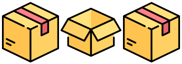
BDP 509: Applied Game Theory



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Summer Session II, 2022

Lecture Three: Lotteries

July 13, 2022

Today's Tasks

1. Announcements: quiz delay and splitting today's lecture
2. Review and 3 player representation
3. Matching pennies and Nash's theorem
4. Lotteries
5. Expected utility hypothesis
6. Risk Aversion and the Allais paradox

Review

A best response is the strategy that yields the highest payoff conditional on a profile of strategies for the other players. e.g. *confess* is the best response for Alice to Bob playing *confess*.

A dominant strategy is a strategy that yields tha highest payoff no matter what profile of strategies the other players are playing. e.g. *confess* is a (strictly) dominant strategy for Alice.

A Nash equilibrium is a profile of strategies for which all players are playing mutually best responses. e.g. (*confess*, *confess*) was the unique Nash equilibrium of the Prisoners Dilema game.

A Pareto optimal outcome is one for which there is no other outcome that yields a higher payoff for all agents. e.g. (*deny*, *deny*) is a Pareto optimal outcome in the Prisoners Dilema game, and so is (*confess*, *deny*) and (*deny*, *confess*).

3 player games

Including a third player in our simultaneous games is straightforward if we introduce another matrix for each additional strategy the third player has and listing their payoff third. Here's a randomly generated example:

Player 3 plays *top*,

		Player 2	
		left	*right*
Player 1	*up*	$10, 1, 2$	$9, 3, 10$
	down	$10, 5, 7$	$2, 5, 8$

Player 3 plays *bottom*,

		Player 2	
		left	*right*
Player 1	*up*	$9, 5, 9$	$6, 1, 3$
	down	$6, 8, 9$	$6, 4, 2$

Equilibria

Player 3 plays *top*,

		Player 2	
		left	*right*
Player 1	*up*	$\underline{10}, 1, 2$	$\underline{9}, \underline{3}, \underline{10}$
	down	$\underline{10}, \underline{5}, 7$	$2, \underline{5}, \underline{8}$

Player 3 plays *bottom*,

		Player 2	
		left	*right*
Player 1	*up*	$\underline{9}, \underline{5}, \underline{9}$	$\underline{6}, 1, 3$
	down	$6, \underline{8}, \underline{9}$	$\underline{6}, 4, 2$

There are two NE: (*up*, *left*, *bottom*) and (*up*, *right*, *top*)

An issue with Nash equilibrium . . .

Let's revisit the matching pennies game:

		US Goalie	
		left	*right*
Aus. Striker	*left*	$-1, 1$	$1, -1$
	right	$1, -1$	$-1, 1$

There are no (pure strategy) Nash equilibrium! This is an issue if we want to use game theory as a predictive tool.

This game an example of a **zero sum** game: a game who's payoffs in each outcome always sum to zero. This does not directly imply that there are no pure strategy Nash equilibrium (consider a game with payoffs that are all zeros!) but does describe the most pure form of a competitive game where one players gain is another players loss. Zero sum games were studied extensively before John Nash, most notably by: **John Von Nueman** and **Oskar Morgenstern**.

Nash's theorem

John Nash, 1950, proved the following result:

Theorem: In any n player, finite game, there always exists at least one (Nash) equilibrium.

This seems to directly contradict our findings about the matching pennies game, however Nash was considering a more general form of strategy called mixed strategies.

To properly appreciate mixed strategies we first need to do some background about lotteries . . .

Review of preferences

Recall that we said our players were rational if their preferences over consequences **A** and **B** were complete and transitive. This allowed us to represent these preferences with a payoff function.

By "represented" we meant the following: if **A** was preferred to **B**, then the payoff associated with **A** was higher than that associated with **B**.If we let $\succsim$ represent the relation "preferred to" and U(**X**) be the "payoff associate with **X**" than we can write this more succinctly:

$$\mathbf{A} \succsim \mathbf{B} \Rightarrow \mathrm{U}(\mathbf{A}) \geq \mathrm{U}(\mathbf{B})$$

Summary of notation:

- $\succsim$ means "weakly preferred to"
- $\succ$ means "strictly preferred to"
- $\sim$ means "indifferent between"
- $\Rightarrow$ means "implies"

Lotteries

Suppose consequence **A** gives our player the following outcome: a coin will be flipped, if the coin lands on *heads* the player receives \$100, and if the coin lands on *tails* the player receives \$20.

A lottery is a consequence that results in a range of different outcomes with a known probability. Suppose lottery **L** gives a probability p of **X** occurring and a probability $(1 - p)$ of **Y** occurring. Then we can represent a (simple) lottery with the following notation:

$$\mathbf{L} = p \circ \mathbf{X} + (1 - p) \circ \mathbf{Y}$$

For example, representing our consequence **A** from above:

$$\mathbf{A} = 0.5 \circ \$100 + 0.5 \circ \$20$$

Question: can we represent the payoff from **L** as:

$$p \times \mathrm{U}(\mathbf{X}) + (1 - p) \times \mathrm{U}(\mathbf{Y})\ ?$$

Expected utility hypothesis

$$\mathbb{E}\mathrm{U}(\mathbf{L}) := p \times \mathrm{U}(\mathbf{X}) + (1 - p) \times \mathrm{U}(\mathbf{Y})$$

This final object is called an expected utility/value. If our player did behave as if they they were maximising their expected utility we could use this characterisation to represent how our players felt about their opponents randomizing.

Von Neumann and **Morgenstern**, 1947, proved that our players preferences can be represented by a payoff function with this property if their preferences were complete, transitive, continuous and independent. That is, if these axioms are met,

$$\mathbf{L} \succsim \mathbf{M} \Rightarrow \mathbb{E}\mathrm{U}(\mathbf{L}) \geq \mathbb{E}\mathrm{U}(\mathbf{M})$$

VnM axioms

1. Completeness: for any lotteries $\mathbf{L}$ and $\mathbf{M}$, either
$$\mathbf{L} \succ \mathbf{M},\ \mathbf{M} \succ \mathbf{L},\text{ or } \mathbf{L} \sim \mathbf{M}$$

2. Transitivity: for any lotteries $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$,
$$\text{if } \mathbf{L} \succ \mathbf{M} \text{ and } \mathbf{M} \succ \mathbf{N}, \text{ then } \mathbf{L} \succ \mathbf{N}$$

3. Continuity: if $\mathbf{L} \succ \mathbf{M} \succ \mathbf{N}$, then there exists a p such that:
$$p \circ \mathbf{L} + (1-p) \circ \mathbf{N} \sim \mathbf{M}$$

4. Independence: if $\mathbf{L} \succ \mathbf{M}$ then it must be for any p and any $\mathbf{N}$,
$$p \circ \mathbf{L} + (1-p) \circ \mathbf{N} \succ p \circ \mathbf{L} + (1-p) \circ \mathbf{N}$$

Risk aversion

Would you rather:

A. \$1 million with a 50% chance and nothing otherwise, or

B. \$490 thousand for sure?

If you chose option B then we would describe you as risk averse.

It may appear at first that that your preferences fail VnM as:

$$0.5\times 1{,}000{,}000 + 0.5\times 0 = 500{,}000$$

and how could anyone value \$500 000 less than \$490 000?

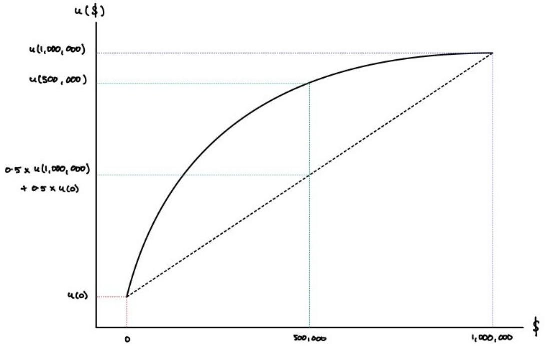
But remember: we are evaluating the expected utility **not** the utility of the expected value. Instead we need to see if there's a utility function consistent with:

$$0.5\times \mathrm{U}(\$1{,}000{,}000) + 0.5\times \mathrm{U}(\$0) < \mathrm{U}(\$490{,}000)$$

e.g. $\mathrm{U}(\$1{,}000{,}000) = 1$, $\mathrm{U}(\$0) = 0$, and $\mathrm{U}(\$490{,}000) = 0.51$

Utility over money

We use utility functions over monetary payoffs that display diminishing marginal utility to represent players with risk aversion:



As such, risk aversion does not pose a problem for the expected utility hypothesis and our subsequent analysis.

Allais paradox

Would you rather:

A. \$1 million for sure, or
B. \$1 million with a 89% chance, \$5 million with a 10% chance and nothing otherwise?

How about:

C. \$1 million with an 11% chance and nothing otherwise, or
D. \$5 million with an 10% chance and nothing otherwise?

This example is called the Allais paradox; most people choose A and D but this contradicts the independence axiom! (Why?) As a result, no expected utility function can be account for both of these choices.

As such, the Allais paradox is an issue for our representation! Accounting for this, and other behavioural phenomenons with respect to risk and uncertainty, is a large and active research field within the social sciences.