Optimal Allocation with Noisy Inspection*

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Abstract

A principal receives an unknown reward from allocating to an agent who has private information about the reward. Prior to allocating, the principal may elicit a report from the agent and inspect them at a cost, but must do so without transfers. When the private information is noisy, the unique separating mechanism that maximizes the principal's expected return segments signals into two groups, inspects high types, allocating to them only if the inspected return is sufficiently positive, and doesn't inspect low types, compensating them with a small probability of allocation. This relates to a number of applied settings such as employer hiring strategies, public grant mechanisms, and portfolio investment rules.

1 Overview

Appraising the value of an asset is an essential precursor to its exchange. Employers interview potential employees, public funds assess grant applications, venture capitalists evaluate investment opportunities. This process is often costly, and information that could be used to lower, and even circumvent, these costs is often privately held.

This paper considers a principal whose return from allocating to (*employing, selecting, supporting*) an agent is uncertain and inherent to the agent they allocate to. The principal has the ability to inspect (*interview, assess, evaluate*) the agent at a cost and learn about the true return, as well as the opportunity to receive a report from the agent of the private information they have about it. The agent, independent of their private information, strictly prefers to be allocated to. Inspection then has two purposes: the verification of private information and the discovery of additional information.

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The environment presented here encompasses many important settings. To fix attention, consider the following three examples.

1. **Hiring**: a manager, the principal, seeks to fill an open position in their firm with a potential employee, the agent. The agent would like to be hired, receiving the benefits of employment at this firm, and has some idea of their future productivity in this position. This information, however, is less than perfect. The principal could interview the agent and discover the true productivity, but this is costly for the principal. What interview and hiring protocol should the firm enact?

2. **Funding**: a governing board, the principal, sets the rules by which it allocates a scarce, publicly owned resource, such as funding for a project and an applicant, the agent. The agent is primarily interested in being approved, valuing their own use above rival uses, and the principal wants to fund positive net value projects, perhaps weighing a combination of private and public preferences for the funding's use. How should a resource restricted board design testing and approval rules to support socially valuable projects?

3. **Investing**: an investor, the principal, sets the mechanism by which it evaluates and finances an early investment opportunity, the agent. The investor may be governed by the motivation to strengthen an existing portfolio or even personal philanthropic concerns, but is restricted in outlining these preferences publicly. The agent wishes to be financed and to continue their startup, and has the most information about the startup. This information doesn't fully determine the investor's value for the opportunity without an appraisal. How much information can the investor elicit through their evaluation and funding decisions?

The three key aspects in each of these examples is that agents have noisy private information about the principal's allocation reward, it is costly for the principal to discover the reward themselves, and the use of transfers to align incentives are limited. It is in this environment that inspection has the potential to serve two different roles. Directly, it informs the principal of the reward they will receive if they carry through with allocating to the agent, and indirectly, it disciplines the agent's disclosure of information by conditioning allocation on the outcome of an inspection. To this end, this paper addresses the tradeoff between these two sources of information: direct acquisition through *discovery*, and indirect acquisition through *screening*.

Modelling the agent's private information as a signal of *favourableness*, the unique separating mechanism that maximizes the expected return for the principal has a simple structure. To elicit truthful reports, the principal pools two types of signals from the agent - *high* and *low*. Agents with high signals are always inspected and only allocated post-inspection if the reward is sufficiently valuable. Agents with low signals are never inspected, compensated for their report with a small probability of unconditional allocation.

This induces three types of inefficiencies for the principal: over-allocation to agents with low types, over-inspection of agents with both high and low types, and under-allocation post-inspection. In practice, the over-inspection of low types and the under-allocation post-inspection requires strong commitment from the principal. This paper demonstrates that when commitment is relaxed, to the net detriment of the principal's objective, the losses from over-allocating to low types are magnified and losses from over-inspecting and under-allocating post-inspection are suppressed.

This paper is related to a branch of the mechanism design literature devoted to costly inspection with no transfers. In a seminal paper, Ben-Porath, Dekel and Lipman (2014) analyse a model of N perfectly informed agents and show that inspection is used to check the highest reported type above a threshold, providing an adequate incentive for low types not to masquerade. Inspection is never used to inform the principal of additional information, a feature that seems to be present in each of our examples. This paper shows that imperfect information recovers this quality, and demonstrates how to model and analyse this feature.

The next section of this paper will further detail this literature in order to provide context for the results presented. Subsequently, the environment and a benchmark for comparison is described in detail (section 3), followed by a proof of the main results on how inspection is used to acquire information (section 4). Finally, the main analysis is concluded with how the results extend to environments where commitment is relaxed (section 5) and a demonstration of when this mechanism prevails over non-preferential, or *pooling*, mechanisms (section 6)

2 Related literature

Verification in mechanism design has received widespread interest, notably studied in Green and Laffont (1986). As mentioned, Ben-Porath, Dekel and Lipman (2014) consider perfectly informed agents and overcome the difficulty of studying a general mechanism design problem without transfers to characterize an optimal policy. The concern about applying this result, however, is that inspection is only used for verification, and this paper broadens the scope of inspection by modelling imperfectly informed agents. The extensions of Mylovanov and Zapechelnyuk (2017), Epitropou and Vohra (2019) and Erlanson and Kleiner (2020) explore alternative timing and actions, though in the single agent environment presented here, they are primarily the same.

There is a sparse, but growing, literature concerning mechanisms with imperfect verification. Pereyra and Silva (2021) is the closest in their treatment, but model scarce allocation, costless and a noisy inspection technology, and primarily focus on efficient mechanisms. Ball and Kattwinkel (2019) explore private information and verification, but consider effort choice and consider transfers. An extension of the results provided here can be applied to noisy verification, however, we will focus on imperfectly informed agents.

There is a conceptual difference between *verifying* an agent's report and *inspecting* their inherent qualities, and this is distinguished in this paper. The literature on scoring rules considers this distinction and, as in the seminal work of McCarthy (1956) and Savage (1971), outlines mechanisms that elicit truthful reports concerning noisy signals.[1] This, though, does not account for the incentives to collect this information and the cost in implementing these scoring rules. The work done here can then be seen as an extension of scoring rules to allocation environments without transfers.

An application of this exercise concerns judicial mechanism design. Silva (2019*a*), and Siegel and Strulovici (2021) explore these topics with similar ideas of noisy inspection, but their models are less general, tailored to the application and vary along the dimensions of costs, transfers, and incentives.

Settings with evidence, disclosure, and audits have a long tradition in mechanism design, from Townsend (1979), to Border and Sobel (1987) and Mookherjee and Png (1989). These primarily study optimal mechanisms with transfers. The contribution of Alaei et al. (2020) unites many of their features, with deferred inspection and payments. Recent work has also looked at environments with limited transfers such as Mylovanov and Zapechelnyuk (2017), Silva (2019*b*), Li (2021). These papers, and their solutions, can then be directly compared to the optimal solution presented here to understand what qualitative changes occur when transfers are prohibited.

3 Environment

A *principal* receives an unknown, real return, R, from allocating to an *agent*. If the agent is allocated to, they receive a payoff of 1, and 0 otherwise. Prior to allocation, the principal may *inspect* the agent at a cost to their final payoff, $c > 0$, and in return learn the true value of the reward, r. The agent has their own private information about the prize, a signal s, which defines their *type*.

Here, we are considering one principal and one agent. An equivalent setup is a single principal with k objects to allocate, among $\ell \leq k$ agents, each of whom has unit demand, the same preferences for each object, and an independent signal of the principal's reward. The problem where $\ell > k$ is left for future research.

The agent has a strict preference to be allocated to, and their payoffs are normalized around this. Changing the intensity of this preference, and even making this intensity type-dependent, makes no difference to the analysis so long as we maintain the strict preference for allocation. As such, this normalization is without loss of generality. It will of course matter when interpreting

[1] For an extensive treatment, see Gneiting and Raftery (2007).

particular applications and extensions, but this is left to the responsibility of the reader.

The principal has the ability to commit to a rule that determines what they do after any report and any subsequent realization of the reward. Following the main analysis, we will explore how the environment changes when we relax this ability, as outlined in section 5. One can then think of the full commitment setting as the most informative case study in understanding this environment.

Direct transfers of value between the principal and agent are *not* permitted. This reflects the observation that transfers are seldom used for direct disclosure of information in practice. A place where we may see transfers occurring is bargaining between the principal and agent once the principal decides to allocate. We can then interpret this restriction as the principal not being able to commit to altering later stage bargaining outcomes prior to making their allocation decision. The model then claims that the principal's reward, and the agent's preference to be allocated to, reflects the expected outcome of this ensuing bargaining game.

An alternative interpretation is that this is an environment where bargaining does not occur, and there are many settings where this is true. In some allocation problems, such as the assignment of public housing, the use of money is seen as counterproductive. In others, there are commitments from one or both of the parties not to use or accept transfers. This could be in the form of an institutional restriction, like that which occurs in many forms of governance that involve public offices. And finally, in some, the entire bargaining power over the value that can be shared may belong to just one of the parties, and so bargaining does not occur in any real sense.

Even if the restriction isn't true of an application, it is still important to understand what inspection offers the principal in isolation of transfers. This may be for predictive and prescriptive analysis, or building an understanding of the ways in which we can gather information in different settings.

The timing and structure of the game is fixed and common knowledge.

1. The principal commits to an inspection and allocation rule; nature assigns signals according to a commonly known generating process.

2. The agent observes their signal (type), s, and submits a report to the principal.

3. The principal implements their rule conditional on the report and any subsequent inspection realizations that are generated by the rule.

4. All remaining uncertainty is resolved, and net payoffs are awarded.

The primary question to address is which inspection and allocation rule the principal should select in order to maximize their expected return, subject to the agent's incentives to report.

This report in practice could be a lengthy and complicated message. Given our objective here is to study the resulting outcomes and, at most, the total information exchanged by this message, however, we will instead work with the *direct mechanism* by appealing to the revelation principle (Myerson, 1981). That is, we restrict attention to the message game where the agent directly reports their type, and require that the agent's expected return from reporting truthfully is weakly greater than that of reporting any other type.

Listing the principal's available actions conditional on report s, let:

- $x(s)$ be the inspection probability,
- $y(s)$ be the allocation probability without inspection, and
- $z(s, r)$ be the allocation probability after inspection and realizing reward r.

To fix language, let's refer to x as the *inspection rule*, y as the *pre-inspection allocation*, and z as the *post-inspection allocation*. Together, (x, y, z) constitute the principal's *mechanism*, and this mechanism is *feasible* if:

$$x(s) \in [0,1],\ y(s) \in [0,1],\ z(s,r) \in [0,1] \quad \forall\ s, r$$

Further, let a *policy* be a particular mechanism that is proposed to maximize some program.

Here we are endowing the principal with the ability to ration the object they're allocating. In the hiring example, this can be thought of as limiting the hours the employee works, and in the funding examples, partially funding the applicant's project. If the object is indivisible, partial allocation can be thought of as the outcome of a lottery over the object. The decision to inspect may also be partial, in which case the lottery interpretation is natural, but could also be a further rationing.[2] It is the lottery interpretation of each action that we'll allude to throughout the paper to fix language. What we are restricting here is that allocation can be at most one, representing the capacity constraint on the principal's allocation.

The principal chooses this mechanism to maximize their ex ante expected return. At the interim stage, after learning s, their payoff is determined by two events. They may allocate without inspecting, receiving the expected return given the signal, and this occurs with the probability that they don't inspect, $(1 - x(s))$, and that they do allocate, $y(s)$. Alternatively, they may allocate after inspecting and learning the return, r, receiving this net of the inspection cost, c, which, conditional on r being the true reward, occurs with the probability they do inspect, $x(s)$,

[2] You would then have to claim that inspecting an agent for part of the object, is only partially costly.

and they do allocate, $z(s,r)$. Let v_s refer to the principal's interim payoff given s.

$$v_s(x,y,z) \coloneqq (1-x(s))y(s)\mathbb{E}(r|s) + x(s)(\mathbb{E}(z(s,r)\cdot r|s) - c)$$

Their ex ante expected return is their expected interim payoff, which we can refer to as the *objective*, v.

$$v(x,y,z) \coloneqq \mathbb{E}_s\left[(1-x(s))y(s)\mathbb{E}(r|s) + x(s)(\mathbb{E}(z(s,r)\cdot r|s) - c)\right]$$

Optimizing involves maximizing the objective subject to the agent's incentives to report their private information truthfully. That is, an agent of type s must receive as high a payoff from reporting s than any other type, $\hat{s}$. Given the normalization, the agent's payoff is the likelihood of being allocated to by the mechanism, and occurs with the net probability of the two events outlined above. Let u_s be the payoff for type s and $u_{s,\hat{s}}$ be the payoff from s reporting $\hat{s}$, so that $u_{s,s} = u_s$.

$$u_s(x,y,z) \coloneqq (1-x(s))y(s) + x(s)\mathbb{E}(z(s,r)|s) \geq (1-x(\hat{s}))y(\hat{s}) + x(\hat{s})\mathbb{E}(z(\hat{s},r)|s) \eqqcolon u_{s,\hat{s}}(x,y,z) \quad \forall\, \hat{s}$$

This *incentive compatibility* constraint for the pair $(s,\hat{s})$ is labelled $IC_{s,\hat{s}}$ for reference. Note that the agent's type only augments their return directly through determining the distribution of r and thus the likelihood of being allocated to conditional on inspection. Also note that given their return is the net probability of being allocated to, there is no need for an individual rationality constraint as all type's receive a weakly positive return.

In total, the mathematical program that the principal solves is:

$$\begin{aligned}
\max_{(x,y,z)} \quad & \mathbb{E}_s\left[(1-x(s))y(s)\mathbb{E}(r|s) + x(s)(\mathbb{E}(z(s,r)\cdot r|s) - c)\right] \\
\text{s.t.} \quad & IC_{s,\hat{s}} : (1-x(s))y(s) + x(s)\mathbb{E}(z(s,r)|s) \geq (1-x(\hat{s}))y(\hat{s}) + x(\hat{s})\mathbb{E}(z(\hat{s},r)|s) \quad \forall\, \hat{s} \quad \forall\, s \\
& F_{s,r} : x(s) \in [0,1],\ y(s) \in [0,1],\ z(s,r) \in [0,1] \quad \forall\, r \quad \forall\, s
\end{aligned}$$

3.1 Signals

In practice, the agent's private information could be complicated and nuanced. For a job applicant, this includes, among many other factors, educational performance, feedback from old colleagues, career ambitions, and observations about the firm's prior hiring decisions. As we are primarily interested in the informational content of this information, let us collapse this into a single parameter and ask what characteristics we'd like this to have.

Let the agent's information be represented by a private signal, $s \in \{s_0, s_1, \ldots, s_N\}$ and suppose $s = s_n$ with probability $p_n \in (0,1)$, so that $\sum_n p_n = 1$. Denote P_n as the cumulative mass function, so that $P_n = \sum_{m \leq n} p_m$.

This paper adopts a discrete formulation of signals for exposition alone. The use of discrete signals allows us to think clearly about the different incentives that are important to the problem, and with proposition 1, the main result is extended to an appropriate limiting environment via Helly's selection theorem. One could instead conduct this exercise entirely with continuous types, and it would not provide any substantive additional insights.

Signals are informative of the principal's allocation reward. If $s = s_n$, let the reward that the principal receives from allocating to the agent is a random variable $R|s_n \sim \Pi_n$ where Π_n is absolutely continuous and admits a density function π_n. Denote the unconditional distribution of the reward R by Π and assume that it is also absolutely continuous, admits a density function π and has support $\mathcal{R} = [\underline{r}, \overline{r}] \subseteq \mathbb{R}$. Finally, assume that each of these distributions has a finite mean.

Here, signals have been introduced first and rewards second, but the order in which nature selects rewards and signals is unimportant as long as the information the players have about these at each stage remains the same. That is, you could think of the agent as having an underlying reward, and receiving a signal about this reward which they then report to the principal, or as receiving as signal and when then they are inspected, or when the game concludes, a reward is generated given the signal.

The signals have an order conducive to analysis and reflective that higher signals are more *favourable* than lower signals. Specifically, we will impose that the signals are completely ordered by the *monotone likelihood ratio property* (MLRP). That is:

$$\frac{\pi_n(r_1)}{\pi_m(r_1)} \geq \frac{\pi_n(r_0)}{\pi_m(r_0)} \quad \text{for all } r_1 > r_0 \text{ and } n > m$$

That is, higher signals generate higher rewards relatively more likely than lower signals. In this sense, an agent with a higher signal is more favourable to the principal, and is a notion outlined by Milgrom (1981) and has been widely adopted since. This is a reasonable characteristic we'd expect of the types of private information we're studying. While the information may be noisy and difficult to communicate, we'd expect agents to be broadly aware of and agree on which information the principal values more highly than others.

Another popular ordering is that of *first order stochastic dominance* (FOSD). A visual demonstration of the differences between MLRP and FOSD, and the reason for selecting the former, is highlighted in section 4.1.1. For now, note that MLRP is a stronger notion than FOSD.

Claim 1 *If the signals are completely ordered by the monotone likelihood ratio property, they are* *ordered by first order stochastic dominance. That is:*

$$\Pi_n(r) \leq \Pi_m(r) \quad \textit{for all } r \textit{ and } n > m$$

Proof: If a higher signal generates higher rewards relatively more likely than lower signals, then it must be true on average for rewards greater than any fixed reward, $\hat{r}$, and the reverse must be true for rewards lower than the fixed reward. Then, $1 - \Pi_n(\hat{r}) \geq 1 - \Pi_m(\hat{r})$ for $n > m$. A complete proof is provided in the Appendix. ■

Given MLRP, it is convenient and unambiguous to relabel the signals by their induced expected reward, so that $s_n = \mathbb{E}(r|s_n)$. As such, our information parameter now has a neat interpretation, and we will call on this interpretation where helpful. We can now state the principal's problem more explicitly.

Principal's problem:
In return for a report of the agent's signal, s_n, the principal may inspect the agent, x_n, allocate to the agent without inspecting, y_n, or allocate to the agent after inspecting and observing r, $z_{n,r}$.

$$\begin{aligned} \max_{(x,y,z)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r|s_n) + x_n (\int r z_{n,r} \pi_{n,r} \, dr - c)] p_n \\ \text{s.t.} \quad & IC_{n,m} : (1 - x_n) y_n + x_n (\int z_{n,r} \pi_{n,r} \, dr) \geq (1 - x_m) y_m + x_m (\int z_{m,r} \pi_{n,r} \, dr) \quad \forall \, n, m \\ & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall \, r \quad \forall \, n \end{aligned}$$

For ease of notation, let:

- $\psi_n(z) \coloneqq \int r z_r \pi_{n,r} \, dr - c$ as the expected net reward for the principal from inspecting given an arbitrary post-inspection allocation rule z, and
- $\phi_n(z) \coloneqq \int z_r \pi_{n,r} \, dr$ as the expected allocation to the agent from being inspected given an arbitrary post-inspection allocation rule z.

3.2 Symmetric information

As a benchmark, consider a problem where the principal has full access to the agent's private information, whose solution we refer to as the *first-best* policy.

$$\begin{aligned} \max_{(x,y,z)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r|s_n) + x_n \psi_n(z_n)] p_n \\ \text{s.t.} \quad & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall \, r \quad \forall \, n \end{aligned}$$

Given the agent has nothing additional to report - information is *symmetric* - the agent has no strategically relevant actions. This problem is then straightforward to optimize and outlines exactly what preference the principal has for treating each type of agent. This provides a basis for which to measure the losses associated with the dispersion of information from the principal's perspective.

The next claim outlines the first-best policy and, as with all claims, theorems and propositions in this paper, a sketch of the proof is provided in the main body and a full proof in the appendix. Let $\mathbb{1}\{\mathcal{Q}\}$ be the indicator function that is equal to 1 if the statement $\mathcal{Q}$ is true given the arguments, and 0 otherwise.[3]

CLAIM 2 *The first-best policy* (x_n^*, y_n^*, z_n^*) *is given by:*

- $z_{n,r}^* = \mathbb{1}\{r \geq 0\}$,
- $y_n^* = \mathbb{1}\{\mathbb{E}(r|s_n) \geq 0\}$, *and*
- $x_n^* = \mathbb{1}\{\psi_n(z_n^*) \geq \max\{\mathbb{E}(r|s_n), 0\}\}$.

Proof: Conditional on the decision to inspect, x_n, $z_{n,r}$ selects when to allocate post-inspection and should then be maximized when $r \geq 0$ and minimized otherwise. Call this the *ideal* post-inspection allocation rule. Similarly, conditional on the decision not to inspect, $1 - x_n$, y_n selects when to allocate pre-inspection without additional information and should then be maximized when $\mathbb{E}(r|s_n) \geq 0$ and minimized otherwise. Finally, x_n selects when to inspect, and should be maximized when the expected net reward for the principal from inspecting given the ideal post-inspection allocation rule is both greater than outright allocating or outright rejecting. ■

There are only three relevant policy combinations of the first-best policy.

1. *no allocation*, **N**, given by $x_n = 0, y_n = 0$,
2. *ideal inspection*, **I**, given by $x_n = 1, z_{n,r} = \mathbb{1}\{r \geq 0\}$, and
3. *full allocation*, **A**, given by $x_n = 0, y_n = 1$.

Let $\psi_n^* \coloneqq \psi_n(z_{n,r}^*)$ be the expected return from ideal inspection. Preferences over these policies have a fixed order with respect to s_n due to the FOSD ordering of the signals.

CLAIM 3 *There exists some* s_α *and* s_β*, with* $s_\alpha \leq s_\beta$*, such that:*

- *if* $s_n \leq s_\alpha$ *then* $0 \geq \max\{\psi_n^*, \mathbb{E}(r|s_n)\}$,
- *if* $s_n \in (s_\alpha, s_\beta)$ *then* $\psi_n^* > \max\{0, \mathbb{E}(r|s_n)\}$, *and*
- *if* $s_n \geq s_\beta$ *then* $\mathbb{E}(r|s_n) \geq \max\{\psi_n^*, 0\}$.

[3] The standard definition of an indicator function is $\mathbb{1}_A(x) \coloneqq 1$ if $x \in A$ and 0 if $x \notin A$. We're more interested in the set A and less in the argument x, so suppress the argument and promote the set.

Proof: By FOSD, the expected return from ideal inspection is increasing in the signal as the cost is fixed and the likelihood of a positive reward is increasing. As such, ψ_n^* has a single crossing with 0, after which ideal inspection is preferred to no allocation. Label the corresponding crossing signal as s_α if it's negative and 0 otherwise. The rate at which ψ_n^* increases is less than $\mathbb{E}(r|s_n)$ increases as the informativeness of the signal must also increase, eventually rendering inspection as little more informative than the signal itself. As such, ψ_n^* and $\mathbb{E}(r|s_n)$ also have a single crossing, after which ideal inspection is less preferred to full allocation. Label the corresponding crossing signal as s_β if it's positive and 0 otherwise. ■

This says that there are three regions of interest with respect to the principal's preferences: *low* signals, who the principal would like to reject outright and not allocate to, *intermediate* signals, who the principal would like to inspect and allocate if they are shown to yield positive returns, and *high* signals, who the principal would like to allocate to outright and save on inspection costs. As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the upper envelope of ψ_n^*, $\mathbb{E}(r|s_n)$ and 0 gives us the first-best objective as a function of the signal, as shown in Figure 1, and the corresponding first-best policy as a function of the signal, as shown in Figure 2.

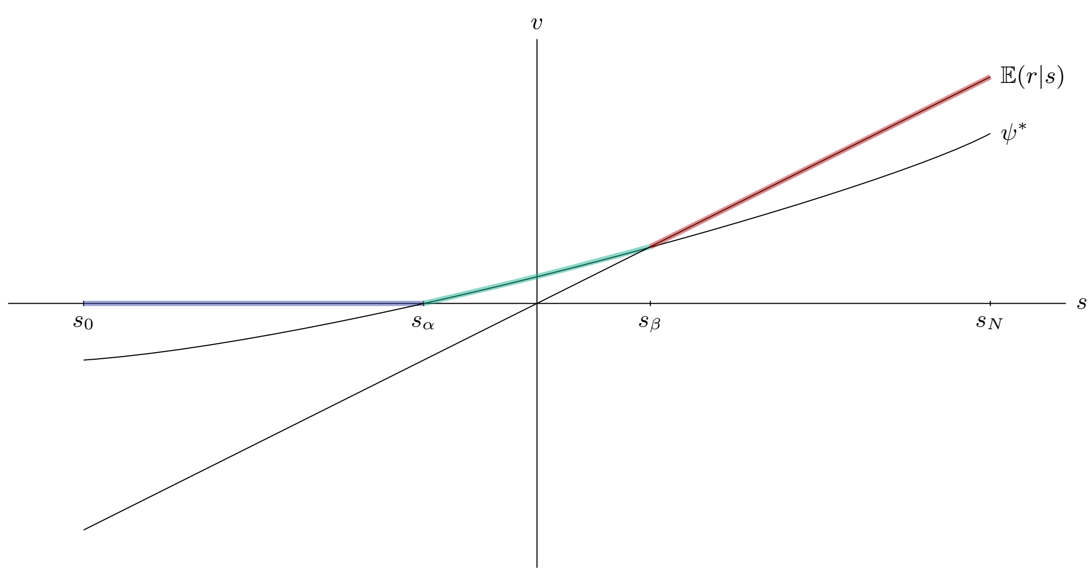

Figure 1: first-best objective, v^*

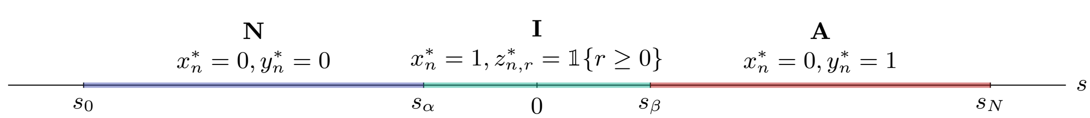

Figure 2: first-best policy, (x^*, y^*, z^*)

If the return from inspection is high enough for all signals, then $\psi_n^* \geq \max\{0, \mathbb{E}(r|s_n)\}$ for all n

and so $s_\alpha < s_0$ and $s_\beta > s_N$. For example, if $c = 0$ then it's trivially the case that inspection is always optimal. Looking forward to the problem where information is privately held, the first-best policy is then incentive compatible too, as all signals are treated equally. As such, we will continue under the assumption that either $s_\alpha > s_0$ or $s_\beta < s_N$.

4 Acquiring information

Now let's return to the problem where the signal is privately held. We will refer to the mechanism that optimizes this problem as the *second-best* policy, in contrast to the symmetric information benchmark. Instead of solving the principal's problem directly, we will solve a relaxation that only requires $IC_{n,m}$ to hold for $m = n + 1$. These constraints are referred to as the *upward local incentive compatibility* (ULIC) constraints.

Relaxed problem:

$$\begin{aligned}
\max_{(x,y,z)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r|s_n) + x_n \psi_n(z_n)] p_n \\
\text{s.t.} \quad & IC_{n,n+1} : (1 - x_n) y_n + x_n \psi_n(z_n) \geq (1 - x_{n+1}) y_{n+1} + x_{n+1} \psi_n(z_{n+1}) \quad \forall\ n < N \\
& F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n
\end{aligned}$$

As we will discover, it is these constraints that matter. The first-best policy tells us that the principal would like to preferentially treat higher type agents, given the favourableness of their information. So we should expect that, under an optimal policy, agent's will primarily have an incentive to lie upwards - claim they have a higher signal in order to receive that preferential treatment - rather than lie downwards. Further, the agent with the strongest incentive to falsely claim they have a particular type is the agent whose signal is the closest to that type. This is because the policy is designed for each type to report truthfully, so it will also be attractive to those that have a similar distribution of rewards.

If the solution to this problem also satisfies the omitted constraints, then it must be an optimal solution to the principal's problem. Proceeding with this relaxation shows us that post-inspection allocations are threshold rules, each of these constraints bind, and the inspection rule itself is a threshold rule. This pins down a second-best policy which indeed satisfies the omitted constraints.

4.1 Threshold post-inspection allocation

To derive the optimal policy for this problem, we will first show that the post-inspection allocations are not only deterministic but only allocate when the realized reward is high.

Claim 4 *Optimal post-inspection allocations are threshold rules. That is, for each n there exists some τ_n such that:*

$$z_{n,r} = \mathbb{1}\{r \geq \tau_n\}$$

Proof: Suppose (x, y, z) is incentive compatible, optimal, and that for some n, z_n is not a threshold rule. Define τ_n such that:

$$\int z_{n,r}\pi_{n,r}\, dr = \int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r}\, dr$$

Consider a new policy which replaces z_n with this threshold rule about τ_n. Clearly this is incentive compatible for n as it's defined such that they receive the same likelihood of allocation given they are inspected as before. That is, $IC_{n,n+1}$ continues to hold.

If $n > 0$, consider $IC_{n-1,n}$:

$$(1 - x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r}\, dr) \geq (1 - x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r}\, dr)$$

This must continue to hold as the transformation of z_n shifts allocation weight away from low rewards and towards high rewards such that s_n is fully compensated. As s_n and s_{n-1} are ordered by MLRP, this compensation is not enough for s_{n-1} to also remain indifferent, and thus the transformation is weakly dominated by the original. As such, the transformation reduces the right-hand side of $IC_{n-1,n}$ which implies that if the initial policy is incentive compatible for s_{n-1}, then the new policy is too.

Finally, the new policy must generate a higher return for the principal, given we've shifted allocation weight from low values of r to high values of r evaluated under the same distribution, Π_n. This implies (x, y, z) cannot have been optimal, a contradiction of the proposition. ■

This says that we can restrict our attention to post-inspection allocations that are *threshold rules*: allocate post-inspection if and only if the reward exceeds some threshold. The next example demonstrates the proof graphically and highlights the role of MLRP over FOSD.

4.1.1 Example: FOSD or MLRP

Consider Figure 3. The top function, z_n is an example of a post-inspection allocation that we would like to transform into a threshold rule. The middle function, π_n, is the reward distribution under the signal that the principal targets with z_n and is normalized to a uniform distribution. Finally, the bottom function, π_{n-1}, is the reward distribution of a lower signal that we're protecting from deviating upwards.

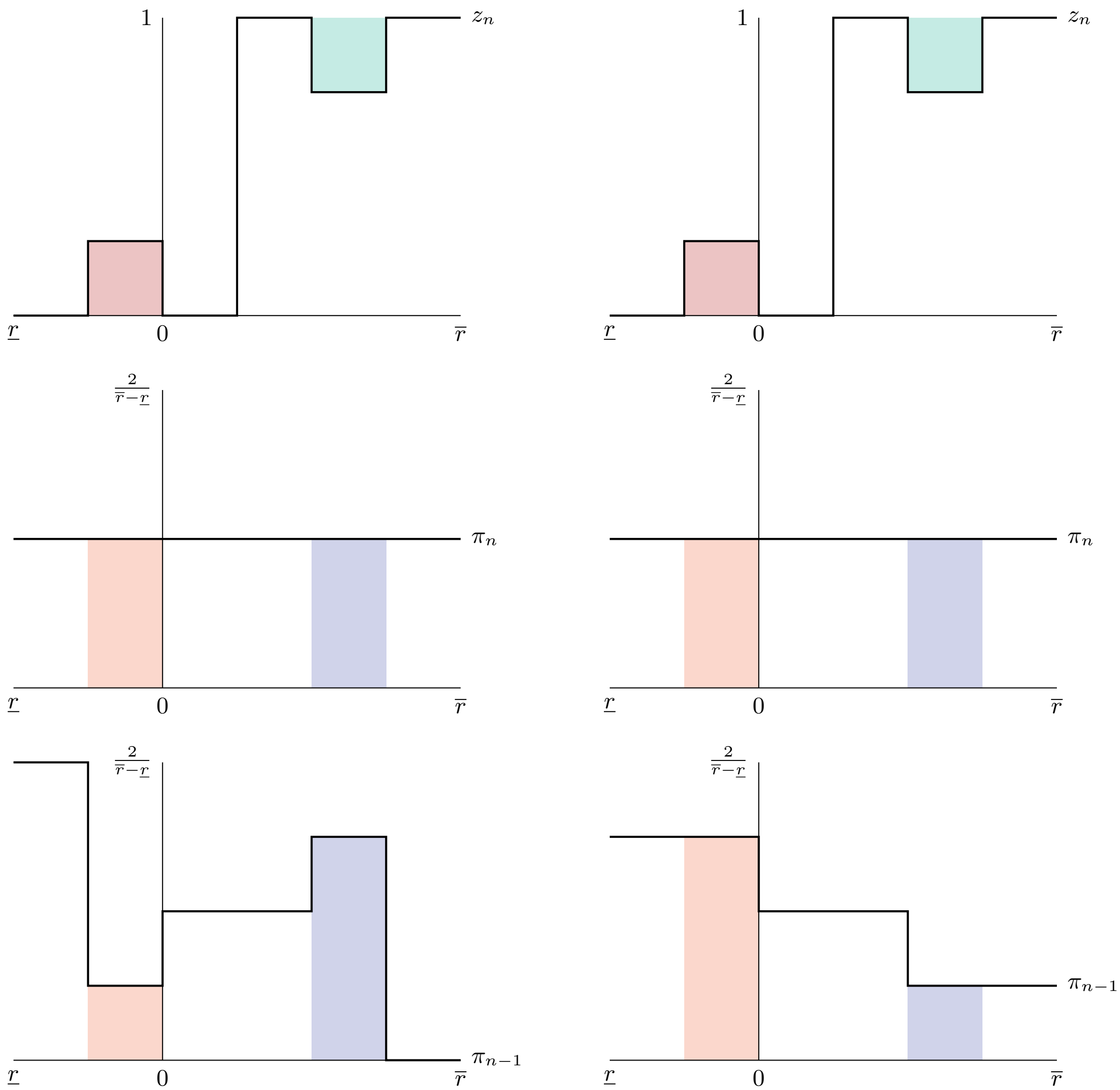

Figure 3: FOSD (left and right) is not enough for the transformation, but MLRP (right only) is.

There are two examples of π_{n-1}: the left, a distribution that is first order stochastically dominated by π_n but does not follow the monotone likelihood ratio property, and the right, a distribution that is both first order stochastically dominated by π_n and follows the monotone likelihood ratio property. Both follow the right mass property given by FOSD: better signals place a greater likelihood on generating rewards that are higher than any fixed threshold. But only the right follows MLRP: the relative likelihood of rewards (relative to the higher signal's likelihood, π_n) is decreasing in the reward.

Transforming z_n involves shifting the red mass up to the green deficit. This is done in a way to make s_n indifferent between the rules, and in our example s_n assigns the same probability to these two events: the orange likelihood and the blue likelihood respectfully.

For this transformation to continue to satisfy s_{n-1}'s incentive constraint, without any additional restrictions on (x, y, z), we must guarantee that deviating to s_n's new policy leaves them no better off than had they deviated prior under the initial policy. FOSD is not enough to guarantee this because lower signals may still put a higher relative likelihood of greater rewards than they had for smaller rewards without violating the total right-mass property given by FOSD. In the left distribution, s_{n-1} puts a low (orange) likelihood on the event that r falls in the range where the principal is reducing the likelihood of allocation, and a high (blue) likelihood on the event where allocation is now being guaranteed.

MLRP prevents this, as the relative likelihood must be decreasing for higher values of r. As such, in the right distribution the orange likelihood is greater than the blue likelihood, and so on net makes s_{n-1} worse off under the deviation.

The takeaway from this is that FOSD is not enough to guarantee, at this stage, that post-inspection allocations must be threshold rules. Instead, some regulation on the shape of the distribution must be made, and in this paper that's the MLRP.

Claim 4 provides us with some additional structure to the functions ψ and ϕ, which help to proceed with determining the optimal policy. In a slight abuse of notation, we will now refer to post-inspection allocation, z_n, by the threshold, τ_n, that defines it, so that $\psi_n(\tau_n)$ and $\phi_n(\tau_n)$, for example, represent the respective expected values under the threshold rule defined by τ_n. Note that $\phi_n(\tau_n)$ is now just the right tail mass above τ_n.

4.2 Binding incentive compatibility

With this characteristic, we can see through variation arguments that each upward local incentive compatibility constraint must hold with equality in any optimal solution. For ease of language, we'll say that these constraints *bind* but do not mean to imply that they have a positive shadow price. Treating all types equally will, in some environments, be optimal and then for some types incentive compatibility is free due to feasibility.

Claim 5 *In any optimal mechanism, every upward local incentive compatibility constraint binds. That is, for all $n < N$, $IC_{n,n+1}$ binds:*

$$(1 - x_n)y_n + x_n\phi_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1})$$

Proof: Take a feasible, incentive compatible policy, (x, y, z), and suppose that under this policy, a particular constraint does not bind. To show this cannot be the optimal policy, we need to find a feasible alternative that is feasible, incentive compatible and give's the principal a higher expected return. As such, we need to know exactly which policy changes improve the objective.

Clearly, raising τ_n if $\tau_n < 0$, or lowering τ_n if $\tau_n > 0$, would constitute such an improvement for any s_n, as this involves allocating to less negative reward agents and allocating to more positive reward agents. Changing x_n and y_n, however, depends on the agent's signal.

For a fixed vector of thresholds, τ, consider the following partition of the signal space:

1. $S_0 \coloneqq \{n \mid 0 \geq \mathbb{E}(r|s_n),\ 0 \geq \psi_n(\tau_n)\}$
2. $S_\alpha \coloneqq \{n \mid 0 \geq \mathbb{E}(r|s_n),\ \psi_n(\tau_n) > 0\}$
3. $S_\beta \coloneqq \{n \mid \mathbb{E}(r|s_n) > 0,\ \psi_n(\tau_n) > \mathbb{E}(r|s_n)\}$
4. $S_1 \coloneqq \{n \mid \mathbb{E}(r|s_n) > 0,\ \mathbb{E}(r|s_n) \geq \psi_n(\tau_n)\}$

This outlines the ideal policy choice given τ: no allocation if $n \in S_0$, inspect if $n \in S_\alpha \cup S_\beta$, and allocate without inspection if $n \in S_1$. If $\tau_n = 0$ for each n, this partition is as described by the first-best policy and displayed in Figure 2. If $\tau_n \neq 0$ then $\psi_n(\tau_n) \leq \psi_n^*$ as the principal is either over or under allocating conditional on the realized reward. As such, the S_0 and S_1 are supersets of their first-best counterparts, while $S_\alpha \cup S_\beta$ is a subset.

Claim 5.1 in the appendix shows that $IC_{n,n+1}$ must bind if $n \in S_0$ or $n+1 \in S_1$. If $n \in S_0$, then reducing y_n and x_n improves the objective as the principal would rather not allocate, and this tightens $IC_{n,n+1}$ while only relaxing $IC_{n-1,n}$. Then, either $IC_{n,n+1}$ binds, or $u_n = 0$ ($x_n = 0$ and $y_n = 0$) and $IC_{n,n+1}$ must trivially bind as $u_{n,n+1} \geq 0$. Similarly, if $n+1 \in S_1$ then expanding y_{n+1} and reducing x_{n+1} improves the objective as the principal would rather allocate outright, and tightens $IC_{n,n+1}$ while only relaxing $IC_{n+1,n+2}$. Then, either $u_n < 1$ but $IC_{n,n+1}$ binds, or $u_n = 1$ ($x_n = 0$ and $y_n = 1$) and $IC_{n,n+1}$ must trivially bind as $u_n \leq 1$.

Claims 5.2, 5.3 and 5.4 in the appendix show that $IC_{n,n+1}$ must bind if $n, n+1 \in S_\alpha \cup S_\beta$. Similar to the previous argument, reducing y_n if $n \in S_\alpha$, or expanding y_{n+1} if $n+1 \in S_\beta$, must improve the objective as the principal would rather not allocate than allocate unconditionally if $n \in S_\alpha$, and vice versa if $n+1 \in S_\beta$. This tightens $IC_{n,n+1}$ while only relaxing the adjacent constraints. Further, if $\tau_n < 0$, or $\tau_{n+1} > 0$, we can improve the objective by raising τ_n, or lowering τ_{n+1}, as this represents rejecting more negative rewards for τ_n and accepting more positive rewards for τ_{n+1}. This tightens $IC_{n,n+1}$ and relaxes the adjacent constraints as the allocation probability is decreasing in τ. Finally, expanding x_{n+1} if $n \in S_\alpha$, or expanding x_n if $n \in S_\beta$, must increase the objective as these are types the principal wants to inspect, and we can show that given $\tau_n \geq 0 \geq \tau_{n+1}$ by the previous argument, this tightens $IC_{n,n+1}$ and relaxes adjacent constraints. By exhausting these adjustments, it cannot be that $IC_{n,n+1}$ holds but does not bind.

Now, we already know $S_0 < S_1$ and $S_\alpha < S_\beta$ as the value of $\mathbb{E}(r|s_n)$ is given by the signal structure and not the policy.[4] Then, the only two types of constraints that haven't been checked are: $n \in S_\alpha$ and $n+1 \in S_0$, and $n \in S_1$ and $n+1 \in S_\beta$.

Claims 5.5 and 5.6 in the appendix show that $IC_{n,n+1}$ must bind if either $n \in S_\alpha$ and $n+1 \in S_0$, or $n \in S_1$ and $n+1 \in S_\beta$. As in claims 5.2 and 5.3, reducing y_n if $n \in S_\alpha$, expanding y_{n+1} if $n+1 \in S_\beta$, raising τ_n if $\tau_n < 0$, and lowering τ_{n+1} if $\tau_{n+1} > 0$, must all improve the objective, tighten $IC_{n,n+1}$ and relax adjacent constraints. Unlike in the previous claims, however, expanding x_{n+1} if $n+1 \in S_0$, and expanding x_n if $n \in S_1$, would make the principal worse off. Instead, we can expand inspection in a way that improves the principal's return by replicating the inspection threshold τ_n if $n \in S_\alpha$, or τ_{n+1} if $n+1 \in S_\beta$. As inspection of these agents using these thresholds is preferable to the principal, it must also be preferable for types $n+1 \in S_0$ and $n \in S_1$ respectfully. As before, this will tighten $IC_{n,n+1}$ and relax adjacent constraints. Finally, by exhausting these adjustments, it cannot be that $IC_{n,n+1}$ holds but does not bind.

Then, by claims 5.1 through 5.6, $IC_{n,n+1}$ must bind for all n. ■

This says that in any optimal mechanism, for any n, the expected allocation to s_n cannot strictly exceed their allocation had they reported s_{n+1} instead. If it does, the principal could do better by varying the mechanism to tighten this constraint. This then implies the IC constraints form a chain of equality conditions from u_0 to u_N, reducing our problem to a simple mathematical program.

4.3 Threshold inspection rules

Given claims 4 and 5, we can rewrite our problem as:

$$\begin{aligned}
\max_{(x,y,\tau)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n\psi_n(\tau_n)]p_n \\
\text{s.t.} \quad & IC_{n,n+1} : (1-x_n)y_n + x_n\phi_n(\tau_n) = (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \quad \forall\, n < N \\
& F : 0 \leq x_n, y_n, \tau_n \leq 1 \quad \forall\, r \quad \forall\, n
\end{aligned}$$

With this, we can conclude that the optimal inspection rule, x, is also a threshold rule and that the inspected agents face the same post-inspection allocation.

Claim 6 *Optimal inspection rules are threshold mechanisms, and the post-inspection allocation is identical for all agents. That is, there exists some ν and τ such that $x_n = \mathbb{1}\{n > \nu\}$ and $\tau_n = \tau$ for all n.*

[4]Here, $<$ refers to the *below* set relation defined by: set A is below set B, $A < B$, if $\forall a \in A$ and $\forall b \in B$, $a < b$.

Proof: First observe that:

$$
\begin{aligned}
u_n &= u_{n,n+1} \\
u_n &= (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \\
u_n &= (1-x_{n+1})y_{n+1} + x_{n+1}\phi_{n+1}(\tau_{n+1}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) + x_n\phi_n(\tau_{n+1}) \\
u_n &= u_{n+1} - x_{n+1}[\phi_{n+1}(\tau_{n+1}) - \phi_n(\tau_{n+1})]
\end{aligned}
$$

As $\phi_{n+1}(\tau_{n+1}) > \phi_n(\tau_{n+1})$ by FOSD, this says that the likelihood of allocation is increasing in s_n and at a rate determined by the inspection rule, x, and the post-inspection allocation threshold, τ.

Note that these two are jointly determined. That is, a choice of x restricts the choice of τ. For example, suppose for some pair $n_0 < n_1$, $x_{n_0} = 1$, $x_{n_1} = 1$, and $x_m = 0$ for $n_0 < m < n_l$. Then by the binding constraint τ_{n_0} and τ_{n_1} must satisfy:

$$\phi_{n_0}(\tau_{n_0}) = \phi_{n_1-1}(\tau_{n_1})$$

If $n_0 = n_1 - 1$ then $\tau_{n_0} = \tau_{n_1}$, and if $n_0 < n_1 - 1$, then $\tau_{n_0} > \tau_{n_1}$ and uniquely determined.

Substituting the arrangement into the objective function we find that, for a fixed y_N and τ the objective is linear in x_n, whose only restriction is that $x_n \in [0,1]$:

$$
\begin{aligned}
\max_{(x_n,\tau_n),y_N} (1-x_N)y_N\mathbb{E}(r) \quad &+ \quad x_N[\phi_{N-1}(\tau_N)\mathbb{E}(r|s \le s_{N-1})P_{N-1} + \psi_N(\tau_N)p_N] \\
&+ \sum_{n=1}^{N-1} x_n[\phi_{n-1}(\tau_n)\mathbb{E}(r|s \le s_{n-1})P_{n-1} - \phi_n(\tau_n)\mathbb{E}(r|s \le s_n)P_n + \psi_n(\tau_n)p_n] \\
&+ \quad x_0[-\phi_0(\tau_0)\mathbb{E}(r|s_0)p_0 + \psi_0(\tau_0)p_0]
\end{aligned}
$$

Let a_n be the coefficient on x_n in the objective, and observe that a_n is only a function of τ_n. We can immediately conclude:

- $x_n = \mathbb{1}\{a_n(\tau_n) \ge 0\}$, and
- $y_N = \mathbb{1}\{\mathbb{E}(r) \ge 0\}$.

This means the restrictions on τ in the previous example are the only relevant restriction to our problem, and as such $x_n = \mathbb{1}\{n > \nu\}$ for some $\nu \in \{-1, 0, \ldots, N\}$ and $\tau_n = \tau$ for all n. ■

4.4 Optimal separating mechanism

Following directly from claim 6, we can now state the main theorem. This, and subsequent results, concern only the optimal separating mechanism, with the pooling mechanisms compared in section 6.

Theorem 1 *The second-best policy* $(x^\star, y^\star, z^\star)$ *is given by:*

- $x_n^\star = \mathbb{1}\{s \geq s_{n_0}\}$,
- $y_n^\star = \phi_{\nu^\star}(\tau^\star)$, *and*
- $z_{n,r}^\star = \mathbb{1}\{r \geq \tau^\star\}$,

where $\nu^\star$ *and* $\tau^\star$ *are the solution to:*

$$\max_{\nu,\tau} \sum_{n=\nu+1}^{N} \psi_n(\tau) p_n + \phi_\nu(\tau) \mathbb{E}(r | s \leq s_\nu) P_\nu$$

Proof: As demonstrated by claim 6, this policy is the solution to the relaxed problem. As such, we are only left to check that this solution satisfies the omitted IC constraints. Note that, for any ν and τ, $u_n = \phi_\nu(\tau)$ for all $n \leq \nu$, and $u_n = \phi_n(\tau)$ for all $n > \nu$. As the marginal signal, ν, is indifferent between the two treatments, by FOSD, all $n < \nu$ must strictly prefer the pre-inspection allocation, and all $n > \nu$ must strictly prefer inspection. As such, the global IC constraints are satisfied. ■

This gives us a simple maximization problem to solve for our two cut-offs with respect to the endowed grid of signals, and can be solved using a linear search algorithm. To get more intuition on what determines these cut-offs, however, consider a continuous extension of this problem. If we take the limit of the environment to a continuos grid, we find the following optimal mechanism.

Proposition 1 *The second-best policy* $(x^\star, y^\star, z^\star)$ *is given by:*

- $x_s^\star = \mathbb{1}\{s \geq s_\gamma\}$,
- $y_s^\star = \phi_{\gamma^\star}(\tau^\star)$, *and*
- $z_{s,r}^\star = \mathbb{1}\{r \geq \tau^\star\}$,

where $\gamma^\star$ *and* $\tau^\star$ *are the solution to:*

$$\phi_\gamma(\tau)\mathbb{E}(r|s_\gamma) = \psi_\gamma(\tau) \quad \textit{and} \quad \tau = \left[\frac{\pi_{s_\gamma,\tau} P_{s_\gamma}}{\int_{s_\gamma}^{s_N} \pi_{s,\tau} p_s \, ds}\right] (-\mathbb{E}(r|s \leq s_\gamma))$$

Proof: By Helly's selection theorem, a uniformly bounded sequence of monotone real functions admits a convergent subsequence. Given the policy of theorem is defined by bounded, monotone real functions, the limiting policy is also an optima of the continuous problem. Finally, the conditions on γ and τ are derived using standard calculus arguments. ■

As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the second-best objective as a function of the signal as shown in Figure 4, and a second-best policy as a function of the signal as shown in Figure 5.

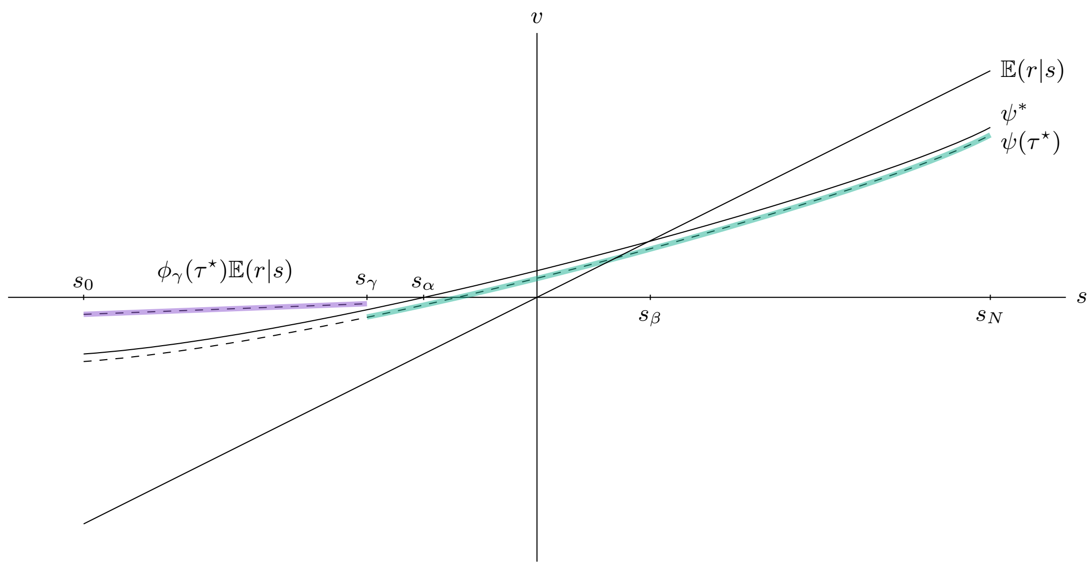

Figure 4: second-best objective, $v^\star$

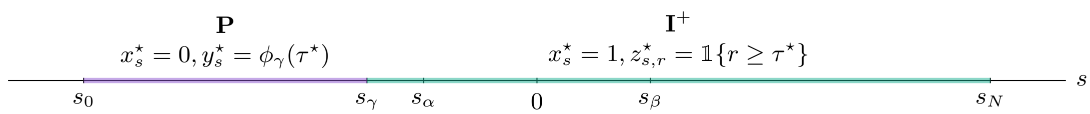

Figure 5: second-best policy, $(x^\star, y^\star, z^\star)$

Comparing the first-best and second-best objectives, we can characterize the losses for the principal associated with private information in terms of the agent's signal.

Proposition 2 *The first-best objective exceeds the second-best objective for all signals. That is,*

- *for* $s \in [s_0, s_\gamma]$, $v_s^* = 0 \geq \phi_\gamma(\tau^\star) = v_s^\star$,
- *for* $s \in (s_\gamma, s_\beta)$, $v_s^* = \psi^* \geq \psi(\tau^\star) = v_s^\star$, *and*
- *for* $s \in [s_\beta, s_N]$, $v_s^* = \mathbb{E}(r|s) \geq \psi(\tau^\star) = v_s^\star$.

This says, that are four types of losses for the principal introduced by the agent's private information:

- *Over-allocation at the bottom*: agents who's private information would be sufficient to reject without inspection, $s \in [s_0, s_\gamma]$, are allocated to with positive probability to elicit truthful reports.
- *Over-inspection at the bottom*: agents who's private information would be marginally sufficient to reject without inspection, $s \in (s_\gamma, s_\alpha)$, are inspected to reduce pre-inspection allocation to lower types.

- *Over-inspection at the top*: agents who's private information would be sufficient to guarantee allocation, $s \in (s_\beta, s_N]$, are inspected in order to elicit truthful reports of lower agents.
- *Under-allocation post inspection*: agents who are inspected, $s \in (s_\gamma, s_N]$, and who generate a marginally positive reward, $r \in [0, \tau^\star)$, are rejected in order to reduce pre-inspection allocation to lower types.

When these losses are less than those associated with the pooling policies, separation is optimal for the principal.

5 Relaxing commitment

There are three natural relaxations to full commitment in this environment.

1. **pre-inspection commitment**, or *partial commitment*: the principal can commit to a pre-inspection allocation, y, and an inspection rule, x, but cannot commit to a post-inspection allocation, z,
2. **pre-assessment commitment**, or *limited commitment*: the principal can commit to a pre-inspection allocation, y, but cannot commit to an inspection rule, x, or a post-inspection allocation, z, and
3. **no commitment**: the principal has no commitment at all, that is they cannot commit to any policy, (x, y, z).

Under no commitment, the principal can only choose between the pooling equilibria, as reports convey no information. This is referred to as the *third-best* policy - the policy that optimizes the principal's objective when they can't elicit, or don't have access to, any additional information. We know what this looks like and will return to this in section 6. For now, let's concider the first two relaxations.

5.1 Pre-inspection commitment

Proposition 3 *The second-best policy $(x^\star, y^\star, z^\star)$ under partial commitment is given by:*

- $x^\star_s = \mathbb{1}\{s \geq s_\delta\}$,
- $y^\star_s = \phi^*_\delta$, *and*
- $z^\star_{s,r} = z^*_{s,r} = \mathbb{1}\{r \geq 0\}$,

where δ is the solution to $\phi^*_\delta \mathbb{E}(r|s_\delta) = \psi^*_\delta$.

As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the second-best objective as a function of the signal as shown in Figure 6, and a second-best policy as a function of the signal as shown in Figure 7.

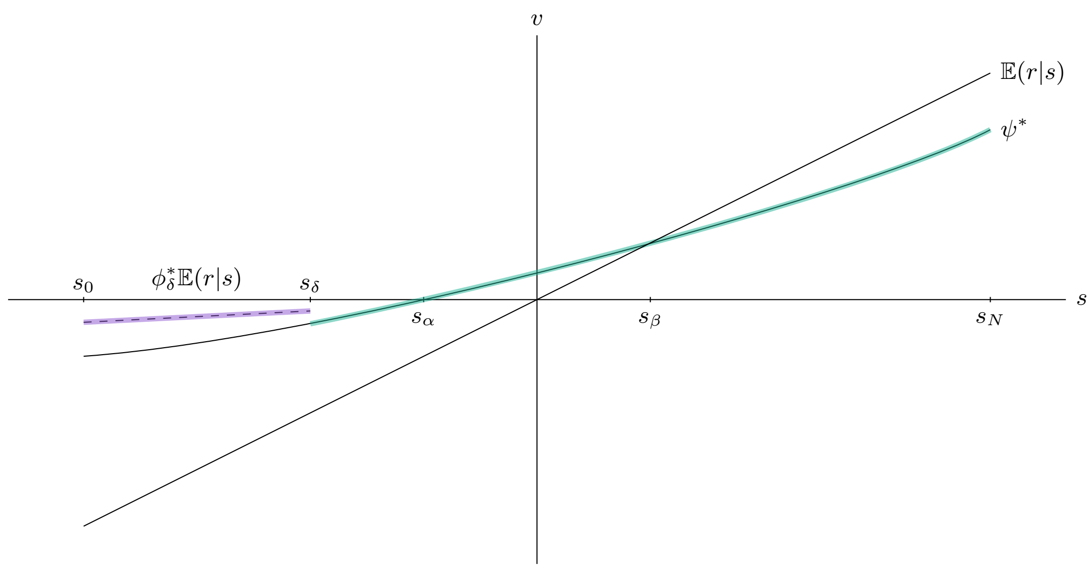

Figure 6: second-best objective under partial commitment, $v^\star$

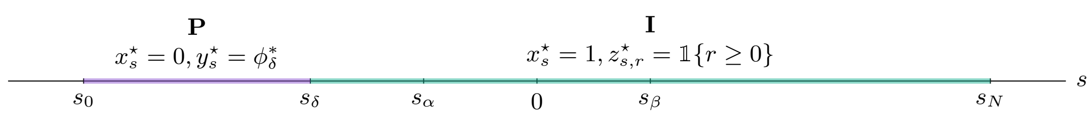

Figure 7: second-best policy under partial commitment, $(x^\star, y^\star, z^\star)$

5.2 Pre-assessment commitment

Proposition 4 *The second-best policy $(x^\star, y^\star, z^\star)$ under limited commitment is given by:*

- $x_s^\star = \mathbb{1}\{s \geq s_\alpha\}$,
- $y_s^\star = \phi_\alpha^*$, *and*
- $z_{s,r}^\star = z_{s,r}^* = \mathbb{1}\{r \geq 0\}$,

where α is the solution to $\psi_\alpha^* = 0$.

As such, if $s_\alpha \in (s_0, 0)$ and $s_\beta \in (0, s_N)$ then tracing out the second-best objective as a function of the signal as shown in Figure 8, and a second-best policy as a function of the signal as shown in Figure 9.

6 Comparative statics

Our attention so far has been directed at the optimal separating mechanism. We may then want to know under what conditions this mechanism is optimal against mechanisms that don't treat

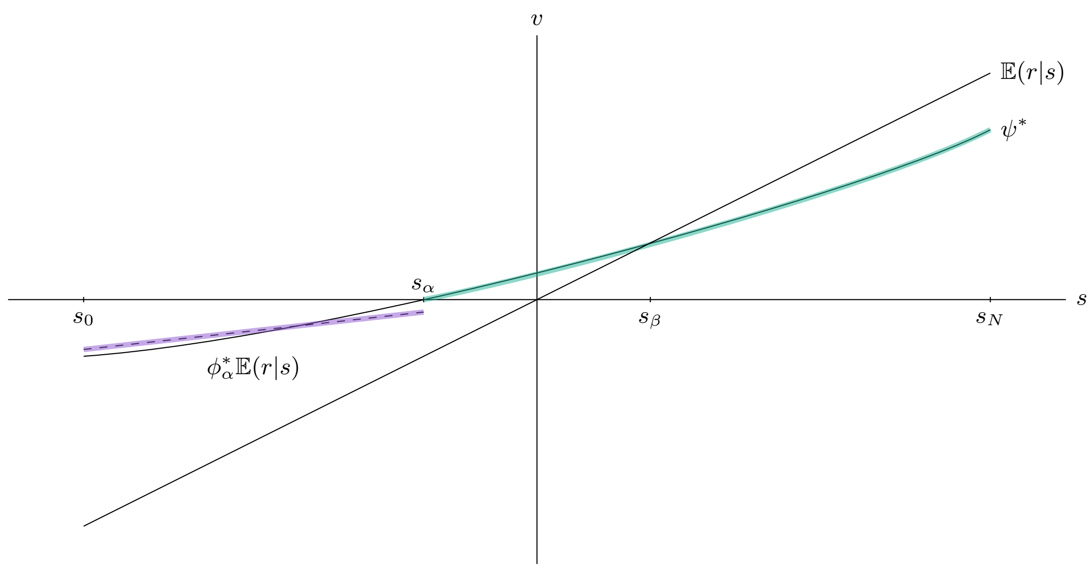

Figure 8: second-best objective under limited commitment, $v^\star$

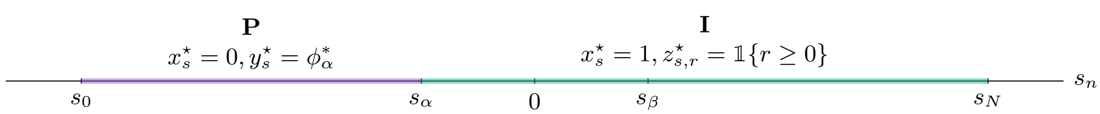

Figure 9: second-best policy under limited commitment, $(x^\star, y^\star, z^\star)$

agents differentially - *pooling* mechanisms. We may also want to know how the separating mechanism itself change under changes to the environment, an exercise known as *comparative statics*.

Let us consider the *Gaussian* environment, which provides enough structure on the setting that we can conduct this exercise in full. As demonstrated in proposition 1, extending these results to a continuous signal space poses no issue so long as the prior and posterior distributions are consistent. The normal distribution in particular satisfies MLRP if the variance is fixed across signals, and so lends itself to providing a clean example for particular comparative statics.

Suppose the prior over the rewards, Π, is given by: $r \sim N(\mu, 1)$, and the agent receives a signal of their reward determined by: $\hat{s} = r + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$. This implies the distribution of signals, $\hat{P}$, is given by: $\hat{s} \sim N(\mu, \sigma^2 + 1)$. Together this generates a posterior distribution of rewards, $\Pi_{\hat{s}}$, that's given by: $r|\hat{s} \sim N(s, \hat{\sigma}^2)$ where:

$$s = \frac{\sigma^2}{\sigma^2 + 1}\left[\mu + \frac{\hat{s}}{\sigma^2}\right] \quad \text{and} \quad \hat{\sigma}^2 = \frac{\sigma^2}{\sigma^2 + 1}$$

It is without loss to relabel the signal $\hat{s}$ by its induced expected value s, which defines our induced distribution of signals, P, given by: $s \sim N(\mu, \frac{1}{\sigma^2+1})$. Finally, define the precision, α, of the signal as $\alpha := 1/\sigma^2$.

Then, the environment is defined by the triple (μ, α, c) where:

- μ is the ex-ante expected reward of allocating to an agent,
- α is the precision of the agent's signal of the reward, and
- c is the inspection cost to the principal.

For each combination of (μ, α, c), it's straightforward to evaluate the four competiting mechanisms the principal entertains.

- *no allocation*, $\mathbf{N}$, given by $x_s = 0, y_s = 0$ for all s,
- *ideal inspection*, $\mathbf{I}$, given by $x_s = 1, z_{s,r} = \mathbb{1}\{r \geq 0\}$ for all s,
- *full allocation*, $\mathbf{A}$, given by $x_s = 0, y_s = 1$ for all s, and
- *seperation*, $\mathbf{S}$, given by proposition 1.

As a baseline, consider the problem where the principal can only use their prior information, whose solution is referred to as the *third-best policy*. Then, the principal has to select, and is the same problem as the no commitment relaxation. For a fixed, reasonable c, this policy is plotted in figure 10 as a function of α and μ.

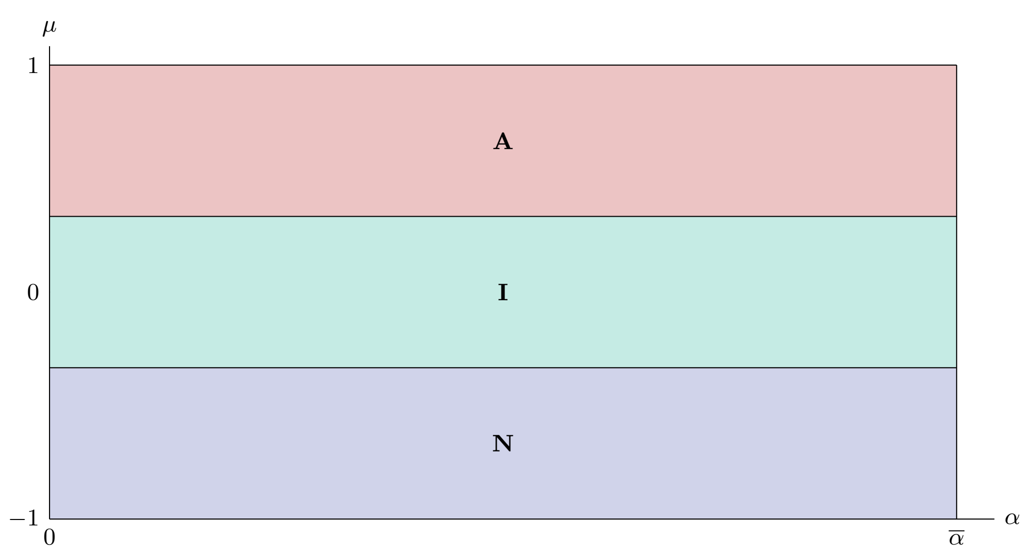

Figure 10: third-best policy as a function of precision, α, and prior mean, μ

In the third-best, the principal cannot condition on the signal, so a change in the signal precision does not change the optimal policy. Changing the prior mean, however, changes the expected return from allocating and the expected return from inspecting.

Now, we can compare this directly with our second-best policy, plotted in figure 11 as a function of α and μ.

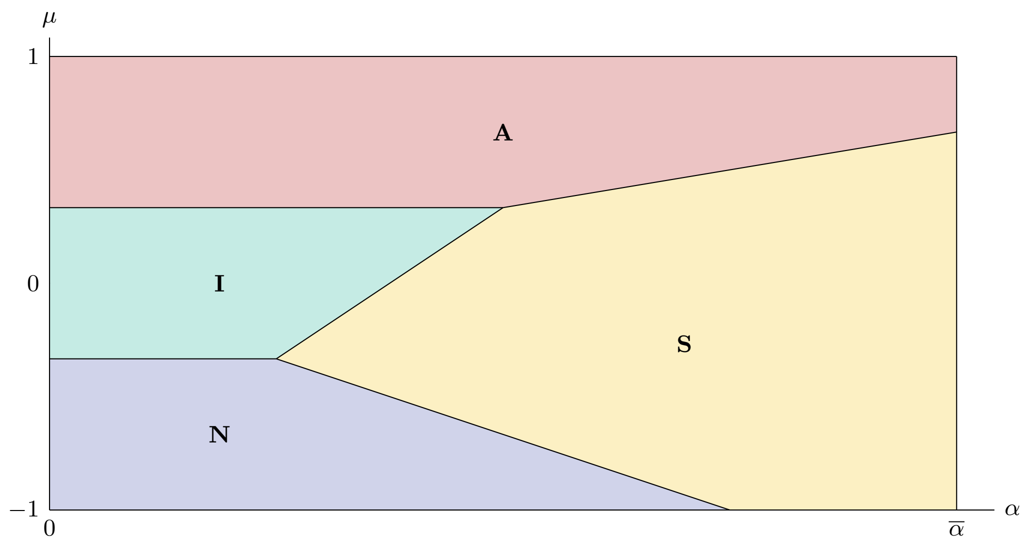

Figure 11: second-best policy as a function of precision, α, and prior mean, μ

7 Summary

In this paper, we've seen that when information is noisy and privately held, costly efforts to acquire this information involve both verification and discovery. This concerns activities such as employer interviews, project assessments and investment evaluation. In these settings, information is useful for an allocation decision that the holders of the private information value, and necessarily, decisions regarding inspection and allocation account for these incentives.

To maximize their expected return, the principal segments signals into two groups, *high* and *low*. Agents with high signals are always inspected and only allocated post-inspection if the reward is sufficiently valuable. Agents with low signals are never inspected, compensated for their report with a small probability of unconditional allocation. This demonstrates that inspection has two purposes in the absence of transfers: the verification of private information by separating signals, and the discovery of additional information by conditioning allocation upon the result of an inspection.

We've discovered the conditions under which this mechanism is optimal, and shown that if these conditions are not met, the principal can only select from non-preferential, or *pooling*, mechanisms: either universally inspecting, unconditionally allocating or blanket rejecting. We've also seen how this extends to many reasonable relaxations of the principal's ability to commitment, and explored the cost that this comes at.

The framework provided here can be extended to many additional problems for economic analysis. Immediately, analogous results with noise on the inspection process can be derived, showing that the principal employs a similar mechanism to discipline reports in an environment with noisy verification. One could also extend the analysis to various meta-games. For example, including a prior selection of signal precision, or allowing the principal to impose an application cost on the agents, requires no additional modelling assumptions and is a straightforward application of this paper's contributions.

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A Environment

One might consider two orderings on the signal space that capture the important features of private information for an economic setting such as this.

A1. The signals are ordered by **first order stochastic dominance (FOSD)**: $\Pi_n(r) \leq \Pi_m(r)$ for all r if $n > m$.

A2. The signals bear the **monotone likelihood ratio property (MLRP)**: $\pi_n(r_1)/\pi_m(r_1) \geq \pi_n(r_0)/\pi_m(r_0)$ for all $r_1 > r_0$ and $n > m$.

Claim 1 *A2 implies A1.*

Proof: Rearranging the condition in A2 for some $r_1 > r_0$ and $n > m$:

$$\pi_n(r_1)\pi_m(r_0) \geq \pi_n(r_0)\pi_m(r_1)$$

We can integrate this expression up to r_1 with respect to r_0 to get:

$$\int^{r_1} \pi_n(r_1)\pi_m(r_0)dr_0 \geq \int^{r_1} \pi_n(r_0)\pi_m(r_1)dr_0$$
$$\pi_n(r_1)\Pi_m(r_1) \geq \Pi_n(r_1)\pi_m(r_1)$$
$$\frac{\pi_n(r_1)}{\pi_m(r_1)} \geq \frac{\Pi_n(r_1)}{\Pi_m(r_1)}$$

Similarly, we can also integrate the original expression down to r_0 with respect to r_1 to get:

$$\int_{r_0} \pi_n(r_1)\pi_m(r_0)dr_1 \geq \int_{r_0} \pi_n(r_0)\pi_m(r_1)dr_1$$
$$(1 - \Pi_n(r_0))\pi_m(r_0) \geq \pi_n(r_0)(1 - \Pi_m(r_0))$$
$$\frac{1 - \Pi_n(r_0)}{1 - \Pi_m(r_0)} \geq \frac{\pi_n(r_0)}{\pi_m(r_0)}$$

Combining and rearranging these last two expressions for any particular $r = r_0 = r_1$ gives us A1:

$$\Pi_m(r) \geq \Pi_n(r)$$

■

A.1 Symmetric information

If the solution to the problem,

$$\begin{aligned}
\max_{(x_n,y_n,z_n)} \quad & \sum_n [(1-x_n)y_n\mathbb{E}(r|s_n) + x_n(\int r z_{n,r}\pi_{n,r}\,dr - c)]p_n \\
\text{s.t.} \quad & F: 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\, r \quad \forall\, n
\end{aligned}$$

is the *first-best* policy, then:

CLAIM 2 *The first-best policy* (x_n^*, y_n^*, z_n^*) *is given by:*

- $z_{n,r}^* = \mathbb{1}\{r \geq 0\}$,
- $y_n^* = \mathbb{1}\{\mathbb{E}(r|s_n) \geq 0\}$, *and*
- $x_n^* = \mathbb{1}\{\psi_n(z_n^*) \geq \max\{\mathbb{E}(r|s_n), 0\}\}$.

Proof: If $z_{n,r} < 1$ for some $r > 0$ then increasing $z_{n,r}$ weakly increases the objective function, and if $z_{n,r} > 0$ for some $r < 0$ then decreasing $z_{n,r}$ weakly increases the objective function. Similarly if $y_n < 1$ for some $\mathbb{E}(r|s_n) > 0$ then increasing y_n weakly increases the objective function, and if $y_n > 0$ for some $\mathbb{E}(r|s_n) < 0$ then decreasing y_n weakly increases the objective function. Then, the only weakly unimprovable policies are $z_{n,r} = \mathbb{1}\{r \geq 0\}$ and $y_n = \mathbb{1}\{\mathbb{E}(r|s_n) \geq 0\}$. Given this, the objective is linear in x_n and so the maximum is obtained by selecting the larger coefficient: setting $x_n = 1$ when $\psi_n(z_n^*) \geq \max\{\mathbb{E}(r|s_n), 0\}$, and 0 otherwise. ■

And there are only three relevant policy combinations of this policy with a fixed order with respect to s_n:

CLAIM 3 *There exists some* s_α *and* s_β*, with* $s_\alpha \leq s_\beta$*, such that:*

- *if* $s_n \leq s_\alpha$ *then* $0 \geq \max\{\psi_n^*, \mathbb{E}(r|s_n)\}$,
- *if* $s_n \in (s_\alpha, s_\beta)$ *then* $\psi_n^* > \max\{0, \mathbb{E}(r|s_n)\}$, *and*
- *if* $s_n \geq s_\beta$ *then* $\mathbb{E}(r|s_n) \geq \max\{\psi_n^*, 0\}$.

Proof: We will prove this claim by constructing thresholds $\tilde{s}_\alpha$ and $\tilde{s}_\beta$ and adjusting them to match the succinct claim.

By FOSD, ψ_n^* is increasing in n, as c is fixed and the likelihood that $r > 0$ is increasing. As such there exists a $\tilde{s}_\alpha$ such that $\psi_n^* > 0$ if $s_n > \tilde{s}_\alpha$. Note that trivially we can set $\tilde{s}_\alpha$ as any value less than s_0 if $\psi_0^* > 0$ and any value greater than s_N if $\psi_N^* < 0$.

Additionally, there exists a $\tilde{s}_\beta$ such that $\psi_n^* < \mathbb{E}(r|s_n)$ if $s_n > \tilde{s}_\beta$. To see this, observe that the following are equivalent:

$$\begin{aligned} \psi_n^* &< \mathbb{E}(r|s_n) \\ \int \mathbb{1}\{r \geq 0\} r\pi_{n,r}\, dr - c &< \int r\pi_{n,r}\, dr \\ -c &< \int \mathbb{1}\{r < 0\} r\pi_{n,r}\, dr \end{aligned}$$

and as $\mathbb{1}\{r < 0\} \cdot r$ is increasing in r, the right hand side is increasing in n by FOSD. As before we can set $\tilde{s}_\beta$ as any value less than s_0 if $\psi_0^* < \mathbb{E}(r|s_0)$ and any value greater than s_N if $\psi_N^* > \mathbb{E}(r|s_N)$.

Finally by definition $s_n = \mathbb{E}(r|s_n)$ and so $\mathbb{E}(r|s_n) > 0$ when $s_n > 0$. This means that $\tilde{s}_\alpha$ is only policy relevant when less than 0 and $\tilde{s}_\beta$ when greater than zero. As such, define $s_\alpha = \min\{\tilde{s}_\alpha, 0\}$ and $s_\beta = \max\{\tilde{s}_\beta, 0\}$. ■

B Acquiring information

Claims 4, 5, and 6 relate to the following relaxation.

Relaxed problem:

$$\begin{aligned} \max_{(x_n, y_n, z_n)} \quad & \sum_n [(1 - x_n) y_n \mathbb{E}(r|s_n) + x_n (\int r z_{n,r} \pi_{n,r}\, dr - c)] p_n \\ \text{s.t.} \quad & IC_{n,n+1} : (1 - x_n) y_n + x_n (\int z_{n,r} \pi_{n,r}\, dr) \geq (1 - x_{n+1}) y_{n+1} + x_{n+1} (\int z_{n+1,r} \pi_{n,r}\, dr) \quad \forall\ n < N \\ & F : 0 \leq x_n, y_n, z_{n,r} \leq 1 \quad \forall\ r \quad \forall\ n \end{aligned}$$

B.1 Threshold post-inspection rules

Claim 4 *Optimal post-inspection allocations are threshold rules. That is, for each n there exists some τ_n such that:*

$$z_{n,r} = \mathbb{1}\{r \geq \tau_n\}$$

Proof: Suppose (x, y, z) is incentive compatible, optimal, and that for some n, z_n is not a threshold rule. Define τ_n such that:

$$\int z_{n,r} \pi_{n,r}\, dr = \int \mathbb{1}\{r \geq \tau_n\} \pi_{n,r}\, dr$$

Given Π_n is absolutely continuous, τ_n is well-defined.

Consider a new policy which replaces z_n with this threshold post-inspection rule about τ_n. Clearly this is incentive compatible for n as it's defined such that they receive the same likelihood of allocation given they are inspected as before. That is, $IC_{n,n+1}$ continues to hold.

Now consider $IC_{n-1,n}$:

$$(1-x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r}\,dr) \geq (1-x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r}\,dr)$$

We'd like to show this continues to hold under the new policy. That is:

$$(1-x_{n-1})y_{n-1} + x_{n-1}(\int z_{n-1,r}\pi_{n-1,r}\,dr) \geq (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr)$$

Note that we can rewrite the right-hand side of the original constraint by decomposing $z_{n,r}$ into the threshold rule and the residual that would reconstitute $z_{n,r}$:

$$\begin{aligned}
&(1-x_n)y_n + x_n(\int z_{n,r}\pi_{n-1,r}\,dr) \\
&= (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr) + x_n(\int^{\tau_n} z_{n,r}\pi_{n-1,r}\,dr - \int_{\tau_n}(1-z_{n,r})\pi_{n-1,r}\,dr)
\end{aligned}$$

By MLRP, $\pi_{n-1,r} \geq \pi_{n,r}\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}$ if $r < \tau_n$ and $\pi_{n-1,r} \leq \pi_{n,r}\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}$ if $r > \tau_n$. As such, the right-hand side of $IC_{n-1,n}$ must be,

$$\begin{aligned}
&\geq (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr) + x_n\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}(\int^{\tau_n} z_{n,r}\pi_{n,r}\,dr - \int_{\tau_n}(1-z_{n,r})\pi_{n,r}\,dr) \\
&= (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr) + x_n\frac{\pi_{n-1,\tau_n}}{\pi_{n,\tau_n}}(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n,r}\,dr - \int z_{n,r}\pi_{n,r}\,dr) \\
&= (1-x_n)y_n + x_n(\int \mathbb{1}\{r \geq \tau_n\}\pi_{n-1,r}\,dr)
\end{aligned}$$

Where the final equality comes from the definition of τ_n. As such, if the initial policy is incentive compatible for $n-1$, then the new policy is also incentive compatible.

Finally, the new policy must generate a higher return for the principal, given we've shifted allocation weight from low values of r to high values of r evaluated under the same Π_n. Another way of showing this is that $\mathbb{1}\{r \geq \tau_n\}\Pi_n$ stochastically dominates $z_n\Pi_n$ and the principal evaluates an increasing function, r, with respect to these censored distributions. This implies (x, y, z) cannot have been optimal, a contradiction of the proposition. ■

B.2 Binding incentive compatibility

CLAIM 5 *In any optimal mechanism, every upward local incentive compatibility constraint binds. That is, for all* $n < N$, $IC_{n,n+1}$ *binds:*

$$(1 - x_n)y_n + x_n\phi_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1})$$

Proof: Consider the following partition of the signal space:

1. $S_0 := \{n \mid 0 \geq \mathbb{E}(r|s_n),\ 0 \geq \psi_n(\tau_n)\}$
2. $S_\alpha := \{n \mid 0 \geq \mathbb{E}(r|s_n),\ \psi_n(\tau_n) > 0\}$
3. $S_\beta := \{n \mid \mathbb{E}(r|s_n) > 0,\ \psi_n(\tau_n) > \mathbb{E}(r|s_n)\}$
4. $S_1 := \{n \mid \mathbb{E}(r|s_n) > 0,\ \mathbb{E}(r|s_n) \geq \psi_n(\tau_n)\}$

In each of the following arguments, the approach is the same: suppose a particular type of constraint does not bind for a solution (x, y, z) and find a feasible policy improvement, contradicting the optimality of the proposed solution.

CLAIM 5.1 *If* $n \in S_0$ *or* $n + 1 \in S_1$, $IC_{n,n+1}$ *binds.*

Proof: Suppose $n \in S_0$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing y_n and x_n improves the objective as $0 \geq \mathbb{E}(r|s_n)$ and $0 \geq \psi_n(\tau_n)$, and decreases the left-hand side so tightens $IC_{n,n+1}$. Further, this only relaxes $IC_{n-1,n}$.

$\therefore$ Either $u_n = 0$ ($x_n = 0$ and $y_n = 0$), or $IC_{n,n+1}$ binds. Note that if $u_n = 0$ then $IC_{n,n+1}$ trivially binds as $u_{n,n+1} \geq 0$.

Now consider $n + 1 \in S_1$ and suppose $IC_{n,n+1}$ doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then expanding y_{n+1} and reducing x_{n+1} improves the objective as $\mathbb{E}(r|s_{n+1}) > 0$ and $\mathbb{E}(r|s_{n+1}) \geq \psi_{n+1}(\tau_{n+1})$, and increases the right-hand side so tightens $IC_{n,n+1}$. Further, this only relaxes $IC_{n+1,n+2}$.

$\therefore$ Either $u_{n+1} = 1$ ($x_{n+1} = 0$ and $y_{n+1} = 1$) or $IC_{n,n+1}$ binds. Note that if $u_{n+1} = 1$, then $IC_{n,n+1}$ trivially binds as $u_n \leq 1$. □

Claim 5.2 *If $n, n+1 \in S_\alpha$, $IC_{n,n+1}$ binds.*

Proof: Suppose $n, n+1 \in S_\alpha$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then reducing y_n improves the objective as $\psi_n(\tau_n) \geq 0 \geq \mathbb{E}(r|s_n)$, and decreases the left-hand side, so tightens $IC_{n,n+1}$ and only relaxes $IC_{n-1,n}$.

Suppose $y_n = 0$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Suppose $\tau_n < 0$, then raising τ_n will improve the objective, decrease the left-hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n-1,n}$. Similarly, suppose $\tau_{n+1} > 0$, then lowering τ_{n+1} will improve the objective, increase the right-hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n+1,n+2}$.

Suppose $y_n = 0$, $\tau_n \geq 0$ and $\tau_{n+1} \leq 0$ and $IC_{n,n+1}$ doesn't bind. Consider expanding x_{n+1}, which would improve the objective as $\psi_{n+1}(\tau_{n+1}) > 0 \geq \mathbb{E}(r|s_{n+1})$. This tightens $IC_{n,n+1}$ if:

$$0 < -y_{n+1} + \phi_n(\tau_{n+1}) \quad \Rightarrow \quad \phi_n(\tau_{n+1}) > y_{n+1}$$

Note that if this is true, then it also relaxes $IC_{n+1,n+2}$ as $\phi_{n+1}(\tau_{n+1}) \geq \phi_n(\tau_{n+1})$ by FOSD.

Suppose by contradiction $\phi_n(\tau_{n+1}) \leq y_{n+1}$, and $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = x_n\phi_n(\tau_n) &> (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ &\geq (1-x_{n+1})\phi_n(\tau_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ &= \phi_n(\tau_{n+1}) \end{aligned}$$

But as $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$ and $\phi_n(\tau_{n+1}) \geq \phi_n(0)$, this is a contradiction. As such, expanding x_{n+1} must tighten $IC_{n,n+1}$.

Finally, suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_{n+1} = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > \phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore$ $IC_{n,n+1}$ binds. $\square$

Claim 5.3 *If $n, n+1 \in S_\beta$, $IC_{n,n+1}$ binds.*

Proof: Suppose $n, n+1 \in S_\beta$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then expanding y_{n+1} improves the objective as $\psi_{n+1}(\tau_{n+1}) \geq 0 \geq \mathbb{E}(r|s_{n+1})$, and increases the right-hand side, so tightens $IC_{n,n+1}$ and only relaxes $IC_{n+1,n+2}$.

Suppose $y_{n+1} = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Suppose $\tau_n < 0$, then raising τ_n will improve the objective, decrease the left-hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n-1,n}$. Similarly, suppose $\tau_{n+1} > 0$, then lowering τ_{n+1} will improve the objective, increase the right-hand side and so tighten $IC_{n,n+1}$ and only relax $IC_{n+1,n+2}$.

Suppose $y_{n+1} = 1$, $\tau_n \geq 0$ and $\tau_{n+1} \leq 0$ and $IC_{n,n+1}$ doesn't bind. Consider expanding x_n, which would improve the objective as $\psi_n(\tau_n) > \mathbb{E}(r|s_n) > 0$. This tightens $IC_{n,n+1}$ if:

$$-y_n + \phi_n(\tau_n) < 0 \quad \Rightarrow \quad \phi_n(\tau_n) < y_n$$

Note that if this is true, then it also relaxes $IC_{n-1,n}$ as $\phi_{n-1}(\tau_n) \leq \phi_n(\tau_n)$ by FOSD.

Suppose by contradiction $\phi_n(\tau_n) \geq y_n$, and $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = (1 - x_n)y_n + x_n\phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ (1 - x_n)\phi_n(\tau_n) + x_n\phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ \phi_n(\tau_n) &> (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \end{aligned}$$

But, $\phi_n(\tau_n) \leq \phi_n(0)$ and $(1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$, a contradiction. As such, expanding x_n tightens $IC_{n,n+1}$.

Finally, suppose $y_{n+1} = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_n = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = \phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$ binds. $\square$

So far, we've shown constraints within the sets are binding, as well as those leading from S_0 or those leading to S_1. Now we're left to check constraints across the inspection sets.

CLAIM 5.4 *If* $n \in S_\alpha$ *and* $n+1 \in S_\beta$, $IC_{n,n+1}$ *binds.*

Proof: Suppose $n \in S_\alpha$, $n+1 \in S_\beta$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as before, reducing y_n, expanding y_{n+1}, raising τ_n if $\tau_n < 0$ and lowering τ_{n+1} if $\tau_{n+1} > 0$ all improve the objective, tighten $IC_{n,n+1}$, relax $IC_{n-1,n}$ and $IC_{n+1,n+2}$.

Suppose, $y_n = 0$, $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

This cannot be the case as $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$ and $(1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$.

$\therefore IC_{n,n+1}$ binds. □

We already know $S_0 < S_1$ and $S_\alpha < S_\beta$ as the value of $\mathbb{E}(r|s_n)$ is given by the signal structure and not the policy.[5] So the only two types of constraints to check are: $n \in S_\alpha$, $n+1 \in S_0$ and $n \in S_1$, $n+1 \in S_\beta$.

CLAIM 5.5 *If* $n \in S_\alpha$ *and* $n+1 \in S_0$, $IC_{n,n+1}$ *binds.*

Proof: Suppose $n \in S_\alpha$, $n+1 \in S_0$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as in the proof of Claim 5.2, reducing y_n, raising τ_n if $\tau_n < 0$ and lowering τ_{n+1} if $\tau_{n+1} > 0$ all improve the objective, tighten $IC_{n,n+1}$ and relax adjacent IC constraints.

Suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Also, as in the proof of Claim 5.2, it must be that $\phi_n(\tau_{n+1}) > y_{n+1}$. Suppose by contradiction

[5] Here, $<$ refers to the *below* set relation defined by: set A is below set B, $A < B$, if $\forall a \in A$ and $\forall b \in B$, $a < b$.

$\phi_n(\tau_{n+1}) \leq y_{n+1}$, and $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = x_n\phi_n(\tau_n) &> (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ &\geq (1 - x_{n+1})\phi_n(\tau_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ &= \phi_n(\tau_{n+1}) \end{aligned}$$

But as $x_n\phi_n(\tau_n) \leq \phi_n(\tau_n) \leq \phi_n(0)$ and $\phi_n(\tau_{n+1}) \geq \phi_n(0)$, this is a contradiction.

Now consider reducing the probability that s_{n+1} is allocated to without inspection, in favour of inspection using the threshold assigned to s_n. In particular, conditional on not inspecting s_{n+1}, instead of allocating with probability y_{n+1}, allocate with probability λy_{n+1} and inspect using the threshold τ_n with probability $(1 - \lambda)$, for some $\lambda \in (0, 1)$.

This improves the objective as $0 \geq \mathbb{E}(r|s_{n+1})$ and $\psi_{n+1}(\tau_n) > 0$. The second fact here comes from the observation that, despite $0 \geq \psi_{n+1}(\tau_{n+1})$, $\psi_n(\tau_n) > 0$ implies $\psi_{n+1}(\tau_n) > 0$. To see this, take a fixed threshold τ and rearrange:

$$\begin{aligned} \psi_n(\tau) &> 0 \\ \int \mathbb{1}\{r \geq \tau\} r\pi_{n,r}\, dr - c &> 0 \\ \int \mathbb{1}\{r \geq \tau\} r\pi_{n,r}\, dr &> c \end{aligned}$$

As $\mathbb{1}\{r \geq \tau\}r$ is an increasing function of r when $\tau > 0$, the left-hand side is increasing in n as ensured by FOSD.

This tightens $IC_{n,n+1}$ as $\phi_n(\tau_n) > y_{n+1}$, and does not contradict for small enough λ. To see this, observe that:

$$\phi_n(\tau_n) > x_n\phi_n(\tau_n) > (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) > y_{n+1}$$

Which follows the condition that $IC_{n,n+1}$ didn't bind and our claim that $\phi_n(\tau_{n+1}) > y_{n+1}$.

And this change only relaxes $IC_{n+1,n+2}$, ensured by FOSD and the same rationale:

$$\phi_{n+1}(\tau_n) > x_n\phi_{n+1}(\tau_n) \geq x_n\phi_n(\tau_n) > y_{n+1}$$

Finally, suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_{n+1} = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = x_n\phi_n(\tau_n) > \phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$ binds. $\square$

Claim 5.6 *If $n \in S_1$ and $n+1 \in S_\beta$, $IC_{n,n+1}$ binds.*

Proof: Suppose $n \in S_1$, $n+1 \in S_\beta$ and $IC_{n,n+1}$ does not bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Then, as in the proof of Claim 5.3, expanding y_{n+1}, raising τ_n if $\tau_n < 0$ and lowering τ_{n+1} if $\tau_{n+1} > 0$ all improve the objective, tighten $IC_{n,n+1}$ and relax adjacent IC constraints.

Suppose $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, and $IC_{n,n+1}$ doesn't bind:

$$u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) > (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

Also as in the proof of Claim 5.3, it must be that $y_n > \phi_n(\tau_n)$. Suppose by contradiction $\phi_n(\tau_n) \geq y_n$, and $y_{n+1} = 1$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$ while $IC_{n,n+1}$ doesn't bind. Then:

$$\begin{aligned} u_n = (1-x_n)y_n + x_n\phi_n(\tau_n) &> (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1} \\ (1-x_n)\phi_n(\tau_n) + x_n\phi_n(\tau_n) &> (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \\ \phi_n(\tau_n) &> (1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \end{aligned}$$

But, $\phi_n(\tau_n) \leq \phi_n(0)$ and $(1-x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) \geq \phi_n(\tau_{n+1}) \geq \phi_n(0)$, a contradiction.

Now consider reducing the probability that s_n is allocated to without inspection, in favour of inspection using the threshold assigned to s_{n+1}. In particular, conditional on not inspecting s_n, instead of allocating with probability y_n, allocate with probability λy_n and inspect using the threshold τ_{n+1} with probability $(1-\lambda)$, for some $\lambda \in (0,1)$.

This improves the objective as $\psi_n(\tau_{n+1}) \geq \mathbb{E}(r|s_n)$. This fact comes from the observation that, despite $\mathbb{E}(r|s_n) \geq \psi_n(\tau_n)$, $\psi_{n+1}(\tau_{n+1}) > \mathbb{E}(r|s_{n+1})$ implies $\psi_n(\tau_n) > \mathbb{E}(r|s_n)$. To see this, take a fixed threshold τ and rearrange:

$$\begin{aligned} \psi_{n+1}(\tau_{n+1}) &> \mathbb{E}(r|s_{n+1}) \\ \int \mathbb{1}\{r \geq \tau\} r\pi_{n+1,r}\, dr - c &> \int r\pi_{n+1,r}\, dr \\ -c &> \int \mathbb{1}\{r < \tau\} r\pi_{n+1,r}\, dr \end{aligned}$$

As $\mathbb{1}\{r < \tau\}r$ is an increasing function of r when $\tau < 0$, the right-hand side is increasing in n as ensured by FOSD.

This tightens $IC_{n,n+1}$ as $y_n > \phi_n(\tau_{n+1})$ and does not contradict for λ small enough. To see this, observe that:

$$y_n > (1 - x_n)y_n + x_n\phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) > \phi_n(\tau_{n+1})$$

Which follows from our claim that $y_n > \phi_n(\tau_n)$ and the condition that $IC_{n,n+1}$ didn't bind.

And this change only relaxes $IC_{n-1,n}$ ensured the same rationale and FOSD:

$$y_n > \phi_n(\tau_{n+1}) > \phi_{n-1}(\tau_{n+1})$$

Finally, suppose $y_n = 0$, $\tau_n \geq 0$, $\tau_{n+1} \leq 0$, $x_n = 1$ and $IC_{n,n+1}$ doesn't bind:

$$u_n = \phi_n(\tau_n) > (1 - x_{n+1}) + x_{n+1}\phi_n(\tau_{n+1}) = u_{n,n+1}$$

As we've already established, this cannot be the case.

$\therefore IC_{n,n+1}$ binds. □

Then, by claims 5.1 through 5.6, $IC_{n,n+1}$ must bind for all n. ■

B.3 Threshold inspection rules

Given claims 4 and 5, we can rewrite the principal's problem as:

$$\begin{aligned}
\max_{(x_n, y_n, \tau_n)} \quad & \sum_n [(1 - x_n)y_n\mathbb{E}(r|s_n) + x_n\psi_n(\tau_n)]p_n \\
\text{s.t.} \quad & IC_{n,n+1} : (1 - x_n)y_n + x_n\phi_n(\tau_n) = (1 - x_{n+1})y_{n+1} + x_{n+1}\phi_n(\tau_{n+1}) \quad \forall\ n < N \\
& F : 0 \leq x_n, y_n, \tau_n \leq 1 \quad \forall\ r \quad \forall\ n
\end{aligned}$$

Claim 6 *Optimal inspection rules are threshold mechanisms. That is, there exists some n_0 such that $x_n = \mathbb{1}\{n \geq n_0\}$.*

Proof: First observe that for each n we can represent $(1 - x_n)y_n$ recursively using the binding

$IC_{n,n+1}$ constraints:

$$\begin{aligned}
(1-x_n)y_n &= (1-x_{n+1})y_{n+1} && + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= (1-x_{n+2})y_{n+2} && + x_{n+2}\phi_{n+1}(\tau_{n+2}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) \\
& && + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= (1-x_{n+3})y_{n+3} && + x_{n+3}\phi_{n+2}(\tau_{n+3}) - x_{n+2}\phi_{n+2}(\tau_{n+2}) \\
& && + x_{n+2}\phi_{n+1}(\tau_{n+2}) - x_{n+1}\phi_{n+1}(\tau_{n+1}) \\
& && + x_{n+1}\phi_n(\tau_{n+1}) - x_n\phi_n(\tau_n) \\
&= \cdots \\
&= (1-x_N)y_N && + \sum_{m=n}^{N-1}[x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]
\end{aligned}$$

Which can also be written as the following arrangement:

$$(1-x_n)y_n = (1-x_N)y_N + x_N\phi_{N-1}(\tau_N) + \sum_{m=n+1}^{N-1} x_m[\phi_{m-1}(\tau_m) - \phi_m(\tau_m)] - x_n\phi_n(\tau_n)$$

Note that this also restricts the choice of τ. For example, suppose for some pair $n_0 < n_1$, $x_{n_0} = 1$, $x_{n_1} = 1$, and $x_m = 0$ for $n_0 < m < n_l$. Then by the binding constraint τ_{n_0} and τ_{n_1} must satisfy:

$$\phi_{n_0}(\tau_{n_0}) = \phi_{n_1-1}(\tau_{n_1})$$

This says that if $n_0 = n_1 - 1$ then $\tau_{n_0} = \tau_{n_1}$, and if $n_0 < n_1 - 1$, then $\tau_{n_0} > \tau_{n_1}$ and uniquely determined. We will return to this example after the substitution.

Substituting the arrangement into the objective function:

$$
\begin{aligned}
&\sum_n [(1-x_n)y_n \mathbb{E}(r|s_n) + x_n\psi_n(\tau_n)]p_n \\
&= \sum_n [\langle (1-x_N)y_N + \sum_{m=n}^{N-1}[x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\rangle \mathbb{E}(r|s_n) + x_n\psi_n(\tau_n)]p_n \\
&= (1-x_N)y_N \sum_n \mathbb{E}(r|s_n)p_n + \sum_n \sum_{m=n}^{N-1}[x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\mathbb{E}(r|s_n)p_n + \sum_n x_n\psi_n(\tau_n)p_n \\
&= (1-x_N)y_N \sum_n \mathbb{E}(r|s_n)p_n + \sum_m^{N-1}[x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\sum_n^m \mathbb{E}(r|s_n)p_n + \sum_n x_n\psi_n(\tau_n)p_n \\
&= (1-x_N)y_N \mathbb{E}(r) + \sum_m^{N-1}[x_{m+1}\phi_m(\tau_{m+1}) - x_m\phi_m(\tau_m)]\mathbb{E}(r|s \leq s_m)P_m + \sum_n x_n\psi_n(\tau_n)p_n
\end{aligned}
$$

This shows us that we that for a fixed y_N and τ the objective is linear in x_n, whose only restriction is that $x_n \in [0,1]$:

$$
\begin{aligned}
\max_{(x_n,\tau_n),y_N} (1-x_N)y_N\mathbb{E}(r) \quad &+ \quad x_N[\phi_{N-1}(\tau_N)\mathbb{E}(r|s \leq s_{N-1})P_{N-1} + \psi_N(\tau_N)p_N] \\
&+ \sum_{n=1}^{N-1} \quad x_n[\phi_{n-1}(\tau_n)\mathbb{E}(r|s \leq s_{n-1})P_{n-1} - \phi_n(\tau_n)\mathbb{E}(r|s \leq s_n)P_n + \psi_n(\tau_n)p_n] \\
&+ \quad x_0[-\phi_0(\tau_0)\mathbb{E}(r|s_0)p_0 + \psi_0(\tau_0)p_0]
\end{aligned}
$$

Let a_n be the coefficient on x_n in the objective, and observe that a_n is only a function of τ_n. We can immediately conclude:

- $x_n = \mathbb{1}\{a_n(\tau_n) \geq 0\}$, and
- $y_N = \mathbb{1}\{\mathbb{E}(r) \geq 0\}$.

This means the restrictions on τ in the previous example are the only relevant restriction to our problem, and as such $x_n = \mathbb{1}\{n \geq n_0\}$ for some $n_0 \in \{0, \ldots, N, N+1\}$ and $\tau_n = \tau$ for all n. ■