Recall classical source coding

XX correlated $\quad$ $X\tilde{X}$

Source

$$\boxed{X^n} \xrightarrow{f} \{0,1\}^{nR} \xrightarrow{g} \boxed{\tilde{X}^n}$$

$$\mathcal{E}_n(R) = \inf P(X^n \neq \tilde{X}^n), \quad R \text{ rate of compression}$$

$$= \| P_{X'X} - P_{X'\tilde{X}} \|_{TV}$$

Shannon's Theorem

$$\lim_n \mathcal{E}_n(R) = \begin{cases} 1 & \text{if } R > H(X) \\ 0 & \text{if } R < H(X) \end{cases}$$

$$P_{X'X}(x,y) = \begin{cases} P_X(x) & x = y \\ 0 & x \neq y \end{cases}$$

Quantum data compression

R reference

φ^{AR} purification $\quad$ $\tilde{\varphi}^{AR}$

$$\boxed{\rho \in D(H_A)} \xrightarrow{E} (M_2)^{\otimes R} \xrightarrow{D} \boxed{\tilde{\rho} \in D(H_A)}$$

$$\mathcal{E}(R) = \inf_{(E,D)} \text{"}Pr(\rho \neq \tilde{\rho})\text{"} = \inf_{(E,D)} \| \varphi^{AR} - \underbrace{E\circ D(\varphi^{AR})}_{\tilde{\varphi}^{AR}} \|_1$$

Classical case: $\quad \|P-Q\|_{TV} = \frac{1}{2}\|P-Q\|_1 = \frac{1}{2}\sum_{x\in\mathcal{X}} |P(x)-Q(x)| = \inf_{X\sim P,\, \tilde{X}\sim Q} P(X \neq \tilde{X})$

Quantum case: $\quad \|\rho - \sigma\|_1 = tr(|\rho-\sigma|) = \sup_{0\le P\le 1} tr((\rho-\sigma)P) :=$ maximal guess probability.

I.I.d Setting

R^n reference

$(\varphi^{AR})^{\otimes n}$ $\quad$ $(\tilde{\varphi}^{AR})^{\otimes n}$

$$\boxed{\rho^{\otimes n}} \xrightarrow{E} (B(\mathbb{C}^2))^{\otimes nR} \xrightarrow{D} \boxed{\tilde{\rho}^{\otimes n}}$$

$$\text{SH } B(\mathbb{C}^{2^{nR}})$$

$$\mathcal{E}_n(R) = \inf_{E,D} \| \varphi^{AR} - DE(\tilde{\varphi}^{AR\otimes n}) \|_1$$

Schumacher Compression (1994)

$$\lim_{n\to\infty} \varepsilon_n(R) = \begin{cases} 0 & \text{if } R > H(\rho) \\ 1 & \text{if } R < H(\rho) \end{cases}$$

Direct coding ($R > H(\rho)$)

$\rho = \sum p_x |x\rangle\langle x|$, $H(\rho) = H(P)$. $\quad P_X = \{p_x\}$ over $x \in \mathcal{X}$

Then if $R > H(\rho) = H(P_X)$, by shannon's coding theorem, $\exists \{f_n, g_n\}$ - dectetable error

$$\boxed{P_X^n} \xrightarrow{f_n} \boxed{\{0,1\}^{nR}} \xrightarrow{g_n} \boxed{P_X^n}$$

$\exists S_n \subseteq \mathcal{X}^n$ s.t. $g_n \circ f_n|_{S_n} = I \qquad \lim_{n\to\infty} P(X^n \neq \hat{X}^n) = \lim_n P(S_n) = 0$

Define partial isometry $V: (\mathbb{C}^{\mathcal{X}})^{\otimes n} \to (\mathbb{C}^2)^{\otimes nR}$

$\quad \cong \mathbb{C}^{\mathcal{X}^n}$

$$V(|x_1 \cdots x_n\rangle) = \begin{cases} |f(x_1 \cdots x_n)\rangle & \text{if } x_1 \cdots x_n \in S_n \\ 0 & \text{otherwise} \end{cases}$$

$V^* V = \Pi_V$ a projection

Define $\mathcal{E}: B(\mathbb{C}^{\mathcal{X}^n}) \to B(\mathbb{C}^{2^{nR}})$

$$\mathcal{E}(\rho) = V \rho V^* + tr((I - V^* V)\rho) |e\rangle\langle e| \quad \text{for some fixed } |e\rangle\langle e| \text{ for error}$$

$\mathcal{D}: B(\mathbb{C}^{2^{nR}}) \to B(\mathbb{C})^{\mathcal{X}^n}$

$$\mathcal{D}(\rho) = V^* \rho V + tr((I - VV^*)\rho) |e\rangle\langle e| \qquad tr(\Pi_V \rho^{\otimes n}) = tr(V \rho V^*) = P(S_n)$$

$$\mathcal{D} \circ \mathcal{E}(\varphi^{AR^{\otimes n}}) = V^* V (\varphi^{AR})^{\otimes n} V^* V + tr((I - V^* V)\rho) \mathcal{D}(|e\rangle\langle e|)$$

$$= \Pi_V (\varphi^{AR})^{\otimes n} \Pi_V + tr(\Pi_V^c \rho) \mathcal{D}(|e\rangle\langle e|)$$

$$\|(\varphi^{AR^{\otimes n}}) - \mathcal{D} \circ \mathcal{E}(\varphi^{AR^{\otimes n}})\|_1 \le \|\varphi^{AR} - \Pi_V (\varphi^{AR})^{\otimes n} \Pi_V\|_1 + tr(\Pi_V^c \rho) \|\mathcal{D}(|e\rangle\langle e|)\|$$

$$P(S_n^c) \|\mathcal{D}(|e\rangle\langle e|)\|$$

$$\le 2\varepsilon_n + \varepsilon_n = 3\varepsilon_n \to 0$$

Weak converse:

Theorem (Fannes-Audenaert) For $\rho, \sigma \in \mathcal{B}(H)$ and $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1$,

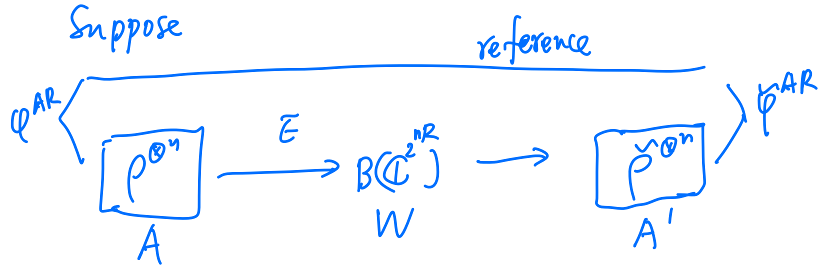
$$|H(\rho) - H(\sigma)| \le \varepsilon \log(\dim H - 1) + h(\varepsilon).$$

Corollary: For $\rho^{AB}, \sigma^{AB} \in D(H_{AB})$ and $\varepsilon = \frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_1$

$$|I(A.B)_\rho - I(A.B)_\sigma| \le 3\varepsilon \log(\dim H - 1) + 3h(\varepsilon)$$

Pf: $I(A.B) = H(A) + H(B) - H(AB)$.

Suppose



$$2nR = 2\log 2^{nR} = 2\log|W| \ge I(W.R^n) \ge I(A'^n.R^n) \overset{\varepsilon}{\ge} I(A^n.R^n)_{\rho^{\otimes n}} - 3\varepsilon\log|AR|^n - 3h(\varepsilon)$$

$$\ge n\, I(A.R) - 3\varepsilon\log|AR|^n - 3h(\varepsilon)$$

$$\ge 2nH(A) - 3\varepsilon\, n\log|AR| - 3h(\varepsilon)$$

$$\Rightarrow \lim_{n\to\infty} \varepsilon_n(R) \ge \frac{2}{3}\,\frac{H(A) - R}{\log|AR|} > 0.$$

If $R < H(A)$, $\exists\, \varepsilon$ s.t. $R + 3\varepsilon < H(A)$, then contradiction.