COSC 6364 Adv. Numerical Analysis

Lecture 1: Matrix-Vector Multiplication; Orthogonal Vectors and Matrices

P. Wu[1]

[1]Computer Science
University of Houston

Spring 2020

To accompany the Numerical Linear Algebra book, Lectures 1 and 2.

Table of Contents

Matrix Vector Multiplication

Matrix Vector Multiplication

- New interpretation that is essential for numerical linear algebra: if $\boldsymbol{b} = A\boldsymbol{x}$, then $\boldsymbol{b}$ is the linear combination of the columns of A.

Matrix Vector Multiplication

- New interpretation that is essential for numerical linear algebra: if $\boldsymbol{b} = A\boldsymbol{x}$, then $\boldsymbol{b}$ is the linear combination of the columns of A.
  - Let $\boldsymbol{a}_i$ denote the i^{th} column of A, $m-$vector. Then we can write

Matrix Vector Multiplication

- New interpretation that is essential for numerical linear algebra: if $\boldsymbol{b} = A\boldsymbol{x}$, then $\boldsymbol{b}$ is the linear combination of the columns of A.
  - Let $\boldsymbol{a}_i$ denote the i^{th} column of A, $m-$vector. Then we can write

$$\boldsymbol{b} = X = \sum_{i=1}^{n} x_i \boldsymbol{a}_i$$

Matrix Vector Multiplication

- New interpretation that is essential for numerical linear algebra: if $\boldsymbol{b} = A\boldsymbol{x}$, then $\boldsymbol{b}$ is the linear combination of the columns of A.
  - Let $\boldsymbol{a}_i$ denote the i^{th} column of A, $m-$vector. Then we can write

$$\boldsymbol{b} = X = \sum_{i=1}^{n} x_i \boldsymbol{a}_i$$

The equation can be displayed schematically as follows:

Matrix Vector Multiplication

- New interpretation that is essential for numerical linear algebra: if $\boldsymbol{b} = A\boldsymbol{x}$, then $\boldsymbol{b}$ is the linear combination of the columns of A.
  - Let $\boldsymbol{a}_i$ denote the i^{th} column of A, $m-$vector. Then we can write

$$\boldsymbol{b} = X = \sum_{i=1}^{n} x_i \boldsymbol{a}_i$$

The equation can be displayed schematically as follows:

$$\begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Vector Multiplication

- New interpretation that is essential for numerical linear algebra: if $\boldsymbol{b} = A\boldsymbol{x}$, then $\boldsymbol{b}$ is the linear combination of the columns of A.
    - Let $\boldsymbol{a}_i$ denote the i^{th} column of A, $m-$vector. Then we can write

$$\boldsymbol{b} = X = \sum_{i=1}^{n} x_i \boldsymbol{a}_i$$

The equation can be displayed schematically as follows:

$$\begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{a}_n$$

Matrix Vector Multiplication

Matrix Vector Multiplication

- Mathematician: $A\boldsymbol{x} = \boldsymbol{b}$ means matrix A (representing linear transformation) acts on vector $\boldsymbol{x}$ to produce $\boldsymbol{b}$.

Matrix Vector Multiplication

- Mathematician: $A\boldsymbol{x} = \boldsymbol{b}$ means matrix A (representing linear transformation) acts on vector $\boldsymbol{x}$ to produce $\boldsymbol{b}$.
- Numerical analyst: $A\boldsymbol{x} = \boldsymbol{b}$ means vector $\boldsymbol{x}$ acts on A to produce $\boldsymbol{b}$ (as linear combination of columns of A).

Matrix Vector Multiplication

- Mathematician: $A\boldsymbol{x} = \boldsymbol{b}$ means matrix A (representing linear transformation) acts on vector $\boldsymbol{x}$ to produce $\boldsymbol{b}$.
- Numerical analyst: $A\boldsymbol{x} = \boldsymbol{b}$ means vector $\boldsymbol{x}$ acts on A to produce $\boldsymbol{b}$ (as linear combination of columns of A).
- Function can be seen as an infinite dimensional vector. Matrix A's columns can be function

Matrix Times a Matrix

Matrix Times a Matrix

- For a matrix product $B = AC$, each column of B is a linear combination of the columns of A.

Matrix Times a Matrix

- For a matrix product $B = AC$, each column of B is a linear combination of the columns of A.
- To see this, write:

Matrix Times a Matrix

- For a matrix product $B = AC$, each column of B is a linear combination of the columns of A.
- To see this, write:

$$AC = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nk} \end{bmatrix}}_{n \times k}$$

Matrix Times a Matrix

- For a matrix product $B = AC$, each column of B is a linear combination of the columns of A.
- To see this, write:

$$AC = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nk} \end{bmatrix}}_{n \times k}$$

$$B = \underbrace{\left[\mathbf{b_1} \mid \mathbf{b_2} \mid \dots \mid \mathbf{b_k}\right]}_{m \times k}$$

Matrix Times a Matrix

Then column $\mathbf{b_i}$ is just

Matrix Times a Matrix

Then column $\mathbf{b_i}$ is just

$$\mathbf{b_i} = \begin{bmatrix} a_{11}c_{1i} + a_{12}c_{2i} + \ldots + a_{1n}c_{ni} \\ a_{21}c_{1i} + a_{22}c_{2i} + \ldots + a_{2n}c_{ni} \\ \vdots \\ a_{m1}c_{1i} + a_{m2}c_{2i} + \ldots + a_{mn}c_{ni} \end{bmatrix}$$

Matrix Times a Matrix

Then column $\mathbf{b_i}$ is just

$$\mathbf{b_i} = \begin{bmatrix} a_{11}c_{1i} + a_{12}c_{2i} + \ldots + a_{1n}c_{ni} \\ a_{21}c_{1i} + a_{22}c_{2i} + \ldots + a_{2n}c_{ni} \\ \vdots \\ a_{m1}c_{1i} + a_{m2}c_{2i} + \ldots + a_{mn}c_{ni} \end{bmatrix}$$

$$= c_{1i} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_{2i} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_{ni} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix Times a Matrix

Then column $\mathbf{b_i}$ is just

$$\mathbf{b_i} = \begin{bmatrix} a_{11}c_{1i} + a_{12}c_{2i} + \ldots + a_{1n}c_{ni} \\ a_{21}c_{1i} + a_{22}c_{2i} + \ldots + a_{2n}c_{ni} \\ \vdots \\ a_{m1}c_{1i} + a_{m2}c_{2i} + \ldots + a_{mn}c_{ni} \end{bmatrix}$$

$$= c_{1i}\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_{2i}\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_{ni}\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= c_{1i}\mathbf{a_1} + c_{2i}\mathbf{a_2} + \ldots + c_{ni}\mathbf{a_n}$$

Outer Product

Outer Product

$$\boldsymbol{u}\boldsymbol{v}^* = \boldsymbol{u}\begin{bmatrix} \overline{v_1} & \overline{v_2} & \dots & \overline{v_n} \end{bmatrix}$$

Outer Product

$$
\begin{aligned}
\boldsymbol{u}\boldsymbol{v}^* &= \boldsymbol{u}\begin{bmatrix} \overline{v_1} & \overline{v_2} & \dots & \overline{v_n} \end{bmatrix} \\
&= \begin{bmatrix} \overline{v_1}\boldsymbol{u} & \overline{v_2}\boldsymbol{u} & \dots & \overline{v_n}\boldsymbol{u} \end{bmatrix}
\end{aligned}
$$

Outer Product

$$
\begin{aligned}
\boldsymbol{u}\boldsymbol{v}^* &= \boldsymbol{u}\begin{bmatrix}\overline{v_1} & \overline{v_2} & \dots & \overline{v_n}\end{bmatrix} \\
&= \begin{bmatrix}\overline{v_1}\boldsymbol{u} & \overline{v_2}\boldsymbol{u} & \dots & \overline{v_n}\boldsymbol{u}\end{bmatrix} \\
&= \begin{bmatrix}
u_1\overline{v_1} & u_1\overline{v_2} & \dots & u_1\overline{v_n} \\
u_2\overline{v_1} & u_2\overline{v_2} & \dots & u_2\overline{v_n} \\
\vdots & \vdots & \ddots & \vdots \\
u_n\overline{v_1} & u_n\overline{v_2} & \dots & u_n\overline{v_n}
\end{bmatrix}
\end{aligned}
$$