Dirac's Bra-ket Notation

$u \in \mathbb{C}^n \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \longrightarrow |u\rangle$ "ket"

$v \in (\mathbb{C}^n)^* \quad v = (\bar{v}_1, \bar{v}_2 \cdots \bar{v}_n) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}^* \longrightarrow \langle v|$ "bra"

$\quad \parallel$

$L(\mathbb{C}^n, \mathbb{C})$

A bracket $\langle v|u\rangle = \langle v, u\rangle$ inner product

$$= (\bar{v}_1 \cdots \bar{v}_n) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_{i=1}^{n} \bar{v}_i u_i$$

$(\bar{v}_1 \cdots \bar{v}_n) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}^*$ so $\underset{L(\mathbb{C}, \mathbb{C}^n)}{\underset{\cap}{(|u\rangle)^*}} = \underset{L(\mathbb{C}^n, \mathbb{C})}{\underset{\cap}{\langle u|}}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots$

$\mathbb{C}^n = \{ a_0|0\rangle + a_1|1\rangle + \cdots a_{n-1}|n-1\rangle \mid a_i \in \mathbb{C} \}$

How it works:

① Inner product : $\langle v|u\rangle$

② Matrix action : $\langle v|A|u\rangle = \langle v, Au\rangle$

③ Rank one operator : $E_{v,u} = |v\rangle\langle u|$ outer bracket

$E_{v,u}(|w\rangle) = (|v\rangle\langle u|)|w\rangle$
$= |v\rangle\langle u|w\rangle$

check linearity $= \underset{\mathbb{C}}{\underset{\cap}{(\langle u|w\rangle)}}|v\rangle$

④ General operator : $A = \sum a_j |v_j\rangle\langle u_j|$ for $|v_j\rangle$ $|u_j|$

e.g. $A = [a_{ij}]$
$= \sum_{i,j} a_{ij} \underset{\text{standard basis}}{\underset{\downarrow}{|i\rangle\langle j|}}$

⑤ Matrix multiplication : $A = \sum a_j |v_j\rangle\langle u_j|$

$B = \sum b_k |w_k\rangle\langle x_k|$

$A \cdot B = \sum_{j,k} a_j b_k |v_j\rangle\langle u_j|w_k\rangle\langle x_k|$

$= \sum_{j,k} a_j b_k \underset{\mathbb{C}}{\underset{\cap}{\langle u_j|w_k\rangle}} |v_j\rangle\langle x_k|$

⑤ diagonal operator : $A = \sum \lambda_j |V_j\rangle\langle V_j|$ $\{|V_j\rangle\}$ O.N.B of eigen values of A

orhogonal decomposition

λ_j eigen value of $|V_j\rangle$

spetrum theorem

A Positive $\Rightarrow \lambda_j \geq 0$ A projection $\Rightarrow \lambda_j \in \{0,1\}$

A Hermitian $\Rightarrow \lambda_j \in \mathbb{R}$ A unitary $\Rightarrow |\lambda_j| = 1 \quad \forall j$

Postulates : Working Assumption / Framework / Model

Watch : Feynman — Difference between Mathematics and Phsysics.

Real World $\xrightarrow{\text{Poseulates}}$ Quatum Mechanic using Mathematics

↑

Experiment / theory

Postulate 1 (State space) Any isolated (quantum) system is associted a $\mathbb{C}$- Hilbertspace as its "state space". The state of system is given by a unit vector

$|\varphi\rangle$, $\quad \| |\varphi\rangle \| = 1$

called the "state vector".

Example (Qubit)

$\mathbb{C}^2 = \{ a|0\rangle + b|1\rangle \mid a, b \in \mathbb{C} \}$ $|a|^2$ prob observing $|\varphi\rangle$ in $|0\rangle$.

logic basis $\quad |0\rangle, |1\rangle \qquad$ Other basis. $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

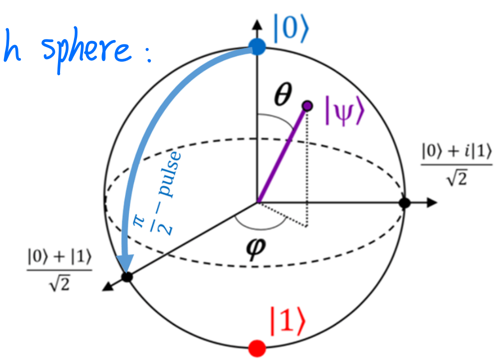
(super position : $|\varphi\rangle = a|0\rangle + b|1\rangle$ unit / state $\Longleftrightarrow |a|^2 + |b|^2 = 1$

$|\varphi\rangle = e^{i\delta}\cos(\theta/2)|0\rangle + e^{i(\delta+\phi)}\sin(\theta/2)|1\rangle$ Bloch sphere :

$0 \leq \delta \leq 2\pi \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$

$e^{i\delta}$ global phase not phsyical relevalent

ϕ relative phase phsyical



$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$

Example: (1-dim Wave function)

Position of a particle O, $X_O \in \mathbb{R}$

prob. particle appear in $(a.b)$

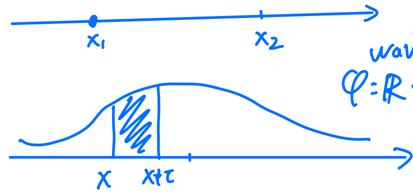
$$P(O\in(a.b)) = \int_a^b |\varphi(x)|^2 dx$$

$$\|\varphi\| = \int_{-\infty}^{\infty} |\varphi(x)|^2 dx = 1 \longrightarrow P((-\infty.\infty)) = 1$$

coordinate

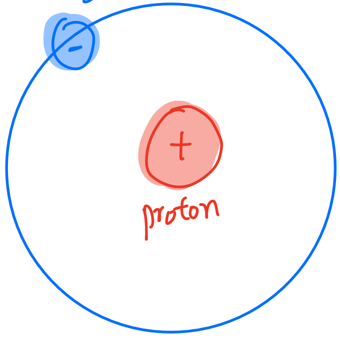
class

quatum

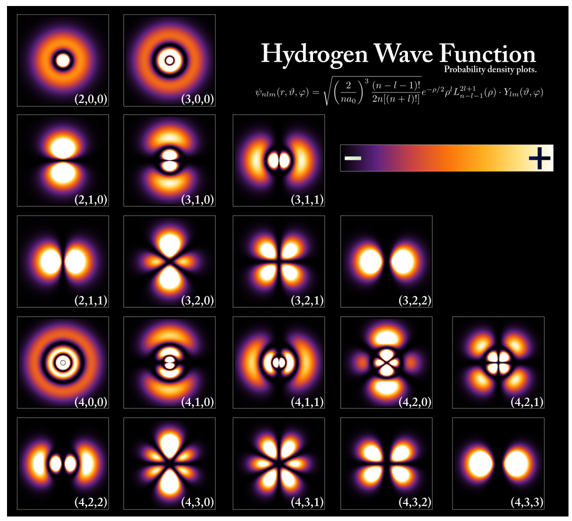


wave function
$\varphi: \mathbb{R} \to \mathbb{C}$

Chemistry



Quantum Mechanic Model



Young Double split, Single photon Inference

Bell inequality

Postulate 2. (Measurement)

Quantum Information Form (fid.)

Every quantum measurement is given by a collection $\{E_m\}$ of Measurement Operator satisfying the completeness equation

$$\sum_m E_m = I \qquad E_m \geq 0$$

$\{E_m\}$ is called a POVM (Positive Operator-Valued Measurement)

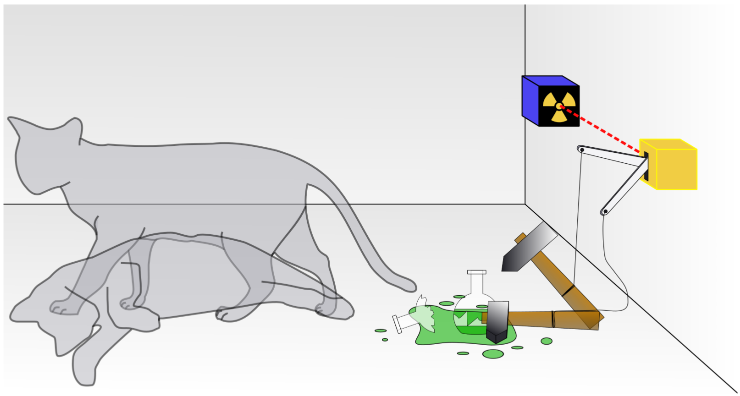
Prob seeing outcome m : $\langle\varphi|E_m|\varphi\rangle$ $\quad \sum_m \langle\varphi|E_m|\varphi\rangle = 1 \quad \forall|\varphi\rangle$

prob. distr

$\Rightarrow \quad \sum E_m = I$

Pose measurement state $\quad \dfrac{E_m|\varphi\rangle}{\|E_m|\varphi\rangle\|} = \dfrac{E_m|\varphi\rangle}{\sqrt{\langle\varphi|E_m|\varphi\rangle}}$

Example. Shördinger's cat



Picture from Wikipedia by Dhatfield.

$|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$E_0 = |0\rangle\langle 0| \quad E_1 = |1\rangle\langle 1|$

After measurement

$\frac{1}{2}$ prob $\quad E_0|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle$

$E_1|\varphi\rangle = \frac{1}{\sqrt{2}}|1\rangle$

Projection Value Measurement (PVM): $\{P_m\}$ $\quad \sum P_m = 1$

von Neumann

P_m, mutually orthogonal projection

Example: Let $\{|v_i\rangle\}$ be a O.N.B $\quad \{E_i = |v_i\rangle\langle v_i|\}$ PVM

$|\varphi\rangle = a|0\rangle + b|1\rangle \qquad |a|^2$-prob seeing $|\varphi\rangle$ in $|0\rangle$

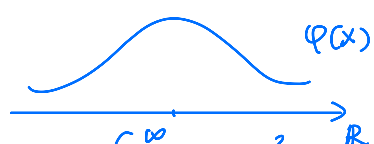
Quantum Mechanics version

Every Observable corresponds to a Hermition operator. (∞-dim)

The only value that will obsered are eigen values.

Example: Position operator: $X\ \varphi(x) = x\varphi(x)$

$e^{-\frac{x^2}{2}} \to x e^{-\frac{x^2}{2}}$



Expected value of position $\quad \langle\varphi|X|\varphi\rangle = \int_{-\infty}^{\infty} X\,|\varphi(x)|^2 dx$

Momentum ... : $P = -i\hbar\frac{\partial}{\partial x}$

$\hbar$ Planck constant

Kinetic Energy : $H = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}$ $\quad$ free Hamiltonian

mass

In f.d. $\quad A = A^*$ Hermitian $\Rightarrow \quad A = \sum \lambda_i E_i \quad \{E_i\}$ PVM

$\lambda_i \in \mathbb{R}$ all physical quantity are real.

$|\varphi_i\rangle$ eign vector of $\lambda_i \Rightarrow \langle\varphi_i|A|\varphi_i\rangle$

$= \lambda_i$

Postulate 3 (Evolution) The evolution of a closed quantum system is described by a unitary.

$$|\varphi\rangle \longrightarrow U|\varphi\rangle$$

Example: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad X|0\rangle = |1\rangle \quad X|1\rangle = |0\rangle$ bit flip

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad Z|0\rangle = |0\rangle \quad Z|1\rangle = -|1\rangle$ phase flip

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ Hadamard gate $\quad H|0\rangle = |+\rangle \quad H|1\rangle = |-\rangle$

Why unitary. $\underset{\||\varphi\|=1}{|\varphi\rangle} \longrightarrow \underset{\||\psi\rangle\|=1}{|\psi\rangle = U|\varphi\rangle}$ So $\|U|\varphi\rangle\| = \||\varphi\rangle\| \;\forall |\varphi\rangle$
$\Downarrow$
unitary.

Postulate 3' (Continuous time). The time evolution of a closed system is given Schrödinger equation:

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = \underset{\substack{\uparrow \\ \text{Hamiltonian}}}{H}|\psi(t)\rangle$$

$\langle\psi|H|\psi\rangle$ expected energy. ① $H(t)$ time dependent

② $H(t) \equiv H$ time independent

② $|\psi(t)\rangle = e^{iHt}|\psi(0)\rangle \qquad H$ hermitian $\Rightarrow e^{iHt}$ unitary.

Example: $H = \sum_E E|\varphi_E\rangle\langle\varphi_E|$ $\quad |\varphi_E\rangle$ energy basis.

$e^{iHt}|E\rangle = e^{iEt}|E\rangle \qquad |\varphi\rangle = \sum a_E|E\rangle \qquad e^{iHt} = \sum a_E e^{iEt}|\varphi_E\rangle$