How to send qubits over a quantum channel ?

$M = B(H_M) \qquad A = B(H_A) \qquad B = B(H_B)$

$\dim H_M = M$

$$|\varphi\rangle \qquad M \xrightarrow{E} A \xrightarrow{\Phi} B \xrightarrow{D} M$$

$$\rho \qquad E(\rho) \qquad \Phi\circ E(\rho) \qquad D\circ\Phi\circ E(\rho)$$

Encoder : quantum channel $E : B(H_M) \longrightarrow B(H_A)$

Decoder : - - — - $D : B(H_B) \longrightarrow B(H_M)$

Error : quantify how $D\circ\Phi\circ E \approx Id_M$

Trace Distance : for $\rho, \sigma \in D(H)$, $\|\rho-\sigma\|_1 = tr(|\rho-\sigma|)$

Channel distance : for $\Phi, \Psi : D(H) \to D(K)$ quantum channel,

$$\|\Phi - \Psi\| = \sup_{tr(|X|)=1} \frac{1}{2}\|\Phi(X) - \Psi(X)\|_1$$

$$\sup_{\rho\in D(H)} \|\Phi(\rho)-\Psi(\rho)\|_1 \le \quad \|\Phi-\Psi\| \le 2\sup_{\rho\in D(H)} \|\Phi(\rho)-\Psi(\rho)\|_1$$

max error over all possible input

$$\varepsilon(E, D) = \|D\circ\Phi\circ E - Id_M\|$$

$$\varepsilon^*(\Phi, M) = \inf_{(E,D)} \|D\circ\Phi\circ E - Id_M\|$$

$$M^*(\Phi, \varepsilon) = \sup\{M \mid \exists\, (M,\varepsilon)\text{-code for } \Phi\}$$

Def (Quantum Capacity of Φ)

$$Q(\Phi) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log M^*(\Phi^{n}, \varepsilon)}{n}$$

Coherent Information

Given $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$.

$$I(A>B)_\rho = H(B)_\rho - H(AB)_\rho = -H(A|B)_\rho$$

Lemma. $I(A>B)_\rho$ is convex over ρ

Pf: $H(A|B)_\rho = \log d_A - D(\rho_{AB} \| \frac{1}{|A|} \otimes \rho_B)$

D is jointly convex $\Rightarrow$ $H(A|B)_\rho$ is concave

Corollary: $I(A>B)_\rho \leq 0$ for all seperable state.

For a quantum channel

$$I_c(\Phi) = \sup_{\substack{\rho_{AA} \\ \in \\ D(\mathcal{H}_A \otimes \mathcal{H}_A)}} I(A>B)_{Id_A \otimes \Phi(\rho)}$$

Theorem: (Lloyd - Shor - Devetak)

$$Q(\Phi) = \lim_{n \to \infty} \frac{I_c(\Phi^{\otimes n})}{n}$$

- $\lim_{n\to\infty} \frac{I_c(\Phi^{\otimes n})}{n}$ is the regularization of $I_c(\Phi)$

Define $M_{sep}(\Phi^n, \varepsilon) = \sup\{M \mid \exists\ (M, \varepsilon)\text{-code for } \Phi^n\}$ s.t. Range (E) are all seperable seates

In general $I_c(\Phi) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log M_{sep}(\Phi^n, \varepsilon)}{n} \leq Q(\Phi) = \sup_n \frac{I_c(\Phi^{\otimes n})}{n}$

capacity for seperable coding $\qquad$ Capacity for general coding

Example: $\Phi : M_2 \to M_2$ $\quad \Phi(\rho) = \lambda\rho + (1-\lambda)\,\rho$

For $\lambda \approx 0.23$, $\quad \dfrac{I_c(\Phi^{\otimes 5})}{5} > I_c(\Phi)$

- Weak converse again by data processing inequality

- A channel Φ is called entanglement breaking if

  $\forall\ \rho_{AR} \in B(H_A \otimes H_R)$

  $\Phi \otimes id_R(\rho_{AR})$ is seperable

  Φ is entanglement-break iff C_Φ is seperable

  $\Phi_\lambda(\rho) = \lambda\rho + (1-\lambda)\frac{1}{2}$ is entanglement breaking iff $0 \le \lambda \le \frac{1}{3}$

  For entanglement breaking Φ

  $I_c(\Phi) = Q(\Phi) = 0$ $\quad$ No qubits can be sent through

- $Q(N)$ is really hard to computed ingeneral

  No control for $\lim_{n\to\infty} \dfrac{I_c(\Phi)^{\otimes n}}{n}$

$\forall n$ $\quad \exists\ \Phi$ s.t. $I_c(\Phi^{\otimes n}) = 0$ $\quad$ but $I_c(\Phi^{\otimes(n+1)}) = 0$

- Superactivation (Smith & Yard)

  $\exists\ \Phi_1, \Phi_2$ s.t. $Q(\Phi_1) = Q(\Phi_2) = 0$, $Q(\Phi_1 \otimes \Phi_2) > 0$

  (Note that $Q(\Phi \otimes \Phi) = 2\,Q(\Phi)$)

- Good case. $\Phi(\rho) = tr_E(V^* \rho V)$ $\quad V: H_A \to H_B \otimes H_E$

  $\Phi^c(\rho) = tr_A(V^* \rho V)$ complementary channel

Note that for a pure state $|\varphi\rangle_{AR}$ $\quad H(A)_\varphi = H(R)_\varphi$

$$I_c(\Phi) = \sup_{|\varphi_{AR}\rangle} H(B) - H(BR) \qquad |\psi_{BER}\rangle = V|\varphi_{AR}\rangle$$

$$= \sup_{\rho} H(B)_{\Phi(\rho)} - H(E)_{\Phi^c(\rho)}$$

Φ is called degradable if $\Phi^c = \Psi \circ \Phi$ for some channel Ψ

Theorem: If Φ_1 & Φ_2 degradable,

$$I_c(\Phi_1 \otimes \Phi_2) = I_c(\Phi_1) + I_c(\Phi_2)$$

Pf: $\quad |\psi\rangle_{B_1B_2E_1E_2R} = V_1 \otimes V_2 |\varphi_{A_1A_2R}\rangle$

$$I_c(\Phi_1 \otimes \Phi_2) = I_c(R > B_1B_2)_\rho$$

$$= H(B_1B_2) - H(RB_1B_2)$$

$$= H(B_1B_2) - H(E_1E_2)$$

$$= H(B_1) + H(B_2) - H(E_1) - H(E_2)$$

$$- [I(B_1 : B_2) - I(E_1 : E_2)] \qquad (DPI)$$

$$\leq H(B_1) - H(E_1) + H(B_2) - H(E_1)$$

$$= I_c(\Phi_1) + I_c(\Phi_2)$$

Cor If Φ degrable, $Q(\Phi) = I_c(\Phi)$

Pf: Φ degrable $\Rightarrow$ Φ^n degrable

Example: (Bit flip) $\Phi_\lambda(\rho) = \lambda X\rho X + (1-\lambda)\rho$ degrable $\quad \lambda \in [0, \tfrac{1}{2})$

$$\Phi_\lambda^c(\rho) = \begin{bmatrix} \lambda & \sqrt{\lambda(1-\lambda)}\, tr(X\rho) \\ \sqrt{\lambda(1-\lambda)}\, tr(X\rho) & 1-\lambda \end{bmatrix} \qquad I_c(\Phi_\lambda) = (1-2\lambda)$$

dephasing

Example: (Schur Multiplier) $\Phi_a(\rho) = [\rho_{ij} \cdot a_{ij}]_{i,j=1}^n$ $\quad [a_{ij}] \geq 0$

$a_{ii} = 1 \ \forall i$