Last time:

Operator $L(\mathbb{C}^n, \mathbb{C}^m) \xleftrightarrow{\text{basis}} M_{m\times n}$ matrix

A $\quad (a_{ij})_{i,j} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

$A+B \quad (a_{ij}) + (b_{ij}) = (a_{ij}+b_{ij})_{i,j}$

$A\cdot B \quad (a_{ij})\cdot(b_{kl}) = (\sum a_{ij} b_{jl})_{i,l}$

$L(\mathbb{C}^n, \mathbb{C}^n) = B(\mathbb{C}^n) \xleftrightarrow{\text{basis}} M_{n\times n} = M_n$

Operator on $\mathbb{C}^n$ $\quad$ square matrix

For $A, B \in B(\mathbb{C}^n) \cong M_n$, we can define

① Addition: $A+B$

$A+B = B+A$ commutitave

$0u = \vec{0}$ zero operator $\quad A + 0 = A$,

② Multiplication: $A\cdot B$

$\boxed{AB \neq BA \text{ non commutitave}}$ e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

Identity operator: $Iu = u$, $\quad I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$A\cdot I = A = I\cdot A$

③ Adjoint: define A^* as $\quad A = [a_{ij}] \quad A^* = [\overline{a_{ji}}]$

$\langle A^* v, u\rangle = \langle v, Au\rangle$

$(A^*)^* = A$ $\quad$ So $A^* = \overline{(A^T)}$ tranpose and conjugate

$(AB)^* = B^* A^* \quad (A+B)^* = B^* + A^*$ $\quad$ e.g. $\begin{bmatrix} 1-i & 2-i \\ 4 & 3 \end{bmatrix}^* = \begin{bmatrix} 1+i & 4 \\ 2+i & 3 \end{bmatrix}$

① + ② is called an algebra

① + ② + ③ is called an $*$-algebra

$B(\mathbb{C}^n)$ Algebra of (linear) operators on $\mathbb{C}^n$

M_n Algebra of $n \times n$ complex matrix

Other example of algebra? $\{u : \Omega \to \mathbb{C}\}$

Function algebras

with $fg = gf$!

Operators / Matrix with special property

1. A is self-adjoint if $A = A^*$

e.g. $A = \begin{bmatrix} 1 & 3+i \\ 3-i & 2 \end{bmatrix}$

$\iff \langle v, Av \rangle \in \mathbb{R} \quad \forall\, v \in \mathbb{C}^n$ $\quad \left(\overline{\langle v, Av \rangle} = \langle Av, v \rangle = \langle A^* v, v \rangle = \langle v, Av \rangle \right)$

2. A is positive if $A = B^* B$ for some B.

e.g. $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $\quad B^* B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$

$\iff \langle v, Av \rangle \geq 0 \quad \forall\, v \in \mathbb{C}^n$ $\quad (\langle v, Av \rangle = \langle v, B^* B v \rangle = \langle Bv, Bv \rangle \geq 0)$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \left\langle \binom{a}{b}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \binom{a}{b} \right\rangle = \left\langle \binom{a}{b}, \binom{a}{2b} \right\rangle = \bar{a} a + 2 \bar{b} b$

$= |a|^2 + 2|b|^2 \geq 0$

$A = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \left\langle \binom{a}{b}, \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \binom{a}{b} \right\rangle = 5|a|^2 + \bar{a} b + a \bar{b} \quad |b|^2$

$= 4|a|^2 + |a|^2 + \bar{a} b + a \bar{b} + |b|^2$

$= 4|a|^2 + |a+b|^2 \geq 0.$

$A \geq 0 \Rightarrow A = A^*$ b/c $\quad A = B^* B \Rightarrow A^* = B^* (B^*)^* = B^* B = A$

3. P is a projection if $P = P^* = P^2$

e.g. $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ or $P(u) = \langle v, u \rangle v$ for some $\|v\| = 1$

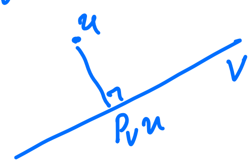
rank one projection

P projection $\iff \forall v, \ \langle Pv, (I-P)v \rangle = 0$

$v = Pv + (I-P)v. \quad Pv \perp (I-P)v$

$\mathcal{V} \subseteq \mathbb{C}^n$ a subspace $(\forall v, u \in \mathcal{V}, \ \alpha v + \beta u \in \mathcal{V})$

$\longleftrightarrow P_V$ s.t. $\forall u \in \mathbb{C}^n, \ \min_{V \in P_V} \|u - v\| = \|u - P_V u\|$



$\mathcal{V} \perp \mathcal{W}$ if $\forall v \in \mathcal{V}, w \in \mathcal{W}, \ \langle v, w \rangle = 0$

$\iff P_V \cdot P_W = 0$ orthogonal

$\mathcal{V} \subseteq \mathcal{W} \iff P_V \cdot P_W = P_V$

4. U is a unitary if $U^* U = U U^* = I$

e.g. $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Pauli Matrix

U Unitary $\iff \langle Uv, Uw \rangle = \langle v, w \rangle \quad \forall v \in \mathbb{C}^n$

$\langle v, \underbrace{U^* U}_{=I} w \rangle$

Change of basis: Given an O.N.B $\{v_i\} \subseteq \mathbb{C}^n$ $\quad \langle v_i, v_j \rangle = \delta_{ij}$

$\{Uv_i\}$ is also O.N.B b/c $\langle Uv_i, Uv_j \rangle$

Given any two basis $\{v_i\}, \{w_i\} \subseteq \mathbb{C}^n$

$\exists!$ Unitary U. s.t. $U v_i = w_i \quad \forall i$

If U unitary. $\{e_i\} \to \{v_i\}$, then U^* unitary $\quad U^* : \{v_i\} \to \{e_i\}. \quad U^* = U^{-1}$

A operator is diagonalizable if $\exists$ some O.N.B $\{v_i\} \subseteq \mathbb{C}^n$ s.t.

$$\langle v_j, A.v_i \rangle = \lambda_i \delta_{ij} = \begin{cases} \lambda_i & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

A matrix $A=(a_{ij})$ is diagonalizable if $\exists\, U$ unitary

$$UAU^* = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ diagonal matrix}$$

U is the unitary $\{e_i\} \rightarrow \{v_i\}$

Spectrum Theorem : A is diagonizable if and only if $AA^* = A^*A$

In particular, $A = \sum \lambda_i E_i$ where $\lambda_i \in \mathbb{C}$

E_i mutual orthogonal projections

s.t. $\sum_{i=1}^{k} E_i = I$

Remark: A satisfy $AA^* = A^*A$ is called a normal operator

What are λ_i and E_i,

Recall that $\lambda \in \mathbb{C}$ is an eigen value of A if $\exists\, u \neq 0$

$$Au = \lambda u$$

Eigen space $V_\lambda = \{u \in \mathbb{C}^n \mid Au = \lambda u\}$ E_λ projection onto V_λ

$\text{Spec}(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigen value of } A\}$ finite set

If $A^*A = AA^*$, $A = \sum_{\lambda \in \text{Spec}(A)} \lambda E_\lambda$ for some E_λ with $E_\lambda \perp E_u$ $\forall \lambda \neq u \in \text{Spec}(A)$

In some basis, $UAU^* = \begin{bmatrix} \lambda_1 E_1 & & & \\ & \lambda_2 E_2 & & \\ & & \ddots & \\ & & & \lambda_n E_n \end{bmatrix} = \begin{bmatrix} \overbrace{\lambda_1 \ddots \lambda_1}^{\dim(E_1)} & & & & & \\ & \overbrace{\lambda_2 \ddots \lambda_2}^{\dim(E_2)} & & & & \\ & & \ddots & & & \\ & & & \lambda_n \ddots \lambda_n \end{bmatrix}$

eigen vector

Example : $\begin{bmatrix}1&0\\0&-2\end{bmatrix} = 1\,E_1 + (-2)E_2$ $\quad V_1=\begin{pmatrix}1\\0\end{pmatrix} \quad V_2=\begin{pmatrix}0\\1\end{pmatrix}$

$$E_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix} \qquad \begin{bmatrix}0&0\\0&2\end{bmatrix}$$

$$\begin{bmatrix}1&-2i\\2i&-2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&0\\0&0\end{bmatrix} + \begin{bmatrix}\frac{1}{2}&1\\1&2\end{bmatrix}$$

$$= (-3)\begin{bmatrix}\frac{1}{5}&\frac{-2i}{5}\\\frac{2i}{5}&\frac{4}{5}\end{bmatrix} + 2\begin{bmatrix}\frac{4}{5}&\frac{2i}{5}\\-\frac{2i}{5}&\frac{1}{5}\end{bmatrix}$$

(Why $\mathbb{C}$? Real matrix can have $\mathbb{C}$ eigenvalues)

Fuctional calculus

Now for a normal A, $\quad A = U^*\overbrace{\begin{bmatrix}\mu_1&&&\\&\mu_2&&\\&&\ddots&\\&&&\mu_n\end{bmatrix}}^{D_\mu}U = \sum \lambda_i E_i$

$\mu_i \in \mathrm{spec}(A)$

For a function $f: \mathrm{spec}(A) \to \mathbb{C}$

$$f(A) = U^*\begin{bmatrix}f(\mu_1)&&\\&\ddots&\\&&f(\mu_n)\end{bmatrix}U = \sum f(\lambda_i)E_i$$

Justification. $\quad A^2 = A\cdot A = U^* D_\mu \underbrace{U\,U^*}_{I} D_\mu U = U^* D_\mu^2 U = U^*\begin{bmatrix}\mu_1^2&&\\&\ddots&\\&&\mu_n^2\end{bmatrix}U$

$$A^k = A\cdot A\cdots A = U^* D_\mu U \cdots U^* D_\mu U = U^* D_\mu U = U^*\begin{bmatrix}\mu_1^k&&\\&\ddots&\\&&\mu_n^k\end{bmatrix}U$$

Thus for any polynomial $f(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots a_1 x + a_0$

$$f(A) = a_k A^k + \cdots a_1 A + a_0 I$$

$$= a_k U^* \begin{bmatrix} u_1^k & & \\ & \ddots & \\ & & u_n^k \end{bmatrix} U + \cdots a_1 U^* \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix} U + a_0 U^* \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} U$$

$$= U^* \begin{bmatrix} a_k u_1^k & & \\ & \ddots & \\ & & a_k u_n^k \end{bmatrix} U + \cdots U^* \begin{bmatrix} a_1 u_1 & & \\ & \ddots & \\ & & a_1 u_n \end{bmatrix} + U^* \begin{bmatrix} a_0 & & \\ & \ddots & \\ & & a_0 \end{bmatrix} V$$

$$= U^* \begin{bmatrix} f(u_1) & & \\ & \ddots & \\ & & f(u_n) \end{bmatrix} U$$ for any polynomial

Now for general $f: spec(A) \to \mathbb{C}$,

$\exists$ polynomial $P_k \to f$ uniformly

$$\begin{array}{ccc} P_k(A) & \longrightarrow & f(A) \\ \parallel & & \\ U^* \begin{bmatrix} P_k(u_1) & & \\ & \ddots & \\ & & P_k(u_n) \end{bmatrix} U & \longrightarrow & U^* \begin{bmatrix} f(u_1) & & \\ & \ddots & \\ & & f(u_n) \end{bmatrix} U \end{array}$$

Example: $f(x) = e^x$ $\quad e^A = \sum_{k=0} \frac{1}{k!} A^k$ $\qquad X = \begin{bmatrix} & t \\ -t & \end{bmatrix}$ $\quad e^X = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

$f(x) = \sqrt{x}$, $x \geq 0$, For $A \geq 0$, $\sqrt{A}$:= unique positive operator $\sqrt{A}$. $\quad A^p$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \sqrt{A} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \qquad A^p = \begin{bmatrix} 1 & 0 \\ 0 & 2^p \end{bmatrix}$$

General A, $\qquad |A| = \sqrt{A^*A}$ $\qquad$ (Note that $A^*A \neq AA^*$)

$\forall f, g : \mathrm{spec}(A) \to \mathbb{C}, \quad f(A)\, g(A) = fg(A) = g(A)\, f(A)$

$f(t) = \log t$, $t > 0$ For $A > 0$, $\log A = U^* \begin{bmatrix} \log u_1 & & \\ & \ddots & \\ & & \log u_n \end{bmatrix} U$

$f(t) = t \log t$ $A \log A = U^* \begin{bmatrix} u_1 \log u_1 & & \\ & \ddots & \\ & & u_n \log u_n \end{bmatrix} U = A \cdot \log A.$

In general, $AB \neq BA$. When $AB = BA$ can happen?

$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $B = \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix}$ diagonal matrix

$AB = BA$ if and only if $\exists$ unitary U

$A = U D_1 U^*$ $B = U D_2 U^*$

D_1, D_2 diagonal matrix