COSC 6364 Adv. Numerical Analysis

Quiz 1 Prep

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Spring 2020

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  - The matrix $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ projects onto $\mathrm{range}(A)$.

Householder Reflector and QR factorization

- How does Householder reflector accomplish QR factorization?

Householder Reflector and QR factorization

- How does Householder reflector accomplish QR factorization?
  - The Householder reflector introduces zeros below the diagonal in the k^{th} column while preserving all the zeroes previously introduced.

$$\underset{A}{\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1 A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}} \xrightarrow{Q_2} \underset{Q_2 Q_1 A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix}} \xrightarrow{Q_3} \underset{Q_3 Q_2 Q_1 A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Then

$$Q = Q_1 Q_2 \dots Q_n$$

Householder Reflector and QR factorization

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Householder Reflector and QR factorization

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  - Write $\boldsymbol{v} = \boldsymbol{x} + \|\boldsymbol{x}\|\boldsymbol{e}_1$. Then the Householder reflector is

$$I - 2\frac{\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}}{\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}}$$

Equivalently, write $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ and our reflector is

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  - $2mn^2 - \frac{2}{3}n^3$

Conditioning and Backward Stability

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  - Given a problem f with input $\boldsymbol{x}$, write $\delta\boldsymbol{x} = (1+\delta)\boldsymbol{x}$ and $\delta f = f(\delta\boldsymbol{x}) - f(\boldsymbol{x})$. The absolute condition number of f is given by
$$\lim_{\delta \to 0} \sup_{\delta} \frac{\|\delta f\|}{\|\delta \boldsymbol{x}\|}$$
    The relative condition number is given by
$$\lim_{\delta \to 0} \sup_{\delta} \frac{\|\delta f\|/\|f\|}{\|\delta \boldsymbol{x}\|/\|\boldsymbol{x}\|}$$

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  $$\lim_{\delta\to 0}\sup_{\delta}\frac{\|\delta f\|/\|f\|}{\|\delta\boldsymbol{x}\|/\|\boldsymbol{x}\|}$$
  - An algorithm is backward stable if, for every $\boldsymbol{x}$, there exists an $\tilde{\boldsymbol{x}}$ with
  $$\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} = \mathcal{O}(\epsilon_{\text{machine}})$$
  such that
  $$\tilde{f}(\boldsymbol{x}) = f(\tilde{\boldsymbol{x}})$$

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  - This follows from the definitions of forward error and κ:

$$\frac{\left\|\tilde{f}(\boldsymbol{x}) - f(\tilde{\boldsymbol{x}})\right\|}{\left\|f(\tilde{\boldsymbol{x}})\right\|} = \frac{\left\|\tilde{f}(\boldsymbol{x}) - f(\tilde{\boldsymbol{x}})\right\| / \|f(\tilde{\boldsymbol{x}})\|}{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\| / \|\boldsymbol{x}\|} \cdot \frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

$$\leq \kappa \frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

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- Is the simple running sum algorithm for computing inner product of two vectors $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$ backward stable?
  - Yes. Let f denote the inner product and $\tilde{f}$ its computation by running sum. When $n = 2$, we have

$$\begin{aligned}
\tilde{f}(\boldsymbol{x}, \boldsymbol{y}) &= x_1 \otimes y_1 \oplus x_2 \otimes y_2 \\
&= (x_1 y_1)(1 + \epsilon_1) \oplus (x_2 y_2)(1 + \epsilon_2) \\
&= [(x_1 y_1)(1 + \epsilon_1) + (x_2 y_2)(1 + \epsilon_2)](1 + \epsilon_3) \\
&= (x_1 y_1 + x_2 y_2 + x_1 y_1 \epsilon_1 + x_2 y_2 \epsilon_2)(1 + \epsilon_3) \\
&= x_1 y_1 + x_2 y_2 + x_1 y_1 \epsilon_1 + x_2 y_2 \epsilon_2 + x_1 y_1 \epsilon_3 + x_2 y_2 \epsilon_3 + x_1 y_1
\end{aligned}$$

$=$ this is unfinished because these proofs are the worst

LU factorization

- How to use LU factorization to solve a linear system?

LU factorization

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  - To solve $A\boldsymbol{x} = \boldsymbol{b}$, solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ then solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$.

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  - A better solution is to write $PA = LU$ for a permutation matrix P. Now, solve $L\boldsymbol{y} = P\boldsymbol{b}$ for $\boldsymbol{y}$ then solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$.

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- How to use LU factorization to invert a square non-singular matrix A?
  - Solve $LU\boldsymbol{x}_i = \boldsymbol{e}_i$ for all i. Then $A^{-1} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \end{bmatrix}$.

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- What's the cost of LU factorization? How does it compare to Householder QR factorization on the same matrix?

QR Algorithm for Eigenvalue Decomposition

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  - Begin with matrix $A^{(0)} = A$ and, for $k = 1, 2, \ldots$, write

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\begin{aligned}
Q^{(k)} R^{(k)} &= A^{(k-1)} \\
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- What's the cost of one iteration in QR algorithm?
  - If we run it on A directly, it is $\mathcal{O}(m^3)$ flops. If instead we perform it on H, the reduced Hessenberg form of A ($\frac{10}{3}m^3$ flops), each phase is $\mathcal{O}(m^2)$ flops.

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- What's the convergence rate of QR algorithm?
  - Without shifting, the QR algorithm exhibits linear convergence. With shifting, it exhibits cubic convergence.