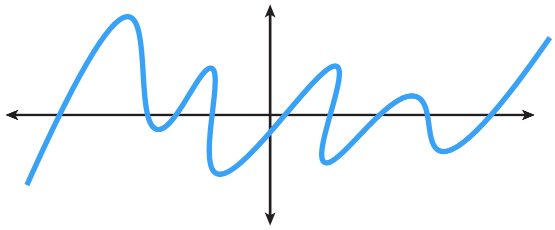
Differ. Equ.

∂f — ∂x — dy — dx — ∂f — ∂x



→Lec. 1: Definitions and Terminology

⁕ An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation**.

1 Type

df/dx | $\partial f/\partial y$ $\quad \partial f/\partial x$

ordinary differential equation (ODE)
contains only ordinary derivatives of one or more functions with respect to a single independent variable

partial differential equation (PDE)
contains only partial derivatives of one or more functions of two or more independent variables

2 Order

The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation.

An **nth-order** ordinary differential equation in one dependent variable expressed in:

general form: $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$

normal form: $\dfrac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$

3 Linearity

An nth-order ordinary differential equation is said to be **linear** when F is linear in y and its derivatives:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

A **nonlinear** ordinary differential equation is simply an ODE that is not linear.
Two characteristic properties of a linear ODE are:

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree; that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

⁕ Any function ϕ, defined on an interval I—this interval is variously called the **interval of definition**, the **interval of validity**, or the **domain of the solution**, and can be an open, closed or an infinite interval—and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval. $F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0$ for all x in I.

Solutions

explicit $y = \phi(x)$

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an explicit solution of the ordinary differential equation.

$G(x, y) = 0$ **implicit**

A relation is said to be an implicit solution of an ODE on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the differential equation.

❋ When solving an nth-order differential equation $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$ we usually seek an n-parameter family of solutions $G(x, y, c_1, c_2, \ldots, c_n) = 0$ that contain n arbitrary constants. This means that a single differential equation can possess an infinite number of solutions corresponding to the unlimited number of choices for the parameter(s). A solution of a differential equation that is free of arbitrary parameters is called a particular solution.

general solution

an n-parameter family from which every solution of an nth-order ODE can be obtained by appropriate choices of the parameters.

singular solution

an extra solution that cannot be obtained by specializing any of the parameters in a family of solutions of an ordinary differential equation.

❋ We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies prescribed side conditions—that is, conditions that are imposed on the unknown $y(x)$ or on its derivatives.

IVP Initial-Value Problem

On some interval I containing x_0, the problem

Solve: $$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$$

Subject to: $$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

some arbitrarily specified real constants, called initial conditions (IC), is called an nth-order initial-value problem (IVP). The term initial conditions derives from physical dynamical systems where values are specified at an initial time.

BVP Boundary-Value Problem

On some interval I containing x_0, $x_1, \ldots, x_{n-1}$, the problem

Solve: $$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$$

Subject to: $$y(x_0) = y_0, y(x_1) = y_1, \ldots, y(x_{n-1}) = y_{n-1}$$

some arbitrarily specified real constants, called boundary conditions (BC), is called an n-point boundary-value problem (BVP). A BVP consists of solving a DE in which the dependent variable or its derivatives are specified at different points.

→ Lec. 2: First-Order Differential Equations

1 Separable Equations

$$\frac{dy}{dx} = g(x)h(y)$$

⁕ method of solution:

① $\dfrac{1}{h(y)}\dfrac{dy}{dx} = g(x)$ ② $\displaystyle\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$ ③ $H(y) = G(x) + c$

2 Exact Equations

⁕ A first order ODE can be written in the differential form

$$M(x,y)dx + N(x,y)dy = 0$$

⁕ If $z = f(x,y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its total differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

multivariable chain rule

Given a one-parameter family of curves $f(x,y) = c$, we can generate a first-order differential equation by computing the differential.

A differential expression $M(x,y)dx + N(x,y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$. A first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is an **exact equation** if the expression on the left side is an exact differential. A necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{since} \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

⁕ method of solution:

$$M(x,y)dx + N(x,y)dy = 0$$

where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

① $\dfrac{\partial f}{\partial x} = M(x,y)$ or $\dfrac{\partial f}{\partial y} = N(x,y)$

② $\displaystyle f(x,y) = \int M(x,y)\,dx + g(y)$

③ $\displaystyle \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$

④ $\displaystyle g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$

⑤ $f(x,y) = c$

solutions are represented as **level curves** of the function, cross sections of the graph of $z = f(x,y)$ taken at an arbitrary constant value

→ Lec. 3: First-Order Differential Equations

3 Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

* A linear first-order ODE can be written in a more useful form by dividing both sides of by the lead coefficient called the *standard form*

$$\frac{dy}{dx} + P(x)y = f(x)$$

We can solve first-order linear ODEs using the concept of an **Integrating Factor** $\mu(x,y)$

* For a **nonexact differential**, it is sometimes possible to find an integrating factor so that after multiplying, the left-hand side of is an exact differential.

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{nonexact}$$

$\times\, \mu(x,y)$

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

a modified exact differential equation

* method of solution:

① $\mu = e^{\int P(x)\,dx}$, $\quad \dfrac{d\mu}{dx} = P(x)e^{\int P(x)\,dx}$

② $e^{\int P(x)\,dx}\dfrac{dy}{dx} + P(x)e^{\int P(x)\,dx}y = e^{\int P(x)\,dx}f(x)$

③ $\mu\dfrac{dy}{dx} + \dfrac{d\mu}{dx}y = \mu f(x)$ $\quad$ using the product rule of differentiation

④ $\dfrac{d}{dx}[\mu y] = \mu f(x)$ $\quad$ integrating w.r.t x

⑤ $\mu y = \displaystyle\int \mu f(x)\,dx + c$ $\quad$ general solution

⑥ $y = ce^{-\int P(x)\,dx} + \dfrac{1}{\mu}\displaystyle\int \mu f(x)\,dx$

If we write the standard linear equation in the differential form:

$$\big(P(x)y - f(x)\big)dx + dy = 0$$

we find a suitable integrating factor

$$\mu = e^{\int P(x)\,dx}$$

multiply both sides by it

$$\mu\big(P(x)y - f(x)\big)dx + \mu dy = 0$$

and solving the exact equation we get

$$\frac{\partial f}{\partial y} = N(x,y) = \mu$$

$$f(x,y) = \int \mu\,dy = \mu y + h(x)$$

$$\frac{\partial f}{\partial x} = P(x)\mu y + h'(x) = M(x,y)$$

$$\frac{\partial f}{\partial x} = M(x,y) = P(x)\mu y - \mu f(x)$$

$$h'(x) = -\mu f(x)$$

$$h(x) = -\int \mu f(x)\,dx$$

the general solution of the linear ODE is

$$f(x,y) = \mu y - \int \mu f(x)\,dx = c$$

* Often the first step in solving a given differential equation consists of transforming it into another differential equation by means of a **substitution**.

4 Homogeneous Equations

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

* If a function $f(x,y)$ possesses the property $f(tx,ty) = t^{\alpha} f(x,y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. A first-order DE in differential form is said to be a **homogeneous equation** if both coefficients M and N are homogeneous of the same degree.

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(tx,ty) = t^{\alpha} M(x,y) \qquad N(tx,ty) = t^{\alpha} N(x,y)$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{x^{\alpha} M\left(1,\frac{y}{x}\right)}{x^{\alpha} N\left(1,\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$

* method of solution:

① $u = \dfrac{y}{x}$ ② $y = ux$ ③ $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$ ④ $x\dfrac{du}{dx} + u = f(u)$ **seperable**

5 Bernoulli's Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

* The following differential equation. where n is any real number, is called **Bernoulli's equation** and is named after the Swiss mathematican **Jacob Bernoulli** (1654 -1705). Note that for $n = 0$ and $n = 1$, the equation is linear, for $n \neq 0,1$ we can use the following substitution.

* method of solution: ② $u = y^{1-n}$ ③ $\dfrac{du}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

① $y^{-n}\dfrac{dy}{dx} + P(x)y^{1-n} = f(x)$ ④ $\dfrac{1}{1-n}\dfrac{du}{dx} + P(x)u = f(x)$ **linear in u**

* Other Special Cases:-

* Note: $\dfrac{dy}{dx} = f(Ax + By + C)$ can be always be reduced to a seperable DE using the subtitution $u = Ax + By + C$

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

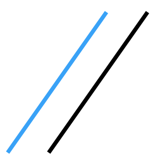
* method of solution:

① find the slope of

$$a_1x + b_1y + c_1 = 0$$

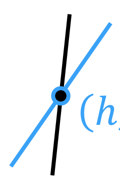
$$a_2x + b_2y + c_2 = 0$$

parallel $m_1 = m_2$



$\dfrac{dy}{dx} = f(Ax + By + C)$ **seperable**

intersecting $m_1 \neq m_2$



(h,k)

$y = Y + k \qquad x = X + h$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y} = f\left(\frac{Y}{X}\right)$$

homogeneous

→ Lec. 4: Higher-Order Linear ODEs

* Linear ordinary differential equations can be classified based on homogeneity.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Homogeneous $g(x) = 0$

$L(y) = g(x)$

Nonhomogeneous $g(x) \neq 0$

* Linear Differential Operators

* Note: $\frac{d^n y}{dx^n} = D^n y$

* A differential operator is an operator defined as a function of the differentiation operator denoted by the capital letter D that is, $dy/dx = Dy$. An nth-order differential operator defined by:

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_2(x)D^2 + a_1(x)D + a_0(x)$$

possesses a linearity property; $L\{\alpha f(x) + \beta g(x)\} = \alpha L\big(f(x)\big) + \beta L\big(g(x)\big)$ as a consequence of two basic properties of differentiation:

① $D\big(cf(x)\big) = cDf(x)$ ② $D\{f(x) + g(x)\} = Df(x) + Dg(x)$

A linear differential operator is a linear transformation between two vector function spaces. If we define $L : C^n(I) \to C(I)$ where $L(y) = g(x)$, then the domain is $C^n(I)$: continuous n times differentiable functions on an invterval I, and the codomain is $C(I)$: all continuous functions on an invterval I.

The problem of finding the kernel of a linear differential operator L is identical to the problem of finding all solutions of a homogeneous linear differential equation, since: $\ker(L) = \{y \in C^n(I) \mid L(y) = 0\} = L^{-1}(0)$.

If we can find a suitable basis for the kernel—a set of linearly independent solutions of a homogeneous linear ODE that spans the kernel—then all linear combinations of the basis functions are also solutions of the ODE. An existence theorem tells us that the solution space of an nth-order linear differential equation is n-dimensional, or equivalently, $\mathrm{Nullity}(L) = \dim(\ker(L)) = n$.

Any set of n linearly independent functions $y_1, y_2, \dots, y_n$ that are solutions of an nth-order homogeneous linear differential equation $L(y) = 0$ on an interval, which is also a basis of the kernel of the nth-order linear differential operator L, is said to be a fundamental set of solutions. $\ker(L) = \mathrm{span}\{y_1, y_2, \dots, y_n\}$

✻ General Solution

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

✻ Let $\{y_1, y_2, \dots, y_n\}$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation, denoted $L(y) = 0$, on an interval I. Then the general solution of the equation on the interval is an n-parameter family of solutions represented by all the linear combinations of the basis functions.

✻ linear independence of a set of functions

✻ Recall from linear algebra that a set of functions $f_1, f_2, \dots, f_n$ is said to be linearly independent on an inteval if the only constants for which the equation $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ is true for some x in the interval are zeros. The question of whether n solutions of a homogeneous linear nth-order ODE are linearly independent can be settled somewhat mechanically using a determinant named after the Polish mathematician Jósef Maria Hoëné-Wronski (1778–1853).

Wronskian

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

✻ Note:

linear dependence implies $W = 0$ for every $x \in I$

$W \neq 0$ for some $x_0 \in I$ implies linear independence

But the converse of both is not true in general

The basic idea is to differentiate the equation $n-1$ times to get the following system of linear equations

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This system of equations has a unique trivial solution (all zeros) if the coefficient matrix is invertible, or equivalently, its determinant is nonzero. This determinant is called the Wronskian. We can now use the following theorem about linear ODEs.

Let $\{y_1, y_2, \dots, y_n\}$ be a set of solutions of a homogeneous linear nth-order differential equation. Then the set is linearly independent on an interval I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for **every** x in the interval.

✻ The previous theorem is due to the fact that the Wronskian of solutions of a linear homogeneous ODE $L(y) = 0$ is either identically zero or never zero on the interval. It follows that if we can show that $W(y_1, y_2, \dots, y_n) \neq 0$ for some x_0 in the interval I, then the solutions $y_1, y_2, \dots, y_n$ are linearly independent on I, and thus form a fundamental set of solutions or a basis for the kernel of L.

⁕ There are various ways of solving linear ODEs. The following are some of the methods used to find explicit solutions of second (and higher) order linear ODEs.

1 Two solutions of y

⁕ The basic idea described here is that a linear second-order equation can be **reduced** to a linear first-order DE by means of a substitution involving one known nontrivial solution y_1. We seek a second solution y_2 so that the two solutings are linearly independent on some interval.

Recall that if y_1 and y_2 are linearly independent, then their ratio y_2/y_1 is nonconstant on the interval; that is, $y_2/y_1 = u(x)$ or $y_2 = uy_1$. The idea is to find $u(x)$ by substituting the expression $y_2 = uy_1$ into the given differential equation. A second solution obtained by choosing $c_1 = 1$ and $c_2 = 0$ is apparent after this first-order DE is solved.

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

By writing the second-order ODE in the standard form, we get

$$y'' + P(x)y' + Q(x)y = 0$$

differerntiating $y_2 = uy_1$ twice

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

by substituting y2 in the ODE

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

by eliminating some terms

$$u'' + \left(\frac{2y_1'}{y_1} + P(x)\right)u' = 0$$

using the substitution $w = u'$ we get a **linear first-order ODE in w**

$$w' + \left(\frac{2y_1'}{y_1} + P(x)\right)w = 0 \Rightarrow w = \frac{c_1 e^{-\int P(x)\,dx}}{y_1^2}$$

solving for a general solution

$$u = \frac{y}{y_1} = c_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx + c_2$$

$$y = c_1 y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx + c_2 y_1$$

2 Constant Coefficient Homogeneous ODEs

⁕ We have seen that the linear first-order DE $y' + ay = 0$, where a is a constant, possesses the exponential solution $y = c_1 e^{-ax}$ on the entire interval $(-\infty, \infty)$. Therefore, it is natural to ask whether exponential solutions exist for homogeneous linear higher-order DEs in the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y^2 + a_1 y' + a_0 y = 0$$

where the coefficients a_i, $i = 0,1,\ldots,n$ are real constants and $a_n \neq 0$. The surprising fact is that all solutions of these higher-order equations are either exponential functions or are constructed out of exponential functions.

* We begin by considering the special case of a linear second-order equation with constnat coefficients.

$$ay'' + by' + cy = 0$$

* method of solution:

① Let $y = e^{mx}$ ② $am^2e^{mx} + bme^{mx} + ce^{mx} = 0$

③ $e^{mx}(am^2 + bm + c) = 0$ ④ $am^2 + bm + c = 0$ since $e^{mx} \neq 0$

solving for m, we have three cases for the discriminant

⑤ $$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* Note: complex roots of real polynomials always appear in conjugate pairs

$b^2 - 4ac > 0$

$m \in \mathbb{R} \quad m_1 \neq m_2$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

For distinct (real or complex) values of m, we can obtain linearly independent solultions that span the entire solution space. Finding the Wronskian of the solutions, we see it is nonzero everywhere.

$b^2 - 4ac = 0$

$m \in \mathbb{R} \quad m = -b/2a$

$$y_2 = e^{mx} \int \frac{e^{\int -\frac{b}{a} dx}}{e^{-\frac{b}{a}x}} dx$$

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

Because the root is repeated twice, we get another linearly independent solution using the method of reduction of order.

$b^2 - 4ac < 0$

$m \in \mathbb{C} \quad m = \alpha \pm i\beta$

$$y = Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x}$$

since $e^{ix} = \cos x + i \sin x$

$$y = e^{\alpha}\left(Ae^{i\beta} + Be^{-i\beta}\right)$$

$$y = e^{\alpha}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Using Euler's formula, we can write the complex exponential as the sum of two trigonometric functions.

Constant Coefficient Homogeneous Linear nth-order ODEs

* We conclude that in general, the n linearly independent solutions of an nth-order equation with constnat coefficients $L(y) = 0$ can be found using the substitution $y = e^{mx}$ which gives us an nth degree polynomial equation in the unkown m called the the auxiliary equation. Solving the polynomial, we get n roots that could be real or complex, distinct or repeated. The general solution is just the linear combination of these different cases:

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0 = 0$$

auxiliary equation

$$\sum_{j=1}^{k} c_j e^{m_j x} = c_1 e^{m_1 x} + \cdots + c_k e^{m_k x}$$

k Distinct Real Roots

$$e^{mx} \sum_{j=1}^{k} c_j x^{j-1} = c_1 e^{mx} + \cdots + c_k x^{k-1} e^{mx}$$

k Repeated Real Roots

$$\sum_{j=1}^{k} e^{\alpha_j x} \left(c_{2j-1} \cos \beta_j x + c_{2j} \sin \beta_j x\right)$$

k Distinct Complex Roots

$$e^{\alpha x} \sum_{j=1}^{k} \left(c_{2j-1} x^{j-1} \cos \beta x + c_{2j} x^{j-1} \sin \beta x\right)$$

k Repeated Complex Roots

→Lec. 5: Higher-Order Linear ODEs

Let y_p be any particular solution, on an interval I, of the nonhomogeneous linear nth-order differential equation $L(y) = g(x)$ where

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_2(x)D^2 + a_1(x)D + a_0(x)$$

and let $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ be the general solution of the associated homogeneous differential equation $L(y) = 0$ on I. Then the general solution of the nonhomogeneous equation on the interval is

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$$

The linear combination $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, which is the general solution of $L(y) = 0$, and the particular solution y_p are called the complementary function, and the particular integral respectively.

General Solution: $y = y_c + y_p$ since $L(y_c + y_p) = L(y_c) + L(y_p) = 0 + g(x)$

⁎ Polynomial Differential Operators

⁎ A linear differential operator L is called an nth order polynomial differential operator with constant coefficients, denoted $P(D)$, if it has the form

$$L = P(D) = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_2D^2 + a_1D + a_0$$

where the coefficients a_i, $i = 0,1,\ldots,n$ are real constants and $a_n \neq 0$.

Polynomial operators can be added, multiplied, factored, and multiplied by a constant, just as if they were ordinary polynomials. Also, these are all valid:

① $P_1P_2 = P_2P_1$ Commutative law

② $(P_1P_2)P_3 = P_1(P_2P_3)$ Associative law

③ $P_1(P_2 + P_3) = P_1P_2 + P_1P_3$ Distributive law

Using induction and building from $D(ue^{ax}) = e^{ax}Du + aue^{ax} = e^{ax}(D + a)u$,

Exponential Shift Theorem: $P(D)(ue^{ax}) = e^{ax}P(D + a)u$.

Superposition principle

Let y_{p_i} be particular solutions of the equations $P(D)y = g_i(x)$ where $i = 1,2,\ldots,k$, then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of the equation

$$P(D)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

⁎Note:

Superposition also applies to linear operators with nonconstant coefficients.

* Inverse Polynomial Differential Operators

Let $P(D)y = g(x)$, then the **inverse operator** of $P(D)$ is defined as

$$\frac{1}{P(D)}g(x) = y_p \qquad \text{or} \qquad P^{-1}(D)g(x) = y_p$$

where y_p is the particular solution of $P(D)y = g(x)$ that contains no constant multiple of a term in y_c (the general solution of the equation $P(D)y = 0$).

* We can use the inverse differential operator to find the particular integral solution of the nonhomogeneous equation $P(D)y = g(x)$ when $g(x)$ is the special function consisting only of such terms as b, x^k, e^{ax}, $\sin ax$, $\cos ax$, and a finite number of combinations of such terms.

* Note:

$D^{-n}g(x) =$ integrating $g(x)$ n times and ignoring constants of integration.

3 Constant Coefficient nonhomogeneous DEs

$$y_p = \frac{1}{P(D)}(bx^k)$$

if D^r is a factor

$$\frac{1}{D^r(a_nD^{n-r} + \cdots + a_{r+1}D + a_r)}(bx^k)$$

$$\frac{1}{D^r}\left[\frac{1}{(a_nD^{n-r} + \cdots + a_{r+1}D + a_r)}(bx^k)\right]$$

expand to get a similar form but then integrate

series expansion using ordinary division

$$\frac{1}{a_0\left(1 + \frac{a_1}{a_0}D + \frac{a_2}{a_0}D^2 + \cdots + \frac{a_n}{a_0}D^n\right)}(bx^k)$$

$$\frac{b}{a_0}(1 + b_1D + b_2D^2 + \cdots + b_kD^k)(x^k)$$

$$y_p = \frac{1}{P(D)}(be^{ax}) = \frac{be^{ax}}{P(a)}$$ if $P(a) \neq 0$ since $P(D)e^{ax} = e^{ax}P(a)$

$$\frac{1}{P(D)}b = \frac{b}{P(0)}$$

$$\frac{1}{P(D)}(ue^{ax}) = e^{ax}\frac{1}{P(D+a)}u$$ shift theorem also works for inverse operators

if $P(a) = 0$ then $\frac{1}{P(D)}(be^{ax}) = \frac{1}{(D-a)^rF(D)}(be^{ax}) = \frac{1}{(D-a)^r}\frac{be^{ax}}{F(a)} = e^{ax}\frac{1}{D^r}\frac{b}{F(a)} = \frac{e^{ax}bx^r}{r!\,F(a)}$

$$y_p = \frac{1}{P(D)}(\sin ax) = \operatorname{Im}\left(\frac{1}{P(D)}e^{iax}\right)$$

$$y_p = \frac{1}{P(D)}(\cos ax) = \operatorname{Re}\left(\frac{1}{P(D)}e^{iax}\right)$$

* Note:
if the polynomial is a function of only derivatives of even order, and $P(-a^2), P(a^2) \neq 0$

$$\frac{1}{P(D^2)}(\sin ax) = \frac{\sin ax}{P(-a^2)}$$ or $\cos ax$

$$\frac{1}{P(D^2)}(\sinh ax) = \frac{\sinh ax}{P(a^2)}$$ or $\cosh ax$

Lec. 6: Higher-Order Linear ODEs

4 Variation of Parameters

* A particular solution of any linear nth-order ODE $L(y) = g(x)$ can be written as a linear combination (with varying coefficients) of the fundamental homogeneous solutions. The method of solution that involves finding those variable coefficients is called variation of parameters.

* method of solution:

① $y'' + P(x)y' + Q(x)y = f(x)$ standard form

② $y_c = c_1y_1 + c_2y_2 \implies y_p = u_1(x)y_1 + u_2(x)y_2$

③ $y_p'' + P(x)y_p' + Q(x)y_p = f(x)$ substitution

④ $\frac{d}{dx}[u_1'y_1 + u_2'y_2] + P(u_1'y_1 + u_2'y_2) + u_1'y_1' + u_2'y_2' = f(x)$

⑤ if $y_1u_1' + y_2u_2' = 0$ then $y_1'u_1' + y_2'u_2' = f(x)$ $\implies u_1' = \frac{W_1}{W} \quad u_2' = \frac{W_2}{W}$ Cramer's rule

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W = W(y_1, y_2)$$

$$u_1 = -\int \frac{y_2 f(x)}{W}\,dx \quad , \quad u_2 = \int \frac{y_1 f(x)}{W}\,dx$$

* In general, the ODE in the form

$$y^{(n)} + \cdots + P_0(x)y = f(x)$$

where

$$y_c = c_1y_1 + \cdots + c_ny_n$$

is its complementary function, has a particular integral solution in the form

$$y_p = u_1(x)y_1 + \cdots + u_n(x)y_n$$

where the coefficients are determined by the n equations

$$\begin{gathered} y_1u_1' + \cdots + y_nu_n' = 0 \\ y_1'u_1' + + \cdots + y_n'u_n' = 0 \\ \vdots \\ y_1^{(n-1)}u_1' + \cdots + y_n^{(n-1)}u_n' = f(x) \end{gathered}$$

Using Cramer's rule

$$u_k' = \frac{W_k}{W} \qquad k = 1,2,\dots,n$$

where W_k is the determinant obtained by replacing the kth column of the Wronskian by the column $(0,0,\dots,f(x))$

We can generalize this to:

$$a_n(x - x_0)^n y^{(n)} + \cdots + a_0y = g(x)$$

5 Cauchy–Euler Equation

$$a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1xy' + a_0y = g(x)$$

* Any linear ODE of this form, where the coefficients $a_n, a_{n-1}, \dots, a_0$ are constants, is known as an Euler-Cauchy equation, named in honor of two of the most prolific mathematicians of all time, Augustin-Louis Cauchy (French, 1789–1857) and Leonhard Euler (Swiss, 1707–1783). Any Cauchy-Euler equation can always be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. If $D = d/dx$ and $\theta = d/dt$ we get:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\cdot\frac{dy}{dt} \implies xD = \theta$$

* Note: $dt/dx = x^{-1}$ $\qquad dx/dt = x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{x}\cdot\frac{d}{dt}\left(\frac{1}{x}\cdot\frac{dy}{dt}\right) = \frac{1}{x}\left(\frac{1}{x}\cdot\frac{d^2y}{dt^2} - \frac{1}{x}\cdot\frac{dy}{dt}\right) \implies x^2D^2 = \theta(\theta - 1)$$

After solving the ODE with t we substitute: $t = \ln x$

using induction: $x^kD^k = \theta(\theta-1)\cdots(\theta-(k-1))$ where $k = 1,2,\dots,n$

→ Lec. 7 : Higher-Order Linear ODEs

6 Reduction of Order

✻ method of solution:

① $y'' + P(x)y' + Q(x)y = f(x)$ standard form

② $\dfrac{y_2}{y_1} = u(x) \Rightarrow y_2 = uy_1$ linearly independent

③ $y_2'' + P(x)y_2' + Q(x)y_2 = f(x)$ substitution

④ $y_1u'' + (2y_1' + P(x)y_1)u' + (y_1'' + P(x)y_1' + Q(x)y_1)u = f(x)$

⑤ $u'' + \left(\dfrac{2y_1'}{y_1} + P(x)\right)u' = \dfrac{f(x)}{y_1}$ $w = u'$

⑥ $w' + \left(\dfrac{2y_1'}{y_1} + P(x)\right)w = \dfrac{f(x)}{y_1}$

We reduced the second-order equation to a first-order ODE

✻ If we know one homogeneous solution, we can reduce any linear second-order ODE using this method. The following are some tests we can use to find one solution of the equation directly.

For the ODE in the form

$$y'' + Py' + Qy = 0$$

if one soluion is $y_1 = e^{mx}$

$$e^{mx}(m^2 + Pm + Q) = 0$$

if $1 + P + Q = 0$ then $y_1 = e^{x}$

if $1 - P + Q = 0$ then $y_1 = e^{-x}$

if one soluion is $y_1 = x^m$

$$m(m-1) + mxP + x^2Q = 0$$

if $P + xQ = 0$ then $y_1 = x$

7 Removal of first derivative

✻ method of solution:

① $y'' + P(x)y' + Q(x)y = f(x)$ standard form

② $y = uv$ write the general solution as a product

③ $uv'' + (2u' + Pu)v' + (u'' + Pu' + Qu)v = f(x)$

④ if $2u' + Pu = 0$ then $uv'' + (u'' + Pu' + Qu)v = f(x)$

⑤ $uv'' + \left(\left(\dfrac{P}{2}\right)^2 u - \left(\dfrac{P}{2}\right)' u - 2\left(\dfrac{P}{2}\right)^2 u + Qu\right)v = f(x)$

⑥ $v'' + \left(\left(\dfrac{P}{2}\right)^2 - \left(\dfrac{P}{2}\right)' - 2\left(\dfrac{P}{2}\right)^2 + Q\right)v = \dfrac{f(x)}{u}$

⑦ $v'' + \left(Q - \left(\left(\dfrac{P}{2}\right)^2 + \left(\dfrac{P}{2}\right)'\right)\right)v = \dfrac{f(x)}{u}$

c $\qquad$ $\dfrac{c}{(ax+b)^2}$

$v'' + cv = f(x)/u$ $\qquad$ $(ax+b)^2v'' + cv = f(x)/u$

✻ Solving the following first-order ODE, and then finding the first and second derivative while ignoring the constant of integration.

$$2u' + Pu = 0$$

$$u = e^{-\int \frac{P}{2}dx}$$

$$u = -\frac{P}{2}e^{-\int \frac{P}{2}dx}$$

$$u' = -\frac{P}{2}u$$

$$u'' = -\frac{P}{2}u' - \left(\frac{P}{2}\right)' u$$

$$u'' = \left(\frac{P}{2}\right)^2 u - \left(\frac{P}{2}\right)' u$$

If we are lucky enough, we get an euler equation or a constant coefficient second-order ODE.

✻ These reduction methods, if suitable, can be used to solve any linear second-order ODE. Not all ODEs have known closed form solutions. Other solving methods include numerical analysis and infinite sereis expansions.

Lec. 8: Higher-Order Linear ODEs

8 Systems of Linear Differential Equations

⁕ A system of ordinary differential equations is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a set of differentiable functions defined on a common interval that satisfy each equation of the system on this interval.

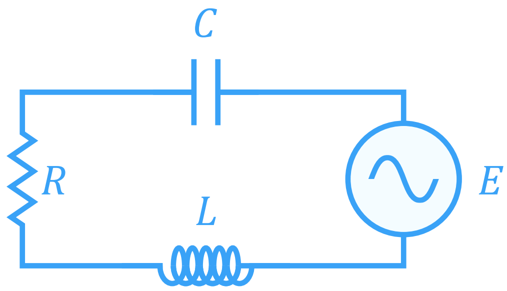
$$\begin{pmatrix} f_1(x, y, y', y'', \dots, y^{(n)}) \\ f_2(x, y, y', y'', \dots, y^{(n)}) \\ \vdots \\ f_3(x, y, y', y'', \dots, y^{(n)}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{y}(x) = \left(y_1(x), y_2(x), \dots, y_m(x)\right)$$

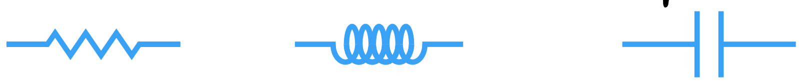
9 Applications of ODEs as Linear models

LRC-Series Circuits

⁕ If $i(t)$ denotes current in an LRC-series electrical circuit, then the voltage drops across the inductor, resistor, and capacitor are:



Resistor	Inductor	Capacitor
Ri	$L\dfrac{di}{dt}$	$\dfrac{1}{C}q$



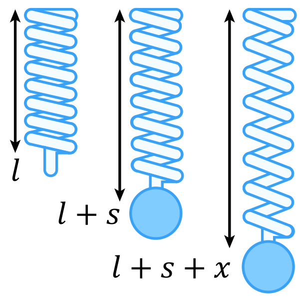
By Kirchhoff's second law, the sum of these voltages equals the voltage $E(t)$ impressed on the circuit, and we get a second-order ODE.

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$$

since $\quad i = \dfrac{dq}{dt}$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Spring/Mass Systems: Free Undamped Motion



At equilibrium $\quad mg - ks = 0$

Newton's 2nd Law $\quad \Sigma F = ma$

$$m\frac{d^2x}{dt^2} = -k(x+s) + mg = -kx$$

Angular frequency $\quad \omega^2 = k/m$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \textit{simple harmonic motion}$$

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

$F_s = -k\Delta l$

Hooke's Law

Newton's Law of Cooling / Warming

$$\dot{T} = k(T - T_m)$$

rate of change of temperature ($\dot{T}$); constant (k); temperature of the medium around an object (T_m)

→ Lec. 9: Laplace Transform $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt$$

is said to be the Laplace transform of $f(t)$, provided the integral converges.

✻ The Laplace transform, named after the famous French astronomer and mathematician Pierre-Simon Marquis de Laplace (1749–1827), is a method of transforming a function of a real variable (usually time) into a function of a complex variable (usually frequency). This integral transform can be used to solve differential equations, analyze systems, and study signals.

To understand what is a complex frequency variable, we note that the laplace transform can be seen as a generalization of the Fourier transform defined as:

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s=i\omega} = \int_{-\infty}^{\infty} e^{-i\omega} f(t)\, dt$$ where ω is angular frequency

The s domain is called complex frequency domain where $s = \sigma + i\omega$

The complex frequency s can be interpreted as a combination of two effects: damping and oscillation. The real part σ represents the damping factor, which determines how fast a signal decays or grows over time. The imaginary part $i\omega$ determines how fast a signal oscillates over time.

Throughout our studies, we assume that s is a real variable. The above definition of the Laplace transform is called the unilateral Laplace transform.

Sufficient Conditions for Existence

A function f is said to be of exponential order if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$

A function f is said to be piecewise continuous if it is continuous on a partition of open intervals of its domain and at the boundaries of the intervals the function has well-defined and finite limits.

If f is piecewise continuous on the interval $[0, \infty)$ and of exponential order,

then $\mathcal{L}\{f(t)\}$ exists for $s > c$, and $\lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0$

The existence conditions are sufficient but not necessary

→Lec. 10: Properties of Laplace Transform

Transforms of Some Basic Functions

$$\mathcal{L}\{1\} = \frac{1}{s}\ ,\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}\ ,\ \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}\ ,\ \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}\ ,\ \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}\ ,\ \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

Gamma Function: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$ when $n > 0$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n},\quad n \in \mathbb{R}/\{0,-1,-2,\dots\}$$

$\Gamma(n+1) = n!$ when $n = 0,1,2,\dots$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Properties of the Laplace Transform

Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then:

$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$ Linearity

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s} F(s)$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$\mathcal{L}\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(x)\,dx$ when $\lim_{t\to 0} \frac{1}{t} f(t)$ *exists*

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ first translation theorem

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$ second translation theorem

Heaviside step function:
unit step function

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t \end{cases}$$

if $f(t) = f(t+T)$ then $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$ where $f * g = \int_0^t f(x) g(t-x)\,dx$

convolution

→ Lec. 11: Inverse Laplace Transform

If $F(s)$ represents the Laplace transform of a function $f(t)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \qquad \text{where} \qquad \mathcal{L}\{f(t)\} = F(s)$$

* The inverse Laplace transform is linear, $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$

If $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if the function $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$.

* Solving Linear ODEs

* Note:

Laplace transforms can also solve **integro-differential equations**, equations involving both integrals and derivatives.

* The Laplace transform is ideally suited for solving linear initial-value problems in which the differential equation has constant coefficients.

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t)$$

$$y(0) = y_0, y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}$$

$$a_n \mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1} \mathcal{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$a_n\left[s^n Y(s) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0)\right]$$

$$+ a_{n-1}\left[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{(n-2)}(0)\right] + \cdots + a_0 Y(s) = G(s)$$

$$P(s) Y(s) = Q(s) + G(s) \qquad Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

1. Find unknown $y(t)$ that satisfies a DE and initial conditions

$\mathcal{L}$

2. Transformed DE becomes an algebraic equation in $Y(s)$

3. Solve transformed equation for $Y(s)$

$\mathcal{L}^{-1}$

4. Solution $y(t)$ of original IVP

→Lec. 12: Partial Differential Equations

* If u denotes a dependent variable and x and y its independent variables, then the general form of a linear second-order partial differential equation is given by.

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

homogeneous $G(x,y)=0$

nonhomogeneous $G(x,y)\neq 0$

where the coefficients $A,B,C,\dots,G$ are constants or functions of x and y .

* Separation of Variables:-

* In this method, if we are seeking a particular solution of, say, a linear second-order PDE in which the independent variables are x and y , then we seek to find a particular solution in the form of product of a function of x and a function of y . With this assumption, it is sometimes possible to reduce a linear PDE in two variables to two ODEs.

$$u(x,y)=X(x)Y(y)$$

$$\frac{\partial u}{\partial x}=X'Y \qquad \frac{\partial u}{\partial y}=XY'$$

$$\frac{\partial^2 u}{\partial x^2}=X''Y \qquad \frac{\partial^2 u}{\partial y^2}=XY''$$

* Note: $u_x=\frac{\partial u}{\partial x}$ $\quad u_{xx}=\frac{\partial^2 u}{\partial x^2}$

Superposition principle

If $u_1,u_2,\dots,u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u=c_1u_1+c_2u_2+\cdots+c_ku_k$$

where the c_i , $i=1,2,\dots,k$ are constants, is also a solution.

* The Laplace operator is a second-order differential operator in Euclidean space.

Three dimensional Lapalcian: $$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

Classical PDEs and Boundary-Value Problems

* **Heat Equation:** It is used in the theory of heat flow to model how a quantity such as heat diffuses through a given region, where T is temperature and α is a constant called thermal diffusivity.

$$\frac{\partial T}{\partial t}=\alpha\nabla^2 T$$

* **Laplace's Equation:** It can be interpreted as the steady-state heat equation that describes the steady-state temperature distribution. It also occurs in other time-independent problems.

$$\nabla^2 T=0$$

* **Wave Equation:** It describes waves or classical standing wave fields such as mechanical waves or electromagnetic waves, where u is the displacement and c is the constant speed of the wave.

$$\frac{\partial^2 u}{\partial t^2}=c^2\nabla^2 u$$

1 One-dimensional heat equation

* Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero for all time $t > 0$. With thermal diffusivity equal to a positive constant k, the temperature $u(x,t)$ in the rod is determined from the following **boundary-value problem**

$$ku_{xx} = u_t$$

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < L$$

"One-dimensional" refers to the fact that x denotes a spatial dimension

By prescribing what happens at time equal zero, we can give an **initial condition (IC)**.

Constrains about the function at the borders are **boundary conditions (BC)**.

* method of solution:

① $u(x,t) = X(x)T(t)$

② $kX''T = XT'$ since $ku_{xx} = u_t$

③ $\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda$ separation constant

can not contain t + can not contain x = must be a constant

④ $X'' + \lambda X = 0$, $T' + k\lambda T = 0$ **two ODEs**

⑤ solving the equations for different cases, where $\alpha > 0$

second-order linear homogeneous ODE

$X(x) = c_1 + c_2 x$ $\quad \lambda = 0$

$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$ $\quad \lambda = -\alpha^2 > 0$

$X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$ $\quad \lambda = \alpha^2 < 0$

first-order linear homogeneous ODE

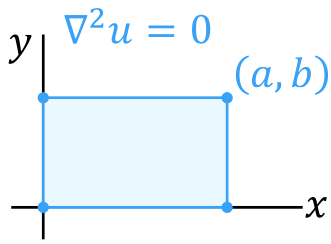
$T(t) = c_3 e^{-k\lambda t}$ in all different cases

⑥ by applying any given BC or IC, we can find the particular solution to the BVP.

2 Laplace's equation in two dimensions

$$u_{xx} + u_{yy} = 0$$

* We use this equation when we wish to find the steady-state temperature distribution $u(x,y)$ along two cartesian coordinates.



3 one-dimensional wave equation

$$c^2 u_{xx} = u_{tt}$$

* The function $u(x,t)$ denotes the vertical displacement of any point on a stretched vibrating string of length L measured from the x-axis.