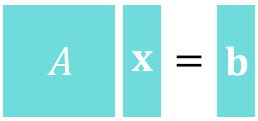
Linear Systems of Equations

$T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$A_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$$

$T(\mathbf{x}) = A\mathbf{x}$

$m = n$ square matrix



inconsistent
$\mathbf{b} \notin C(A)$
$\rho(A) \neq \rho(augA)$
$|A| = 0$
$\rho(A) < n$

consistent
$\mathbf{b} \in C(A)$
$\rho(A) = \rho(augA)$

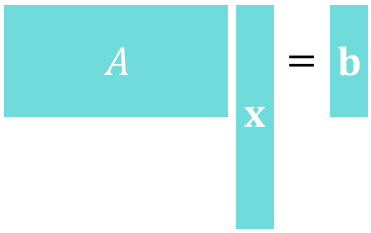
unique
$|A| \neq 0$
$\mathbf{x} = A^{-1}\mathbf{b}$
$\rho(A) = n$

infinite
$|A| = 0$
$\rho(A) < n$

full rank
onto
one-to-one

isomorphism

$m < n$ under-determined



$\rho(A) < n$

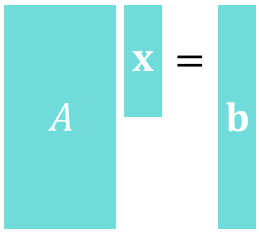
inconsistent
$\mathbf{b} \notin C(A)$
$\rho(A) \neq \rho(augA)$
$\rho(A) < m$

consistent
$\mathbf{b} \in C(A)$
$\rho(A) = \rho(augA)$

infinite
$\rho(A) < m$

infinite
$\rho(A) = m$
full rank
onto

$m > n$ over-determined



$\rho(A) < m$

inconsistent
$\mathbf{b} \notin C(A)$
$\rho(A) \neq \rho(augA)$

$\rho(A) = n$
full rank
one-to-one

$\rho(A) < n$

consistent
$\mathbf{b} \in C(A)$
$\rho(A) = \rho(augA)$

unique
$\rho(A) = n$
full rank
one-to-one

infinite
$\rho(A) < n$

Important Notes:

m = number of equations

n = number of unkowns

$C(A) = \text{range}(T)$

$\mathbf{rank}(A) = \dim(C(A)) = \rho(A)$ = number of leading 1's in the row-echelon form

$\mathbf{nullity}(A) = \dim(\ker(T))$ = number of free variables in the row-echelon form

$\text{rank}(A) + \text{nullity}(A) = n$

$augA = [A : \mathbf{b}]$

full rank: $\mathbf{rank}(A) = \min(m, n)$

Onto Transformation: $\mathbf{rank}(A) = m$ (all rows have pivots in row-echelon form)

One-to-one Transformation: $\mathbf{rank}(A) = n$ (all columns have pivots in row-echelon form)