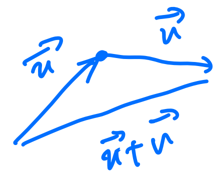
Preliminary on Linear Algebra

1. Complex vector space

n-dim (complex) v.s. $\mathbb{C}^n = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \middle| \ u_i \in \mathbb{C} \right\}$

Two operations:

① Addition: $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix}$



② Scalar multiplication: $\alpha \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{pmatrix}$

$\alpha u + \beta v$ is a linear combination of u and v

Standard basis:

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots\cdots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = u_1 e_1 + u_2 e_2 + \cdots \quad u_n e_n$

$u \xleftrightarrow{\{e_i\}} (u_1 \cdots u_n)$

In general, let I be a finite set. We define

$\mathbb{C}^I = \{u: I \to \mathbb{C}\}$ all complex valued functions on I

Addition: $\forall\, u, v \in \mathbb{C}^I$, $\quad u+v \in \mathbb{C}^I$

$$u+v\,(a) = u(a) + v(a)$$

Scalar multiplication: $\quad \forall\, u \in \mathbb{C}^I,\ \alpha \in \mathbb{C}, \quad \overset{\alpha u}{\overset{\shortparallel}{\alpha \cdot u}} \in \mathbb{C}^I$

$$\alpha u\,(a) = \alpha\ u(a)$$

$\mathbb{C}^n := \mathbb{C}^{\{1,\dots,n\}}$ n-dimension complex v.s.

$$u \in \mathbb{C}^{\{1,\dots,n\}} \longleftrightarrow \begin{pmatrix} u(1) \\ u(2) \\ \vdots \\ u(n) \end{pmatrix}$$

Thm: $\quad \mathbb{C}^I \cong \mathbb{C}^n$ iff $n = |I|$

$$I = \{x_1, \dots, x_n\} \longleftrightarrow \{1, \dots, n\}$$

$$e_x(y) = \delta_{xy} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \qquad e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{- ith}$$

Example $\mathbb{X} = \{0, 1\}^2 = \{00, 01, 10, 11\}$

$$u \in \mathbb{C}^{\mathbb{X}} \Leftrightarrow \begin{pmatrix} u(00) \\ u(10) \\ u(01) \\ u(11) \end{pmatrix} \in \mathbb{C}^4$$

Inner product : $\langle \cdot , \cdot \rangle : \mathbb{C}^I \times \mathbb{C}^I \to \mathbb{C}$

$$\langle u, v\rangle = \sum_{i\in I} \overline{u(i)}\, v(i)$$

$\overline{a+bi} = a-bi$
$a, b \in \mathbb{R}$

Definition - Properties

① Linearity in second input

$$\langle u, \alpha v+\beta w\rangle = \alpha\langle u, v\rangle + \beta\langle u, w\rangle$$

② Conjugate symmetry

$$\langle u, v\rangle = \overline{\langle v, u\rangle} \qquad \forall\, u, v$$

③ Positivity:

$$\langle u, u\rangle \geq 0$$

with equality iff $u=0$.

Note: ① + ② implies anti-linear in first input

$$\langle \alpha u+\beta v, w\rangle = \bar{\alpha}\langle u, w\rangle + \bar{\beta}\langle v, w\rangle$$

$\bar{a}a = |a|^2$

Norm: $\|u\| := \sqrt{\langle u, u\rangle} = \sqrt{\sum_{i\in I} |u(i)|^2}$ $\quad$ u is a unit vector if $\|u\|=1$

distance $d(u, v) = \|u-v\| = \sqrt{\sum_i |u(i)-v(i)|^2}$ $\quad$ Euclidean distance

Defining - Properties

1. Positivity: $\|u\| \geq 0$ with equality iff $u=0$
2. $\|\alpha u\| = |\alpha|\, \|u\| \quad \forall\, \alpha\in\mathbb{C}$
3. $\|u+v\| \leq \|u\| + \|v\| \quad \forall\, u, v \in \mathbb{C}^n$ triangle inequality.

$\Leftrightarrow$ Cauchy-Schwarz inequality ("The inequality" in Hilbert space)

$\forall\, u, v\in\mathbb{C}^n \qquad |\langle u, v\rangle| \leq \|u\|\,\|v\|$

with equality iff $u = \alpha v$ for some $\alpha\in\mathbb{C}$.

CS inequality $\Leftrightarrow$ triangle inequality.

Other example of norms: $\|u\|_p = \left(\sum_{i\in I} |u(i)|^p\right)^{\frac{1}{p}}$ (How to prove triangle inequality?)

$\|u\|_\infty = \max\{ u(i) \mid i \in I\}$

Hibert space norm $\|u\| := \|u\|_2$ $p=2$

unit: $\|u\|=1$

A set $\{u_1, u_2, \cdots u_k\}$ is orthogonal if $\langle u_i ; u_j \rangle = 0$ if $i \neq j$.

is orthonormal if orthogonal and $\|u_i\|=1$

is a orthonormal basis if orthonormal and basis

E.g. $e_1 = (1, 0 \cdots 0)$, $e_2 = (0, 1, 0 \cdots 0)$ $\cdots$ $e_n = (0, 0 \cdots 1)$

O.N.B. $\mathbb{C}^n$

$X = \{x_1 \cdots x_n\}$ $\quad e_{x_1} \cdots$ O.N.B $\mathbb{C}^X$

$(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is called n-dim complex Hilbert space

(or Euclidean)

Thm Every n-dim $\mathbb{C}$ Hibert space is isomorphic to $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$

In general, Hibert space := vector space + inner product. "Completeness"

infinite dim

E.g. $L_2(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C} \mid \int |f(x)|^2 dx < +\infty \}$. $\langle f, g \rangle = \int \overline{f(x)} g(x)\, dx$

measurable

Linear operator:

Let V, W be complex. V.S. A map $L: V \to W$ is linear if

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \quad \forall \alpha, \beta \in \mathbb{C},\ u, v \in V$$

$L(V, W) :=$ the space of all linear operator

$L(V, W)$ is a complex vector space with

Addition: $(A+B)u := Au + Bu$

Scalar multiplication: $(\alpha A)u = \alpha Au$

$\dim V = n \qquad \dim W = m$

Given an basis $\{v_1 \cdots v_n\}$ of V

basis $\{w_1 \cdots w_m\}$ of W,

$a_{ij} \in \mathbb{C}$

$$A v_1 = a_{11} w_1 + a_{12} w_2 + \cdots a_{1m} w_m = \begin{pmatrix} a_{11} \\ \vdots \\ a_{1m} \end{pmatrix}$$

$$A v_2 = a_{21} w_1 + a_{22} w_2 + \cdots a_{2m} w_m$$

$$\vdots$$

$$A v_n = a_{n1} w_1 + \cdots a_{nm} w_m = \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nm} \end{pmatrix}$$

$$\begin{tikzcd}
\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \arrow[r, "M"] \arrow[d, leftrightarrow, "\{v_i\}"] & \boxed{\begin{pmatrix} \sum a_{1j} b_j \\ \sum a_{2j} b_j \\ \vdots \\ \sum a_{nj} b_j \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \ddots \\ \ddots & & a_{nm} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}} \arrow[d, leftrightarrow, "\{w_j\}"] \\
V = b_1 v_1 + \cdots b_n v_n \arrow[r, "A"] & AV = A(b_1 v_1 + \cdots b_n v_n)
\end{tikzcd}$$

Definition of matrix multiply vector.

$$= b_1 A v_1 + \cdots b_n A v_n$$

$$= \sum_{i=1} \sum_{j=1} (a_{ij} b_j) w_i$$

$M = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} =$ is $m \times n$ complex matrix

$M_{ij} = \langle w_i, A v_j \rangle$

$M_{n\times m}$ space of all $n\times m$ complex matrices

$\cong L(\mathbb{C}^n, \mathbb{C}^m)$ Space of Linear operator

$$\text{matrix } M \xleftrightarrow[\{w_j\} \text{ basis of } \mathbb{C}^m]{\{v_i\} \text{ basis of } \mathbb{C}^n} A \text{ linear operator}$$

$$M\begin{pmatrix} u(1) \\ \vdots \\ u(n) \end{pmatrix} \qquad Au$$

For $A \in L(V, W)$ $B \in L(W, Z)$. $AB \in L(V, Z)$

$A \circ B(u) := A(B(u))$ "$\circ$" often omitted

$$A\circ B(\alpha u + \beta v) := A(B(\alpha u + \beta v))$$
$$= A(\alpha Bu + \beta Bv)$$
$$= \alpha ABu + \beta ABv. \quad \text{Linear}$$

$$\begin{tikzcd}
V \arrow[r, "A"] \arrow[d, leftrightarrow, "\{v_i\}"'] & W \arrow[r, "B"] \arrow[d, leftrightarrow, "\{w_i\}"] & Z \arrow[d, leftrightarrow, "\{z_i\}"] \\
\mathbb{C}^n \arrow[r, "M"] & \mathbb{C}^m \arrow[r, "N"] & \mathbb{C}^d
\end{tikzcd}$$

$A \circ B \longleftrightarrow MN$

$$\boxed{(MN)_{ik} = \sum_{j=1}^{m} M_{ij} N_{jk} \quad \text{Matrix multiplication}}$$

$1 \le i \le d,\ 1 \le k \le n$

Basis for $M_{n\times m}$

$$E_{i,j}(k,l) = \begin{cases} 1 & \text{if } (k,l)=(i,j) \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{cc} \begin{matrix} & & j\text{th} & & \end{matrix} & \\ \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ \vdots & & 1 & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} & i\text{th} \end{array}$$

$M = \sum M(i,j)\, E_{i,j}$ $\qquad$ $\{E_{ij}\}$ basis for $M_{n\times m}$

Basis for $L(V,W)$

For $v \in V$, $w \in W$, $\qquad E_{w,v}(u) = \langle v,u \rangle w$

Given a basis $\{v_i\} \subseteq V$, $\{w_j\} \subseteq W$

$$E_{w_j,v_i}(v_k) = \delta_{ik} w_j = \begin{cases} w_j & \text{if } i=k \\ 0 & \text{otherwise} \end{cases}$$

$\left\{ E_{w_j,v_i} \;\middle|\; \begin{matrix} i=1\cdots n \\ j=1\cdots m \end{matrix} \right\}$ forms a basis for $L(V,W)$.

So $\dim(M_{n\times m}) = nm$

$\dim(L(V,W)) = \dim V \dim W$

Direct sum of V.S. and Operators

$V_1 = \mathbb{C}^{X_1}, \quad V_2 = \mathbb{C}^{X_2} \quad \cdots \quad V_n = \mathbb{C}^{X_n} \qquad X$

$$V_1 \oplus V_2 \oplus \cdots V_n = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| \ v_i \in V_i \right\}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} := v_1 \oplus v_2 \oplus \cdots \oplus v_n$$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{pmatrix}$$

E.g. $\mathbb{C}^2 \oplus \mathbb{C}^4 \oplus \mathbb{C}^3 = \mathbb{C}^{2+4+3} = \mathbb{C}^9$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} -5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 1 \\ 1 \\ -5 \\ 6 \\ 7 \end{pmatrix}$$

Inner product $\left\langle \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \right\rangle = \sqrt{\sum_i \langle u_i, v_i \rangle}$

$\forall i, \ \{e_k^i\}_{k=1}^{|V_i|}$ O.N.B of $V_i \Rightarrow \{e_k^i\}_{k,i}$ O.N.B of $V_1 \oplus \cdots V_n$

Thus, $\mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \oplus \cdots \mathbb{C}^{d_n} \cong \mathbb{C}^{d_1 + \cdots d_n}$

Let $A_1 \in L(V_1, W_1) \cdots A_n \in L(V_n, V_m)$

Define $A_1 \oplus A_2 \oplus \cdots A_n \in L(V_n . V_m)$

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} A_1 u_1 \\ A_2 v_2 \\ \vdots \\ A_n u_n \end{pmatrix}$$

$$\text{If } A_1 \text{ has matrix } M_1 \qquad A_1 \oplus \cdots A_n \longleftrightarrow \begin{bmatrix} \underset{\downarrow}{M_{n_1 \times m_1}} & & & \\ M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{bmatrix} \begin{pmatrix} \underset{\downarrow}{\mathbb{C}^{m_1}} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$\vdots$$

$$A_n - - - - M_n$$

Tensor product

$$V = \mathbb{C}^X \qquad W = \mathbb{C}^Y$$

$$V \otimes W := \mathbb{C}^{X \times Y} = \{ u : \underset{\overset{(a,b)}{\cup}}{X \times Y} \to \mathbb{C} \}$$

$$v \otimes w(a,b) = v(a) w(b)$$

$$V \otimes W = \text{Span} \{ v \otimes w \mid u \in V, v \in W \}$$

$$= \{ \sum \alpha_i v_i \otimes w_i \mid \forall \alpha_i \in \mathbb{C}, v_i \in V, w_i \otimes W \}$$

all linear combinations of elementary tensor.

$$v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w \qquad v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$$

$$\alpha (v \otimes w) = \alpha v \otimes w = v \otimes \alpha w.$$

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$$

If $\{e_i\} \subseteq V$ basis

$\{f_j\} \subseteq W$ basis.

$$v \otimes w = (a_1 e_1 + \cdots a_n e_n) \otimes (b_1 f_1 + \cdots b_m f_m)$$

$$= \sum_{i,j} a_i b_j \; e_i \otimes f_j$$

$\{e_i \otimes f_j\}_{i,j}$ is a basis for $V \otimes W$

$$\dim (V \otimes W) = \dim (V) \dim (W)$$

One can similarly define $V_1 \otimes V_2 \otimes \cdots \otimes V_n$

General rule

$$V \oplus W = W \oplus V \qquad V \otimes W \otimes Z \cong (V \otimes W) \otimes Z$$

$$V \otimes W = W \otimes V \qquad V \oplus W \oplus Z = (V \oplus W) \oplus Z$$

$$(V \oplus W) \otimes Z = V \otimes Z + W \otimes Z$$

Tensor product operator

$$A \in L(V_1, W_1) \qquad B \in L(V_2, W_2)$$

Define $A \otimes B \in L(V_1 \otimes V_2,\ W_1 \otimes W_2)$

$$A \otimes B\,(v \otimes w) = Av \otimes Bw$$

$$M_{n_1 \times m_1} \otimes M_{n_2 \times m_2} = M_{n_1 n_2 \times m_1 m_2}$$