Last time: $|\varphi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ induce $\varphi \in B(\mathbb{C}^n)^* \cong L(B(\mathbb{C}^n), \mathbb{C})$

$\varphi(A) = \langle \varphi | A \otimes I | \varphi \rangle$ $\qquad \overset{\shortparallel}{M_n^*}$

Fact: $\forall f \in M_n^*$, $\exists$ operator $\rho \in M_n$ s.t. $f(A) = tr(A\rho)$

$\varphi \in M_n^*$ is called a linear functional of M_n

What property φ have $\qquad \rho \in B(\mathbb{C}^n)$, $\varphi(A) = tr(A\rho)$

① $\varphi(I) = \langle \varphi | I \otimes I | \varphi \rangle = \langle \varphi | \varphi \rangle = 1 \iff tr(\rho I) = tr(\rho) = 1$

called, $\boxed{\text{unital}}$ $\qquad$ trace 1

② If $A \geq 0$, $\varphi(A) = \langle \varphi | A \otimes I | \varphi \rangle \geq 0 \iff \forall A \geq 0 \ tr(\rho A) \geq 0$

called, $\boxed{\text{positive}}$ $\qquad \rho \geq 0$

Lemma $tr(\rho A) \geq 0$ for $\forall A \geq 0 \iff \rho \geq 0$

Pf: (See Homework)

$\varphi \in M_n^*$ is called a state $\qquad \rho \in M_n$ is a density operator

if φ is positive and unital $\qquad$ if $\rho \geq 0$ and $tr(\rho) = 1$

$$S(M_n) = \{\varphi \in M_n \mid \varphi \text{ positive unital}\} \underset{1 \text{ to } 1}{\cong} D(M_n) = \{\rho \in M_n \mid \rho \geq 0, \ tr(\rho)\}$$

$S(M_n) \ni \varphi \longrightarrow \exists!\ d_\varphi \in M_n$ s.t. $\varphi(A) = tr(A d_\varphi)$

$D(M_n) \ni \rho \rightarrow \varphi_\rho(A) := tr(A\rho)$

Fact. $D(M_n) \cong S(M_n)$ is a convex set.

If $\rho_i \in D(M_n)$, $\underset{p_i \geq 0}{\sum p_i = 1}$, then $\sum \lambda_i \rho_i \in D(M_n)$

A state of a quantum system $\mathbb{C}^n$ can be equivalently described by one of the following

① A state $\varphi: B(\mathbb{C}^n) \to \mathbb{C}$, i.e. $\varphi(I)=1$. and $\varphi(A)\geq 0$ if $A\geq 0$

② A density operator $\rho \in B(\mathbb{C}^n)$ s.t. $tr(\rho)=1$, $\rho\geq 0$

③ An ensemble of pure state $\{p_i, |\psi_i\rangle\}$ $\Sigma p_i=1$. $p_i\geq 0$ $|\psi_i\rangle \in \mathbb{C}^n$

④ A state vector $|\varphi\rangle \in \mathbb{C}^n \times \mathbb{C}^m$ for some m

$$\{p_i, |\psi_i\rangle\} \underset{\text{orthogonal decomposition}}{\overset{\rho=\Sigma p_i|\psi_i\rangle\langle\psi_i|}{\rightleftarrows}} \rho \overset{\varphi(A)=tr(\rho A)}{\longleftrightarrow} \varphi \underset{\varphi(A)=\langle\varphi|A\otimes I|\varphi\rangle}{\overset{\text{purification}}{\rightleftarrows}} |\varphi\rangle$$

density operator

State (unital positive linear function)

pure state on larger system

Examples

① Pure state = Vector state: $|\varphi\rangle \leftrightarrow \varphi(A)=\langle\varphi|A|\varphi\rangle \longleftrightarrow \rho=|\varphi\rangle\langle\varphi|$ density operator

② A mixed state $\rho = \Sigma p_i|\varphi_i\rangle\langle\varphi_i|$ $\{|\varphi_i\rangle\}$ orthonormal set

$\Sigma p_i = 1$ $p_i \geq 0$

Mixed state are convex combination of pure state

If $p_i=1$. $p_j=0$ $\forall j\neq i$ $\Rightarrow$ $\rho=|\varphi_i\rangle\langle\varphi_i|$ pure state

For $\mathbb{C}^n$, $p_i=\frac{1}{n}$, $\rho=\Sigma\frac{1}{n}|\varphi_i\rangle\langle\varphi_i| = \frac{1}{n}I$, completely mixed state

like uniform distribution $(\frac{1}{n}, \frac{1}{n} \cdots \frac{1}{n})$

③ Flat state: P projection. $\rho=\frac{P}{tr(P)} = \frac{1}{k}\sum_{i=1}^{k}|\varphi_i\rangle\langle\varphi_i|$ $\{|\varphi_i\rangle\}$ O.N.B of $Ran(P)$.

$k=tr(P)\in\mathbb{N}$

④ Ensemble of pure states $\{p_i, |\varphi_i\rangle\} \to \rho=\Sigma p_i|\varphi_i\rangle\langle\varphi_i|$

e.g. $\lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1| = \begin{pmatrix}\lambda & \\ & 1-\lambda\end{pmatrix}$ $\lambda|+\rangle\langle+| + (1-\lambda)|-\rangle\langle-| = \begin{bmatrix}\frac{1}{2} & 2\lambda-1 \\ 2\lambda-1 & \frac{1}{2}\end{bmatrix}$

e.g. $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{I}{2} = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$

Postulate 1: Quantum system $\mathbb{C}^n$

State $\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i|$ density operator

$\{p_i, |\varphi_i\rangle\}$ ensemable

$\varphi(\cdot) = tr(\rho\ \cdot)$ "state"

Postulate 2: Observables $A = A^*$ expected value

$\varphi(A) = tr(\rho A) = \sum_i p_i \langle\varphi_i| A |\varphi_i\rangle$

Measurement $\{E_m\}$: prob. of out come m

POVM $\varphi(E_m) = tr(\rho E_m) = \sum_i p_i \langle\varphi_i|E_m|\varphi_i\rangle$

$E_m \geq 0 \Rightarrow \varphi(E_m) \geq 0.$, $\sum_m E_m = I \Rightarrow \sum \varphi(E_m) = \varphi(\sum E_m)$

$= \varphi(I) = 1$

So $\{\varphi(E_m)\}$ prob. density function

Postulate 3: Transformation of closed system: Unitary U

$|\varphi\rangle \rightarrow U|\varphi\rangle$

$\{p_i, |\varphi_i\rangle\}$'s $\longrightarrow \{p_i, U|\varphi_i\rangle\}$ $\quad (U|\varphi_i\rangle)^* = \langle\varphi_i|U$

$\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i| \longrightarrow \sum p_i\ U|\varphi_i\rangle\langle\varphi_i|U^*$

$= U(\sum p_i |\varphi_i\rangle\langle\varphi_i|)U^*$

$= U\rho U^*$ unitary conjugate

Postulate 4: Composite System $\longleftrightarrow$ Tensor product space $\mathbb{C}^n \otimes \mathbb{C}^m$

What type of density operators we can have on $\mathbb{C}^n \otimes \mathbb{C}^m$?

Joint States

Denote $H_A = \mathbb{C}^n$ and $H_B = \mathbb{C}^m$, A density operator $\rho_{AB} \in B(H_A \otimes H_B)$ is called a joint state over composite system AB.

① Product state

Let $\rho \in D(\mathbb{C}^n)$ and $\sigma \in D(\mathbb{C}^m)$. Then $\rho \otimes \sigma \in D(\mathbb{C}^{n \times m})$

Fact: a) $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$. b) $\rho \geq 0$, $\sigma \geq 0 \Rightarrow \rho \otimes \sigma \geq 0$

Example:

$$\rho = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1| = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

$$\sigma = \frac{1}{3}|+\rangle\langle +| + \frac{2}{3}|-\rangle\langle -| = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$\rho \otimes \sigma = \begin{bmatrix} \frac{1}{4}\cdot\sigma & 0\sigma \\ 0\cdot\sigma & \frac{3}{4}\cdot\sigma \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{24} & 0 & 0 \\ -\frac{1}{24} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

Completely mixed state

$$\frac{1_n}{n} \otimes \frac{1_m}{m} = \begin{bmatrix} \frac{1}{n} & & \\ & \ddots & \\ & & \frac{1}{n} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{m} & & \\ & \ddots & \\ & & \frac{1}{m} \end{bmatrix} = \begin{bmatrix} \frac{1}{nm} & & \\ & \ddots & \\ & & \frac{1}{nm} \end{bmatrix} = \frac{1_{nm}}{nm}$$

Classical analog: $P_{XY} = P_X \times P_Y$ independent distribution

2. Seperable state

$w = \sum_i \lambda_i \rho_i \otimes \sigma_i$ $\quad \sum \lambda_i = 1 \quad \lambda_i \geq 0$

convex combination of product state $\quad \rho_i \in D(H_A)$. $\sigma_i \in D(H_B)$

Example. a) $\rho \otimes \sigma$ product state

b) $w = \sum \lambda_i |i\rangle\langle i| \otimes \sigma_i$ $\quad \sigma_i \in D(H_B)$, $\sum \lambda_i = 1$ $\lambda_i \geq 0$ $\quad$ classical-quantum state

If $[\sigma_i, \sigma_j] = 0 \ \forall i, j = 0$ Then $\exists$ a O.N.B $|\varphi_i\rangle$ s.t. $\forall i$, $\sigma_i = \sum P_{i,k} |\varphi_k\rangle\langle\varphi_k|$

Then $w = \sum \lambda_i |i\rangle\langle i| \otimes \sigma_i = \sum_i \lambda_i |i\rangle\langle i| \otimes \sum_k P_{ik} |\varphi_k\rangle\langle\varphi_k|$

$$= \sum_{i,k} \underbrace{\boxed{\lambda_i P_{ik}}}_{= Q_{i,k}} |i\rangle\langle i| \otimes |\varphi_k\rangle\langle\varphi_k|$$

Definition: A joint state ρ_{AB} is called entangled if it is not seperable

Example: $|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \in \mathbb{C}^2 \otimes \mathbb{C}^2$

density $|\Phi^+\rangle\langle\Phi^+| = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$

$= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$

O.N.B $\{|00\rangle \quad |01\rangle \quad |10\rangle \quad |11\rangle\}$

$$|\Phi^+\rangle\langle\Phi^+| = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

How to see this is not seperable state

In general, $|\varphi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$. if $|\varphi\rangle \neq |h_1\rangle \otimes |h_2\rangle$ produce vector

then $|\varphi\rangle\langle\varphi|$ is an entangled state

Marginal state / Reduced density

Given a joint state $\rho_{AB} \in B(H_{AB})$. it induce a state on A.

$\varphi_A : B(H_A) \to \mathbb{C}$. $\quad \varphi_A(x) = tr_{AB}((X_A \otimes I_B)\, \rho_{AB})$

$= tr_A(X_A\, \rho_A)$ for some ρ_A

$\rho_A = Id \otimes tr_B(\rho_{AB})$

More explicitly, $\quad \rho_{AB} = \sum a_{ij,kl}\, e_{ij} \otimes e_{kl}$

$\rho_A = id_A \otimes tr_B(\rho_{AB}) = \sum a_{ij,kl}\, Id(e_{ij}) \otimes tr_B(e_{kl})$

$= \sum_{k,i,j} a_{ij,kk}\, e_{ij}$

For example: ① $\omega_{AB} = \rho_A \otimes \sigma_B \Rightarrow \omega_A = id_A \otimes tr_B = \rho_A\, tr_B(\sigma_B) = \rho_A$

$W_B = \sigma_B$

② $W_{AB} = \sum \lambda_i |i\rangle\langle i| \otimes \sigma_i$ $\quad W_A = \sum \lambda_i |i\rangle\langle i|$ $\quad W_B = \sum \lambda_i \sigma_i$

③ $\varphi_{AB} = |\varphi\rangle\langle\varphi|$. $\quad |\varphi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad \varphi_A = \varphi_B = \frac{1}{2}$

$|\varphi\rangle = a|00\rangle + b|11\rangle$ $\quad \varphi_A = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$

$\varphi_B = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$

Now: back to our equivalence:

Given $|\varphi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$, $\longrightarrow$ $\varphi : B(\mathbb{C}^n) \to \mathbb{C}$ $\longrightarrow$ $\rho_\varphi \in B(\mathbb{C})^n$

vector state $\qquad$ state $\varphi(X) = \langle\varphi| X \otimes I |\varphi\rangle$ $\qquad \varphi(X) = tr(\rho_\varphi X)$

How to compute ρ_φ from $|\varphi\rangle$

$$\begin{aligned}
\langle\varphi| X \otimes I |\varphi\rangle &= tr_{AB}(X \otimes I\, |\varphi\rangle\langle\varphi|) \\
&= tr_A \otimes tr_B (X \otimes I\, |\varphi\rangle\langle\varphi|) \\
&= tr_A(X\ tr_B(|\varphi\rangle\langle\varphi|)) \\
&= tr_A(X\ tr_B(\varphi_{AB})) = tr_A(X\, \varphi_A)
\end{aligned}$$

$\rho = \varphi_A = tr_B(|\varphi\rangle\langle\varphi|)$

Can we go back! $\quad \rho \in B(\mathbb{C})^n$ $\longrightarrow$ find $|\rho\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ s.t.

density $\qquad tr_B(|\rho\rangle\langle\rho|) = \rho$

Purification.

Given a mixed state $\rho \in D(H_A)$

① Does there exist a joint state W_{AB} such that $W_A = \rho$?

② . . . . a pure joint state $\varphi_{AB} = |\varphi\rangle\langle\varphi|$ such that $\varphi_A = \rho$?

Such $|\varphi\rangle$ is called a purification of ρ.

① Yes. $\omega_{AB} = \rho_A \otimes \sigma_B$

② Given $\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i|$. Define $|\varphi\rangle_{AA'} = \sum_i \sqrt{p_i}\, |\varphi_i\rangle \otimes |i\rangle$ $\quad A \cong A'$

Then $\varphi = |\varphi\rangle\langle\varphi|$ has reduced density ρ on A.

$$\mathrm{id}_A \otimes \mathrm{tr}_{A'} (|\varphi\rangle\langle\varphi|) = \mathrm{id}_A \otimes \mathrm{tr}_{A'} \Big(\sum_{i,j} \sqrt{p_i}\sqrt{p_j}\, |\varphi_i\rangle\langle\varphi_j| \otimes |i\rangle\langle j| \Big)$$

$$= \sum_i p_i |\varphi_i\rangle\langle\varphi_i| = \rho.$$

Theorem (Ulmann): Let $|\psi\rangle_{AC}$ be any purification of ρ. Then there exises an isometry $V \in L(H_{A'}, H_C)$ such that $V|\varphi\rangle = |\psi\rangle$.

$H_{A'} \cong H_A$