Optimal Allocation with Noisy Inspection

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Inspection

A core economic activity

- employers **interview** potential employees
- public funds **assess** grant applications
- venture capitalists **evaluate** investment opportunities

Why **inspect**?

1. discovery or *information acquisition*
2. verification or *screening*

A class of problems

A **principal** receives an unknown reward from allocating to an **agent**.

The agent has imperfect private information about this unknown reward; they receive a unit reward from being allocated to.

The principal may elicit a report from the agent, as well as inspect the reward at a cost.

The principal can commit to a mechanism, but must do so without transfers.

How should the principal design the inspection and allocation mechanism to maximize their ex ante expected return?

Applications

Mechanism design problems with noisy information, costly inspection, and limited transfers are widespread.

1. **Hiring**: a firm seeks to fill an open position in their operation with a potential employee.

2. **Grant assignment**: a public fund is tasked with assessing a grant application.

3. **Impact investment**: a venture capitalist sets the mechanism by which it reviews and invests in startups.

Literature

Perfect information: Green and Laffont (1986), Ben-Porath, Dekel and Lipman (2014), Mylovanov and Zapechelnyuk (2017), Epitropou and Vohra (2019).

Transfers: Townsend (1979), Border and Sobel (1987), Mookherjee and Png (1989), Alaei et al. (2020).

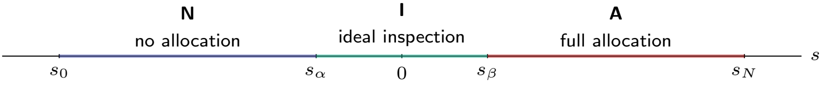
Limited transfers: Mylovanov and Zapechelnyuk (2017), Silva (2019*b*), Li (2021).

Efficient mechanisms: Ball and Kattwinkel (2019), Silva (2019*a*), Siegel and Strulovici (2021), Pereyra and Silva (2021), Erlanson and Kleiner (2020).

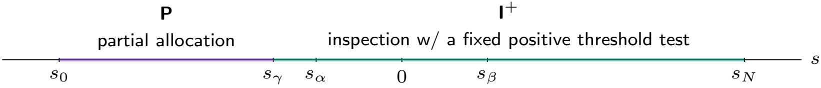
A simple solution

Let s be the agent's **type**, sorted and labelled by the expected value of the reward given the private information.

Symmetric information benchmark:



Optimal (separating) mechanism:

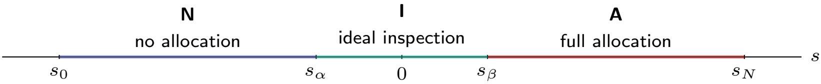


Losses

Three types of losses from private information:

1. over-allocation at the bottom,
2. over-inspection at the top and bottom, and
3. under-allocation post-inspection.

Symmetric information benchmark:



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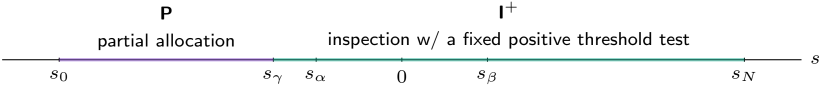


Environment

The agent is endowed with a signal, s, about the reward that defines the agents **type**.

The principal receives a reward, r, from **allocating** to the agent, and 0 otherwise.

The agent's payment is 1 if allocated to, and 0 otherwise.

The principal can **inspect** the agent to reveal the true reward, r, which costs a fixed $c > 0$ to their final payoff.

Direct transfers of value between the principal and agent are prohibited.

Signals

Suppose $s \in \{s_0, s_1, \ldots, s_N\}$, where $s = s_n$ with probability $p_n \in (0, 1)$, $\sum_n p_n = 1$, and P_n is the cmf.

If $s = s_n$, then the reward $r_n \sim \Pi_n$ where Π_n is absolutely continuous and admits a pdf π_n.

Suppose that the signals are ordered by the monotone likelihood ratio property, **MLRP**.

$$\pi_n(r_1)/\pi_m(r_1) \geq \pi_n(r_0)/\pi_m(r_0) \text{ for all } r_1 > r_0 \text{ and } n > m$$

Note that MLRP $\Rightarrow$ FOSD.

It's without loss to relabel the signals by their induced expected reward, so that $s_n = \mathbb{E}(r|s_n)$.

Timing

$(x, y, z),\ s$	$\hat{s}$	$x(\hat{s}), y(\hat{s}), z(\hat{s})$	$v_s(\hat{s}), u_s(\hat{s})$
1	2	3	4

The timing of the game is as follows:

1. The principal commits to a mechanism, and nature assigns signals.
2. The agent observes their signal and submits a report to the principal.
3. The principal implements the mechanism conditional on the report and any reward realizations.
4. All remaining uncertainty is resolved, and rewards are distributed.

Mechanism

Listing the principal's available actions, let:

- x_s be the inspection probability given report s
- y_s be the allocation probability without inspection given report s
- $z_{s,r}$ be the allocation probability after inspection given report s and observing r

Together, (x, y, z) constitutes a **mechanism** where,

- x is the inspection rule,
- y is the pre-inspection allocation, and
- z is the post-inspection allocation.

Optimal allocation

The principal's problem:

$$\max_{(x,y,z)} \quad \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n\psi_n(z_n)]p_n$$

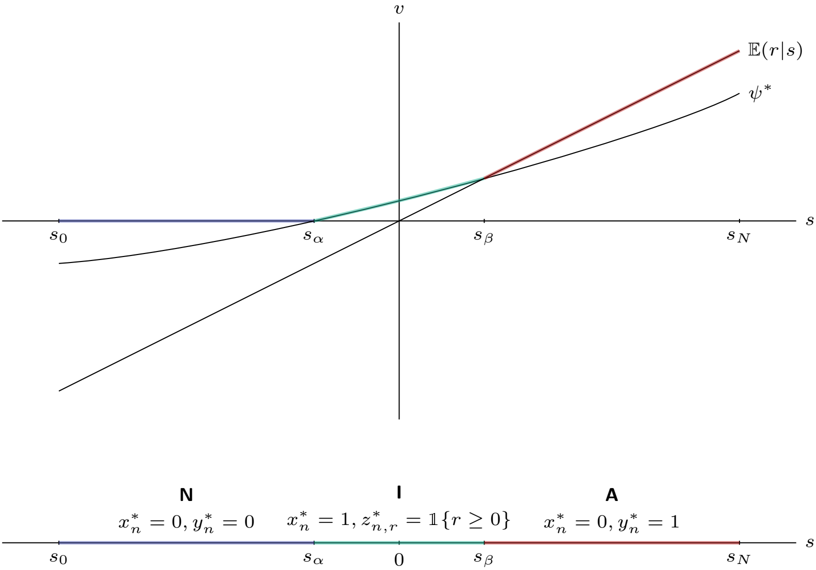
$$\text{s.t.} \quad IC_{n,m} : (1-x_n)y_n + x_n\mathbb{E}(z_{n,r}|n) \geq (1-x_m)y_m + x_m\mathbb{E}(z_{m,r}|n) \quad \forall\ n,m$$

$$F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n$$

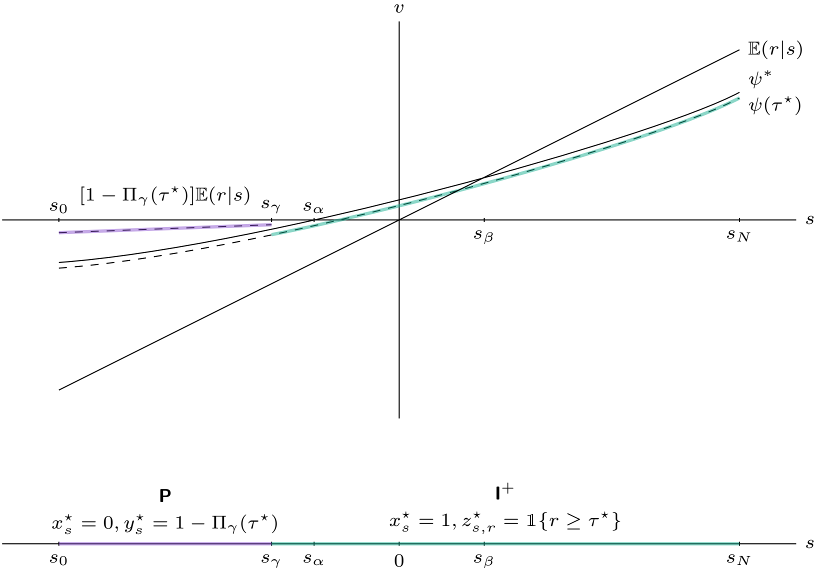
where:

- $\psi_n(z_n) \coloneqq \mathbb{E}(z_{s,r}.r|s) - c = \int r z_{n,r}\pi_{n,r}\, dr - c$, is the expected reward from inspecting n with post-inspection allocation rule z_n.

First best policy, $*$



Second best policy, $\star$



A relaxation

Consider a **relaxation** of the principal's problem that only requires the upward local IC constraints to be satisfied. That is:

$$IC_{n,n+1} : (1-x_n)y_n + x_n\mathbb{E}(z_{n,r}|n) \geq (1-x_{n+1})y_{n+1} + x_{n+1}\mathbb{E}(z_{n+1,r}|n) \quad \forall\, n < N$$

To derive our mechanism, we will prove the following claims about the relaxed problem are sequentially true:

1. The post-inspection allocation, z_n, is a threshold rule.
2. All upward local IC constraints bind.
3. The inspection rule, x_n, is a threshold rule.

Then we are only left to derive the optimal thresholds for the relaxed problem, calculate the pre-inspection allocation, and verify the solution satisfies the global IC constraints.

1. Threshold post-inspection allocation

Claim 1: Optimal post-inspection rules are threshold mechanisms. That is, for each n there exists some τ_n such that:

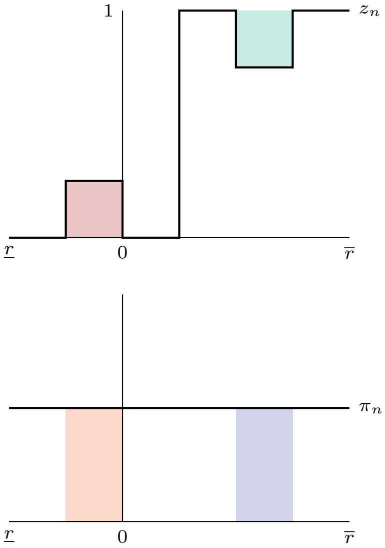
$$z_{n,r} = \mathbb{1}\{r \geq \tau_n\}$$

Idea: For each n find the τ_n such that:

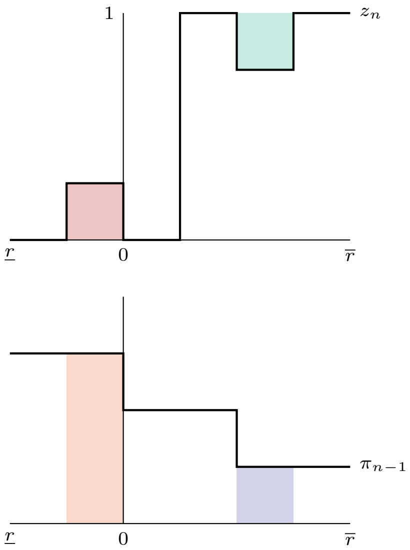
$$\int z_{n,r} \pi_{n,r} \, dr = \int \mathbb{1}\{r \geq \tau_n\} \pi_{n,r} \, dr$$

This transformation will always improve the objective, maintain the expected payoff for n, and weakly reduce the expected deviation payoff for $n-1$.

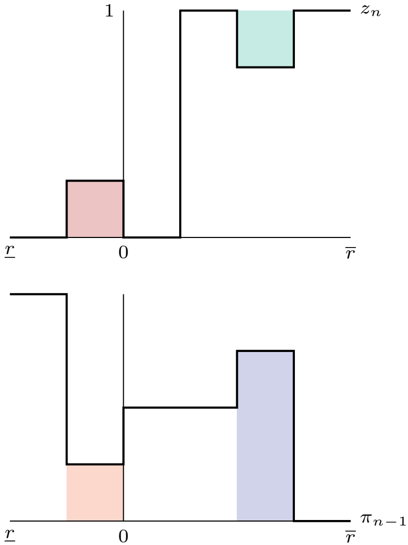
The transformation



MLRP



FOSD



Threshold tests

Post-inspection allocations are then determined by a simple threshold test.

For the agent:

$$\mathbb{E}(z_{n,r}|n) = \int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r}\, dr = 1 - \Pi_n(\tau_n)$$

For convenience, let's denote:

$$\overline{\Pi}_n(\tau) \coloneqq 1 - \Pi_n(\tau)$$

2. Binding ULIC

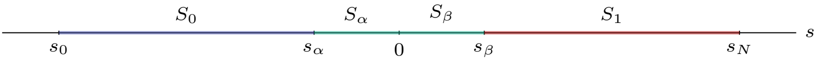
Claim 2: Each upward local incentive compatibility constraint binds. That is, for each $n < N$:

$$(1 - x_n)y_n + x_n\overline{\Pi}_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\overline{\Pi}_n(\tau_{n+1})$$

Idea: Consider the following partition:

1. $S_0 \coloneqq \{n \mid 0 \geq \mathbb{E}(r|s_n),\ 0 \geq \psi_n(\tau_n)\}$
2. $S_\alpha \coloneqq \{n \mid 0 \geq \mathbb{E}(r|s_n),\ \psi_n(\tau_n) > 0\}$
3. $S_\beta \coloneqq \{n \mid \mathbb{E}(r|s_n) > 0,\ \psi_n(\tau_n) > \mathbb{E}(r|s_n)\}$
4. $S_1 \coloneqq \{n \mid \mathbb{E}(r|s_n) > 0,\ \mathbb{E}(r|s_n) \geq \psi_n(\tau_n)\}$

Note, that if $\tau_n = 0$ for each n, this corresponds with our first best policy:



3. Threshold inspection rules

Claim 3: Optimal inspection rules are threshold mechanisms. That is, there exists n_0 such that $x_n = \mathbb{1}\{n \geq n_0\}$.

Idea: We can now rewrite $(1 - x_n)y_n$ recursively:

$$
\begin{aligned}
(1 - x_n)y_n &= (1 - x_{n+1})y_{n+1} \quad + x_{n+1}\overline{\Pi}_n(\tau_{n+1}) - x_n\overline{\Pi}_n(\tau_n) \\
&= (1 - x_{n+2})y_{n+2} \quad + x_{n+2}\overline{\Pi}_{n+1}(\tau_{n+2}) - x_{n+1}\overline{\Pi}_{n+1}(\tau_{n+1}) \\
&\qquad\qquad\qquad\qquad + x_{n+1}\overline{\Pi}_n(\tau_{n+1}) - x_n\overline{\Pi}_n(\tau_n) \\
&= \cdots \\
&= (1 - x_N)y_N \qquad + \sum_{m=n}^{N-1} [x_{m+1}\overline{\Pi}_m(\tau_{m+1}) - x_m\overline{\Pi}_m(\tau_m)]
\end{aligned}
$$

A linear objective

Our value function becomes:

$$
\begin{aligned}
v = \quad & (1 - x_N) y_N \mathbb{E}(r) \\
+ \quad & x_N[\overline{\Pi}_{N-1}(\tau_N)\mathbb{E}(r|s \leq s_{N-1})P_{N-1} + \psi_N(\tau_N)p_N] \\
+ \sum_{n=1}^{N-1} \quad & x_n[\overline{\Pi}_{n-1}(\tau_n)\mathbb{E}(r|s \leq s_{n-1})P_{n-1} - \overline{\Pi}_n(\tau_n)\mathbb{E}(r|s \leq s_n)P_n + \psi_n(\tau_n)p_n] \\
+ \quad & x_0[-\overline{\Pi}_0(\tau_0)\mathbb{E}(r|s_0)p_0 + \psi_0(\tau_0)p_0]
\end{aligned}
$$

This is a linear function in x_n. Similar to the proof of claim 2, we can then use variation arguments to prove that, $x_n \in \{0, 1\}$.

For example, our constraint directly implies that for any consecutive signals such that $x_n = x_{n+1} = 1$, then $\tau_n = \tau_{n+1}$, as $\overline{\Pi}_n(\tau_{n+1}) = \overline{\Pi}_n(\tau_n)$.

Optimal separating policy

Given Claims 1-3, we are only left to optimize our objective by selecting:

- n_0: the first type to inspect, and
- τ: the threshold to which to test those who are inspected.

This is given by:

$$\max_{n_0,\tau} \sum_{n=n_0}^{N} \psi_n(\tau)p_n + \overline{\Pi}_{n_0-1}(\tau)\mathbb{E}(r|s \leq s_{n_0-1})P_{n_0-1}$$

This satisfies the **global** IC constraints for all n_0 and τ, and thus must be a solution to the original problem.

Optimal seperating policy

Appealing to the limit case, as the signal grid becomes finer, the equivalent result is given by:

Theorem: The second-best (separating) policy $(x_s^\star, y_s^\star, z_s^\star)$ is given by:

- $x_s^\star = \mathbb{1}\{s \geq s_\gamma\}$,
- $y_s^\star = \overline{\Pi}_\gamma^*$, and
- $z_{s,r}^\star = \mathbb{1}\{r \geq \tau^\star\}$,

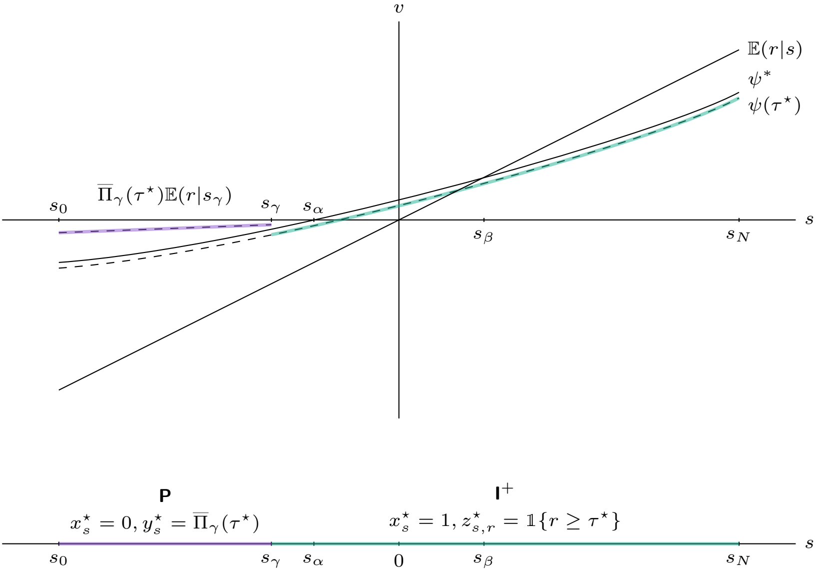
where $(\gamma, \tau^\star)$ solve:

$$\overline{\Pi}_\gamma(\tau)\mathbb{E}(r|s_\gamma)P_\gamma = \psi_\gamma(\tau)p_\gamma \tag{1}$$

and,

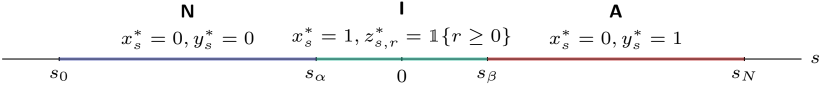
$$\tau = \left[\frac{\pi_{s_\gamma,\tau}P_{s_\gamma}}{\int_{s_\gamma}^{s_N} \pi_{s,\tau}p_s\,ds}\right](-\mathbb{E}(r|s \leq s_\gamma)) \tag{2}$$

Second best solution

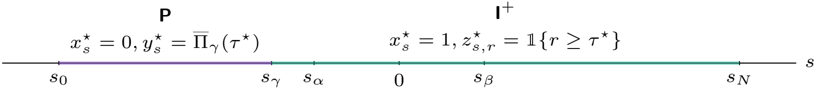


Losses

Public information benchmark:



Optimal (separating) mechanism:



Three types of losses from private information:

1. over-allocation: for $s \in [s_0, s_\gamma]$, $y^\star_s = \overline{\Pi}_\gamma(\tau^\star) > 0$,
2. over-inspection: for $s \in [s_\gamma, s_\alpha] \cup [s_\beta, s_N]$, $x^\star_s = 1$, and
3. under-allocation post-inspection: for $s \in [s_\gamma, s_N]$ and $r \in [0, \tau^\star]$, $z^\star_{s,r} = 0$.

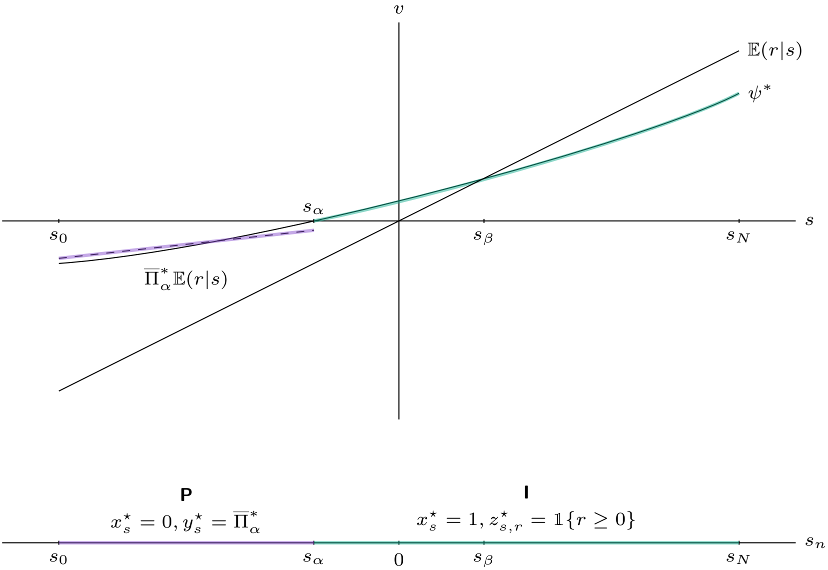
Relaxing commitment

There are three natural relaxations to the commitment assumption:

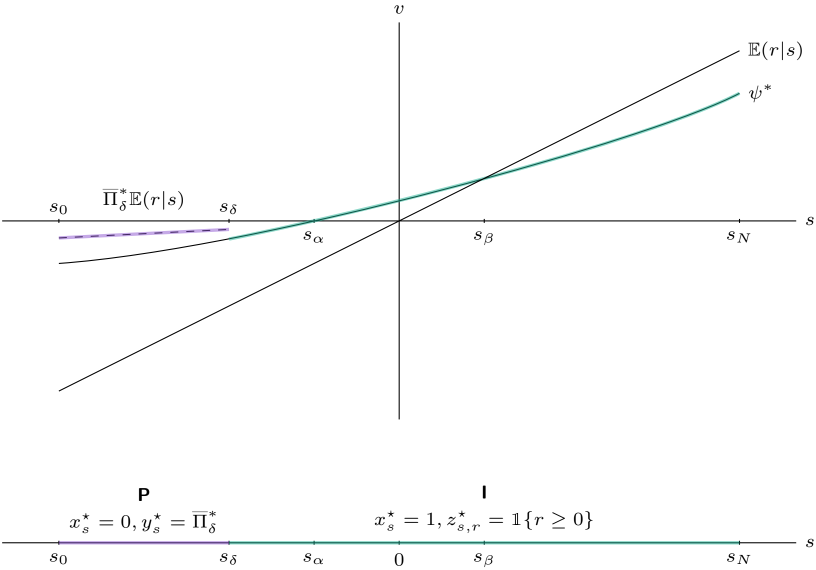
1. **partial commitment**, or *pre-inspection commitment*: the principal can commit to pre-inspection allocations and an inspection rule but cannot commit to post-inspection allocations,
2. **limited commitment**, or *pre-assessment commitment*: the principal cannot commit to either an inspection rule or post-inspection allocations, but can commit to pre-inspection allocations, and
3. **no commitment**: the principal cannot commit to allocations or an inspection rule.

For **no commitment**, the principal can only choose between the pooling equilibria and reports convey no information. We know what this looks like, so let's turn to the first two relaxations.

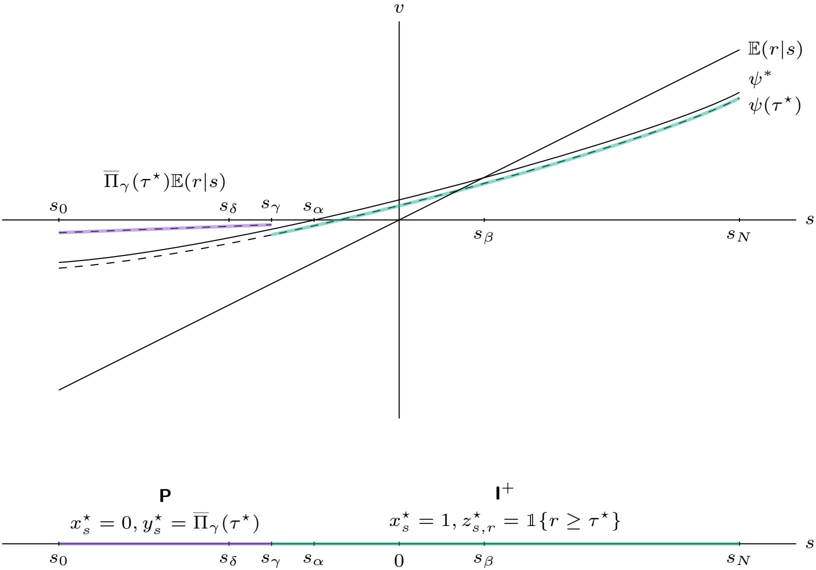
Limited commitment



Partial commitment



Full commitment



Noisy inspection

Optimal inspection balances *information acquisition* and *screening*.

When agents have *noisy private information*, the principal:

- over-inspects high and low types,
- under-allocates to agents who are inspected, and
- over-allocates to agents who are not inspected.

Weakening commitment magnifies the losses from over-allocating to agents who aren't inspected.

Outstanding questions:

1. a single object among many agents or *noisy search*
2. flexible inspection technology or *optimally noisy search*

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