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Notes

COSC 6364
Advanced Numerical Analysis

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Contents

Chapter 1

Matrix-Vector Multiplication

We interpret a matrix-vector multiplication $\mathbf{b} = A\mathbf{x}$ as follows: if $\mathbf{b} = A\mathbf{x}$, then $\mathbf{b}$ is a linear combination of columns of A. In particular, letting $\mathbf{a_i}$ denote the i^{th} column of A and x_i the i^{th} element of $\mathbf{x}$, we can write this equation as

$$\begin{aligned} b &= \begin{bmatrix} \mathbf{a_1} \mid \mathbf{a_2} \mid \ldots \mid \mathbf{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n} \end{aligned}$$

A matrix product $B = AC$ can be interpreted as: each column of B is a linear combination of the columns of A. Write:

$$\begin{aligned} AC &= \underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1k} \\ c_{21} & c_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nk} \end{bmatrix}}_{n\times k} \\ B &= \underbrace{\begin{bmatrix} \mathbf{b_1} \mid \mathbf{b_2} \mid \ldots \mid \mathbf{b_k} \end{bmatrix}}_{m\times k} \end{aligned}$$

Then column $\mathbf{b_i}$ is just

$$\begin{aligned} \mathbf{b_i} &= \begin{bmatrix} a_{11}c_{1i} + a_{12}c_{2i} + \ldots + a_{1n}c_{ni} \\ a_{21}c_{1i} + a_{22}c_{2i} + \ldots + a_{2n}c_{ni} \\ \vdots \\ a_{m1}c_{1i} + a_{m2}c_{2i} + \ldots + a_{mn}c_{ni} \end{bmatrix} \\ &= c_{1i} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_{2i} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_{ni} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\ &= c_{1i}\mathbf{a_1} + c_{2i}\mathbf{a_2} + \ldots + c_{ni}\mathbf{a_n} \end{aligned}$$

Chapter 2

Practice Quizzes

Quiz 1

1. Given a matrix that is both triangular and unitary, is it non-diagonal?

   Solution. A triangular, unitary matrix must be diagonal. To see this, consider an upper-triangular, normal matrix[1], A. Write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

$$A^* = \begin{bmatrix} \overline{a_{11}} & 0 & 0 & \dots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & \dots & 0 \\ \overline{a_{13}} & \overline{a_{23}} & \overline{a_{33}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \overline{a_{3n}} & \dots & \overline{a_{nn}} \end{bmatrix}$$

   Consider the 1st diagonal element of AA^*, writen explicitly:

$$(AA^*)_{11} = a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} + \ldots + a_{1n}\overline{a_{1n}}$$

   And similarly for A^*A:

$$(A^*A)_{11} = \overline{a_{11}}a_{11}$$

   These two values must be equal, forcing

$$a_{12}\overline{a_{12}} + \ldots + a_{1n}\overline{a_{1n}} = 0$$

   However, $z\overline{z} = |z|$ is strictly non-negative, hence these values must be identically 0. In particular, this means the first row of A is

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \end{bmatrix}$$

   The same argument applies for each row of the matrix A.

   If A is lower-triangular, then $B = A^*$ is upper-triangular, and B is diagonal, by the above argument, hence A is diagonal. □

2. Can the absolute value of an eigenvalue of a unitary matrix be 1?

[1] A normal matrix is one which satisfies $AA^* = A^*A$. Clearly, every unitary matrix is normal.

Solution. Clearly, the answer is yes. Take $A = I_n$ and note that it has characteristic equation $(1-\lambda)^n = 0$, which has eigenvalues of 1. However, the stronger result is that *all* eigenvalues of a unitary matrix have modulus 1:

Consider some unitary matrix A, i.e., A satisfies $AA^* = A^*A = I$, and any eigenvalue, λ. We have

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some vector $\mathbf{x}$. Taking the conjugate transpose of both sides gives

$$\mathbf{x}^*A^* = \overline{\lambda}\mathbf{x}^*$$

Multiplying these equations yields

$$\begin{aligned}(\mathbf{x}^*A^*)(A\mathbf{x}) &= \left(\overline{\lambda}\mathbf{x}^*\right)(\lambda\mathbf{x}) \\ \mathbf{x}^*(A^*A)\mathbf{x} &= \lambda\overline{\lambda}\mathbf{x}^*\mathbf{x} \\ \mathbf{x}^*\mathbf{x} &= \lambda\overline{\lambda}\mathbf{x}^*\mathbf{x}\end{aligned}$$

This forces $\lambda\overline{\lambda} = |\lambda|^2 = 1$. □

3. If W is an arbitrary nonsingular matrix, then is the function $\|.\|_W$ defined by $\|\mathbf{x}\|_W = \|W\mathbf{x}\|$ (weighted norm) a vector norm?

Solution. In order for $\|.\|_W$ to be a vector norm, it must satisfy:

1. $\|\mathbf{u}+\mathbf{v}\|_W \leq \|\mathbf{u}\|_W + \|\mathbf{v}\|_W$ (triangle inequality)
2. $\|c\mathbf{u}\|_W = |c|\,\|\mathbf{u}\|_W$ (scalable/homogenous)
3. if $\|\mathbf{u}\|_W = 0$ then $\mathbf{u} = \mathbf{0}$ (positivity)

2 and 3 are obvious. To see 1, note that

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_W &= \|W(\mathbf{u}+\mathbf{v})\| \\ &= \|W\mathbf{u}+W\mathbf{v}\| \\ &\leq \|W\mathbf{u}\| + \|W\mathbf{v}\| \text{ by the triangle inequality} \\ &= \|\mathbf{u}\|_W + \|\mathbf{v}\|_W\end{aligned}$$

Thus, $\|.\|_W$ is a vector norm. □

4. If E is an outer product $E = \mathbf{u}\mathbf{v}^*$, then $\|E\|_2 = \|\mathbf{u}\|_2\,\|\mathbf{v}\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|\mathbf{u}\|_F\,\|\mathbf{v}\|_F$?

Solution. Write $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then

$$\begin{aligned}E &= \mathbf{u}\mathbf{v}^* \\ &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\begin{bmatrix} \overline{v_1} & \overline{v_2} & \dots & \overline{v_n} \end{bmatrix} \\ &= \begin{bmatrix} u_1\overline{v_1} & u_1\overline{v_2} & \dots & u_1\overline{v_n} \\ u_2\overline{v_1} & u_2\overline{v_2} & \dots & u_2\overline{v_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_n\overline{v_1} & u_n\overline{v_2} & \dots & u_n\overline{v_n} \end{bmatrix}\end{aligned}$$

We therefore have

$$\begin{aligned}
\|E\|_F &= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |u_i\overline{v_j}|^2} \\
&= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |u_i|^2|\overline{v_j}|^2} \\
&= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |u_i|^2|v_j|^2}
\end{aligned}$$

And further

$$\begin{aligned}
\|\mathbf{u}\|_F \|\mathbf{v}\|_F &= \left(\sqrt{\sum_{i=1}^{n} |u_i|^2}\right)\left(\sqrt{\sum_{i=1}^{n} |v_i|^2}\right) \\
&= \sqrt{\left(\sum_{i=1}^{n} |u_i|^2\right)\left(\sum_{i=1}^{n} |v_i|^2\right)} \\
&= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |u_i|^2|v_j|^2} \\
&= \|E\|_F
\end{aligned}$$

□

Quiz 2

1. What is the SVD of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$?
2. Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not exactly a standard mathematical transformation). Do A and B have the same singular values?
3. Two matrices, $A, B \in \mathbb{R}^{m\times m}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{R}^{m\times m}$. Is it true that A and B are unitarily equivalent if and only if they have the same singular values?
4. Given $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and A has full rank, is A^*A nonsingular?

Glossary

basis

A **basis** of a vector space is a set of linearly independent vectors that spans the entire space.

column rank

The dimension of the vector space spanned by the columns of a matrix.

dimension

The size of a basis of a vector space. Equivalently, the greatest number of linearly independent vectors, or the least number of vectors which spans the vector space (see also: rank).

eigenvalue

A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for non-zero $\mathbf{x}$.

eigenvector

A non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

Frobenius norm

$$\|\mathbf{x}\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$$

full rank

An $m \times n$ matrix has **full rank** when its rank is equal to $\min(m, n)$.

induced norm

A set of vectors $\mathbf{x_1}$, $\mathbf{x_2}$, …, $\mathbf{x_n}$ is linearly independent of the only solution to the equation

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + \ldots + c_n$$

description

∞-norm

$$\|\mathbf{x}\|_\infty = \max_{1\leq i\leq m}|x_i|$$

inner product

The **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}^{\mathrm{T}}\mathbf{v}$. Also called the dot product.

kernel

The set of all vectors that satisfy $A\mathbf{x} = \mathbf{0}$ (see also: nullspace).

linearly independent

A set of vectors $\mathbf{x_1}$, $\mathbf{x_2}$, …, $\mathbf{x_n}$ is linearly independent of the only solution to the equation

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + \ldots + c_n\mathbf{x_n} = \mathbf{0}$$

is

$$c_1 = c_2 = \ldots = c_n = 0$$

lower triangular matrix

A matrix where all entries above the diagonal are zero, i.e.,

$$L = \begin{bmatrix} \ell_{1,1} & 0 & 0 & 0 & 0 \\ \ell_{2,1} & \ell_{2,2} & 0 & 0 & 0 \\ \ell_{3,1} & \ell_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \ell_{n,1} & \ell_{n,2} & \dots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

non-singular matrix

A matrix that has an inverse. Equivalently, a matrix with non-zero determinant.

norm

A norm is a real-valued function $\|.\|$ such that

1. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)
2. $\|c\mathbf{x}\| = |c| \, \|\mathbf{x}\|$
3. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$

nullspace

The set of all vectors that satisfy $A\mathbf{x} = \mathbf{0}$ (see also: kernel).

orthogonal matrix

A real square matrix A is orthogonal if $AA^{\mathrm{T}} = I$, i.e., $A^{-1} = A^{\mathrm{T}}$.

orthogonal vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if their inner product is 0, i.e., if $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$.

outer product

The matrix produced by the product of a column vector and its transpose, i.e., $\mathbf{u}\mathbf{v}^*$

***p*-norm**

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

range

The set of all vectors that can be expressed as $A\mathbf{x}$ for some $\mathbf{x}$.

rank

The dimension of the vector space spanned by a matrix.

row rank

The dimension of the vector space spanned by the rows of a matrix.

singular matrix

A matrix that has no inverse. Equivalently, a matrix with 0 determinant.

singular value

The diagonal entries of the matrix Σ in the Singular Value Decomposition. Equivalently, a non-negative real number σ is a **singular value** for M if and only if there exist unit-length vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$M\mathbf{u} = \sigma\mathbf{v}$$
$$M^*\mathbf{v} = \sigma\mathbf{u}$$

Singular Value Decomposition

For any $m \times n$ matrix A, a **Singular Value Decomposition** (SVD) of A is a factorization

$$A = U \Sigma V^*$$

where

- U is an $m \times m$ unitary matrix
- V is an $n \times n$ unitary matrix
- Σ is an $m \times n$ diagonal matrix with non-negative real numbers on the digonal

The columns of U and V are called the left-singular and right-singular vectors, respectively, and the diagonal entries σ_{ii} of Σ are known as the **singular values**.

span

The set of all linear combinations of the columns of a matrix.

symmetric

A matrix A such that $A = A^{\mathrm{T}}$.

unitary matrix

A square matrix Q is unitary if $QQ^* = I$, i.e., $Q^{-1} = Q^*$.

upper triangular matrix

A matrix where all entries below the diagonal are zero, i.e.,

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix}$$

Acronyms

SVD

Singular Value Decomposition