Quantum Channel II

Stinspring Dilation

Theorem. The following are equivalent

① A linear map $\Phi: B(H) \to B(H)$ is a quantum channel (CPTP map)

② There exists a partial isometry $V: H \to K \otimes H_E$ s.t.

$$\Phi(\rho) = tr_E(V \rho V^*) \qquad \text{(Stinespring Dilation)}$$

③ There exists a family of operators $\{a_i\} \subseteq B(H, K)$ s.t.

$$\sum a_i^* a_i = I \quad \text{and} \quad \Phi(\rho) = \sum_{i=1}^{k} a_i \rho a_i^* \qquad \text{(Kraus operators)}$$

Pf (③ ⇒ ② ⇒ ①)

Given $\Phi(\rho) = \sum_{i=1}^{k} a_i \rho a_i^*$

Define $H_E = \mathbb{C}^k \qquad H \otimes H_E = H \otimes \mathbb{C}^k \cong H \oplus H \cdots \oplus H$

$V: H \to H \otimes H_E$

$V(|\varphi\rangle) = \sum_i a_i |\varphi\rangle \otimes |i\rangle$

V is a isometry b/c $\langle\varphi|V^* V|\varphi\rangle = \sum_{i=1}^{k} \langle\varphi| a_i^* a_i |\varphi\rangle = \langle\varphi| \sum_{i=1}^{k} a_i^* a_i |\varphi\rangle$
$= \langle\varphi|\varphi\rangle$

Then for any $|\varphi\rangle\langle\varphi|$, purestate

$$tr_E(V|\varphi\rangle\langle\varphi|V^*) = tr_E\left(\left(\sum_i a_i|\varphi\rangle \otimes |i\rangle\right)\left(\sum_j \langle\varphi|a_j^* \otimes \langle j|\right)\right)$$
$$= tr_E\left(\sum a_i|\varphi\rangle\langle\varphi|a_j^* \otimes |i\rangle\langle j|\right)$$
$$= \sum_i a_i|\varphi\rangle\langle\varphi|a_i^* = \Phi(|\varphi\rangle\langle\varphi|)$$

② ⇒ ① is obvious b/c

$V \cdot V^*$: $\rho \to V\rho V^*$
tr_E : $\mathcal{G}_{AE} \to \mathcal{G}_A$ are CPTP

So does their composition $\Phi(\rho) = tr_E(V\rho V^*)$

Now for the hard part ① ⇒ ③

CPTP ⇒ Kraus operators.

We introduce several interesting tools.

Let $H_A \cong H_{A'} \cong \mathbb{C}^d$. Fix a O.N.B $\{|i\rangle\} \subseteq H_A$

Maximally entangled state $_{AA'} = |\Phi\rangle\langle\Phi|$

$$|\varphi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle$$

Lemma: $\forall a \in B(H)$, $a \otimes 1 |\varphi\rangle = 1 \otimes a^t |\varphi\rangle$

where a^t is the transpose of a w.r.t to $\{|i\rangle\}$

Pf: Sufficient to consider $a = |k\rangle\langle l|$

$$|k\rangle\langle l| \otimes 1 |\varphi\rangle = (|k\rangle\langle l| \otimes 1) \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} |k\rangle \otimes |l\rangle$$

$$= (1 \otimes |l\rangle\langle k|)|\varphi\rangle = (1 \otimes |k\rangle\langle l|^t)|\varphi\rangle$$

The same extends to general $a = \sum a_{kl} |k\rangle\langle l|$

① $\Rightarrow$ ③ $\quad \Phi : B(H_A) \to B(H_B)$ CP

$\Rightarrow \mathrm{id}_{A'} \otimes \Phi : B(H_{A'} \otimes H_A) \to B(H_{A'} \otimes H_B)$ CP $\qquad H_{A'} \cong H_A$

Take (non normalized) maximal entangled state (MES)

$|\varphi\rangle = \sum |i\rangle|i\rangle \qquad \varphi_{A'A} = |\varphi\rangle\langle\varphi| \qquad \varphi_A = \varphi_{A'} = 1$

Then $\omega = \mathrm{id} \otimes \Phi(\varphi) \geq 0$. Note that $\forall \rho \in D(H_A)$

Define $|\rho\rangle = 1 \otimes \sqrt{\rho}\, |\varphi\rangle \qquad |\rho\rangle$ is a purification of ρ

$= \sqrt{\rho}^t \otimes 1 |\varphi\rangle \qquad \mathrm{tr}_{A'}(|\rho\rangle\langle\rho|) = \mathrm{tr}_{A'}(1 \otimes \sqrt{\rho}|\varphi\rangle\langle\varphi| 1 \otimes \sqrt{\rho})$

$= \sqrt{\rho} \;\; 1 \sqrt{\rho} = \rho$

Then $\Phi(\rho) = \Phi(\mathrm{tr}_{A'}(|\rho\rangle\langle\rho|))$

$= \mathrm{tr}_{A'}(\mathrm{id}_{A'} \otimes \Phi(|\sqrt{\rho}\rangle\langle\sqrt{\rho}|))$

$= \mathrm{tr}_{A'}(\mathrm{id}_{A'} \otimes \Phi(\sqrt{\rho}_{A'}^t \otimes 1 |\varphi\rangle\langle\varphi| \sqrt{\rho}_{A'}^t \otimes 1))$

$= \mathrm{tr}_{A'}(\sqrt{\rho_{A'}}^t \omega_{AA'} \sqrt{\rho_{A'}}^t) = \mathrm{tr}_{A'}(\rho_{A'}^t \omega_{AA'})$

$\mathrm{tr}(AB) = \mathrm{tr}((AB)^t) = \mathrm{tr}(B^t A^t) = \mathrm{tr}(A^t B^t)$

Recall $H_A \otimes H_{A'} \cong B(H_A) \qquad \mathrm{Vec}(a) \leftarrow a$

$e_i \otimes e_j \leftrightarrow e_{ij} \qquad |h\rangle \to \mathrm{op}(|h\rangle)$

Then $\mathrm{tr}(\rho_{A'}^t |h\rangle\langle k|) = \mathrm{op}(|h\rangle)\, \rho\, \mathrm{op}(|k\rangle)^*$

Indeed, $\mathrm{tr}_{A'}(\rho\, (|i\rangle |j\rangle)(\langle k| \langle l|)) = |i\rangle\langle k| \langle j|\rho|l\rangle$

$= |i\rangle\langle j|\rho|l\rangle\langle k|$

$= e_{ij}\, \rho\, e_{kl}^*$

Assume $W_{AA'} = \sum |\psi_i\rangle\langle\psi_i|$

$$tr_{A'}(\rho_{A'}^t W_{AA'}) = tr(\rho_{A'}^t |\psi_i\rangle\langle\psi_i|) = \sum a_i \rho a_i^*$$

As $a_i = op(|\psi_i\rangle)$

Since $\Phi(\rho) = \sum a_i \rho a_i^*$ and $tr(\Phi(\rho)) = tr(\rho)$

$$tr(\rho \sum_i a_i^* a_i) = tr(\rho)$$

$$\Rightarrow \sum_i a_i^* a_i = I$$

$\square$

Note that in the above proof,

We used Φ CP $\Rightarrow$ $W = id_{A'} \otimes \Phi(\varphi) \geq 0 \Rightarrow$ ③ $\Rightarrow$ ② $\Rightarrow$ Φ CP

So Φ CP $\Leftrightarrow$ $W = id_{A'} \otimes \Phi(\varphi) \geq 0$

Theorem (Choi) TFAE

① $\Phi : B(H_A) \to B(H_B)$ is CP

② $id_{A'} \otimes \Phi_A : B(H_{A'} \otimes H_A) \to B(H_{A'} \otimes H_B)$ is positive

③ $W_{A'B} = id_{A'} \otimes \Phi_{A \to B}(\varphi_{AA'})$ is positive

where $\varphi_{AA'} = \frac{1}{\sqrt{d}} \sum |i\rangle|i\rangle$ is the m.e.s.

In particular $W_{A'B}$ is called the Choi density of Φ

If $\dim(H_A) = d$, $\tilde{W}_{A'B} = id_{A'} \otimes \Phi_{A \to B}(d\varphi_{AA'}) = d\, W_{A'B}$ Choi matrix of Φ

$\tilde{W}_{A'B} = \sum |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$ determines Φ.

Pf: $\underbrace{① \Rightarrow ②}_{\text{obvious}} \Rightarrow ③ \Rightarrow ①$ by argument in previous proof

What is really a quantum channel?

Mathematically. ① CP map

② Embedding and Partial trace

$\rho \to V\rho V^*$ $\qquad \sigma_{AE} \to \sigma_A$

V isometry

Physically: ① A linear map sending all states (including entangled state with enviroment) to states

② Unitary evolution with environment