Distributed Algorithms for Connectivity and MST in Large Graphs with Efficient Local Computation

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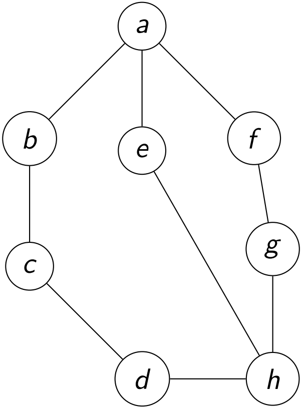
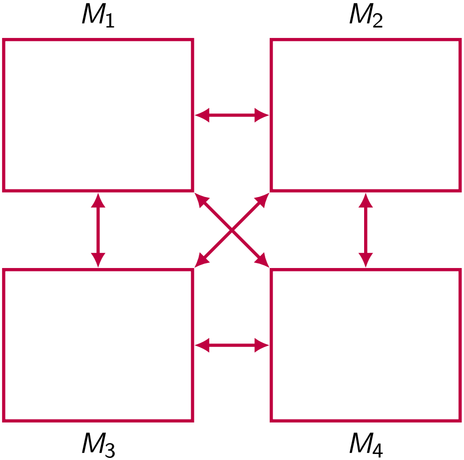
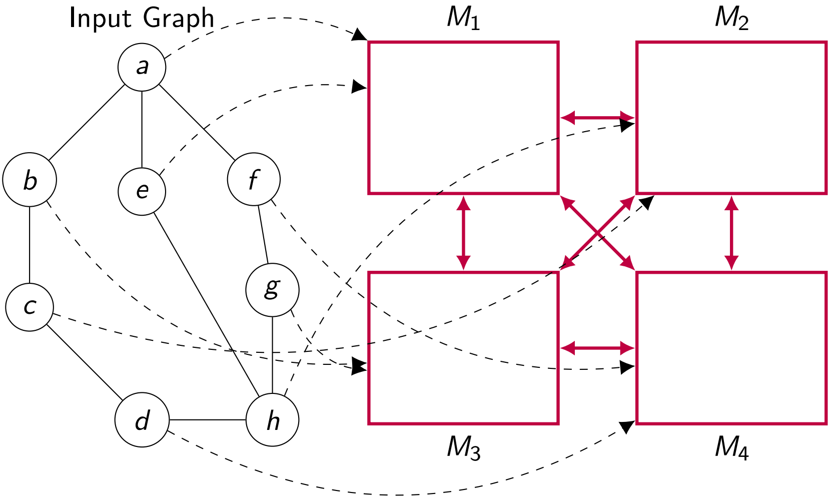
The k-Machine Model

- study distributed algorithms for large-scale graphs
- focus on connectivity and MST
- in k-machine model
  - $k \geq 2$ machines jointly perform computations on input graph
  - input graph distributed uniformly at random to k machines
    - n nodes, m edges
    - when a node is given to a machine, its adjacency list is also
    - commonly called vertex centric model
  - assume $n \gg k$ (e.g. $k = \mathcal{O}(n^{\varepsilon})$)

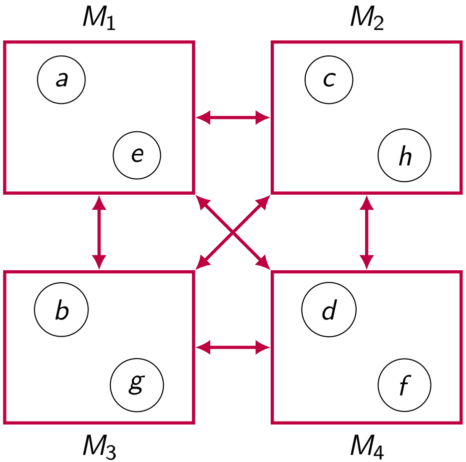
Input Graph







Input Graph
a
b
e
f
g
c
d
h
M_1
M_2
M_3
M_4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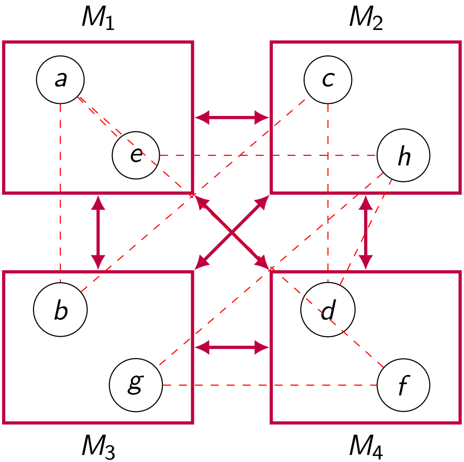

M_1
M_2
a
c
e
h
b
d
g
f
M_3
M_4

- typically, only consider communication cost
  - message complexity — number of communication rounds
- traditionally done because network communication is much slower than local computation
  - less true as network speeds increase

Our Results

- we posit a new complexity measure
  - local computation cost — T_ℓ
  - measures the worst-case local computation cost among k machines
- we refer to traditional communication complexity by T_c
- lower bounds

$$T_\ell = \Omega\left(\frac{m+n}{k} + \Delta + k\right)$$

$$T_c = \Omega\left(\frac{n}{k^2}\right)$$

- in practical implementations of k-machine model algorithms, often see suboptimal speedup
  - e.g. PageRank with $T_c = \mathcal{O}(n/k)$ but whose wall-clock time is approximately $n/k^{0.8}$

- analyze several algorithms under the new complexity measure

Algorithm	Round complexity	Local runtime
Flooding	$\tilde{O}\left(\frac{n}{k} + D\right)$	$\tilde{O}\left(\frac{m}{k} + \Delta + k\right)$
Filtering	$\tilde{O}\left(\frac{n}{k}\right)$	$\tilde{O}\left(\frac{m}{k} + n\right)$
Improved Local Borůvka	$\tilde{O}\left(\frac{n}{k}\right)$	$\tilde{O}\left(\frac{m+n}{k} + \Delta + k\right)$
Randomized CC	$\tilde{O}\left(\frac{n}{k^2}\right)$	$\tilde{O}\left(\frac{m+n}{k} + \Delta + k\right)$

- many distributed algorithms that are optimal are often only optimal under communication complexity
- as a byproduct of our analysis, we have two general results
  - the Node Distribution Lemma
  - the Mapping Lemma

Lemma (Node Distribution Lemma)

Consider a graph G of nodes $v_1, v_2, \ldots, v_n$ with associated non-negative real-valued "weights" $w(v)$ for each node v. Given a uniform, random distribution of the n nodes to k machines, as in the k-machine model, then, with probability at least $1 - 1/n^a$ for any $a > 0$, the total weight of nodes at every machine is bounded above by

$$\mathcal{O}(T_{avg} + \log n \cdot w_{max})$$

where $T_{avg} = \frac{1}{k}\sum_{i=1}^{n} w(v_i)$ and $w_{max} = max\{w(v_i)\}$.

Lemma (Mapping Lemma)

Let an n-node, m-edge graph G be partitioned among the k machines as $N = \{p_1, \ldots, p_k\}$. Then with probability at least $1 - 1/n^{\alpha}$, where $\alpha > 1$ is an arbitrary fixed constant, the following bounds hold:

1. *The number of vertices mapped to any machine is $\mathcal{O}(n/k)$.*
2. *The number of edges mapped to any machine is $\mathcal{O}(m/k + \Delta \log n)$.*
3. *The number of edges mapped to any link of the network is $\mathcal{O}\left(m/k^2 + n/k\right)$.*

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- the algorithm can be described from the perspective of a node
  - each node chooses an ID uniformly at random in $[1, n^4]$
  - each node floods its ID
  - upon receiving a message (with an ID), if the new ID is greater than the current ID, the node updates its ID and floods
- in order to simulate this in the k-Machine Model, each machine M
  - maintains list of nodes on the machine in descending-order of ID
  - iterate through list and simulate each node individually
    - when u located on M sends a message to v on M', M sends the appropriate message to M'
  - after some $\mathcal{O}(n \log n/k + D)$ rounds, aggregate all IDs located on machine M and send them to machine M_1
- Finally, machine M_1 counts the number of distinct IDs (which is the number of connected components) and broadcasts it to all machines

Theorem

With high probability, the above algorithm correctly counts the number of connected components with

$$T_\ell = \mathcal{O}((m/k + \Delta \log n) \log n + k)$$
$$T_c = \mathcal{O}(n \log n/k + D)$$

Proof.

- with Chernoff Bounds, can show that a node v will update max-ID $\mathcal{O}(\log n)$ times with probability $1 - 1/n^a$
- v will receive $\mathcal{O}(\deg(v) \log n)$ higher IDs
- when max-ID is updated, v sends to $\deg(v)$ nodes, totaling $\mathcal{O}(\deg(v) \log n)$
- maximum runtime is therefore $\mathcal{O}(\Delta \log n)$.

$\square$

Proof.

- apply node distribution lemma on the degree of nodes

$$T_\ell = \mathcal{O}\left(\frac{1}{k}\left(\sum_{i=1}^{n}(d(v_i)\log n)\right) + \log n \cdot \Delta \log n\right)$$
$$= \mathcal{O}\left(\left(\frac{m}{k} + \Delta \log n\right)\log n\right)$$

- in the worst case, a machine sends/receives messages from all other machines, totaling

$$\mathcal{O}\left(\left(\frac{m}{k} + \Delta \log n\right)\log n + k\right)$$

□

Lemma

The communication complexity of the algorithm is $\mathcal{O}(n \log n/k + D)$ with high probability.

Proof.

- by conversion theorem (see), total number of broadcasts is $\mathcal{O}(B \log n/(kW) + D)$ where W is the bandwidth
- taking $W = \mathcal{O}(\log n)$
- by mapping lemma, $\mathcal{O}(\log n)$ broadcasts for a particular node with probability $1 - 1/n^2$
- by union bound, $\mathcal{O}(\log n)$ broadcasts for all nodes with probability $1 - 1/n$
- thus, $\mathcal{O}(n \log n)$ broadcasts initiated with high probability

$\square$

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Preliminaries

- uses three procedures:
- Kruskal's algorithm — outputs an MST in $\mathcal{O}(m \log m)$ rounds
- maximal matching a clique of size n
  - done by simply sorting IDs and matching node $2i-1$ to node $2i$
- distributed routing in a clique
  - node u wants to send f messages to a different node v
  - phase 1:
    - divide messages into groups of $f/(n-1)$
    - append destination id to messages
    - send one group per edge
  - phase 2:
    - each node forwards its message to the destination

The Algorithm

- each machine M maintains
  - a graph G_M
    - initially comprised of subgraph induced by edges in M
  - ordered list H_M comprised of IDs of all machines
- each machine M executes at most $2 \log k$ phases of:
  - if M received information about new edges in previous phase, it updates G_M accordingly, then runs Kruskal's to remove cycles
  - if $|H_M| = 1$, terminate. Use matching procedure on H_M to match with machine M'. If the ID of M is greater than that of M', use routing procedure to send G_M to M'
  - For any machine M'' in H_M, if M'' matched with a machine with a lower ID than M'', remove M'' from H_M. If M is such a machine, terminate
- at the end of the algorithm, the machine with the lowest ID will have the entire MST

- Correctness is fairly straightforward
  - if e is a cut edge, then it will never be removed by Kruskal's
  - the matching algorithm guarantees that after $2 \log k$ phases, the lowest ID machine will contain e

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- we present two deterministic algorithms
- however, a quick review of Borøuvka's
- each machine stores a disjoint-set (union-find) $\mathcal{M}$
- in each phase:
  - an MOE is found for each fragment
  - fragments are merged along MOEs

Algorithm 1

- each phase consists of two steps
  - for any node v in M, M iterates through the adjacency list of v and performs FIND until it finds an edge belonging to another fragment, after which it broadcasts this MOE
  - M merges fragments: if (u, v) is an MOE, M updates $\mathcal{M}$ with UNION(u, v).

Algorithm 2 — The Improved Algorithm

- can improve T_ℓ
- rather than broadcasting MOEs, simulate unicast version (as in GHS)
- additionally, we filter to reduce the number of edges in each machine to $\mathcal{O}(n)$
  - machines create MSFs of their local subgraphs using Kruskal's
  - "discard" edges that are not part of local MSF
  - the remaining (at most $k(n-1)$ edges whp) are the only MST edges reamining
- algorithm then continues on remaining subgraphs

- for a fixed machine M and fragment f
- set $\texttt{LOE} = \infty$
  - iterate through edges in f — for edge (u, v) with u in M, send a message to the machine containing v with IDs of u and v and fragment ID of f
  - for each message (u, v, f), if the fragment ID of v is different than the received fragment ID, respond with original fragment ID and fragment ID of f
  - update $\texttt{LOE} = \textsc{min}(\texttt{LOE}, w)$ for each message received corresponding to f, where w is the weight of the outgoing edge
  - broadcast the $\texttt{LOE}$ and fragment ID
- from the broadcast of the $\texttt{LOE}$s, each machine locally determines the global MOE for any fragment it contains
- the machine then merges fragments

Merging

- Note that we cannot merge all fragments at once, as that takes time proportial to the length of the fragment chain
- use a technique similar to controlled GHS
  - create rooted tree F — each node is a fragment and there is an edge between nodes if they share an MOE
  - construct maximal matching (using e.g. Cole-Vishkin)
  - merge all matches edges and any edge where exactly one endpoint is matched

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- we analyze the 2018 algorithm of Pandurangan, Robinson, and Scquizzato
- optimal up to polylog factors in terms of round complexity
- but $\tilde{O}(n^2)$ local computation complexity
- the algorithm is similar to Borøuvka's algorithm, with the output being a labeling of the nodes, such that nodes in the same component have the same label
- component also have a component proxy that handles finding MOEs and merging for the component
- initially, each node has label equal to its ID (forming its own component) and is its own component proxy
- to load balance communication and computation, component proxies are chosen randomly among the machines
- given a component f and a machine M, the set of nodes belonging to f in M are called the component part
- relies on linear graph sketches to efficiently merge multiple components

- sketches are found by forming a vector of length $\binom{n}{2}$ for each node v and and appropriate choice of an $\mathcal{O}(\log^2 n) \times \binom{n}{2}$ matrix. This requires $\tilde{\Omega}(n^2)$ local computation time.
- can be improved to $\tilde{\mathcal{O}}((m+n)/k + \Delta + k)$ by using the 2011 sketches of Tardos, Saglam, and Jowhari
- instead of storing vectors of length $\binom{n}{2}$, a leader machine broadcasts an odd hash function

- each component uses this hash function to find an MOE as follows
  - each node in a component part evaluates $\sum_{\text{incident edges } e} h(e) \bmod 2$
  - this sum is aggregated among the nodes in a component part by the corresponding machine
  - the sums of each part of the component are aggregated to the corresponding component proxy machine.
  - If the sum is 1, we conclude with some constant probability ε that an outgoing edge exists out of the component
    - By restricting the edges considered to be those within some range $[i, j]$, can (like binary search) determine an MOE with constant probability
    - repeating this process $\Omega(\log n)$ times yields an MOE whp for a single component
    - by union bound, we have MOE for all components whp