Posulate 1. Quantum System $\longrightarrow \mathbb{C}^n$

State $\longrightarrow |\varphi\rangle$ unit vector

Posulate 2. Measurement $\longrightarrow$ POVM $\sum_m E_m = I \quad E_m \geq 0$

Posulate 3. Evolution $\longrightarrow$ unitary $U : |\varphi\rangle \longrightarrow U|\varphi\rangle$

Postulate 4 (Composite System) The state space of a composite system is the tensor product of the state space.

Tensor product space

$V = \mathbb{C}^X \quad W = \mathbb{C}^Y$

$$V \otimes W = \mathbb{C}^{X \times Y} = \{\varphi : X \times Y \to \mathbb{C}\}$$

$$\varphi \otimes \psi (x, y) = \varphi(x)\psi(y)$$

$$|\varphi\rangle \otimes |\psi\rangle$$

$$V \otimes W = \text{span}\{|\varphi\rangle \otimes |\psi\rangle \mid |\varphi\rangle \in V, |\psi\rangle \in |\psi\rangle\}$$

$$\alpha(|\varphi\rangle \otimes |\psi\rangle) = \alpha|\varphi\rangle \otimes |\psi\rangle = |\varphi\rangle \otimes \alpha|\psi\rangle$$

$$(\alpha|\varphi_1\rangle \otimes |\psi\rangle + \beta|\varphi_2\rangle \otimes |\psi\rangle) = (\alpha|\varphi_1\rangle + \beta|\varphi_2\rangle) \otimes |\psi\rangle$$

$$(\langle\varphi_1| \otimes \langle\varphi_2|)(|\psi\rangle_1 \otimes |\psi_2\rangle) = \langle\varphi_1|\psi_1\rangle \langle\varphi_2|\psi_2\rangle$$

Given $\{|V_i\rangle\} \leq V \quad \{|W_j\rangle\} \leq W$ O.N.B

$\{|V_i\rangle \otimes |W_j\rangle \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ O.N.B of $V \otimes W$

So $\dim(V \otimes W) = \dim(V) \dim W$

Now back to state space

Example (product state) $V_1 = V_2 = \mathbb{C}^2$.

$|0\rangle \otimes |1\rangle = |01\rangle \qquad |1\rangle \otimes |0\rangle = |10\rangle$

$|0\rangle \otimes |0\rangle = |00\rangle \qquad |1\rangle \otimes |1\rangle = |11\rangle$

$|\varphi\rangle = a|0\rangle + b|1\rangle \qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\underbrace{|\varphi\rangle}_{\text{state of 1st system}} \otimes \underbrace{|\psi\rangle}_{\text{state of 2nd system}} = a\alpha|00\rangle + a\beta|01\rangle + b\alpha|10\rangle + b\beta|11\rangle$

Example (Entangled state / non product state)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

O.N.B of $\mathbb{C}^2 \times \mathbb{C}^2 \cong \mathbb{C}^4$

$|\Phi^+\rangle \neq |\varphi\rangle \otimes |\psi\rangle$ for any $|\varphi\rangle \otimes |\psi\rangle$, called entangled state.

super position of produse state.

Operations on Tensor produce system

$A \in B(\mathbb{C}^n) \qquad B \in B(\mathbb{C}^m)$

Define $(A \otimes B)\,|\varphi\rangle \otimes |\psi\rangle = A|\varphi\rangle \otimes B|\psi\rangle$. $\quad A \otimes B \in B(\mathbb{C}^n \otimes \mathbb{C}^m)$

$$= \operatorname{span}\left\{ A \otimes B \;\middle|\; \begin{array}{l} A \in B(\mathbb{C}^n) \\ B \in B(\mathbb{C}^m) \end{array} \right\}$$

## Product measurement

$A = A^* \in B(\mathbb{C}^n) \quad B = B^* \in B(\mathbb{C}^m)$

$(A \otimes B)^* = A^* \otimes B^* = A \otimes B$ is Hermitian in $B(\mathbb{C}^n \otimes \mathbb{C}^m)$

$\langle\varphi| \otimes \langle\psi| (A \otimes B) |\varphi\rangle \otimes |\psi\rangle = \langle\varphi|A|\varphi\rangle \; \langle\psi|B|\psi\rangle$

Expectation of observable A given $\varphi$ ... B — -- $\psi$

POVM $\sum_m E_m = I_1 \qquad \sum_n F_n = I_2 \qquad \sum_{m,n} E_m \otimes F_n = I_1 \otimes I_2$

$E_m \geq 0. \; F_n \geq 0 \Rightarrow E_m \otimes F_n \geq 0$

prob of $|\varphi\rangle \otimes |\psi\rangle$ measured in outcome $(m.n)$

$\langle\varphi| \otimes \langle\psi| E_m \otimes F_n \; |\varphi\rangle \otimes |\psi\rangle = \langle\varphi|E_m|\varphi\rangle \quad \langle\psi|F_n|\psi\rangle$

produce probability

## Partial Measurement : $A \in B(\mathbb{C}^n)$

$\langle\varphi| \otimes \langle\psi| A \otimes I |\varphi\rangle \otimes |\psi\rangle$
$= \langle\varphi|A|\varphi\rangle \langle\psi|\psi\rangle = \langle\varphi|A|\varphi\rangle$

or $\sum_m E_m = I \quad \langle\varphi|\langle\psi| (E_m \otimes I) |\psi\rangle|\varphi\rangle = \langle\varphi|E_m|\varphi\rangle$

Product state ~ independent measurement outcome

Entangled state ~ corrolated measurement outcome

Example: $E_0 = |0\rangle\langle 0| \quad E_+ = |+\rangle\langle +| \quad |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$E_1 = |1\rangle\langle 1| \quad E_- = |-\rangle\langle -|$

$\langle\Phi^+| E_0 \otimes I |\Phi^+\rangle = \frac{1}{2} = \langle\Phi^+| I \otimes E_0 |\Phi^+\rangle$

$\langle\Phi^+| E_1 \otimes I |\Phi^+\rangle = \frac{1}{2} = \langle\Phi^+| I \otimes E_1 |\Phi^+\rangle$

$\langle\Phi^+| (E_i \otimes E_j |\Phi^+\rangle = \begin{cases} \frac{1}{2} & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ not independent on $i.j$

$\langle\Phi^+| E_i \otimes E_\pm |\Phi^+\rangle = 0$ for $i = 1.2$, + or −

$$\langle \Phi^+ | A \otimes I | \Phi^+ \rangle = \frac{1}{2}\langle 0|A|0\rangle + \frac{1}{2}\langle 1|A|1\rangle$$

$$\neq \langle \varphi|A|\varphi\rangle \quad \forall\ |\varphi\rangle \in \mathbb{C}^2$$

The observation of $|\Phi^+\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ on the first system does not match any vector state. Does this violates Postulate 1?

No. B/C A $\boxed{\text{closed system}}$ has a vector state $|\varphi\rangle$

The state of a general system is given by a $\boxed{\text{density operator}}$

A vector state $|\varphi\rangle \in \mathbb{C}^n$ is also called pure state

In general, a quantum system can have mixed state. given by

$\{p_i, |\psi_i\rangle\}$ ensemble of pure states. $p_i$ is the prob. system in $|\psi_i\rangle$

Then $\rho \equiv \sum p_i |\psi_i\rangle\langle\psi_i|$ is the $\boxed{\text{density operator}}$

Example: $\rho = |\varphi\rangle\langle\varphi|$. $\lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|$ $\quad u|+\rangle\langle +| + (1-u)|-\rangle\langle -|$. $\sum p_i |i\rangle\langle i|$

Postulate 1 (State) The state of a quantum system is completely described by a density operator $\rho$ acting on the state space of the system.

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

if the system is of prob. $p_i$ in the pure state $|\psi_i\rangle$.

Postulate 2 (Measurement) A POVM $\{E_m\}$ has prob. of

$$p(m) = \operatorname{tr}(\rho E_m)$$

to be outcome $E_m$.

(Observable) The expected value of an observable $A = A^\dagger$ given state $\rho$ is $\operatorname{tr}(\rho A)$

Postulate 3 (Evolution) The unitary evolution of a closed quantum system is given by

$$\rho \to U \rho U^*$$