Quantum channel $\Phi : B(H_A) \longrightarrow B(H_B)$

How we transmit $\underbrace{\text{information}}$ through quantum channel?

classical bits / quantum qubits

Classical communication over quantum channel

$A = B(H_A) \qquad B = B(H_A) \qquad \mathcal{M} = \{1, \cdots, M\} = \hat{\mathcal{M}}$

$$
\begin{array}{ccccccc}
\boxed{\mathcal{M}} & \xrightarrow{f} & \boxed{A} & \xrightarrow{\Phi} & \boxed{B} & \xrightarrow{D} & \boxed{\hat{\mathcal{M}}} \\
m & & \rho_m & & \Phi(\rho_m) & & p(\hat{m}) = tr(\Phi(\rho_m) E_{\hat{m}})
\end{array}
$$

Encoder : $f : \mathcal{M} \to D(H_A)$. $\quad m \longmapsto \rho_m \quad$ state preparation

Decoder : POVM measurement $\{E_{\hat{m}}\}_{\hat{m}=1}^{M}$

$$D : D(H_B) \longrightarrow C(\mathcal{M})$$

$$\rho \longrightarrow p_{\hat{M}}(\hat{m}) = tr(\rho E_m)$$

Error probability : Given input m, $P(\hat{M} = m \mid M = m) = tr(\Phi(\rho_m) E_m)$

$$P(\hat{M} \neq M \mid M = m) = 1 - tr(\Phi(\rho_m) E_m)$$

Averaged error : $P_e(f.D) = P(M \neq \hat{M}) = 1 - \frac{1}{M} \sum_{m=1}^{M} tr(\Phi(\rho_m) E_m)$

Max error : $P_{e,max}(f.D) = \max_{m} P(M \neq \hat{M} \mid M = m) = \max_{m} 1 - tr(\Phi(\rho_m) E_m)$

$(f.D)$ is an M code if $f : \mathcal{M} \to A$, $D : B \to \mathcal{M}$; $(M\text{-}\varepsilon)$ code if $P_e(f.D) \leq \varepsilon$

Optimal error : $\varepsilon^*(\Phi, M) = \inf_{\substack{(f,D) \\ M\text{-code}}} P_e(f, D)$

Optimal code size : $M^*(\Phi, \varepsilon) = \sup\{M \mid \exists (f,D),\ (M,\varepsilon)\text{-code}\}$
$P_e(f,D) \le \varepsilon$

In the i.i.d setting :

$$M \xrightarrow{f_n} A^n \xrightarrow{\Phi^{\otimes n}} B^n \xrightarrow{D_n} \hat{M}$$

Def: (Classical Capacity of Quantum Channel.)

$$C(\Phi) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log M(\Phi^n, \varepsilon)}{n}$$

$$= \{R \mid \forall \varepsilon > 0 \ \exists \ (2^{nR}, \varepsilon)\text{-code for } \Phi^{\otimes n}\}$$
for some n

Holevo information

<u>Def</u>: Let $\mathcal{Y} = \{P_x, \rho_x\}$ be ensemble of quan states where $(P_x) \in \mathcal{P}(X)$ $\{\rho_x\} \subseteq D(\mathcal{H}_B)$.

$$\chi(\{P_x, \rho_x\}) = H(\sum_x P_x \rho_x) - \sum_x P_x H(\rho_x)$$

- $\chi(\{P_x, \rho_x\}) \ge 0$ by concavity of H
- Define c-q state : $\rho_{XB} = \sum P_x |x\rangle\langle x| \otimes \rho_x^B \longleftrightarrow \mathcal{Y} = \{P_x, \rho_x\}$

$$\chi(\{P_x, \rho_x\}) = I(X:B)_{\rho_{XB}} \qquad \rho_X = \sum P_x |x\rangle\langle x| \quad \rho_B = \sum P_x \rho_x^B$$

Indeed, $I(X:B)_\rho = H(X) + H(B) - H(XB)$
$= H(B) - H(X|B)$
$= H(\sum p_x \rho_x) - \sum p_x H(\rho_x)$

<u>Def</u> (Holevo Information of a Channel)

$$\chi(\Phi) = \sup_{\substack{\{p_x, \rho_x\} \\ \rho_x \in D(\mathcal{H}_A)}} \chi(\{p_x, \Phi(\rho_x)\})$$

$$= \sup_{\substack{\rho_{XB} = Id_X \otimes \Phi(\rho_{XX'}) \\ \rho_{XX'} = \sum p_x(x) |x\rangle\langle x| \otimes |x\rangle\langle x|}} I(X:B)_\rho$$

Theorem (Holevo - Schumacher - Westmore, 1997)

$$C(\Phi) = \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n}$$

① Acheivability : $\chi(\Phi) \leq C(\Phi)$

② By definition, $C(\Phi^{\otimes n}) = n\, C(\Phi)^n$

(i) $C(\Phi^{\otimes n}) \geq n\, C(\Phi)$, R is an achievable rate for Φ

$\forall \varepsilon \; \exists (f_\varepsilon, D_\varepsilon)$ $\quad (2^{kR}, \varepsilon)$ code for $\Phi^{\otimes k}$ some k)

$\Rightarrow \quad (f_\varepsilon^{\otimes n}, D_\varepsilon^{\otimes n}) \quad (2^{nkR}, \underset{\substack{\uparrow \\ \text{union bound}}}{n\varepsilon})$ code for $\underset{\substack{\shortparallel \\ (\Phi^{n})^{\otimes k}}}{\Phi^{\otimes nk}}$

$\Rightarrow nR$ is achievable for $C(\Phi^{\otimes n})$

(ii) $C(\Phi^{n}) \leq n\, C(\Phi)$, if nR - - - - - - for $\Phi^{\otimes n}$

$\forall \varepsilon \; \exists (f_\varepsilon, D_\varepsilon)$. $(2^{nRk}, \varepsilon)$ code for $\Phi^{\otimes nk}$ - ~

$\Rightarrow R$ achievable for Φ.

① + ② $\Rightarrow$ $\dfrac{\chi(\Phi^{\otimes n})}{n} \leq C(\Phi)$

③ (Weak Converse) : Suppose (f, D) is (M, ε) code for Φ

$$\boxed{M=2^{nR}} \xrightarrow{f} \boxed{A} \xrightarrow{\Phi} \boxed{B} \xrightarrow{D} \boxed{\hat{M}}$$

$P_M(m)$ $\quad \{P_M(m), \rho_m\}$ $\quad \{P_M(m), \Phi(\rho_m)\}$ $\quad P_{M\hat{M}}(m, \hat{m}) = tr(\Phi(\rho_m) E_{\hat{m}})$

$$\chi(\Phi) = \sup_{P_X} I(X:B) \geq I(M:B) \geq I(M:\hat{M}) \geq D(B_\varepsilon \| B_{\frac{1}{M}})$$

$$= \varepsilon \log \frac{\varepsilon}{\frac{M-1}{M}} + (1-\varepsilon) \log \frac{(1-\varepsilon)}{\frac{1}{M}}$$

$$= \log M - \varepsilon \log(M-1) - h(\varepsilon)$$

$$\log M \leq \frac{\chi(\Phi) + h(\varepsilon)}{1-\varepsilon}$$

I.I.d Suppose $R > \lim\limits_{n\to\infty} \dfrac{\chi(\Phi^{\otimes n})}{n}$. $\exists\, \varepsilon > 0$ s.t. $(1-\varepsilon)R > \lim\limits_{n\to\infty} \dfrac{\chi(\Phi^{\otimes n})}{n}$

For any $(2^{nR}, \varepsilon)$ code for $\Phi^{\otimes n}$

$$nR = \log 2^{nR} \leq \frac{\chi(\Phi^{\otimes n}) + h(\varepsilon)}{1-\varepsilon}$$

$$(1-\varepsilon) R \leq \frac{\chi(\Phi^{\otimes n})}{n} + \frac{h(\varepsilon)}{n} \longrightarrow \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n} \quad \text{Contradiction}$$

In terms of optimal error ε^*

Theorem: $\lim_{n\to\infty} \varepsilon^*(2^{nR}, \Phi^n) = \begin{cases} 0 & \text{if } R < C(\Phi) = \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n} \\ 1 & \text{if } R > C(\Phi) \end{cases}$

$\lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n}$ called regularization of $\chi(\Phi)$.

In general, $\chi(\Phi^n) \geq n\,\chi(\Phi)$

(Hastings '07) $\exists\, \Phi$ s.t. $\chi(\Phi \otimes \Phi) > 2\chi(\Phi)$

probabilistically proof. No explicit construction.

Why does this result mean?

$$M \longrightarrow A \xrightarrow{\Phi} B \longrightarrow \hat{m}$$

m $\quad \rho_m \in D(H_A)$

$$M \longrightarrow A^n \xrightarrow{\Phi^{\otimes n}} B^n \longrightarrow \hat{m}$$

$\rho_m \in B(H_A^{\otimes n}) \cong B(H_A) \otimes B(H_A) \cdots B(H_A)$

$M^*(\Phi^n, \varepsilon) = \max\{M \mid \exists\, (M, \varepsilon) \text{ code for } \Phi^n\}$

$M^*_{pr}(\Phi^n, \varepsilon) = \max\{M \mid \exists\, (M, \varepsilon) \text{ code for } \Phi^n \;\&\; f(m) = \rho_{1,m} \otimes \rho_{2,m} \otimes \cdots \rho_{n,m}\}$ product state

(or $f(m) = \sum_\lambda B_\lambda \rho_{1,\lambda} \otimes \rho_2 \otimes \cdots \rho_{n\lambda}\}$
separate state

$\chi(\Phi) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log M^*_{pr}(\Phi^n, \varepsilon)}{n}$ capacity if we only use product code state

$C(\Phi)$ capacity use all possible code state (including entangled state)

Will entanglement help? Yes, by (Hastings '05)

$$C(\Phi) \geq \frac{\chi(\Phi \otimes \Phi)}{2} > \chi(\Phi)$$

Capacity using all possible code words.

capacity of allowing using entanglement between two input system

$\rho_1 \otimes \rho_2 \otimes \cdots \rho_{\frac{n}{2}}$

$\rho_1 \in B(H_A \otimes H_A)$, $\rho_2 \in B(H_A \otimes H_A)$

Capacity using only product code word