UNIVERSITY of HOUSTON
DEPARTMENT OF COMPUTER SCIENCE

COSC6364 Adv. Numerical Analysis

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COSC 6364 Adv. Numerical Analysis

Quiz 1 Prep

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Table of Contents

Orthogonal Projector

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Householder Reflector and QR factorization

- How does Householder reflector accomplish QR factorization?

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- Given a vector $\boldsymbol{x}$, what is the Householder reflector that maps $\boldsymbol{x}$ to the first axis?
  - Write $\boldsymbol{v} = \boldsymbol{x} + \|\boldsymbol{x}\|\boldsymbol{e}_1$. Then the Householder reflector is

$$I - 2\frac{\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}}{\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}}$$

Equivalently, write $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ and our reflector is

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- What's the FLOP count of Householder QR factorization?

Conditioning and Backward Stability

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  - Given a problem f with input $\boldsymbol{x}$, write $\delta\boldsymbol{x} = (1+\delta)\boldsymbol{x}$ and $\delta f = f(\delta\boldsymbol{x}) - f(\boldsymbol{x})$. The absolute condition number of f is given by

$$\lim_{\delta\to 0}\sup_{\delta}\frac{\|\delta f\|}{\|\delta\boldsymbol{x}\|}$$

    The relative condition number is given by

$$\lim_{\delta\to 0}\sup_{\delta}\frac{\|\delta f\|/\|f\|}{\|\delta\boldsymbol{x}\|/\|\boldsymbol{x}\|}$$

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  - An algorithm is backward stable if, for every $\boldsymbol{x}$, there exists an $\tilde{\boldsymbol{x}}$ with

    $$\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} = \mathcal{O}(\epsilon_{\text{machine}})$$

    such that

    $$\tilde{f}(\boldsymbol{x}) = f(\tilde{\boldsymbol{x}})$$

Conditioning and Backward Stability

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  - $$\frac{\left\|\tilde{f}(\boldsymbol{x})-f(\tilde{\boldsymbol{x}})\right\|}{\left\|f(\tilde{\boldsymbol{x}})\right\|}=\frac{\left\|\tilde{f}(\boldsymbol{x})-f(\tilde{\boldsymbol{x}})\right\|/\|f(\tilde{\boldsymbol{x}})\|}{\|\tilde{\boldsymbol{x}}-\boldsymbol{x}\|/\|\boldsymbol{x}\|}\cdot\frac{\|\tilde{\boldsymbol{x}}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$
    $$\leq\kappa\frac{\|\tilde{\boldsymbol{x}}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

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    $$\leq \kappa \frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$
- Is the simple running sum algorithm for computing inner product of two vectors $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$ backward stable?

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- Is the simple running sum algorithm for computing inner product of two vectors $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$ backward stable?
  - Yes. Let f denote the inner product and $\tilde{f}$ its computation by running sum. Then

$$\begin{aligned}\tilde{f}(\boldsymbol{x}, \boldsymbol{y}) &= x_1 \otimes y_1 \oplus x_2 \otimes y_2 \oplus \ldots \oplus x_n \otimes y_n \\ &= (x_1 y_1)(1+\epsilon_1) \oplus (x_2 y_2)(1+\epsilon_2) \oplus \ldots \oplus (x_n y_n)(1+\epsilon_n)\end{aligned}$$

LU factorization

- How to use LU factorization to solve a linear system?

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- How to use LU factorization to invert a square non-singular matrix A?

LU factorization

- How to use LU factorization to solve a linear system?
- How to use LU factorization to invert a square non-singular matrix A?
- What's the cost of LU factorization? How does it compare to Householder QR factorization on the same matrix?

QR Algorithm for Eigenvalue Decomposition

- How does QR algorithm work?

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- What's the cost of one iteration in QR algorithm?

QR Algorithm for Eigenvalue Decomposition

- How does QR algorithm work?
- What's the cost of one iteration in QR algorithm?
- What's the convergence rate of QR algorithm?