COSC6464
Lecture 4 Projection

NLA Chapter 6

Orthogonality

- In this lecture and the next few, we are going to study the QR factorization and related orthogonal transformation and its use in least square problem.
- This is kind of unusual approach (usually LU factorization comes first) but I’m convinced that QR factorization (and orthogonality) is the more important and should be taught first.
- In this lecture we are focusing on a specific class of linear transformations (represented by a matrix) called <u>projectors</u>
- <u>Projector</u> can be either *orthogonal* or *oblique (non-orthogonal)*; for now orthogonal projection is more important.

Projector (projection matrix)

Definition: a projector is square matrix that satisfies:

$$P^2 = P$$

Geometrically, a projector P maps a vector v to Pv in range(P).

What if v is already in range(P)?

Then P maps v to v. Why?
Since $v \epsilon$range(P), there exists x such that $v = Px$. So the map of v is

$$Pv = P^2x = Px = v$$

Projector Geometry

- If you were to shine a light at the "right" direction (which is Pv-v) then Pv is the shadow of v.

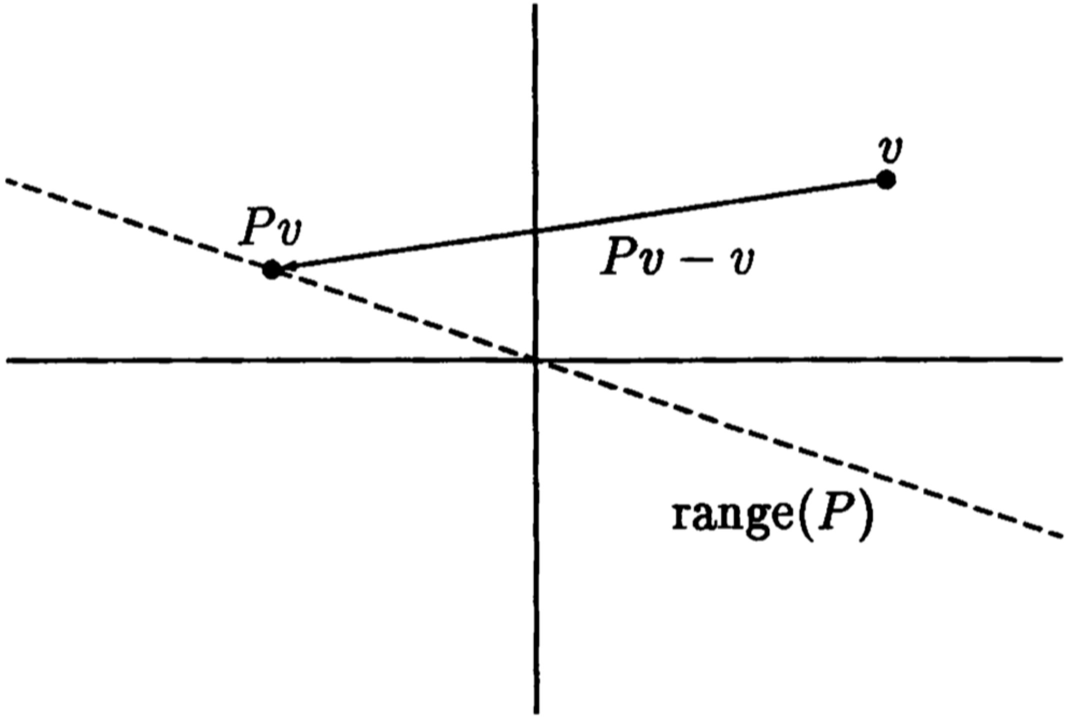

Figure 6.1. *An oblique projection.*

- The "right" direction **depends** on v.
- The "right" direction is a vector
$$Pv - v = (P - I)v$$
- What is the image of this direction under P?
$$P(Pv - v) = P^2 v - Pv = 0$$
- The direction is within the null space of P!
$$Pv - v \in \mathrm{null}(P)$$
- Take-way: the direction of light is different for different v, but it's always described by a vector in null(P).

Complementary Projector

- If P is a projector, (I-P) is also a projector. Why?
  $(I-P)^2 = I - 2P + P^2 = I - P$
- $I - P$ is called the <u>complementary projector</u>
- Now, what does this complementary projector project to (range(I-P))?
- It's exactly the null(P)!!
  $$\mathrm{range}(I-P) = \mathrm{null}(P)$$

We have seen in previous slide that

$$(I-P)v = -(Pv - v) \in \ \mathrm{null}(P)$$

So $\mathrm{range}(I-P) \in \mathrm{null}(P)$

The reverse is also true:

$$\mathrm{null}(P) \in \ \mathrm{range}(I-P)$$

Why?

If $v \in \mathrm{null}(P)$, then $Pv = 0$, so

$$(I-P)v = v - Pv = v$$

Thus $v \in \mathrm{range}(I-P)$.

Complementary Projector (cont'd)

Following last slide,

$$\mathrm{range}(I - P) = \mathrm{null}(P)$$

By some magic (how?) we claim

$$\mathrm{range}(P) = \mathrm{null}(I - P)$$

Now, what's the relationship between range(I-P) and range(P)

Or, the relationship between range of projector and its complementary?

Let's denote:

$$S_1 = \mathrm{range}(P), S_2 = \mathrm{range}(I - P)$$

Then

$$S_1 + S_2 = \mathbb{R}^{\mathrm{m}}$$
$$S_1 \cap S_2 = \{0\}$$

Basically subspace S_1 and S_2 are complementary subspaces. I.e., for any vector v we can uniquely write it as a sum of two vectors from these two <u>complementary subspaces</u>:

$$v = \underbrace{Pv}_{\in S_1} + \underbrace{(I - P)v}_{\in S_2}$$

Complementary Projector (cont’d)

We say that

P is the projector *onto* S_1 *along* S_2.

Quiz:

1) a, b is a vector. When is the matrix $P = ab^T$ a projector?

2) If $a = [1; 2], b = [-3; 2]$, what is $\mathrm{range}(P)$ and $\mathrm{range}(I - P)$?

3) P is an <u>orthogonal projection</u> iff $S_1 \perp S_2$. Is the above projector orthogonal?

Orthogonal projector

As we said earlier, a projector P is orthogonal iff $S_1 \perp S_2$; i.e.

$$\mathrm{range}(P) \perp \mathrm{range}(I - P)$$

Or, i.e. “the light direction” is always perpendicular to the range of P.

Algebraically, orthogonal projector is defined as a project that

$$P^T = P$$

The two definitions are equivalent. Why? (hint: SVD for the rescue)

Warning: orthogonal projector is NOT orthogonal matrix!

Orthogonal Projector, $P^T = P$

Prove: If a projector $P \in \mathbb{R}^{m \times m}$, is orthogonal ($S_1 \perp S_2$), then it's symmetric $P^T = P$.

Say S_1 has dimension n ($< m$), and $\{q_1, q_2, \dots, q_n\}$ is an orthogonal basis for S_1, and $\{q_{n+1}, \dots, q_m\}$ are the orthogonal basis for S_2. (Why can we do this?)

Denote $Q = [q_1, \dots, q_n, \dots, q_m]$. Then $PQ = [Pq_1, \dots, Pq_n, \dots, Pq_m] = [q_1, \dots, q_n, 0, \dots, 0]$. Therefore, $Q^T PQ = \mathrm{diag}\{1, \dots, 1, 0, \dots, 0]$. We get a SVD (also a EVD) of P:

$$P = \mathrm{Q\Sigma Q^T}$$

It follows that $P^T = P$

Projection with Orthogonal Basis

From last slide, we see that

$$P = Q\,\mathrm{diag}\{\underbrace{1, \dots, 1}_{n}, \underbrace{0, \dots, 0}_{m-n}\}Q^T$$

We can drop the "silent" $m - n$ columns of Q and get the skinny hat Q: $\hat{Q} = [q_1, \dots, q_n]$.

$$P = \hat{Q}\hat{Q}^T$$

Where columns of $\hat{Q}$ are orthonormal. This formula is very important! In fact the $\hat{Q}$ does not have to come from SVD.

For **any** orthonormal vectors $Q \in \mathbb{R}^{m \times n}$, $P = QQ^T$ is an orthogonal projector onto range(Q). (Why?)

The complement, $I - QQ^T$ is also an orthogonal projector (onto $\mathrm{range}(Q)^{\perp}$).

Example: rank-1 orthogonal projector

What does this matrix

$$P_v = vv^T$$

Do? Is it a projector? If so, what's $\mathrm{range}(P_v)$?

$$P_v^2 = v(v^T v)v^T = (v^T v)vv^T$$

So $P_v^2 = P_v$ if and only if $v^T v = 1$ (v is unit vector). And it maps any vector x to $P_v x = vv^T x = (v^T x)v \parallel v$. Thus,

$$\mathrm{range}(P_v) = < v >$$

If a vector a is not a unit vector, normalizing it gives us orthogonal matrix

$$P_a = \frac{aa^T}{a^T a}$$

Quiz: what does the complement project $P_{a\perp} = I - \frac{aa^T}{a^T a}$ do?

Orthogonal Projection From Arbitrary Basis

If we want the orthogonal projector onto a subspace spanned by an orthogonal basis $Q = [q_1, \ldots, q_n]$ it's easy; just construct:

$$P = QQ^T$$

What if we have a subspace spanned by (non-orthonormal) arbitrary basis $A = [a_1, \ldots, a_n] \in \mathbb{R}^{m \times n}$? What's the orthogonal projector P onto $\mathrm{range}(A)$?

For a given vector v, denote the image of v as $y = Pv \in \mathrm{range}(A)$. So there exist x s.t. $y = Pv = Ax$.

As the projector P is orthogonal, we must have $y - v \perp [a_1, \ldots, a_n]$, or i.e. $A^T(y - v) = 0$

Therefore $A^T(Ax - v) = 0$, $x = (A^T A)^{-1} A^T v$, thus

$$y = Ax = A(A^T A)^{-1} A^T v = Pv$$

So the orthogonal projector we seek is $P = A(A^T A)^{-1} A^T$ (does it look familiar?)

Questions

In 2d space, can we have an example of oblique (non-orthogonal projector)?

An example of orthogonal projector?

What are the eigenvalues/eigenvectors of a projector?

How about singular values/vectors of a projector?