$\rho \in B(H)_+$ $\quad S(\rho) :=$ minimal projection s.t $S(\rho)\rho S(\rho) = \rho$

Def (Umegaki). Given $\rho, \sigma \in D(H)$, the relative entropy from ρ to σ is

$$D(\rho\|\sigma) = \begin{cases} tr(\rho\log\rho - \rho\log\sigma) & \text{if } S(\rho) \leq S(\sigma) \\ +\infty & \text{otherwise} \end{cases}$$

- Measuring how different ρ is w.r.t to sigma

E.g. $\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$ $\qquad \sigma = \frac{3}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$

$$= \begin{bmatrix} \frac{1}{3} & \\ & \frac{2}{3} \end{bmatrix} \qquad = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\rho\log\rho = \begin{bmatrix} \frac{1}{3}\log\frac{1}{3} & \\ & \frac{2}{3}\log\frac{2}{3} \end{bmatrix}$$

$$\log\sigma = \log\frac{3}{4}|+\rangle\langle +| + \log\frac{1}{4}|-\rangle\langle -|$$

$$= \log\frac{3}{4}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \log\frac{1}{4}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\log\frac{3}{16} & \frac{1}{2}\log 3 \\ \frac{1}{2}\log 3 & \frac{1}{2}\log\frac{3}{16} \end{bmatrix}$$

$$D(\rho\|\sigma) = tr(\rho\log\rho - \rho\log\sigma) = tr\left(\begin{bmatrix} \frac{1}{3}\log\frac{1}{3} & 0 \\ 0 & \frac{2}{3}\log\frac{2}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \\ & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{2}\log\frac{3}{16} & \frac{1}{2}\log 3 \\ \frac{1}{2}\log 3 & \frac{1}{2}\log\frac{3}{16} \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} \frac{1}{3}\log\frac{1}{3} - \frac{1}{6}\log\frac{3}{16} & * \\ * & \frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{3}{16} \end{bmatrix}\right)$$

$$= \frac{1}{6}\log\frac{16}{27} + \frac{1}{3}\log\frac{64}{27} \simeq 0.2$$

- Non symmetric $D(\rho\|\sigma) \neq D(\sigma\|\rho)$ (e.g. if $s(\rho) \nsubseteq S(\sigma)$)
- If $\rho = \sum P_x |\varphi_x\rangle\langle\varphi_x|$ $\sigma = \sum Q_x |\varphi_x\rangle\langle\varphi_x|$ for the same O.N.B $|\varphi_x\rangle$

$$D(\rho\|\sigma) = D(P\|Q) \quad \text{classical RE}$$

Relation to other entropies.

① $\rho \in D(H)$, $H(\rho) = \log d_H - D(\rho \| \frac{1}{d_H})$

② $\rho_{AB} \in D(H_A \otimes H_B)$, $I(A:B)_\rho = D(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0$

$$H(A|B)_\rho = \log d_A - D(\rho_{AB} \| 1_A \otimes \rho_B)$$

③ $\rho_{ABC} \in D(H_A \otimes H_B \otimes H_C)$

$$I(A:B|C) = I(A:BC) - I(A:C)$$
$$= D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) - D(\rho_{AC} \| \rho_A \otimes \rho_C) \geq 0$$

Data Process Inequality for quantum Entropy

For any quantum channel Φ:

$$D(\rho\|\sigma) \geq D(\Phi(\rho) \| \Phi(\sigma)) \qquad \forall \rho, \sigma \in D(H).$$

- Lindblad '75 & Ulmann '77
- Super important in QIT ("The" inequality in my opinion)

- Later many simplied alternative proof. We consider two for final projects
  1. Operator montone/convex function
  2. Complex interpolation

Many Corollaries

Cor. ① $D(\rho\|\sigma)\geq 0$ with equality iff $\rho=\sigma$

② Joint convexity. $D(t\rho_1+(1-t)\rho_2\| t\sigma_1+(1-t)\sigma_2)\leq t D(\rho_1\|\sigma_1)+(1-t)D(\rho_2\|\sigma_2)$

③ $I(A:B|C)\geq 0$.

Pf ① Consider $tr: B(H)\to \mathbb{C}$

$$D(\rho\|\sigma)\geq D(tr\rho\| tr\sigma)= D(1\|1)=0$$

② Consider channel $id\otimes tr_2 \otimes B(H)\otimes M_2\to B(H)$

$$tD(\rho_1\|\sigma_1)+(1-t)D(\rho_2\|\sigma_2) = D\left(\begin{bmatrix} t\rho_1 & \\ & (1-t)\rho_2\end{bmatrix} \middle\| \begin{bmatrix} t\sigma_1 & \\ & (1-t)\sigma_2\end{bmatrix}\right)$$

$$(\text{DPI of } id\otimes tr_2) \geq D(t\rho_1+(1-t)\rho_2 \| t\sigma_1+(1-t)\sigma_2)$$

③ $$I(A:B|C)=I(A:BC)-I(A:C)$$
$$= D(\rho_{ABC}\|\rho_A\otimes\rho_{BC})-D(\rho_{AC}\|\rho_A\otimes\rho_C)\geq 0$$
$$id_A\otimes tr_B\otimes id_C$$

Cor 2

Given quantum Channel $\Phi : B(H_B) \to B(H_{B'})$

① $I(A:B)_\rho \geq I(A:B')_{id_A \otimes \Phi(\rho)}$

② $H(A|B)_\rho \leq H(A|B')_{id_A \otimes \Phi(\rho)}$

Pf: $I(A:B)_\rho = D(\rho_{AB} \| \rho_A \otimes \rho_B) \geq D(id \otimes \Phi(\rho_{AB}) \| \rho_A \otimes \Phi(\rho_B))$

$= I(A:B')_{Id \otimes \Phi(\rho)}$

$I(A:B)_\rho = H(A)_\rho - H(A|B)_\rho$

$\vee|$

$I(A:B')_{id \otimes \Phi(\rho)} = H(A)_\rho - H(A|B')_{\Phi(\rho)}$

Cor 3

If $H_B \cong H_{B'}$, and $\Phi(I) = I$ unital

③ $H(\rho) \leq H(\Phi(\rho))$

④ $H(B|A)_\rho \geq H(B'|A)_{\Phi \otimes id(\rho)}$

Pf: ③ $H(\rho) = \log d - D(\rho \| \frac{1}{d})$

$\leq \log d - D(\Phi(\rho) \| \Phi(\frac{1}{d}))$

$= \log d - D(\Phi(\rho) \| \frac{1}{d})$ b/c $\Phi(1) = 1$

④ is similar

Operational meaning of $D(\cdot\|\cdot)$

Given $\rho, \sigma \in D(H)$. We want dinstinguish ρ from σ by an measurement.

Ideal case: $\rho = |0\rangle\langle 0|$ $\quad \sigma = |1\rangle\langle 1|$,

we do $E_0 = |0\rangle\langle 0|$ $\quad E_1 = |1\rangle\langle 1|$ POVM

Then $\mathrm{tr}(\rho E_0) = 1$ $\quad \mathrm{tr}(\rho E_0) = 0$

$\mathrm{tr}(\rho E_1) = 0$ $\quad \mathrm{tr}(\sigma E_1) = 1$

In general, such perfect tese is no avaliable. For a general $\{T, I-T\}^{\geq 0}$

we want $\mathrm{tr}(\rho T)$ large $\quad \beta(T) = \mathrm{tr}(\sigma T)$ small

$\alpha(T) = \mathrm{tr}(\rho(I-T))$ small $\quad \mathrm{tr}(\sigma(I-T))$ large

$\alpha(T)$ type I error $\quad \beta(T)$ type II error

We can consider $\quad P_e^* = \min_T \alpha(T) + \beta(T)$

or $\beta^*(\varepsilon) = \min_T \beta(T)$ given $\alpha(T) \leq \varepsilon$

For general (ρ, σ), $\beta^*(\varepsilon) \neq 0$ for $0 < \varepsilon < 1$.

In the i.i.d setting. $\rho^{\otimes n}$ and $\sigma^{\otimes n} \in B(H^{\otimes n})$ $\subset B(H^{\otimes n})$

$$\beta_n^*(\varepsilon) = \min_{T_n} \left\{ \mathrm{tr}(\sigma^{\otimes n} T_n) \;\middle|\; 0 \leq T_n \leq I, \ \mathrm{tr}(\rho^{\otimes n} T_n) \geq 1-\varepsilon \right\}$$

Intutively, $\beta_n^*(\varepsilon) \to 0$, but how?

Theorem (Quantum Stein's Lemma)

$$\beta_n^*(\varepsilon) \simeq e^{-nD(\rho\|\sigma)}$$

More precisely, $\limsup_{n\to\infty} \frac{1}{n} \log \beta_n^*(\varepsilon) = -D(\rho\|\sigma)$

Final project: give an proof of this.