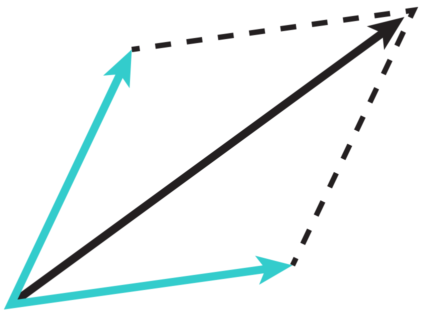
Linear Algebra



Lec. 1: Algebra of Matrices

If m and n are positive integers, then the **matrix** A is a rectangular array in which each **entry**, a_{ij}, of that matrix is a number. An $m\times n$ matrix (read "m by n") has m **rows** (horizontal lines) and n **columns** (vertical lines).

$$A_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=[a_{ij}]$$

$$i=1,2,\dots m, \qquad j=1,2,\dots n$$

If the **size** $(m\times n)$ of the matrix is $n\times n$, the matrix is called **square** of **order** n. For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \dots$ are called the **main diagonal** entries. A matrix with only **one column**, is called a **column matrix** or a **column vector**. A matrix that has only **one row** is called a **row matrix** or **row vector**.

column vector: $\mathbf{v_1}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}=[v_1 \quad v_2 \quad \cdots \quad v_m]^T$ **row vector:** $\mathbf{v_2}=[v_1 \quad v_2 \quad \cdots \quad v_m]$

1 Matrix Operations

Matrix Addition

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are matrices of size $m\times n$, then their **sum** is the $m\times n$ matrix given by $A+B=[a_{ij}+b_{ij}]$.

Matrix Multiplication

If $A=[a_{ij}]$ is an $m\times n$ matrix and $B=[b_{ij}]$ is an $n\times p$ matrix, then their **product** AB is an $m\times p$ matrix $AB=[c_{ij}]$ where:

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{in}b_{nj}$$

Matrix Transpose

If $A=[a_{ij}]$ is an $m\times n$ matrix, then the **transpose** of A is an $n\times m$ matrix formed by switching the matrix row and column indices denoted by $A^T=[b_{ij}]$ where $b_{ij}=a_{ji}$.

Matrix Equality

Two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ are **equal** if they have the same size $(m\times n)$ and $a_{ij}=b_{ij}$ for $1\le i\le m$ and $1\le j\le n$.

Scalar Multiplication

If $A=[a_{ij}]$ is an $m\times n$ matrix and c is a scalar (real number), then the **scalar multiple** of A by c is the $m\times n$ matrix given by $cA=[ca_{ij}]$.

Conjugate Transpose

If $A=[a_{ij}]$ is an $m\times n$ **complex matrix** (entries are complex numbers) then its **conjugate** (**Hermitian**) **transpose** is an $n\times m$ matrix formed by transposing A and applying **complex conjugate** on each entry (the conjugate of $a+bi$ is $a-bi$), denoted by $A^H=(\bar{A})^T=\overline{A^T}$ where $\bar{A}$ denotes the matrix with complex conjugated enteries.

2 Properties of Matrix Operations

If A and B are matrices and c and d are scalars, then these properties are true:

$A + B = B + A$	$A(BC) = (AB)C$	$(A^T)^T = A$
$A + (B + C) = (A + B) + C$	$A(B + C) = AB + AC$	$(A + B)^T = A^T + B^T$
$(cd)A = c(dA)$	$(A + B)C = AC + AB$	$(cA)^T = c(A)^T$
$c(A + B) = cA + cB$	$c(AB) = (cA)B = A(cB)$	$(AB)^T = B^T A^T$

3 Special Matrices

Triangular Matrices

A **square matrix** is **lower triangular**, when all the entries **above** its main diagonal are **zero**.

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

A **square matrix** is **upper triangular**, when all the entries **below** its main diagonal are **zero**.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

A matrix that is both upper and lower triangular is **diagonal**.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

The Zero Matrix

The matrix $O_{m\times n} = [0]$ is called a **zero matrix**, and it serves as the **additive identity** for the set of all $m\times n$ matrices.

If A is an $m\times n$ matrix and c is a scalar, then the following properties are true.

$$O_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A + O_{mn} = A$$

$$A + (-A) = O_{mn}$$

If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$

The Identity Matrix

A diagonal matrix I_n that has 1's on the main diagonal is called the **identity matrix of order n**, and it serves as the **identity** for matrix multiplication.

If A is an $m\times n$ matrix and then the following properties are true.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$AI_n = A$$

$$I_m A = A$$

An $n \times n$ square matrix A is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix A^{-1} called the **inverse** of A such that $AA^{-1} = A^{-1}A = I_n$.

✳ If $A = [a_{ij}]$ is a **square matrix** then:

Symmetric Matrices

A is **symmetric** if $A = A^T$ and $a_{ij} = a_{ji}$.

Skew-symmetric Matrices

A is **skew-symmetric** or **anti-symmetric** if $A = -A^T$ and $a_{ij} = -a_{ji}$.

Unitary Matrices

A is **unitary** if $A^{-1} = A^H$. If A is a **real matrix** (all entries are real numbers), then the matrix is said to be **orthogonal** and $A^{-1} = A^T = A^H$.

→Lec. 2: Systems of Linear Equations

A **linear equation in n variables** $x_1, x_2, x_3, \dots, x_n$ has the form:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

The coefficients $a_1, a_2, a_3, \dots, a_n$ and the constant term b are real numbers.

A **system of m linear equation in n variables** is a set of m equations, each of which is **linear** in the same n variables. A **solution** of a system of m linear equations is a sequence of n numbers that is a solution of each of the m linear equations in the system. A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution. A system of linear equations can be written as the matrix equation $A\mathbf{x} = \mathbf{b}$ where A is the cofficient matrix of the system, and $\mathbf{x}$ and $\mathbf{b}$ are column matrics. For a system of 3 equations in 3 variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \longrightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

We can derive a matrix from the coefficients and constant terms of a system of linear equations, denoted $[A \vdots \mathbf{b}]$, such a matrix is called **augmented**. An augmented matrix is a matrix obtained by appending the columns of two given matrices.

$$[A \vdots \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & a_{12} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Two systems of linear equations are called **equivalent** if they have precisely the same **solution set**. Each of the following three operations can be used on a system of linear equations to produce equivalent systems.

In matrix terminology these three operations correspond to **elementary row operations**—operations on an augmented matrix that produces a new augmented matrix corresponding to an equivalent system of linear equations—.

Interchange two equations ⟶ **Interchange two rows** 1 $R_i \leftrightarrow R_j$

Multiply an equation by a nonzero constant ⟶ **Multiply a row by a nonzero constant** 2 $kR_i \rightarrow R_i$

Add a multiple of an equation to another equation ⟶ **Add a multiple of a row to another row** 3 $R_i + kR_j \rightarrow R_i$

Solving Systems of Linear Equations

1 Gaussian Elemination (row reduction)

A system of linear equations is in row-echelon form when it follows a stair-step pattern and has leading coefficients of 1. The process of Using forward elimination with the elementary row operations to find an equivalent system that is in row-echelon form and then using back-substitution to solve the easier equivalent system is called Gaussian elimination, after the German mathematician Carl Friedrich Gauss.

A matrix in row-echelon form has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).
3. For two successive nonzero rows, the leading 1 in the higher row is farther to the left than the one in the lower row.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{12} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & u_{12} & u_{12} & c_1 \\ 0 & 1 & u_{23} & c_2 \\ 0 & 0 & 1 & c_3 \end{bmatrix}$$

forward elimination

$$\begin{aligned} x_1 + u_{12}x_2 + u_{13}x_3 &= c_1 \\ x_2 + u_{23}x_3 &= c_2 \\ x_3 &= c_3 \end{aligned}$$

back-substitution

2 Gauss-Jordan Elimination

A second method of elimination, called Gauss-Jordan elimination after Carl Gauss and Wilhelm Jordan, continues the reduction process until a reduced row-echelon form is obtained. A matrix in row-echelon form is in reduced row-echelon form if every column that has a leading 1 has zeros in every position above and below its leading 1. No matter which order you use, the reduced row-echelon form will be the same.

$$\begin{bmatrix} 1 & 0 & 0 & s_1 \\ 0 & 1 & 0 & s_2 \\ 0 & 0 & 1 & s_3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= s_1 \\ x_2 &= s_2 \\ x_3 &= s_3 \end{aligned}$$

For a system of linear equations, use the following steps to solve it.

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

→ Lec. 3: The Inverse of a Matrix $AA^{-1} = I$

1 Determinant

* Every square matrix can be associated with a real number called its **determinant**.

If A is a square matrix, then the **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$. If A is a square matrix of order n:

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + \cdots + a_{in}C_{in} = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

If A and B are square matrices of order n and c is a scalar, then:

$$\det(AB) = \det(A)\det(B) \qquad \det(cA) = c^n \det(A) \qquad \det(A) = \det(A^T)$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

If A is a triangular matrix:

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

If any one of the following conditions is true, then $\det(A) = 0$:

1. An entire row (or an entire column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

2 The Inverse of a Matrix

If A is a square matrix, then the transpose of the **matrix of cofactors** is called the **classical adjoint** of A and is denoted by $\operatorname{adj}(A)$. Multiplying $\operatorname{adj}(A)$ by A , we get:

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \xrightarrow{\text{transpose}} \operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$A[\operatorname{adj}(A)] = \det(A)\, I$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

* A square matrix is **invertible** if and only if $\det(A) \neq 0$. If $\det(A) = 0$, then the matrix A is called **noninvertible** (or **singular**).

* we can also **find the Inverse of a Matrix using Gauss-Jordan Elimination:**

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix $[A \vdots I]$ that you obtain by adjoining the $n \times n$ identity matrix I to the given matrix A .
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A \vdots I]$. The result will be the matrix $[I \vdots A^{-1}]$. If this is not possible, then A is noninvertible (or singular).

3 Properties of Inverse Matrices

If A and B are invertible matrices of size n, k is a positive integer, and c is a scalar, then A^{-1}, A^k (the kth power of A where $A^k = AA\cdots A$), cA, A^T and AB are invertible and the following are true:

$$(A^{-1})^{-1} = A \qquad (A^k)^{-1} = (A^{-1})^k = A^{-k} \qquad (A^T)^{-1} = (A^{-1})^T$$

$$(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0 \qquad (AB)^{-1} = B^{-1}A^{-1}$$

4 Elementary Matrices

An $n \times n$ matrix is called an **elementary matrix** if it can be obtained from the identity matrix I_n by a single elementary row operation.

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

Let A and B be $m \times n$ matrices. Matrix A is **row-equivalent** to B if there exists a finite number of elementary matrices $E_1, E_2, \dots, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A$

* If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

To find the inverse, simply reverse the elementary row operation used to obtain it. If the matrix E corresponds to the following elementary row operation, then:

1. E: $R_i \leftrightarrow R_j$ ⟶ $\det(E) = -1$ E^{-1}: $R_i \leftrightarrow R_j$ $E^{-1} = E$
2. E: $kR_i \to R_i$ ⟶ $\det(E) = k$ E^{-1}: $\frac{1}{k}R_i \to R_i$
3. E: $R_i + kR_j \to R_i$ ⟶ $\det(E) = 1$ E^{-1}: $R_i + (-k)R_j \to R_i$

* A square matrix is invertible if and only if it can be written as the product of elementary matrices $E_k \cdots E_2 E_1 A = I$, $A = E_1{}^{-1} E_2{}^{-1} \cdots E_k{}^{-1} I$.

5 The LU-Factorization

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

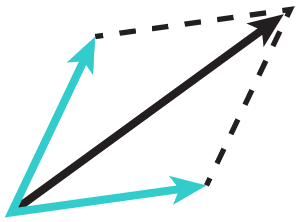
If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an **LU-factorization** of A. If A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row, then A has an LU-factorization. $E_k \cdots E_2 E_1 A = U$, $A = E_1{}^{-1} E_2{}^{-1} \cdots E_k{}^{-1} U$, $A = LU$. This algorithm can be used to solve the linear system $A\mathbf{x} = \mathbf{b}$ by writing $A\mathbf{x} = LU\mathbf{x}$ and letting $U\mathbf{x} = \mathbf{y}$, you can then solve for $\mathbf{x}$ in two stages. First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$; then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Each system is easy to solve because they do not require any row operations.

→Lec. 4: Vector Spaces — Subspaces

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every vector $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space**.

Vector Addition:

1. $\mathbf{u}+\mathbf{v}$ is in V. — **Closure under addition**
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ — **Commutative property**
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ — **Associative property**
4. V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$. — **Additive identity**
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. — **Additive inverse**



Scalar Multiplication:

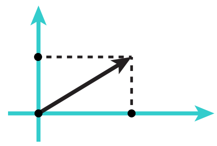
6. $c\mathbf{u}$ is in V. — **Closure under scalar multiplication**
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ — **Distributive property**
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ — **Distributive property**
9. $c(d\mathbf{u})=(cd)\mathbf{u}$ — **Associative property**
10. $1(\mathbf{u})=\mathbf{u}$ — **Scalar identity**

⁕ Some of the known important vector spaces include:

① Real Coordinate Space $\mathbb{R}^n$

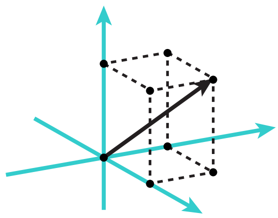
the set of all $n-$tuples of real numbers $(x_1, x_2, \cdots, x_n)$ that is the set of all sequences of n real numbers. In standard matrix notation, each element of $\mathbb{R}^n$ is typically written as a column vector $[x_1, x_2, \cdots, x_n]^T$. $\mathbb{R}^n$ is also called $n-$ space, so 1-space is the set of all real numbers $\mathbb{R}^1$, 2-space is $\mathbb{R}^2$ and 3-space is $\mathbb{R}^3$

Real coordinate plane



$\mathbb{R}^2$

Three-dimensional space



$\mathbb{R}^3$

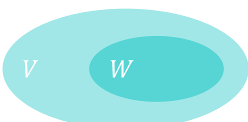
② Matrix Space $M_{m,n}$

the set of all $m \times n$ matrices

③ Polynomial Space P_n

the set of all polynomials of degree $\leq n$

⁕ A nonempty subset W of a vector space V is called a **subspace** of V if it is also a vector space, which is true if and only if it is closed under both vector addition and scalar multiplication. Subspaces other than the zero subspace $W=\{\mathbf{0}\}$ and the vector space $W=V$ itself are called proper (or nontrivial) subspaces.



$W \subseteq V$

Geometrical interpretation of possible subspaces of $\mathbb{R}^2$: