COSC 6364
Advanced Numerical Analysis

Panruo Wu

Khalid Hourani

Contents

Chapter 1

Matrix-Vector Multiplication

We interpret a matrix-vector multiplication $\mathbf{b} = A\mathbf{x}$ as follows: if $\mathbf{b} = A\mathbf{x}$, then $\mathbf{b}$ is a linear combination of columns of A. In particular, letting $\mathbf{a_i}$ denote the ith column of A and x_i the ith element of $\mathbf{x}$, we can write this equation as

$$b = \begin{bmatrix} \mathbf{a_1} \mid \mathbf{a_2} \mid \ldots \mid \mathbf{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n}$$

A matrix product $B = AC$ can be interpreted as: each column of B is a linear combination of the columns of A. Write:

$$AC = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1k} \\ c_{21} & c_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nk} \end{bmatrix}}_{n \times k}$$

$$B = \underbrace{\begin{bmatrix} \mathbf{b_1} \mid \mathbf{b_2} \mid \ldots \mid \mathbf{b_k} \end{bmatrix}}_{m \times k}$$

Then column $\mathbf{b_i}$ is just

$$\mathbf{b_i} = \begin{bmatrix} a_{11}c_{1i} + a_{12}c_{2i} + \ldots + a_{1n}c_{ni} \\ a_{21}c_{1i} + a_{22}c_{2i} + \ldots + a_{2n}c_{ni} \\ \vdots \\ a_{m1}c_{1i} + a_{m2}c_{2i} + \ldots + a_{mn}c_{ni} \end{bmatrix}$$

$$= c_{1i}\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_{2i}\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_{ni}\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= c_{1i}\mathbf{a_1} + c_{2i}\mathbf{a_2} + \ldots + c_{ni}\mathbf{a_n}$$

Chapter 2

Practice Quizzes

Quiz 1

1. Given a matrix that is both triangular and unitary, is it non-diagonal?

   Solution. A triangular, unitary matrix must be diagonal. To see this, consider an upper-triangular, normal matrix[1], A. Write

   $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

   $$A^* = \begin{bmatrix} \overline{a_{11}} & 0 & 0 & \dots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & \dots & 0 \\ \overline{a_{13}} & \overline{a_{23}} & \overline{a_{33}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \overline{a_{3n}} & \dots & \overline{a_{nn}} \end{bmatrix}$$

   Consider the 1st diagonal element of AA^*, writen explicitly:

   $$(AA^*)_{11} = a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} + \dots + a_{1n}\overline{a_{1n}}$$

   And similarly for A^*A:

   $$(A^*A)_{11} = \overline{a_{11}}a_{11}$$

   These two values must be equal, forcing

   $$a_{12}\overline{a_{12}} + \dots + a_{1n}\overline{a_{1n}} = 0$$

   However, $z\overline{z} = |z|$ is strictly non-negative, hence these values must be identically 0. In particular, this means the first row of A is

   $$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \end{bmatrix}$$

   The same argument applies for each row of the matrix A.

   If A is lower-triangular, then $B = A^*$ is upper-triangular, and B is diagonal, by the above argument, hence A is diagonal. $\qquad \square$

2. Can the absolute value of an eigenvalue of a unitary matrix be 1?

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[1] A normal matrix is one which satisfies $AA^* = A^*A$. Clearly, every unitary matrix is normal.

Solution. Clearly, the answer is yes. Take $A = I_n$ and note that it has characteristic equation $(1 - \lambda)^n = 0$, which has eigenvalues of 1. However, the stronger result is that *all* eigenvalues of a unitary matrix have modulus 1:

Consider some unitary matrix A, i.e., A satisfies $AA^* = A^*A = I$, and any eigenvalue, λ. We have

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some vector $\mathbf{x}$. Taking the conjugate transpose of both sides gives

$$\mathbf{x}^*A^* = \bar{\lambda}\mathbf{x}^*$$

Multiplying these equations yields

$$(\mathbf{x}^*A^*)(A\mathbf{x}) = (\bar{\lambda}\mathbf{x}^*)(\lambda\mathbf{x})$$
$$\mathbf{x}^*(A^*A)\mathbf{x} = \lambda\bar{\lambda}\mathbf{x}^*\mathbf{x}$$
$$\mathbf{x}^*\mathbf{x} = \lambda\bar{\lambda}\mathbf{x}^*\mathbf{x}$$

This forces $\lambda\bar{\lambda} = |\lambda| = 1$.   □

3. If W is an arbitrary nonsingular matrix, then is the function $\|.\|_W$ defined by $\|\mathbf{x}\|_W = \|W\mathbf{x}\|$ (weighted norm) a vector norm?

   Solution. In order for $\|.\|_W$ to be a vector norm, it must satisfy:

   1. $\|\mathbf{u} + \mathbf{v}\|_W \leq \|\mathbf{u}\|_W + \|\mathbf{v}\|_W$ (triangle inequality)
   2. $\|c\mathbf{u}\|_W = |c|\,\|\mathbf{u}\|_W$ (scalable/homogenous)
   3. if $\|\mathbf{u}\|_W = 0$ then $\mathbf{u} = 0$ (positivity)

   2 and 3 are obvious. To see 1, note that

   $$\begin{aligned}
   \|\mathbf{u} + \mathbf{v}\|_W &= \|W(\mathbf{u} + \mathbf{v})\| \\
   &= \|W\mathbf{u} + W\mathbf{v}\| \\
   &\leq \|W\mathbf{u}\| + \|W\mathbf{v}\| \quad \text{by the triangle inequality} \\
   &= \|\mathbf{u}\|_W + \|\mathbf{v}\|_W
   \end{aligned}$$

   Thus, $\|.\|_W$ is a vector norm.   □

4. If E is an outer product $E = \mathbf{u}\mathbf{v}^*$, then $\|E\|_2 = \|\mathbf{u}\|_2\,\|\mathbf{v}\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|\mathbf{u}\|_F\,\|\mathbf{v}\|_F$?

   Solution. Write $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then

   $$\begin{aligned}
   E &= \mathbf{u}\mathbf{v}^* \\
   &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} \overline{v_1} & \overline{v_2} & \ldots & \overline{v_n} \end{bmatrix} \\
   &= \begin{bmatrix} u_1\overline{v_1} & u_1\overline{v_2} & \ldots & u_1\overline{v_n} \\ u_2\overline{v_1} & u_2\overline{v_2} & \ldots & u_2\overline{v_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_n\overline{v_1} & u_n\overline{v_2} & \ldots & u_n\overline{v_n} \end{bmatrix}
   \end{aligned}$$

We therefore have

$$\|E\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i \overline{v_j}|^2}$$

$$= \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i|^2 |\overline{v_j}|^2}$$

$$= \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i|^2 |v_j|^2}$$

And further

$$\|\mathbf{u}\|_F \|\mathbf{v}\|_F = \left(\sqrt{\sum_{i=1}^{n} |u_i|^2} \right) \left(\sqrt{\sum_{i=1}^{n} |v_i|^2} \right)$$

$$= \sqrt{\left(\sum_{i=1}^{n} |u_i|^2 \right) \left(\sum_{i=1}^{n} |v_i|^2 \right)}$$

$$= \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i|^2 |v_j|^2}$$

$$= \|E\|_F$$

$\square$

Glossary

column rank

    The dimension of the vector space spanned by the columns of a matrix.

full rank

    An $m \times n$ matrix has **full rank** when its rank is equal to $\min(m, n)$.

inner product

    The **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}^{\mathrm{T}}\mathbf{v}$. Also called the dot product.

lower triangular matrix

    A matrix where all entries above the diagonal are zero, i.e., $L = \begin{bmatrix} \ell_{1,1} & 0 & 0 & 0 & 0 \\ \ell_{2,1} & \ell_{2,2} & 0 & 0 & 0 \\ \ell_{3,1} & \ell_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$

non-singular

    A matrix that has an inverse. Equivalently, a matrix with non-zero determinant.

nullspace

    The set of all vectors that satisfy $A\mathbf{x} = \mathbf{0}$.

orthogonal

    Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if their inner product is 0, i.e., if $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$.

outer product

    The matrix produced by the product of a column vector and its transpose, i.e., $\mathbf{u}\mathbf{v}^{*}$

range

    The set of all vectors that can be expressed as $A\mathbf{x}$ for some $\mathbf{x}$.

rank

    The dimension of the vector space spanned by a matrix.

row rank

    The dimension of the vector space spanned by the rows of a matrix.

singular

    A matrix that has no inverse. Equivalently, a matrix with 0 determinant.

symmetric

    A matrix A such that $A = A^{\mathrm{T}}$.

upper triangular matrix

A matrix where all entries below the diagonal are zero, i.e., $U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix}$

Acronyms

SVD

Singular Value Decomposition