Quantum Channels

Transformation of closed quantum system

$$\Phi : B(\mathbb{C}^n) \longrightarrow B(\mathbb{C})^n$$

$$\Phi(\rho) = U\rho U^*$$

How about open quantum systems?

Two appraches $\begin{cases} \text{Mathematical: CPTP map} \\ \text{Phsyiscal : interaction with environment} \end{cases}$ > Quantum channels

Mathematical approach

$$H = \mathbb{C}^n \quad K = \mathbb{C}^m, \quad \Phi : B(H_A) \rightarrow B(H_B) \text{ linear}$$

$$\rho \in D(H_A) \Rightarrow \Phi(\rho) \in D(H_B)$$

Def: A linear map $\Phi : B(H) \rightarrow B(K)$ is positive if

$$\forall A \geq 0, \quad \Phi(A) \geq 0$$

Example: ① $a \in L(K, H)$ $\quad \Phi(X) = a^* X a$

$\forall |k\rangle \in K$, $\quad \langle k| aXa^* |k\rangle = \langle k| a^* X a |k\rangle$

$= \langle ak| X |ak\rangle \geq 0$

In particular: Unitary U, $\quad \Phi(\rho) = U\rho U^*$

isometry. $V: H \rightarrow K$. $\quad V^* V = I_H$

$$\Phi(\rho) = V\rho V^*$$

② Trace map: $tr: B(H) \to \underset{\underset{B(\mathbb{C})}{\cong}}{\mathbb{C}}$, $X \to tr(x)$

If $X \geq 0$, $tr(X) = \sum_{i=1}^{n} \langle e_i | X | e_i \rangle \geq 0$

③ Transpose: Fixed a basis $\{|i\rangle\} \subseteq H$

$T: B(H) \longrightarrow B(H)$, $(\rho_{ij})_{i,j} \longrightarrow (\rho_{ji})_{i,j}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$A \geq 0 \iff A = B^*B \Rightarrow A^t = B^t (B^*)^t = \underset{\geq 0}{B^t (B^t)^*}$

④ $P(B(H), B(K)) = \{$positive map from $B(H)$ to $B(K)\}$

is a convex cone

If Φ_1, Φ_2 positive, $\forall$ $\lambda \Phi_1 + u \Phi_2$ positive

if $\lambda, u \geq 0$

So $\Phi(X) = \sum a_i X a_i^*$ is positive

Def: A quantum channel from system H to K is a linear map $\Phi: B(H) \to B(K)$ satisfying

① Φ is completely positive: $\forall$ Hilbert space H' s.t. $id_{B(H')} \otimes \Phi$ is positive

② Φ is trace preserving: $tr(\rho) = tr(\Phi(\rho))$

Idea: If $\rho \geq 0$, $tr(\rho)=1$, then $\Phi(\rho) \geq 0$, $tr(\Phi(\rho))=1$

$tr(\rho) = tr(\Phi(\rho)) = 1$ by TP

$\rho \geq 0 \Rightarrow \Phi(\rho) \geq 0$ by positivity

So $\Phi(D(H)) \subseteq D(K)$.

Why complete positivity instead of positivity?

It's all about entanglement

$\exists$ Φ positive, $Id_{B(H')} \otimes \Phi$ is not positive for some H'.

So Φ positive $\not\Rightarrow$ Φ CP

$\exists$ $W \in B(H \otimes H')$, $W \geq 0$, but $\Phi \otimes id_{H'}(W)$ not positive

Note that if $W = \sum p_i \rho_i \otimes \sigma_i$ separable state

$$\Phi \otimes id(W) = \sum p_i \underset{D(K)}{\underbrace{\Phi(\rho_i)}} \otimes \sigma_i \geq 0 \quad \text{seperable seates}$$

This implies if $W \geq 0$ & $\Phi \otimes id_{H'}(W)$ is not positive for some Φ positive

W is entangled!

Example: $\Phi = t$ tranpose, on the $\{|i\rangle\}$ basis

$$\Phi \otimes id \left(\sum a_{ij.kl} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \right)$$
$$= \sum a_{ij.kl} \, |i\rangle\langle j|^t \otimes |k\rangle\langle l|$$
$$= \sum a_{ij.kl} \, |j\rangle\langle i| \otimes |k\rangle\langle l|$$

Now consider $|\varphi\rangle = \frac{1}{\sqrt{n}} \sum |i\rangle \otimes |i\rangle$ maximally entangled state on $\mathbb{C}^n \otimes \mathbb{C}^n$

density

$$\varphi = |\varphi\rangle\langle\varphi| = \frac{1}{n} \sum_{i,j} |i\rangle|i\rangle \, \langle j|\langle j|$$
$$= \frac{1}{n} \sum \overset{e_{ij}}{|i\rangle\langle j|} \otimes \overset{e_{ij}}{|i\rangle\langle j|}$$

Apply partial transpose,

$$\Phi \otimes id(\varphi) = \frac{1}{n} \sum \left(|i\rangle\langle j| \right)^t \otimes |i\rangle\langle j|$$
$$= \underbrace{\frac{1}{n} \sum_{i,j} |j\rangle\langle i| \otimes |i\rangle\langle j|}_{F}$$

One can verify $F = F^*$, $F^2 = 1$ self-adjoint unitary

$F \neq I \Rightarrow$ F has spectrum $\{-1, 1\}$

Not positive!

So transpose t map is positive, but not completely positive

it does not perserve positivity for all entangled state.

What are CP?

① $\Phi(\rho) = a\rho a^*$ b/c $\Phi \otimes id(\rho) = (a \otimes 1)\rho(a^* \otimes 1)$

$\Phi(\rho) = \sum a_i \rho a_i^*$ CP

② Trace: $tr: B(H) \to \mathbb{C}$, $tr(\rho) = 1 \quad \forall \rho \in D(H)$

Phsyically, this is juse igonre the system

called forgot ful channel

Partial trace $tr \otimes id_K : B(H \otimes K) \to B(H)$

forget pare of the system.

Pf: $\forall \underset{B(K)}{\underset{\cap}{A}} \geq 0$, $tr_K(tr_H \otimes id_K(\rho) A)$

$= tr_K \otimes tr_H (\rho\, A \otimes I)$