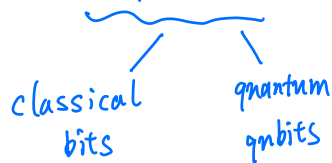


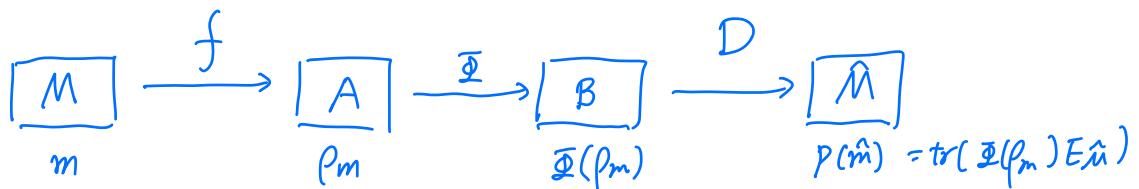
Quantum channel $\Phi: B(H_A) \rightarrow B(H_B)$

How we transmit information through quantum channel?



Classical communication over quantum channel

$$A = B(H_A) \quad B = B(H_B) \quad M = \{1, \dots, M\} = \hat{M}$$



Encoder: $f: M \rightarrow D(H_A)$. $m \mapsto \rho_m$ state preparation

Decoder: POVM measurement $\{E_{\hat{m}}\}_{\hat{m}=1}^M$

$$D: D(H_B) \longrightarrow C(M)$$

$$\rho \longrightarrow P(\hat{m}) = \text{tr}(\rho E_{\hat{m}})$$

Error probability: Given input m , $P(\hat{m} = m | M = m) = \text{tr}(\Phi(\rho_m) E_m)$
 $P(\hat{m} \neq m | M = m) = 1 - \text{tr}(\Phi(\rho_m) E_m)$

Averaged error: $P_e(f, D) = P(M \neq \hat{M}) = 1 - \frac{1}{M} \sum_{m=1}^M \text{tr}(\Phi(\rho_m) E_m)$

Max error: $P_{e, \max}(f, D) = \max_m P(M \neq \hat{M} | M = m) = \max_m 1 - \text{tr}(\Phi(\rho_m) E_m)$

(f, D) is an M code if $f: M \rightarrow A$, $D: B \rightarrow M$; $(M - \epsilon)$ code if $P_e(f, D) \leq \epsilon$

Optimal error : $\varepsilon^*(\Phi, M) = \inf_{\substack{(f,D) \\ M\text{-code}}} P_e(f,D)$

Optimal code size : $M^*(\Phi, \varepsilon) = \sup \{M \mid \exists (f,D), (M, \varepsilon)\text{-code} \mid P_e(f,D) \leq \varepsilon\}$

In the i.i.d setting:

$$M \xrightarrow{f_n} A^n \xrightarrow{\Phi^{\otimes n}} B^n \xrightarrow{D_n} \hat{M}$$

Def: (Classical Capacity of Quantum Channel.)

$$C(\Phi) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\log M(\Phi^n, \varepsilon)}{n}$$

$$= \{R \mid \forall \varepsilon > 0 \exists (2^{nR}, \varepsilon)\text{-code for } \Phi^{\otimes n} \text{ for some } n\}$$

Holevo information

Def: Let $\gamma = \{P_x, p_x\}$ be ensemble of quan states where $(P_x) \in \mathcal{P}(X)$
 $\{P_x\} \subseteq \mathcal{DCH}_B$.

$$\chi(\{P_x, p_x\}) = H(\sum_x p_x P_x) - \sum_x p_x H(P_x)$$

• $\chi(\{P_x, p_x\}) \geq 0$ by concavity of H

• Define c-q state : $P_{XB} = \sum p_x |x\rangle\langle x| \otimes P_x^B \iff \gamma = \{P_x, p_x\}$

$$\chi(\{P_x, p_x\}) = I(X:B)_{P_{XB}} \quad P_X = \sum p_x |x\rangle\langle x| \quad P_B = \sum p_x P_x^B$$

$$\begin{aligned}
 \text{Indeed, } I(X:B)_\rho &= H(X) + H(B) - H(XB) \\
 &= H(B) - H(X|B) \\
 &= H(\sum p_x B_x) - \sum p_x H(p_x)
 \end{aligned}$$

Def (Holevo Information of a Channel)

$$\chi(\Phi) = \sup_{\substack{\{p_x, \rho_x\} \\ \rho_x \in D(\mathcal{H})}} \chi(\{p_x, \Phi(\rho_x)\})$$

$$\begin{aligned}
 &= \sup_{\substack{\rho_{XB} = \mathbb{I}_X \otimes \rho_{X'} \\ \rho_{X'} = \sum p_x(x) |x\rangle\langle x| \otimes |x\rangle\langle x|}} I(X:B)_\rho
 \end{aligned}$$

Theorem (Holevo - Schumacher - Westmore, 1997)

$$C(\Phi) = \lim_{n \rightarrow \infty} \frac{\chi(\Phi^{\otimes n})}{n}$$

① Achievability : $\chi(\Phi) \leq C(\Phi)$

② By definition, $C(\Phi^{\otimes n}) = n C(\Phi)$

(i) $C(\Phi^{\otimes n}) \geq n C(\Phi)$, R is an achievable rate for Φ

$\forall \varepsilon \exists (f_\varepsilon, D_\varepsilon) \quad (2^{nR}, \varepsilon)$ code for $\Phi^{\otimes k}$ some k

$\Rightarrow (f_\varepsilon^{\otimes n}, D_\varepsilon^{\otimes n}) \quad (2^{nR}, n\varepsilon)$ code for $\Phi^{\otimes nk}$
 \uparrow
 min bound $(\Phi^n)^{\otimes k}$

$\Rightarrow nR$ is achievable for $C(\Phi^{\otimes n})$

(ii) $C(\Phi^n) \leq n C(\Phi)$, if nR ----- for $\Phi^{\otimes n}$

$\forall \varepsilon \exists (f_\varepsilon, D_\varepsilon) \quad (2^{nR}, \varepsilon)$ code for $\Phi^{\otimes nk}$ - ~

$\Rightarrow R$ achievable for Φ .

$$①+② \Rightarrow \frac{\chi(\Phi^{\otimes n})}{n} \leq C(\Phi)$$

③ (Weak Converse): Suppose (f, D) is (M, ϵ) code for Φ

$$\begin{array}{ccccccc}
 \boxed{M=2^{nR}} & \xrightarrow{f} & \boxed{A} & \xrightarrow{\Phi} & \boxed{B} & \xrightarrow{D} & \boxed{\hat{M}} \\
 P_M^{(m)} & & \{P_M^{(m)}, P_m\} & & \{P_M^{(m)}, \Phi(P_m)\} & & P_{M\hat{M}}(m, \hat{m}) = \text{tr}(\Phi(P_m) E_{\hat{m}}^M)
 \end{array}$$

$$\begin{aligned}
 \chi(\Phi) &= \sup_{P_X} I(X=B) \geq I(M=B) \geq I(M:\hat{M}) \geq D(B_\epsilon \| B_{\frac{1}{M}}) \\
 &= \epsilon \log \frac{\epsilon}{\frac{M-1}{M}} + (1-\epsilon) \log \frac{(1-\epsilon)}{\frac{1}{M}} \\
 &= \log M - \epsilon \log(M-1) - h(\epsilon)
 \end{aligned}$$

$$\log M \leq \frac{\chi(\Phi) + h(\epsilon)}{1-\epsilon}$$

I.I.d Suppose $R > \lim_{n \rightarrow \infty} \frac{\chi(\Phi^{\otimes n})}{n}$. $\exists \epsilon > 0$ s.t. $(1-\epsilon)R > \lim_{n \rightarrow \infty} \frac{\chi(\Phi^{\otimes n})}{n}$
 For any $(2^{nR}, \epsilon)$ code for $\Phi^{\otimes n}$

$$nR = \log 2^{nR} \leq \frac{\chi(\Phi^{\otimes n}) + h(\epsilon)}{1-\epsilon}$$

$$(1-\epsilon)R \leq \frac{\chi(\Phi^{\otimes n})}{n} + \frac{h(\epsilon)}{n} \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\chi(\Phi^{\otimes n})}{n} \quad \text{contradiction}$$

In terms of optimal error ϵ^*

$$\text{Theorem: } \lim_{n \rightarrow \infty} \mathcal{E}^*(2^{nR}, \Phi^n) = \begin{cases} 0 & \text{if } R < C(\Phi) \\ 1 & \text{if } R > C(\Phi) \end{cases} \quad C(\Phi) = \lim_{n \rightarrow \infty} \frac{\chi(\Phi^{\otimes n})}{n}$$

$\lim_{n \rightarrow \infty} \frac{\chi(\Phi^{\otimes n})}{n}$ called regularization of $\chi(\Phi)$.

In general, $\chi(\Phi^n) \geq n \chi(\Phi)$

(Hassings '07) $\exists \Phi$ s.t. $\chi(\Phi \otimes \Phi) > 2\chi(\Phi)$

probabilistically proof. No explicit construction.

Why does this result mean?

$$\begin{array}{ccccc} M & \longrightarrow & A & \xrightarrow{\Phi} & B \longrightarrow \hat{M} \\ m & & p_m \in D(A) & & \end{array}$$

$$\begin{array}{ccccc} M & \longrightarrow & A^n & \xrightarrow{\Phi^{\otimes n}} & B^n \longrightarrow \hat{M} \\ & & p_m \in B(A^{\otimes n}) \cong B(A) \otimes B(A) \dots B(A) & & \end{array}$$

$$M^*(\Phi^n, \varepsilon) = \max \{M \mid \exists (M, \varepsilon) \text{ code for } \Phi^n\}$$

$$M_{pr}^*(\Phi^n, \varepsilon) = \max \{M \mid \exists (M, \varepsilon) \text{ code for } \Phi^n \text{ \& } f(m) = p_{1,m} \otimes p_{2,m} \otimes \dots p_{n,m}\}$$

product state

(or $f(m) = \sum_i p_{i,1} \otimes p_{i,2} \otimes \dots p_{i,n}$)

separate state

$$\chi(\Phi) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\log M_{pr}^*(\Phi^n, \varepsilon)}{n} \quad \text{capacity if we only use product code state}$$

$C(\Phi)$ capacity use all possible code state (including entangled state)

Will entanglement help? Yes, by (Hastings '05)

$$C(\Phi) \geq \frac{\chi(\Phi \otimes \Phi)}{2} >$$

Capacity using
all possible
code words.

capacity of allowing using entanglement

$$\underbrace{P_1}_{B(\mathcal{H}_A \otimes \mathcal{H}_A)} \otimes \underbrace{P_2}_{B(\mathcal{H}_A \otimes \mathcal{H}_A)} \otimes \dots \otimes P_{\frac{n}{2}}$$

between two
input system

$$\chi(\Phi)$$

Capacity using
only product
code word