_{n,r} < 1$  for some  $r > 0$  then increasing  $z_{n,r}$  weakly increases the objective function, and if  $z_{n,r} > 0$  for some  $r < 0$  then decreasing  $z_{n,r}$  weakly increases the objective function. Similarly if  $y_n < 1$  for some  $\mathbb{E}(r \mid s_n) > 0$  then increasing  $y_n$  weakly increases the objective function, and if  $y_n > 0$  for some  $\mathbb{E}(r \mid s_n) < 0$  then decreasing  $y_n$  weakly increases the objective function. Then, the only weakly unimprovable policies are  $z_{n,r} = \mathbb{1}\{r \geq 0\}$  and  $y_n = \mathbb{1}\{\mathbb{E}(r \mid s_n) \geq 0\}$ . Given this, the objective is linear in  $x_n$  and so the maximum is obtained by selecting the larger coefficient: setting  $x_n = 1$  when  $\psi_n(z_n^*) \geq \max\{\mathbb{E}(r \mid s_n), 0\}$ , and 0 otherwise. ■

And there are only three relevant policy combinations of this policy with a fixed order with respect to  $s_n$ :

CLAIM 3 *There exists some  $s_\alpha$  and  $s_\beta$ , with  $s_\alpha \leq s_\beta$ , such that:*

- if  $s_n \leq s_\alpha$  then  $0 \geq \max\{\psi_n^*, \mathbb{E}(r \mid s_n)\}$ ,
- if  $s_n \in (s_\alpha, s_\beta)$  then  $\psi_n^* > \max\{0, \mathbb{E}(r \mid s_n)\}$ , and
- if  $s_n \geq s_\beta$  then  $\mathbb{E}(r \mid s_n) \geq \max\{\psi_n^*, 0\}$ .

**Proof:** We will prove this claim by constructing thresholds  $\tilde{s}_\alpha$  and  $\tilde{s}_\beta$  and adjusting them to match the succinct claim.

By FOSD,  $\psi_n^*$  is increasing in  $n$ , as  $c$  is fixed and the likelihood that  $r > 0$  is increasing. As such there exists a  $\tilde{s}_\alpha$  such that  $\psi_n^* > 0$  if  $s_n > \tilde{s}_\alpha$ . Note that trivially we can set  $\tilde{s}_\alpha$  as any value less than  $s_0$  if  $\psi_0^* > 0$  and any value greater than  $s_N$  if  $\psi_N^* < 0$ .

Additionally, there exists a  $\tilde{s}_\beta$  such that  $\psi_n^* < \mathbb{E}(r \mid s_n)$  if  $s_n > \tilde{s}_\beta$ . To see this, observe that the following are equivalent:

$$\begin{aligned}\psi_n^* &< \mathbb{E}(r \mid s_n) \\ \int \mathbb{1}\{r \geq 0\} r \pi_{n,r} dr - c &< \int r \pi_{n,r} dr \\ -c &< \int \mathbb{1}\{r < 0\} r \pi_{n,r} dr\end{aligned}$$

and as  $\mathbb{1}\{r < 0\} \cdot r$  is increasing in  $r$ , the right hand side is increasing in  $n$  by FOSD. As before we can set  $\tilde{s}_\beta$  as any value less than  $s_0$  if  $\psi_0^* < \mathbb{E}(r \mid s_0)$  and any value greater than  $s_N$  if  $\psi_N^* > \mathbb{E}(r \mid s_N)$ .

Finally by definition  $s_n = \mathbb{E}(r \mid s_n)$  and so  $\mathbb{E}(r \mid s_n) > 0$  when  $s_n > 0$ . This means that  $\tilde{s}_\alpha$  is only policy relevant when less than 0 and  $\tilde{s}_\beta$  when greater than zero. As such, define  $s_\alpha = \min\{\tilde{s}_\alpha, 0\}$  and  $s_\beta = \max\{\tilde{s}_\beta, 0\}$ . ■

## B Acquiring information

Claims 4, 5, and 6 relate to the following relaxation.

**Relaxed problem:**

$$\begin{aligned}\max_{(x_n, y_n, z_n)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r \mid s_n) + x_n (\int r z_{n,r} \pi_{n,r} dr - c)] p_n \\ \text{s.t.} \quad & IC_{n,n+1} : (1 - x_n) y_n + x_n (\int z_{n,r} \pi_{n,r} dr) \geq (1 - x_{n+1}) y_{n+1} + x_{n+1} (\int z_{n+1,r} \pi_{n,r} dr) \quad \forall n < N \\ & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall r \quad \forall n\end{aligned}$$

### B.1 Threshold post-inspection rules

**CLAIM 4** *Optimal post-inspection allocations are threshold rules. That is, for each  $n$  there exists some  $\tau_n$  such that:*

$$z_{n,r} = \mathbb{1}\{r \geq \tau_n\}$$

**Proof:** Suppose  $(x, y, z)$  is incentive compatible, optimal, and that for some  $n$ ,  $z_n$  is not a threshold rule. Define  $\tau_n$  such that:

$$\int z_{n,r} \pi_{n,r} dr = \int \mathbb{1}\{r \geq \tau_n\} \pi_{n,r} dr$$

Given  $\Pi_n$  is absolutely continuous,  $\tau_n$  is well-defined.

Consider a new policy which replaces  $z_n$  with this threshold post-inspection rule about  $\tau_n$ . Clearly this is incentive compatible for  $n$  as it's defined such that they receive the same likelihood of allocation given they are inspected as before. That is,  $IC_{n,n+1}$  continues to hold.

Now consider  $IC_{n-1,n}$ :

$$(1 - x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r} dr) \geq (1 - x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r} dr)$$

We'd like to show this continues to hold under the new policy. That is:

$$(1 - x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r} dr) \geq (1 - x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r} dr)$$

Note that we can rewrite the right-hand side of the original constraint by decomposing  $z_{n,r}$  into the threshold rule and the residual that would reconstitute  $z_{n,r}$ :

$$\begin{aligned} & (1 - x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r} dr) \\ &= (1 - x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r} dr) + x_n(\int_{\tau_n}^{\tau_n} z_{n,r}\pi_{n-1,r} dr - \int_{\tau_n} (1 - z_{n,r})\pi_{n-1,r} dr) \end{aligned}$$

By MLRP,  $\pi_{n-1,r} \geq \pi_{n,r} \frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}$  if  $r < \tau_n$  and  $\pi_{n-1,r} \leq \pi_{n,r} \frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}$  if  $r > \tau_n$ . As such, the right-hand side of  $IC_{n-1,n}$  must be,

$$\begin{aligned} & \geq (1 - x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r} dr) + x_n \frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}} (\int_{\tau_n}^{\tau_n} z_{n,r}\pi_{n,r} dr - \int_{\tau_n} (1 - z_{n,r})\pi_{n,r} dr) \\ &= (1 - x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r} dr) + x_n \frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}} (\int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r} dr - \int z_{n,r}\pi_{n,r} dr) \\ &= (1 - x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r} dr) \end{aligned}$$

Where the final equality comes from the definition of  $\tau_n$ . As such, if the initial policy is incentive compatible for  $n - 1$ , then the new policy is also incentive compatible.

Finally, the new policy must generate a higher return for the principal, given we've shifted allocation weight from low values of  $r$  to high values of  $r$  evaluated under the same  $\Pi_n$ . Another way of showing this is that  $\mathbb{1}\{r \geq \tau_n\}\Pi_n$  stochastically dominates  $z_n\Pi_n$  and the principal evaluates an increasing function,  $r$ , with respect to these censored distributions. This implies  $(x, y, z)$  cannot have been optimal, a contradiction of the proposition.  $\blacksquare$



## B.2 Binding incentive compatibility

CLAIM 5 *In any optimal mechanism, every upward local incentive compatibility constraint binds. That is, for all  $n < N$ ,  $IC_{n,n+1}$  binds:*

$$(1 - x_n)y_n + x_n\phi_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1})$$

**Proof:** Consider the following partition of the signal space:

1.  $S_0 := \{n \mid 0 \geq \mathbb{E}(r \mid s_n), 0 \geq \psi_n(\tau_n)\}$
2.  $S_\alpha := \{n \mid 0 \geq \mathbb{E}(r \mid s_n), \psi_n(\tau_n) > 0\}$
3.  $S_\beta := \{n \mid \mathbb{E}(r \mid s_n) > 0, \psi_n(\tau_n) > \mathbb{E}(r \mid s_n)\}$
4.  $S_1 := \{n \mid \mathbb{E}(r \mid s_n) > 0, \mathbb{E}(r \mid s_n) \geq \psi_n(\tau_n)\}$

In each of the following arguments, the approach is the same: suppose a particular type of constraint does not bind for a solution  $(x, y, z)$  and find a feasible policy improvement, contradicting the optimality of the proposed solution.

CLAIM 5.1 *If  $n \in S_0$  or  $n + 1 \in S_1$ ,  $IC_{n,n+1}$  binds.*

**Proof:** Suppose  $n \in S_0$  and  $IC_{n,n+1}$  does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing  $y_n$  and  $x_n$  improves the objective as  $0 \geq \mathbb{E}(r \mid s_n)$  and  $0 \geq \psi_n(\tau_n)$ , and decreases the left-hand side so tightens  $IC_{n,n+1}$ . Further, this only relaxes  $IC_{n-1,n}$ .

$\therefore$  Either  $u_n = 0$  ( $x_n = 0$  and  $y_n = 0$ ), or  $IC_{n,n+1}$  binds. Note that if  $u_n = 0$  then  $IC_{n,n+1}$  trivially binds as  $u_{n,n+1} \geq 0$ .

Now consider  $n + 1 \in S_1$  and suppose  $IC_{n,n+1}$  doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then expanding  $y_{n+1}$  and reducing  $x_{n+1}$  improves the objective as  $\mathbb{E}(r \mid s_{n+1}) > 0$  and  $\mathbb{E}(r \mid s_{n+1}) \geq \psi_{n+1}(\tau_{n+1})$ , and increases the right-hand side so tightens  $IC_{n,n+1}$ . Further, this only relaxes  $IC_{n+1,n+2}$ .

$\therefore$  Either  $u_{n+1} = 1$  ( $x_{n+1} = 0$  and  $y_{n+1} = 1$ ) or  $IC_{n,n+1}$  binds. Note that if  $u_{n+1} = 1$ , then  $IC_{n,n+1}$  trivially binds as  $u_n \leq 1$ .  $\square$

CLAIM 5.2 *If  $n, n+1 \in S_\alpha$ ,  $IC_{n,n+1}$  binds.*

**Proof:** Suppose  $n, n+1 \in S_\alpha$  and  $IC_{n,n+1}$  does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing  $y_n$  improves the objective as  $\psi_n(\tau_n) \geq 0 \geq \mathbb{E}(r \mid s_n)$ , and decreases the left-hand side, so tightens  $IC_{n,n+1}$  and only relaxes  $IC_{n-1,n}$ .

Suppose  $y_n = 0$  and  $IC_{n,n+1}$  doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Suppose  $\tau_n < 0$ , then raising  $\tau_n$  will improve the objective, decrease the left-hand side and so tighten  $IC_{n,n+1}$  and only relax  $IC_{n-1,n}$ . Similarly, suppose  $\tau_{n+1} > 0$ , then lowering  $\tau_{n+1}$  will improve the objective, increase the right-hand side and so tighten  $IC_{n,n+1}$  and only relax  $IC_{n+1,n+2}$ .

Suppose  $y_n = 0$ ,  $\tau_n \geq 0$  and  $\tau_{n+1} \leq 0$  and  $IC_{n,n+1}$  doesn't bind. Consider expanding  $x_{n+1}$ , which would improve the objective as  $\psi_{n+1}(\tau_{n+1}) > 0 \geq \mathbb{E}(r \mid s_{n+1})$ . This tightens  $IC_{n,n+1}$  if:

$$0 < -y_{n+1} + \phi_n(\tau_{n+1}) \Rightarrow \phi_n(\tau_{n+1}) > y_{n+1}$$

Note that if this is true, then it also relaxes  $IC_{n+1,n+2}$  as  $\phi_{n+1}(\tau_{n+1}) \geq \phi_n(\tau_{n+1})$  by FOSD.

Suppose by contradiction  $\phi_n(\tau_{n+1}) \leq y_{n+1}$ , and  $y_n = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$  while  $IC_{n,n+1}$  doesn't bind. Then:

$$\begin{aligned} u_n &= x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ &\geq (1 - x_{n+1})\phi_n(\tau_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ &= \phi_n(\tau_{n+1}) \end{aligned}$$

But as  $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$  and  $\phi_n(\tau_{n+1}) \geq \phi_n(0)$ , this is a contradiction. As such, expanding  $x_{n+1}$  must tighten  $IC_{n,n+1}$ .

Finally, suppose  $y_n = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ ,  $x_{n+1} = 1$  and  $IC_{n,n+1}$  doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > \phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$  binds. □

CLAIM 5.3 If  $n, n+1 \in S_\beta$ ,  $IC_{n,n+1}$  binds.

**Proof:** Suppose  $n, n+1 \in S_\beta$  and  $IC_{n,n+1}$  does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then expanding  $y_{n+1}$  improves the objective as  $\psi_{n+1}(\tau_{n+1}) \geq 0 \geq \mathbb{E}(r \mid s_{n+1})$ , and increases the right-hand side, so tightens  $IC_{n,n+1}$  and only relaxes  $IC_{n+1,n+2}$ .

Suppose  $y_{n+1} = 1$  and  $IC_{n,n+1}$  doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Suppose  $\tau_n < 0$ , then raising  $\tau_n$  will improve the objective, decrease the left-hand side and so tighten  $IC_{n,n+1}$  and only relax  $IC_{n-1,n}$ . Similarly, suppose  $\tau_{n+1} > 0$ , then lowering  $\tau_{n+1}$  will improve the objective, increase the right-hand side and so tighten  $IC_{n,n+1}$  and only relax  $IC_{n+1,n+2}$ .

Suppose  $y_{n+1} = 1$ ,  $\tau_n \geq 0$  and  $\tau_{n+1} \leq 0$  and  $IC_{n,n+1}$  doesn't bind. Consider expanding  $x_n$ , which would improve the objective as  $\psi_n(\tau_n) > \mathbb{E}(r \mid s_n) > 0$ . This tightens  $IC_{n,n+1}$  if:

$$-y_n + \phi_n(\tau_n) < 0 \quad \Rightarrow \quad \phi_n(\tau_n) < y_n$$

Note that if this is true, then it also relaxes  $IC_{n-1,n}$  as  $\phi_{n-1}(\tau_n) \leq \phi_n(\tau_n)$  by FOSD.

Suppose by contradiction  $\phi_n(\tau_n) \geq y_n$ , and  $y_{n+1} = 1$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$  while  $IC_{n,n+1}$  doesn't bind. Then:

$$\begin{aligned} u_n &= (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ (1 - x_n)\phi_n(\tau_n) + x_n\phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ \phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \end{aligned}$$

But,  $\phi_n(\tau_n) \leq \phi_n(0)$  and  $(1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$ , a contradiction. As such, expanding  $x_n$  tightens  $IC_{n,n+1}$ .

Finally, suppose  $y_{n+1} = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ ,  $x_n = 1$  and  $IC_{n,n+1}$  doesn't bind:

$$u_n = \phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$  binds. □

So far, we've shown constraints within the sets are binding, as well as those leading from  $S_0$  or those leading to  $S_1$ . Now we're left to check constraints across the inspection sets.

**CLAIM 5.4** *If  $n \in S_\alpha$  and  $n + 1 \in S_\beta$ ,  $IC_{n,n+1}$  binds.*

**Proof:** Suppose  $n \in S_\alpha$ ,  $n + 1 \in S_\beta$  and  $IC_{n,n+1}$  does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as before, reducing  $y_n$ , expanding  $y_{n+1}$ , raising  $\tau_n$  if  $\tau_n < 0$  and lowering  $\tau_{n+1}$  if  $\tau_{n+1} > 0$  all improve the objective, tighten  $IC_{n,n+1}$ , relax  $IC_{n-1,n}$  and  $IC_{n+1,n+2}$ .

Suppose,  $y_n = 0$ ,  $y_{n+1} = 1$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ , and  $IC_{n,n+1}$  doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

This cannot be the case as  $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$  and  $(1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$ .

$\therefore IC_{n,n+1}$  binds. □

We already know  $S_0 < S_1$  and  $S_\alpha < S_\beta$  as the value of  $\mathbb{E}(r \mid s_n)$  is given by the signal structure and not the policy.<sup>5</sup> So the only two types of constraints to check are:  $n \in S_\alpha$ ,  $n + 1 \in S_0$  and  $n \in S_1$ ,  $n + 1 \in S_\beta$ .

**CLAIM 5.5** *If  $n \in S_\alpha$  and  $n + 1 \in S_0$ ,  $IC_{n,n+1}$  binds.*

**Proof:** Suppose  $n \in S_\alpha$ ,  $n + 1 \in S_0$  and  $IC_{n,n+1}$  does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as in the proof of Claim 5.2, reducing  $y_n$ , raising  $\tau_n$  if  $\tau_n < 0$  and lowering  $\tau_{n+1}$  if  $\tau_{n+1} > 0$  all improve the objective, tighten  $IC_{n,n+1}$  and relax adjacent IC constraints.

Suppose  $y_n = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ , and  $IC_{n,n+1}$  doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Also, as in the proof of Claim 5.2, it must be that  $\phi_n(\tau_{n+1}) > y_{n+1}$ . Suppose by contradiction

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<sup>5</sup>Here,  $<$  refers to the *below* set relation defined by: set  $A$  is below set  $B$ ,  $A < B$ , if  $\forall a \in A$  and  $\forall b \in B$ ,  $a < b$ .

$\phi_n(\tau_{n+1}) \leq y_{n+1}$ , and  $y_n = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$  while  $IC_{n,n+1}$  doesn't bind. Then:

$$\begin{aligned} u_n &= x_n \phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ &\geq (1 - x_{n+1})\phi_n(\tau_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ &= \phi_n(\tau_{n+1}) \end{aligned}$$

But as  $x_n \phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$  and  $\phi_n(\tau_{n+1}) \geq \phi_n(0)$ , this is a contradiction.

Now consider reducing the probability that  $s_{n+1}$  is allocated to without inspection, in favour of inspection using the threshold assigned to  $s_n$ . In particular, conditional on not inspecting  $s_{n+1}$ , instead of allocating with probability  $y_{n+1}$ , allocate with probability  $\lambda y_{n+1}$  and inspect using the threshold  $\tau_n$  with probability  $(1 - \lambda)$ , for some  $\lambda \in (0, 1)$ .

This improves the objective as  $0 \geq \mathbb{E}(r \mid s_{n+1})$  and  $\psi_{n+1}(\tau_n) > 0$ . The second fact here comes from the observation that, despite  $0 \geq \psi_{n+1}(\tau_{n+1})$ ,  $\psi_n(\tau_n) > 0$  implies  $\psi_{n+1}(\tau_n) > 0$ . To see this, take a fixed threshold  $\tau$  and rearrange:

$$\begin{aligned} \psi_n(\tau) &> 0 \\ \int \mathbb{1}\{r \geq \tau\} r \pi_{n,r} dr - c &> 0 \\ \int \mathbb{1}\{r \geq \tau\} r \pi_{n,r} dr &> c \end{aligned}$$

As  $\mathbb{1}\{r \geq \tau\}r$  is an increasing function of  $r$  when  $\tau > 0$ , the left-hand side is increasing in  $n$  as ensured by FOSD.

This tightens  $IC_{n,n+1}$  as  $\phi_n(\tau_n) > y_{n+1}$ , and does not contradict for small enough  $\lambda$ . To see this, observe that:

$$\phi_n(\tau_n) > x_n \phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) > y_{n+1}$$

Which follows the condition that  $IC_{n,n+1}$  didn't bind and our claim that  $\phi_n(\tau_{n+1}) > y_{n+1}$ .

And this change only relaxes  $IC_{n+1,n+2}$ , ensured by FOSD and the same rationale:

$$\phi_{n+1}(\tau_n) > x_n \phi_{n+1}(\tau_n) \geq x_n \phi_n(\tau_n) > y_{n+1}$$

Finally, suppose  $y_n = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ ,  $x_{n+1} = 1$  and  $IC_{n,n+1}$  doesn't bind:

$$u_n = x_n \phi_n(\tau_n) > \phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$  binds. □

CLAIM 5.6 *If  $n \in S_1$  and  $n+1 \in S_\beta$ ,  $IC_{n,n+1}$  binds.*

**Proof:** Suppose  $n \in S_1$ ,  $n+1 \in S_\beta$  and  $IC_{n,n+1}$  does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as in the proof of Claim 5.3, expanding  $y_{n+1}$ , raising  $\tau_n$  if  $\tau_n < 0$  and lowering  $\tau_{n+1}$  if  $\tau_{n+1} > 0$  all improve the objective, tighten  $IC_{n,n+1}$  and relax adjacent IC constraints.

Suppose  $y_{n+1} = 1$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ , and  $IC_{n,n+1}$  doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Also as in the proof of Claim 5.3, it must be that  $y_n > \phi_n(\tau_n)$ . Suppose by contradiction  $\phi_n(\tau_n) \geq y_n$ , and  $y_{n+1} = 1$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$  while  $IC_{n,n+1}$  doesn't bind. Then:

$$\begin{aligned} u_n &= (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ (1 - x_n)\phi_n(\tau_n) + x_n\phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ \phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \end{aligned}$$

But,  $\phi_n(\tau_n) \leq \phi_n(0)$  and  $(1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$ , a contradiction.

Now consider reducing the probability that  $s_n$  is allocated to without inspection, in favour of inspection using the threshold assigned to  $s_{n+1}$ . In particular, conditional on not inspecting  $s_n$ , instead of allocating with probability  $y_n$ , allocate with probability  $\lambda y_n$  and inspect using the threshold  $\tau_{n+1}$  with probability  $(1 - \lambda)$ , for some  $\lambda \in (0, 1)$ .

This improves the objective as  $\psi_n(\tau_{n+1}) \geq \mathbb{E}(r \mid s_n)$ . This fact comes from the observation that, despite  $\mathbb{E}(r \mid s_n) \geq \psi_n(\tau_n)$ ,  $\psi_{n+1}(\tau_{n+1}) > \mathbb{E}(r \mid s_{n+1})$  implies  $\psi_n(\tau_n) > \mathbb{E}(r \mid s_n)$ . To see this, take a fixed threshold  $\tau$  and rearrange:

$$\begin{aligned} \psi_{n+1}(\tau_{n+1}) &> \mathbb{E}(r \mid s_{n+1}) \\ \int \mathbb{1}\{r \geq \tau\} r \pi_{n+1,r} dr - c &> \int r \pi_{n+1,r} dr \\ -c &> \int \mathbb{1}\{r < \tau\} r \pi_{n+1,r} dr \end{aligned}$$

As  $\mathbb{1}\{r < \tau\} r$  is an increasing function of  $r$  when  $\tau < 0$ , the right-hand side is increasing in  $n$  as

ensured by FOSD.

This tightens  $IC_{n,n+1}$  as  $y_n > \phi_n(\tau_{n+1})$  and does not contradict for  $\lambda$  small enough. To see this, observe that:

$$y_n > (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) > \phi_n(\tau_{n+1})$$

Which follows from our claim that  $y_n > \phi_n(\tau_n)$  and the condition that  $IC_{n,n+1}$  didn't bind.

And this change only relaxes  $IC_{n-1,n}$  ensured the same rationale and FOSD:

$$y_n > \phi_n(\tau_{n+1}) > \phi_{n-1}(\tau_{n+1})$$

Finally, suppose  $y_n = 0$ ,  $\tau_n \geq 0$ ,  $\tau_{n+1} \leq 0$ ,  $x_n = 1$  and  $IC_{n,n+1}$  doesn't bind:

$$u_n = \phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$  binds. □

Then, by claims 5.1 through 5.6,  $IC_{n,n+1}$  must bind for all  $n$ . ■

### B.3 Threshold inspection rules

Given claims 4 and 5, we can rewrite the principal's problem as:

$$\begin{aligned} \max_{(x_n, y_n, \tau_n)} \quad & \sum_n [(1 - x_n)y_n \mathbb{E}(r \mid s_n) + x_n \psi_n(\tau_n)] p_n \\ \text{s.t.} \quad & IC_{n,n+1} : (1 - x_n)y_n + x_n\phi_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \quad \forall n < N \\ & F : 0 \leq x_n, y_n, \tau_n \leq 1 \quad \forall r \quad \forall n \end{aligned}$$

**CLAIM 6** *Optimal inspection rules are threshold mechanisms. That is, there exists some  $n_0$  such that  $x_n = \mathbb{1}\{n \geq n_0\}$ .*

**Proof:** First observe that for each  $n$  we can represent  $(1 - x_n)y_n$  recursively using the binding

$IC_{n,n+1}$  constraints:

$$\begin{aligned}
(1 - x_n)y_n &= (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= (1 - x_{n+2})y_{n+2} + x_{n+2}\phi_{n+1}(\tau_{n+2}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) \\
&\quad + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= (1 - x_{n+3})y_{n+3} + x_{n+3}\phi_{n+2}(\tau_{n+3}) - x_{n+2}\phi_{n+2}(\tau_{n+2}) \\
&\quad + x_{n+2}\phi_{n+1}(\tau_{n+2}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) \\
&\quad + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= \dots \\
&= (1 - x_N)y_N + \sum_{m=n}^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]
\end{aligned}$$

Which can also be written as the following arrangement:

$$(1 - x_n)y_n = (1 - x_N)y_N + x_N\phi_{N-1}(\tau_N) + \sum_{m=n+1}^{N-1} x_m[\phi_{m-1}(\tau_m) - \phi_m(\tau_m)] - x_n\phi_n(\tau_n)$$

Note that this also restricts the choice of  $\tau$ . For example, suppose for some pair  $n_0 < n_1$ ,  $x_{n_0} = 1$ ,  $x_{n_1} = 1$ , and  $x_m = 0$  for  $n_0 < m < n_1$ . Then by the binding constraint  $\tau_{n_0}$  and  $\tau_{n_1}$  must satisfy:

$$\phi_{n_0}(\tau_{n_0}) = \phi_{n_1-1}(\tau_{n_1})$$

This says that if  $n_0 = n_1 - 1$  then  $\tau_{n_0} = \tau_{n_1}$ , and if  $n_0 < n_1 - 1$ , then  $\tau_{n_0} > \tau_{n_1}$  and uniquely determined. We will return to this example after the substitution.



Substituting the arrangement into the objective function:

$$\begin{aligned}
& \sum_n [(1 - x_n)y_N \mathbb{E}(r \mid s_n) + x_n \psi_n(\tau_n)] p_n \\
&= \sum_n [(1 - x_N)y_N + \sum_{m=n}^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)] \mathbb{E}(r \mid s_n) + x_n \psi_n(\tau_n)] p_n \\
&= (1 - x_N)y_N \sum_n \mathbb{E}(r \mid s_n) p_n + \sum_n \sum_{m=n}^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)] \mathbb{E}(r \mid s_n) p_n + \sum_n x_n \psi_n(\tau_n) p_n \\
&= (1 - x_N)y_N \sum_n \mathbb{E}(r \mid s_n) p_n + \sum_m^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)] \sum_n^m \mathbb{E}(r \mid s_n) p_n + \sum_n x_n \psi_n(\tau_n) p_n \\
&= (1 - x_N)y_N \mathbb{E}(r) + \sum_m^{N-1} [x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)] \mathbb{E}(r \mid s \leq s_m) P_m + \sum_n x_n \psi_n(\tau_n) p_n
\end{aligned}$$

This shows us that for a fixed  $y_N$  and  $\tau$  the objective is linear in  $x_n$ , whose only restriction is that  $x_n \in [0, 1]$ :

$$\begin{aligned}
\max_{(x_n, \tau_n), y_N} & (1 - x_N)y_N \mathbb{E}(r) + x_N [\phi_{N-1}(\tau_N) \mathbb{E}(r \mid s \leq s_{N-1}) P_{N-1} + \psi_N(\tau_N) p_N] \\
& + \sum_{n=1}^{N-1} x_n [\phi_{n-1}(\tau_n) \mathbb{E}(r \mid s \leq s_{n-1}) P_{n-1} - \phi_n(\tau_n) \mathbb{E}(r \mid s \leq s_n) P_n + \psi_n(\tau_n) p_n] \\
& + x_0 [-\phi_0(\tau_0) \mathbb{E}(r \mid s_0) p_0 + \psi_0(\tau_0) p_0]
\end{aligned}$$

Let  $a_n$  be the coefficient on  $x_n$  in the objective, and observe that  $a_n$  is only a function of  $\tau_n$ . We can immediately conclude:

- $x_n = \mathbb{1}\{a_n(\tau_n) \geq 0\}$ , and
- $y_N = \mathbb{1}\{\mathbb{E}(r) \geq 0\}$ .

This means the restrictions on  $\tau$  in the previous example are the only relevant restriction to our problem, and as such  $x_n = \mathbb{1}\{n \geq n_0\}$  for some  $n_0 \in \{0, \dots, N, N+1\}$  and  $\tau_n = \tau$  for all  $n$ . ■