University of Houston

Classical and Quantum Information Theory

Math 6397

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Contents

1 Overview

Information theory studies the processing, quantification, storage, and communication of information.

- 1948 — Claude Shannon defines *Shannon Entropy* in “The Mathematical Theory of Communication.” Answers questions:
    1. What is information?
    2. How do we quantify information?
    3. How do we transmit information?
- 2001 — Shannon Award is created, with Shannon the first recipient.
- 1900 — Max Plank describes Black-body Radiation
- 1920s — Heisenberg, Bohr, and Schrödinger, Matrix Mechanics
- 1930s — Hilbert, Dirac, Von Neumann describe the Hilbert Space, Mathematical foundation of Quantum Mechanics, and Von Neumann Entropy
- Interaction: Quantum Information
- 1950s – 1970s — Mathematical Quantities of Information
- 1970s
    - Information Transmission by Coherent Laser
    - Alexander Holevo — Holevo Bound
        - 1998 — Holevo et al show bound is tight (receive 2017 Shannon Award)
- 1980s — Richard Feynman: Computing with Quantum Mechanical Model
- 1990s — Peter Schor: Quantum Algorithm for Prime Factorization
    - In general, the only known algorithm for determining the prime factors of a number is naïve factorization. For example, given $n = 4801 \times 35317 = 169556917$, to retrieve the factors 4801 and 35317 requires substantially more time than to simply construct the number via multiplication.
- let’s finish the rest of the trivia chapter later

2 Probability Theory

A discrete probability space $(\Omega, \mathbb{P})$ is given by

- a finite or countably infinite set Ω
  - e.g. $\{a, b, c, d\}$, $\mathbb{N} = \{0, 1, 2, \dots\}$
- a probability mass function $\mathbb{P} : \Omega \to [0, 1]$, such that

  (1) For all $\omega \in \Omega$, $\mathbb{P}(\omega) \geq 0$

  (2) $\displaystyle\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

  For $\omega \in \Omega$, $\mathbb{P}(\omega)$ is the probability that ω "occurs"

Definition 2.1 ▶ Event

Given a probability space $(\Omega, \mathbb{P})$, an *event* E is a subset $E \subseteq \Omega$, with corresponding probability

$$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$$

The function $\mathbb{P} : \Omega \to [0, 1]$ induces a *probability distribution*,

$$\mathbb{P} : 2^{\Omega} \to [0, 1]$$

also denoted by $\mathbb{P}$, with properties:

(1) if $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

(2) $\mathbb{P}(\Omega) = 1$

As an abuse of notation, we write $\mathbb{P}(\omega)$ and $\mathbb{P}(\{\omega\})$ interchangeably.

Example 2.1 ▶ Rolling a fair die

TBD

Definition 2.2 ▶ Conditional Probability

Let $A, B \subseteq \Omega$. The *conditional probability* of A given B, denoted by $\mathbb{P}(A \mid B)$, is defined

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Example 2.2 ▶ Fair Die Revisited

TBD

Theorem 2.1 ▶ Bayes' Rule

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\,\mathbb{P}(B)}{\mathbb{P}(A)}$$

Proof. By definition,

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

hence

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(B \mid A)\,\mathbb{P}(A) \\ &= \mathbb{P}(A \mid B)\,\mathbb{P}(B)\end{aligned}$$

from which the result follows. □

Example 2.3 ▶ Flipping a fair coin twice

TBD

Definition 2.3 ▶ Random Variable

A *Random Variable* X is a function

$$X : \Omega \to \mathcal{X}$$

from probability space $(\Omega, \mathbb{P})$ to a target space $\mathcal{X}$. We say X is discrete if $\mathcal{X}$ is discrete and call

$$\mathcal{X} = \{x_1, x_2, \dots\}$$

the *alphabet* of X.

Notice that X induces a distribution on $\mathcal{X}$. For any $x \in \mathcal{X}$

$$\mathbb{P}_X(x) = \mathbb{P}(\{\omega \mid X(\omega) = x\})$$

In many cases, $(X.\mathbb{P}_x)$ captures all information needed from random variable X. We write $X \sim \mathbb{P}_x$ to indicate that X has distribution $\mathbb{P}_x$ on $\mathcal{X}$.

Example 2.4 ▶ 52 Card Deck

TBD

Definition 2.4 ▶ Joint Distribution

Let $X : \Omega \to \mathcal{X}$, $Y : \Omega \to \mathcal{Y}$ be two random variables. The *joint distribution* on $\mathcal{X} \times \mathcal{Y}$ is given by

$$\mathbb{P}_{XY}(X = x, Y = y) = \mathbb{P}(\{X(\omega) = x, Y(\omega) = y\})$$

For subsets $E_1 \subseteq \mathcal{X}$, $E_1 \subseteq \mathcal{Y}$

$$\mathbb{P}_{XY}(X \in E_1, Y \in E_2) = \mathbb{P}(\{X(\omega) \in E_1, Y(\omega) \in E_2\})$$

Notice that $\mathbb{P}_{XY}$ is a distribution on the product space $(\mathcal{X} \times \mathcal{Y}, \mathbb{P}_{XY})$.

Example 2.5 ▶ Fair Die Joint Distribution

TBD

Example 2.6 ▶ Flipping a fair coin twice joint distribution

TBD

Definition 2.5 ▶ Independent Random Variables

Two random variables X and Y are *independent* if, for any x, y

$$\mathbb{P}_{XY}(X = x, Y = y) = \mathbb{P}_X(X = x)\,\mathbb{P}_Y(Y = y)$$

Equivalently, if for any subsets E_1 and E_2

$$\mathbb{P}_{XY}(X \in E_1, Y \in E_2) = \mathbb{P}_X(X \in E_1)\,\mathbb{P}_Y(Y \in E_2)$$

Definition 2.6 ▶ Product Probability

Given two probability spaces $(\Omega_1, \mathbb{P}_1)$, $(\Omega_2, \mathbb{P}_2)$

$$\mathbb{P}_1 \times \mathbb{P}_2(E_1 \times E_2) = \mathbb{P}_1(E_1)\,\mathbb{P}_2(E_2)$$

is the product probability on $\Omega_1 \times \Omega_2$.

Thus, we have the property that X and Y are independent random variables if and only if $\mathbb{P}_{XY} = \mathbb{P}_X \times \mathbb{P}_Y$.

Example 2.7 ▶ Rank and Suit of a card

TBD

Definition 2.7 ▶ Real Random Variable

A *Real Random Variable* is a function

$$X : \Omega \to \mathbb{R}$$

For example, the height of a randomly sampled person, the value of a die, and the rank of a playing card (where Ace is 1, Jack is 11, Queen is 12, and King is 13) are all real random variables. On the other hand, the suit of a playing card is *not* a real random variable.

In the discrete case, if $X : \Omega \to \mathcal{X}$ is a random variable, then

$$\mathbb{P}_X : X \to [0, 1]$$

is a real random variable.

Definition 2.8 ▶ Conditional Distribution

Given two random variables X and Y, the conditional distribution is the real random variable given by

$$\mathbb{P}_{X|Y} : \mathcal{X} \times \mathcal{Y} \to [0, 1]$$

where

$$\mathbb{P}_{X|Y}(x \mid y) = \mathbb{P}(X = x \mid Y = y)$$

Given two real random variables X and Y, we can define

- $X + Y$
- $X \cdot Y$
- $f(X)$ (where $f : \mathbb{R} \to \mathbb{R}$)

as new random variables.

Definition 2.9 ▶ Expectation and Variance

The *expected value* (or expectation or mean) of a real random variable X is defined as the real number

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x\,\mathbb{P}_X(X = x) = \sum_{\omega \in \Omega} X(\omega)\,\mathbb{P}(X = \omega)$$

The *variance* is defined as

$$\mathrm{Var}[X] = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big]$$

Example 2.8 ▶ Expected Value and Variance of a Fair Die

TBD

Theorem 2.2 ▶ Linearity of Expectation

Let X and Y be real random variables and $a, b \in \mathbb{R}$. Then

$$\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$$

Proof. By definition,

$$\begin{aligned}
\mathbb{E}[aX + bY] &= \sum_{\omega\in\Omega} (aX(\omega) + bY(\omega))\,\mathbb{P}(\omega) \\
&= \sum_{\omega\in\Omega} aX(\omega)\,\mathbb{P}(\omega) + \sum_{\omega\in\Omega} bY(\omega)\,\mathbb{P}(\omega) \\
&= a \sum_{\omega\in\Omega} X(\omega)\,\mathbb{P}(\omega) + b \sum_{\omega\in\Omega} Y(\omega)\,\mathbb{P}(\omega) \\
&= a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]
\end{aligned}$$

□

Corollary 2.3

$$\mathrm{Var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$$

Proof. By definition,

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\
&= \mathbb{E}\left[X^2 - 2X\,\mathbb{E}[X] + \mathbb{E}[X]^2\right] \\
&= \mathbb{E}\left[X^2\right] - \mathbb{E}[2X\,\mathbb{E}[X]] + \mathbb{E}\left[\mathbb{E}[X]^2\right] \\
&= \mathbb{E}\left[X^2\right] - 2\,\mathbb{E}[X]^2 + \mathbb{E}[X]^2 \\
&= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2
\end{aligned}$$

□

If X and Y are independent, we have the following

Theorem 2.4

Let X and Y be independent real random variables. Then

(1) $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$

(2) $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

Proof. First, Item (1):

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} xy\,\mathbb{P}_{XY}(X = x, Y = y) \\
&= \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} xy\,\mathbb{P}_X(X = x)\,\mathbb{P}_Y(Y = y) \text{ since } X \text{ and } Y \text{ are independent} \\
&= \sum_{x\in\mathcal{X}} x\,\mathbb{P}_X(X = x) \sum_{y\in\mathcal{Y}} y\,\mathbb{P}_Y(Y = y) \\
&= \mathbb{E}[X]\,\mathbb{E}[Y]
\end{aligned}$$

Now,

$$\begin{aligned}
\mathrm{Var}[X + Y] &= \mathbb{E}\left[(X + Y)^2\right] - \mathbb{E}[X + Y]^2 \\
&= \mathbb{E}\left[X^2 + 2XY + Y^2\right] - (\mathbb{E}[X] + \mathit{expectation}Y)^2 \\
&= \mathbb{E}\left[X^2\right] + 2\,\mathbb{E}[XY] + \mathbb{E}\left[Y^2\right] - \mathbb{E}[X]^2 - 2\,\mathbb{E}[X]\,\mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y] \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] \text{ by Item (1)}
\end{aligned}$$

□

Definition 2.10

A sequence of random variables X_1, X_2, ..., X_n is independent and identically distributed from $\mathbb{P}_X$ (i.i.d $\sim \mathbb{P}_X$) if

(1) for all i, $X_i \sim \mathbb{P}_x$

(2) X_1, X_2, ..., X_n are mutually independent, i.e., for any $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$

$$\mathbb{P}(X_{i_1} X_{i_2} \ldots X_{i_k}) = \mathbb{P}(X_{i_1})\,\mathbb{P}(X_{i_2}) \ldots \mathbb{P}(X_{i_k})$$

Theorem 2.5 ▶ The Weak Law of Large Numbers (WLLN)

Let X_n be an infinite i.i.d. sequence drawn from $\mathbb{P}_X$. Write

$$\hat{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

and suppose $\mathrm{Var}[X]$ and $\mathbb{E}[X]$ are both finite. Then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\hat{X}_n - \mathbb{E}[X]\right| < \varepsilon\right) = 1$$

We first show the following two lemmas.

Lemma 2.6 ▶ Markov's Inequality

Let X be any non-negative random variable and $a > 0$. Then

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

Proof. Define the indicator random variable

$$1_{X\geq a} = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{if } X < a \end{cases}$$

and notice that $\mathbb{E}[1_{X\geq a}] = \mathbb{P}(X \geq a)$. Clearly, $X \geq a1_{X\geq a}$, hence

$$\mathbb{E}[X] \geq a\,\mathbb{E}[1_{X\geq a}] = a\,\mathbb{P}(X \geq a)$$

from which the result follows. □

Lemma 2.7 ▶ Chebyshev's Inequality

Let X be any random variable with finite variance. Then

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \geq \varepsilon^2\big) \leq \frac{\mathrm{Var}[X]}{\varepsilon}$$

for any $\varepsilon > 0$.

Proof. Set $Y = (X - \mathbb{E}[X])^2$ and notice that $\mathbb{E}[Y] = \mathrm{Var}[X]$. Then,

$$\begin{aligned} \mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) &= \mathbb{P}\big(Y \geq \varepsilon^2\big) \\ &\leq \frac{\mathbb{E}[Y]}{\varepsilon^2} \text{ by Markov's Inequality} \\ &= \frac{\mathrm{Var}[X]}{\varepsilon^2} \end{aligned}$$

□

Now, we prove Theorem 2.5.

Proof. First, notice that

$$\begin{aligned}\mathbb{E}\left[\hat{X}_n\right] &= \mathbb{E}\left[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right] \\ &= \frac{1}{n} \cdot n\,\mathbb{E}[X] \text{ by Linearity of Expectation} \\ &= \mathbb{E}[X]\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}\left[\hat{X}_n\right] &= \mathrm{Var}\left[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right] \\ &= \frac{1}{n^2}(\mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_n]) \\ &= \frac{1}{n^2} \cdot n\,\mathrm{Var}[X] \\ &= \frac{1}{n}\,\mathrm{Var}[X]\end{aligned}$$

then, by Chebyshev's Inequality,

$$\begin{aligned}\mathbb{P}\left(\left|\hat{X}_n - \mathbb{E}[X]\right| \geq \varepsilon\right) &\leq \frac{\mathrm{Var}\,\hat{X}_n}{\varepsilon^2} \\ &= \frac{\mathrm{Var}[X]}{n\varepsilon^2} \to 0 \text{ as } n \to \infty\end{aligned}$$

hence

$$\mathbb{P}\left(\left|\hat{X}_n - \mathbb{E}[X]\right| < \varepsilon\right) = 1 - \mathbb{P}\left(\left|\hat{X}_n - \mathbb{E}[X]\right| \geq \varepsilon\right) \to 1 \text{ as } n \to \infty$$

□

Example 2.9 ▸ Bernoulli Random Variable

TBD

Definition 2.11 ▸ Vector Valued Random Variable

Let

$$X = (X_1, X_2, \ldots, X_n) : \Omega \to \mathbb{R}^n$$

finish this part — part in notes is a bit cryptic

3 Entropy

Definition 3.1 ▶ Entropy

Let $\mathbb{P}$ be a probability distribution on a discrete space Ω. The Shannon Entropy (hereby simply Entropy) of $\mathbb{P}$ is defined

$$H(\mathbb{P}) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) \log \frac{1}{\mathbb{P}(\omega)}$$

If X is a discrete random variable, we define

$$\begin{aligned} H(X) &= H(\mathbb{P}_X) \\ &= \sum_{x \in X} \mathbb{P}_X(x) \log \frac{1}{\mathbb{P}_X(x)} \\ &= \mathbb{E}\left[\log \frac{1}{\mathbb{P}_X(X)}\right] \end{aligned}$$

noting that $\log \frac{1}{\mathbb{P}_X(X)}$ is a real random variable.

We can think of $\log \frac{1}{\mathbb{P}_X(x)}$ as the level of "surprise" that $X = x$ occurs and $H(X)$ as the uncertainty or randomness of $\mathbb{P}_X$.

Note that, in Definition 3.1, log refers to $\log_2$, and $\log_2(X)$ is the number of bits of X. Additionally, since a byte is 8 bits, $\log_{256}(X)$ is the number of bytes of X. Additionally, we define $0 \log \frac{1}{0} = 0$, which can be motivated by the fact that

$$\lim_{x \to 0^+} x \log \frac{1}{x} = 0$$

Example 3.1 ▶ Bernoulli Distribution

The Bernoulli Distribution is the discrete random variable

$$\begin{aligned} \mathbb{P}(X = 1) &= p \\ \mathbb{P}(X = 0) &= 1 - p \end{aligned}$$

and has entropy

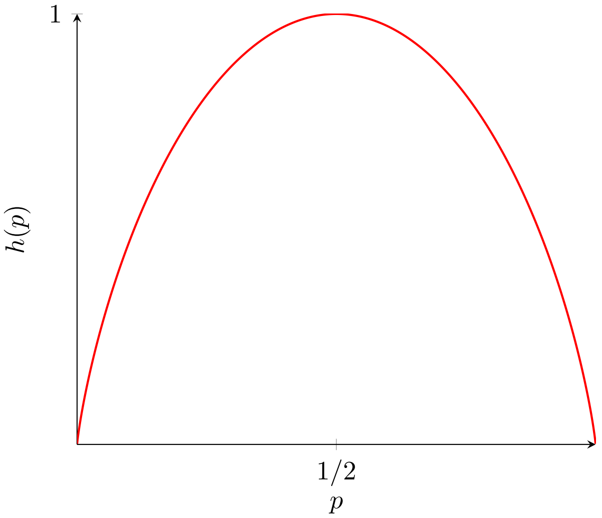
$$H(X) = p \log \frac{1}{p} + (1 - p) \log \frac{1}{1 - p}$$

Definition 3.2 ▶ Binary Entropy

The binary entropy of p, $h(p)$, is the entropy of the Bernoulli Distribution with parameter p, i.e.,

$$h(p) = p \log \frac{1}{p} + (1 - p) \log \frac{1}{1 - p}$$

Notice that $h(0) = h(1) = 0$ and $h\left(\frac{1}{2}\right) = 1$. More generally, the graph of $h(p)$ is given in Figure 1.



add some flourishes

Figure 1: Binary Entropy as a function of p. Notice that the entropy is maximized when $p = 1/2$ and 0 when $p = 0$ or $p = 1$. When $p = 0$ or $p = 1$, the Bernoulli Distribution is non-random, and thus there is no uncertainty.

Figure 2: Drawing of ant nest used to empirically verify ...

Example 3.2 ▶ Geometric Distribution

The Geometric Distribution is the positive, integer-valued random variable that describes the number of Bernoulli trials performed until a success. That is,

$$\mathbb{P}(X = k) = p(1-p)^{k-1}$$

is the probability that it will require k trials until a success.
The entropy of the Geometric Distribution is given by

$$\begin{aligned}
H(X) &= \sum_{k=1}^{\infty} p(1-p)^k \log \frac{1}{p(1-p)^k} \\
&= \sum_{k=1}^{\infty} p(1-p)^k \left(\log \frac{1}{p} + k \log \frac{1}{1-p} \right) \\
&= p \log \frac{1}{p} \sum_{k=1}^{\infty} (1-p)^k + p \log \frac{1}{1-p} \sum_{k=1}^{\infty} k(1-p)^k \\
&= p \frac{1}{p} \log \frac{1}{p} + p \log \frac{1}{1-p} \frac{1-p}{p^2} \\
&= \frac{1}{p} \left(p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p} \right) \\
&= \frac{h(p)}{p} \to 0 \text{ as } p \to 0^+
\end{aligned}$$

Example 3.3 ▶ Distribution with ∞ Entropy

TBD

An empirical justification for the use of $\log_2$.

Finish figure, caption, and description.

Definition 3.3 ▶ Convexity

Let $V \cong \mathbb{R}^n$ be a vector space. A subset $S \subseteq V$ is convex if, for any pair $\mathbf{x}, \mathbf{y} \in S$

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in S \text{ for all } \lambda \in [0, 1]$$

figure demonstrating convexity

Example 3.4

The following are convex

(1) $\mathbb{R}^n$

(2)

(3)

Definition 3.4 ▶ Convex Function

A function $f : S \to \mathbb{R}$ is

(i) convex if $f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S$ and $\lambda \in [0, 1]$

(ii) *strictly* convex if $f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S$ and $\lambda \in [0, 1]$

Definition 3.5 ▶ Concave Function

A function $f : S \to \mathbb{R}$ is (strictly) concave if $-f$ is (strictly) convex.

Example 3.5

Notice

(1) The function $x \to x \log x$ is strictly convex

(2) The function $x \to \log x$ is strictly concave

(3) The function $X \to \mathbb{E}[X]$ is convex (but not strictly)

Theorem 3.1 ▶ Jensen's Inequality

Let X be a real vector valued random variable. Then, if f is any convex function,

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

If f is strictly convex, then $f(\mathbb{E}[X]) = \mathbb{E}[f(X)]$ if and only if $X = \mathbb{E}[X]$, i.e., X is a constant random variable.

Proof. Since f is convex,

$$\begin{aligned} f(\mathbb{E}[X]) &= f\left(\sum_{x \in X} x \, \mathbb{P}(X = x)\right) \\ &\leq \sum_{x \in X} f(x) \, \mathbb{P}(X = x) \text{ since } f \text{ is convex and } \mathbb{P}(X = x) \in [0, 1] \\ &= \mathbb{E}[f(X)] \end{aligned}$$

□

Theorem 3.2 ▶ Properties of Entropy

The Entropy function satisfies

(1) $H(X) \geq 0$ with equality if and only if X is constant

(2) if $\mathcal{X}$ is finite, then $H(X) \leq \log|\mathcal{X}|$ with equality if and only if $\mathbb{P}_X$ is uniform on $\mathcal{X}$

(3) For any injective f, $H(X) = H(f(X))$

(4) $\mathbb{P} \to H(\mathbb{P})$ is strictly concave

Proof.

(1) $H(X) = \mathbb{E}\left[\log \frac{1}{\mathbb{P}_X}\right] \geq 0$ with equality if and only if $\log \frac{1}{\mathbb{P}_X} = 0$, which occurs only when $\mathbb{P}_X \equiv 1$.

(2) If $\mathcal{X}$ is finite, then

$$\begin{aligned} H(X) &= \mathbb{E}\left[\log \frac{1}{\mathbb{P}_X}\right] \\ &\leq \log \mathbb{E}\left[\frac{1}{\mathbb{P}_X}\right] \\ &= \log \sum_{x \in X} \mathbb{P}(x) \frac{1}{\mathbb{P}(X)} \\ &= \log|\mathcal{X}| \end{aligned}$$

with equality if and only if $\log \frac{1}{\mathbb{P}_x}$ is constant, which forces $\mathbb{P}(X) = \frac{1}{|\mathcal{X}|}$.

(3) If f is injective, then $\mathbb{P}_{f(X)}(f(x)) = \mathbb{P}_X(x)$, and the result follows.

(4) Take $\lambda \in [0,1]$ and write $f(x) = x \log \frac{1}{x}$, then

$$\begin{aligned} H(\lambda\mathbb{P}_1 + (1-\lambda)\mathbb{P}_2) &= \sum_{\omega \in \Omega} f(\lambda\, \mathbb{P}_1(\omega) + (1-\lambda)\, \mathbb{P}_2(\omega)) \\ &\geq \sum_{\omega \in \Omega} \lambda f(\mathbb{P}_1(\omega)) + (1-\lambda) f(\mathbb{P}_2(\omega)) \\ &= \lambda \sum_{\omega \in \Omega} f(\mathbb{P}_1(\omega)) + (1-\lambda) \sum_{\omega \in \Omega} f(\mathbb{P}_2(\omega)) \\ &= \lambda H(\mathbb{P}_1) + (1-\lambda) H(\mathbb{P}_2) \end{aligned}$$

□

4 Conditional Entropy

Definition 4.1 ▶ Joint Entropy

Given random variables X and Y, the *Joint Entropy*, $H(XY)$, is defined

$$H(XY) = \mathbb{E}\left[\log \frac{1}{\mathbb{P}_{XY}}\right] = \sum_{x\in X}\sum_{y\in Y} \mathbb{P}_{XY}(X = x, Y = y) \log \frac{1}{\mathbb{P}_{XY}(X = x, Y = y)}$$

Definition 4.2 ▶ Conditional Entropy

Let X and Y be random variables. Then

$$H(X \mid Y) = \mathop{\mathbb{E}}_{y\sim\mathbb{P}_Y}\left[H\big(\mathbb{P}_{X|Y=y}\big)\right] = \mathbb{E}\left[\log \frac{1}{\mathbb{P}_{X|Y}}\right]$$

This can be thought of as the expected uncertainty $H\big(\mathbb{P}_{X|Y=y}\big)$ over $y \sim \mathbb{P}_Y$.

Definition 4.3 ▶ Conditional Probability Notation

Some notation:

(1) $\mathbb{P}_{X|Y=y}$ is a distribution on X, with

$$\begin{aligned}\mathbb{P}_{X|Y=y}(x) &= \mathbb{P}(X = x \mid Y = y)\\ &= \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}\end{aligned}$$

(2) $\mathbb{P}_{X|Y}$ is a random variable on $\mathcal{X} \times \mathcal{Y}$ with

$$\mathbb{P}_{X|Y}(x, y) = \mathbb{P}(X = x \mid Y = y)$$

Example 4.1 ▶ Joint and Conditional Entropy of a Fair Die

TBD

Theorem 4.1 ▶ Properties of Conditional Entropy

Let X and Y be random variables. Then

(1) $H(X \mid Y) \leq H(X)$ with equality if and only if X and Y are independent

(2) $H(XY) = H(Y) + H(X \mid Y) \leq H(Y) + H(X)$ with equality if and only if X and Y are independent

(3) $H(XY) \geq \max\{H(X), H(Y)\}$

Proof.

(1)

$$\begin{aligned}H(X \mid Y) &= \mathop{\mathbb{E}}_{y\sim\mathbb{P}_Y}\left[H\big(\mathbb{P}_{X|Y=y}\big)\right]\\ &\leq H\left(\mathop{\mathbb{E}}_{y\sim\mathbb{P}_Y}\left[\mathbb{P}_{X|Y=y}\right]\right)\\ &= H(\mathbb{P}_X)\\ &= H(X)\end{aligned}$$

(2)

(3) $H(XY) = H(X) + H(Y \mid X) \geq H(X)$ The same argument shows $H(XY) \geq H(Y)$, hence it must be greater than or equal to the maximum of the two.

□

Corollary 4.2

For any function f

(1) $H(X) = H(Xf(X))$

(2) $H(f(X) \mid X) = 0$

(3) $H(X) \geq H(f(X))$ with equality if and only if f is injective

Proof. □

Dec	Hex	Char	Dec	Hex	Char	Dec	Hex	Char	Dec	Hex	Char
000	000	Null	032	020	Space	064	040	@	096	060	‘
001	001	Start of Heading	033	021	!	065	041	A	097	061	a
002	002	Start of Text	034	022	”	066	042	B	098	062	b
003	003	End of Text	035	023	#	067	043	C	099	063	c
004	004	End of Transmission	036	024	$	068	044	D	100	064	d
005	005	Enquiry	037	025	%	069	045	E	101	065	e
006	006	Acknowledgement	038	026	&	070	046	F	102	066	f
007	007	Bell	039	027	’	071	047	G	103	067	g
008	008	Backspace	040	028	(	072	048	H	104	068	h
009	009	Horizontal Tab	041	029	)	073	049	I	105	069	i
010	00a	Line Feed	042	02a	*	074	04a	J	106	06a	j
011	00b	Vertical Tab	043	02b	+	075	04b	K	107	06b	k
012	00c	Form Feed	044	02c	,	076	04c	L	108	06c	l
013	00d	Carriage Return	045	02d	-	077	04d	M	109	06d	m
014	00e	Shift Out	046	02e	.	078	04e	N	110	06e	n
015	00f	Shift In	047	02f	/	079	04f	O	111	06f	o
016	010	Data Link Escape	048	030	0	080	050	P	112	070	p
017	011	Device Control 1	049	031	1	081	051	Q	113	071	q
018	012	Device Control 2	050	032	2	082	052	R	114	072	r
019	013	Device Control 3	051	033	3	083	053	S	115	073	s
020	014	Device Control 4	052	034	4	084	054	T	116	074	t
021	015	Negative	053	035	5	085	055	U	117	075	u
022	016	Synchronous Idle	054	036	6	086	056	V	118	076	v
023	017	End of Trans. Block	055	037	7	087	057	W	119	077	w
024	018	Cancel	056	038	8	088	058	X	120	078	x
025	019	End of Medium	057	039	9	089	059	Y	121	079	y
026	01a	Substitute	058	03a	:	090	05a	Z	122	07a	z
027	01b	Escape	059	03b	;	091	05b	[	123	07b	{
028	01c	File Separator	060	03c	<	092	05c	\	124	07c	|
029	01d	Group Separator	061	03d	=	093	05d	]	125	07d	}
030	01e	Record Separator	062	03e	>	094	05e	ˆ	126	07e	˜
031	01f	Unit Separator	063	03f	?	095	05f	_	127	07f	

Figure 3: ASCII code for characters.

5 Lossless Data Compression

“Today is a Wednesday” is a sequence of letters, which can be converted into bytes (8 bits) via Figure 3.

One may ask if this is optimal. In fact, if using only English words, then we need only

$$2^5 = 32 < 26 \times 2 = 52 < 64 = 2^6$$

6 bits.

Definition 5.1 ▶ Lossless Compression

Let $\mathcal{X}$ denote some alphabet, and let f and g be functions:

$$\mathcal{X} \xrightarrow[\text{compressor}]{f} \{0,1\}^* \xrightarrow[\text{decompressor}]{g} \mathcal{X}$$

where $\{0,1\}^*$ is the set of all binary strings (including the empty string)[1]. The functions f and g are also often called the *encoder* and *decoder*, respectively.

We say that f and g form a *lossless compression scheme* if $g \circ f \equiv I_{\mathcal{X}}$ is the identity function on the alphabet. For each $x \in \mathcal{X}$, we call $f(x)$ the *code word* or *encoding* of x and refer to the set $f(X)$ as the *code book*.

Definition 5.2 ▶ Length of a Code Word

The length of a code word $\omega \in \{0,1\}^*$ is the number of bits in ω and is denoted $\ell(\omega)$. Note that

$$\ell : \{0,1\}^* \to \mathbb{N}$$

Notice that, given an alphabet $\mathcal{X}$, the maximal length of a compression must be $\log|\mathcal{X}|$, by the pigeonhole principle: enumerate the alphabet of $\mathcal{X}$, say $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Then we can map

$$\begin{aligned} x_1 &\to \emptyset \\ x_2 &\to 0 \\ x_3 &\to 1 \\ x_4 &\to 00 \\ &\vdots \end{aligned}$$

and clearly our maximum length is $\log|\mathcal{X}|$. However, given a distribution (a list of frequencies of the occurrences of the alphabet), we can reduce the *expected* code word length.

Example 5.1

Say $\mathcal{X} = \{a, b, c, d\}$. Of course, we can map

$$a \to 00 \quad b \to 01 \quad c \to 10 \quad d \to 11$$

and our expected codeword length will obviously be 2. On the other hand, say our alphabet has the following frequencies

Character	Frequency
a	1/2
b	1/8
c	1/4
d	1/8

We can map

$$a \to 0 \quad b \to 110 \quad c \to 10 \quad d \to 111$$

(called a variable length encoding, in constrast to the fixed length encoding given above), and see that our expected codeword length is

$$\frac{1}{2} \cdot 1 + \frac{1}{8} \cdot 3 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 = \frac{7}{4} < 2$$

i.e., we can do better than 2 bits per character.

In general, we determine the frequency of letters empirically, which we then model by a probability distribution. Our objective is to minimize both $\sup \ell(f(\mathcal{X})$ and $\mathbb{E}[\ell(f(\mathcal{X}))]$. In fact, there is an optimal compressor f^* which minimizes both. The core idea is to assign shorter code words to more frequently occurring characters.

[1] The * is called the Kleene Star.

Theorem 5.1 ▶ Optimal Compressor

Let $\mathcal{X}$ be an alphabet. Without loss of generality, say

$$\mathcal{X} = \{1, 2, \ldots, n\}$$

and that $\mathbb{P}_X(i) \geq \mathbb{P}_X(i+1)$, i.e., that the frequencies are in decreasing order. Then

(1) $\ell(f^*(i)) = \lfloor \log i \rfloor$

(2) For all k and any encoder f,

$$\mathbb{P}(\ell(f(x)) \leq k) \leq \mathbb{P}(\ell(f^*(x)) \leq k)$$

Example 5.2 ▶ Telegraph and Morse Code

TBD

First, we show the following lemma

Lemma 5.2

Let Z be a positive integer valued random variable with finite expectation. Then $H(Z) \leq \mathbb{E}[Z] H\left(\frac{1}{\mathbb{E}[Z]}\right)$.

Proof. Let Q_p denote the geometric distribution with parameter p, i.e., Q_p is the distribution with positive integer random variable X given by

$$\mathbb{P}(X = i) = p(1-p)^{i-1}$$

and recall that

$$H(Q_p) = \frac{h(p)}{p}$$

The relative entropy from Q to P, $D(P \parallel Q)$, is given by

$$D(P \parallel Q) = \sum_{\omega \in \Omega} P(\omega) \log \frac{P(\omega)}{Q(\omega)}$$

Notice that

$$\begin{aligned}
D(P \parallel Q) &= \sum_{\omega \in \Omega} P(\omega) \log \frac{P(\omega)}{Q(\omega)} \\
&= \sum_{\omega \in \Omega} P(\omega) \log P(\omega) - \sum_{\omega \in \Omega} P(\omega) \log Q(\omega) \\
&= \sum_{\omega \in \Omega} P(\omega) \log \frac{1}{Q(\omega)} - \sum_{\omega \in \Omega} P(\omega) \log \frac{1}{P(\omega)} \\
&= H(P, Q) - H(P)
\end{aligned}$$

where

$$H(P, Q) = \sum_{\omega \in \Omega} P(\omega) \log \frac{1}{Q(\omega)}$$

is the cross-entropy of P and Q. Further, $D(P \parallel Q) \geq 0$: since log is concave, we must have

$$
\begin{aligned}
D(P \parallel Q) &= \sum_{\omega\in\Omega} P(\omega) \log \frac{P(\omega)}{Q(\omega)} \\
&= -\sum_{\omega\in\Omega} P(\omega) \log \frac{Q(\omega)}{P(\omega)} \\
&\geq -\log\left(\sum_{\omega\in\Omega} P(\omega) \frac{Q(\omega)}{P(\omega)}\right) \\
&= -\log\left(\sum_{\omega\in\Omega} Q(\omega)\right) \\
&\geq -\log 1 \\
&= 0
\end{aligned}
$$

Now, set $p = 1/\mathbb{E}[Z]$[2] and notice that

$$
\begin{aligned}
H(Z, Q_p) &= \sum_{z=1}^{\infty} P(z) \log \frac{1}{p(1-p)^z} \\
&= \sum_{z=1}^{\infty} P(z) \left(\log \frac{1}{p} + z \log \frac{1}{1-p}\right) \\
&= \log \frac{1}{p} \sum_{z=1}^{\infty} P(z) + \log \frac{1}{1-p} \sum_{z=1}^{\infty} zP(Z) \\
&= \log \frac{1}{p} + \log \frac{1}{1-p} \, \mathbb{E}[Z] \\
&= \log \frac{1}{p} + \frac{1}{p} \log \frac{1}{1-p} \\
&= H(Q_p) \\
&= \mathbb{E}[Z] h\left(\frac{1}{\mathbb{E}[Z]}\right)
\end{aligned}
$$

Now, since $D(P \parallel Q) \geq 0$, we conclude that

$$
\begin{aligned}
D(Z \parallel Q_p) &= H(Z, Q_p) - H(Z) \\
&= H(Q_p) - H(Z) \\
&= \mathbb{E}[Z] h\left(\frac{1}{\mathbb{E}[Z]}\right) - H(Z)
\end{aligned}
$$

is greater than or equal to 0, hence

$$
H(Z) \leq \mathbb{E}[Z] h\left(\frac{1}{\mathbb{E}[Z]}\right) \qquad \square
$$

Now, we prove Theorem 5.1.

Proof. □

[2] Since $\mathbb{E}[Z] < \infty$ and Z is positive-valued, it is easy to see that

$$
\mathbb{E}[Z] = \sum_{z=1}^{\infty} z\,\mathbb{P}(Z = z) \geq \sum_{z=1}^{\infty} \mathbb{P}(Z = z) = 1
$$

hence $Q_{1/\mathbb{E}[Z]}$ is a well-defined distribution.