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MIR KHALID HASSAN

TABLE OF CONTENTS

ABSTRACT

An important area of mathematics for understanding the physical sciences is Differential equations. The majority of differential equations are created as a result of physics, engineering, and other scientific difficulties, and these equations are used as mathematical models to address a wide range of scientific and technical issues. A total of three chapters make up this report. The theory of differential equations is introduced and certain foundational ideas are covered within chapter 2; they will be beneficial in the report's main body.

The methods for solving some differential equations are the main topic of the third chapter., particularly separable, linear first order differential equations, homogeneous equations, and Bernoulli's Equation, which will be applied to a real-world system in unit three of this report. Chapter three's analyzes many real contexts where first order differential equations are used. To support the applications under this context, sample examples are used.

CHAPTER 1
INTRODUCTION

1.1 Differential Equations

A differential equation in mathematics is an equation that contains one or more functions and their derivatives. The rate of change of a function at a particular point is determined by its derivatives. It is primarily employed in disciplines like physics, engineering, biology, and others. The main goal of differential equations and the characteristics of the solutions.

Definition An equation that combines one or more terms and the derivatives of one variable (the dependent variable) with respect to some other variable is described to by the term differential equation (i.e. independent variable).

$$\frac{dy}{dx} = f(x)$$

In this case, x and y are independent variables.

$For\ Example, \frac{dy}{dx} = 5\text{x}$

A differential equation comprises derivatives, either partial or regular derivatives. A relationship between a quantity that is constantly varying with respect to a change in another quantity is described by a differential equation, where the derivative denotes the rate of change. There are several alternative formulas for solving differential equations using derivatives. [1].

Differential Equations and Their Types

The numerous types of differential equations include:

1. Ordinary Differential Equations
2. Partial Differential Equations
3. Linear Differential Equations
4 Non-linear differential equations
5. Homogeneous Differential Equations
6. Non-homogenous Differential Equations.

1.2 Ordinary Differential Equations (O.D.E.)

Definition. A differential equation is an equation that connects the variables used to create the function, constants, the function's derivations, and the function itself. If the unknown function depends only on one real variable, the differential equation is referred to as an ordinary differential equation.

These are some illustrations of ordinary differential equations:

1. $dy = (x + \sin x)dx$

2. $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$

3. $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{dy/dx}$

4. $k(d^2y/dx^2) = \{1 + (dy/dx)^2\}^{3/2}$ [2]

1.2.1 Order of a Ordinary Differential Equation:

Differential equations can also be categorized based on the sequence in which they appear. Simply put, the highest derivative that appears in a differential equation determines the order of the equation. For instance,

Sample 1:

$$\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} = e^y$$

This equation is a third order differential equation since the highest order derivative in it is three.

Sample 1:

$$\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} = 3$$

This equation represents a second order differential equation.[3]

1.2.2 Degree of a Ordinary Differential Equation:

The highest order derivative in the given differential equation has a power that indicates the degree of the problem.

The degree can only be defined if the differential equation is a polynomial equation with derivatives.

Sample 1 $$\frac{d^4y}{dx^4} + \left(\frac{d^2y}{dx^2}\right)^2 - 3\frac{dy}{dx} = 9$$

Here, the exponent of the highest order derivative is one and the given differential equation is a polynomial equation in derivatives. Hence, the degree of this equation is 1.

Sample 2:

$$\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^4 = k^2\left(\frac{d^3y}{dx^3}\right)^2$$

In this instance, the differential equation is a polynomial equation in derivatives, and the exponent of the highest order derivative is one. As a result, this equation has a degree of 1.[3]

1.2.3 Linear and non-linear a Ordinary Differential Equation:

The linear polynomial equation, which is derived from numerous variables, is what defines a linear differential equation. When the function is variable-dependent and partial derivations are used, it is often referred to as a linear partial differential equation.

First-order linear differential equations are defined as differential equations of the following form, where P and Q are only constants or functions of the independent variable (in this example, x). Moreover, the differential equation of the form dy/dx+ Py = Q is a first-order linear equation in which P and Q are either constants or exclusive functions of y (the independent variable).

A differential equation is considered to be nonlinear if the unknown function and its derivatives are not linear. It offers numerous options for dealing with turmoil. [2]

1.2.4 Solution of a differential equation

Any specified and differentiable function that can satisfy a differential equation is said to be that equation's solution.

$\frac{dy}{dx} = 2x$ is a differential equcation wich is satisfy with

$$y = x^2 + c$$

Explicit Solution A differential equation with the variables x and y has an explicit solution if the function y = f(x) is the answer.

Example : $\frac{d^2y}{dx^2} = 2x$ is a differential equcation wich has a explicit solution is $y = x^3 + Ax + B$

A and B are arbitrary constants in this situation.

Implicit Solution

It is referred to as an implicit solution if the function F(x, y) = 0 solves the differential equation involving the variables x and y.

Example : $2x\,dx + 2ydy = 0$
There exists a differential equation with an implicit solution.

$$x^2 + y^2 = c$$

General or complete solution

A differential equation is said to have a general solution if it has a solution that comprises n arbitrary constants for every differential equation of nth order.

Example : $\frac{dy}{dx} = y$ is a differential equcation wich

has a complete solution is $y = Ae^x$

Where A is a arbitrary constant.

Particular Solution

It is referred to as the particular solution of the differential equation when it is possible to find solutions for definite values of the arbitrary constants in the general solution.

Example : $\dfrac{dy}{dx} = y$ is a differential equcation wich

has a particular solution is $y = e^x$.

Singular solution

It is known as a singular solution of the differential equation if there is any other solution to the differential equation than the general or particular solutions.

Example : $\left(\frac{dy}{dx}\right)^2 = y$ is a differential equcation

Wich has a Singular solution is $y = 0$. [8]

1.2.5 Formation of Ordinary Differential Equation

As an example, consider the function

$$f\ (x,\ y, c_1) = 0 \qquad (1)$$

where c_1 is an arbitrary constant. From this equation, we create the differential equation. To accomplish this, distinguish between equation (1) and the independent variable that is present in the equation.

Remove the arbitrary constant c from (1) and its derivative. The necessary differential equation is then obtained.

Assume that f (x, y, c_1 and c_2) = 0. These two constants, $c_1$and c_2, are completely arbitrary. Hence, identify the first two successive derivatives. The given function and its subsequent derivatives must be free of the variables $c_1$and c_2. We obtain the necessary differential equation.

Note:

The amount of arbitrary constants included in the differential equation forming the family of curves determines the sequence in which it must be constructed.[9]

1.2.6 Initial value problem

When a differential equation's specific solution is determined by applying the conditions of a dependent variable to one point of an independent variable, the conditions are referred to as starting conditions, and the differential equation in question is referred to as an initial value problem.(1) Form of first order initial value problem

$$f\left(x, y, \frac{dy}{dx}\right) = 0, y(x_0) = y_0$$

Here $y(x_0) = y_0$ is initial condition

(2) Form of second order initial value problem

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0; y(x_0) = y_0, y'(x_0) = y_1$$

Here $y(x_0) = y_0, y'(x_0) = y_1$ is the initial condition

(3) Form of n order initial value problem

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \dots .. \frac{d^ny}{dx^n}\right) = 0; y(x_0) = y_0, y'(x_0) = y_1$$
$$\dots\dots\dots y^{n-1}(x_0) = y_{n-1}$$

Here $y(x_0) = y_0, y'(x_0) = y_1 \dots\dots\dots y^{n-1}(x_0) = y_{n-1}$ are the initial condition.[8]

2.2.7 Boundary value problem

The differential equation with boundary conditions is known as a boundary value problem if the specific solution of a higher than first order differential equation can be found by applying the conditions of the dependent variable and its derivative at two locations of the independent variable.

Example : $\frac{d^2y}{dx^2}+y=0, y(0) = 1, y\left(\frac{\pi}{2}\right) = 5$

is a boundary value problem.[8]

1.2.8 Lipschitz condition

Consider the rectangular region

$R = \{(x, y): |x - x_0| \leq a, |x - x_0| \leq b\}$ in the xy plane. If $|f(x, y) - f(x, y)| \leq A(y_1 - y_2)$where A > 0 is a constant and ∀ (x, y_1) (x, y_2)ϵR, then we can say the function $f(x, y)$ satisfy the Lipschitz condition over the region R.[4]

2.2.9 Basic existence and uniqueness theorems

(1) Existence Theorem over a Rectangular region

Let $R = \{(x, y): |x - x_0| \leq a, |x - x_0| \leq b\}$be the rectangular region. There exists at least one solution to the initial value problem if f(x, y) is continuous and bounded, that is, if |f(x, y)| ≤K. ∀(x, y)ϵR. $y' = \frac{dy}{dx} = f(x, y), y(x_0) = y_0$ which is defined for all values of x in the interval $|x - x_0| < \alpha$, where α =min$\left(a, \frac{b}{k}\right)$.

(2) Uniqueness Theorem over a Rectangle

Let $R = \{(x, y): |x - x_0| \leq a, |x - x_0| \leq b\}$ be the rectangular region. If f(x, y) and $\frac{\partial f}{\partial x}$ are continuous and bounded i. e. if $|f(x, y)| \leq K$, $\left|\frac{\partial f}{\partial x}\right| \leq M$, ∀ f(x, y)ϵR, then there exist uniqueness solution of the initial value problem y'= $\frac{dy}{dx}$= f(x, y). $y(x_0) = y_0$. which is defined for all values of x in the interval $|x - x_0| \leq \alpha$ where $\alpha = \min\left(a, \frac{b}{k}\right)$. [4]

2.3 Partial Differential Equations (P.D.E.)

Definition: Partial differential equations are those that contain one or more partial derivatives of an unknown function of two or more independent variables. We provide the following items as partial differential equation examples:

$$\frac{\partial a}{\partial x} + \frac{\partial a}{\partial y} = a + xy \tag{1}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x. \tag{2}$$

$$a\left(\frac{\partial a}{\partial x}\right) + \frac{\partial a}{\partial y} = x \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz \tag{4}$$

$$\frac{\partial^2 a}{\partial x^2} = \left(1 + \frac{\partial a}{\partial y}\right)^{\frac{1}{2}} \tag{5}$$

$$y\left\{\left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2\right\} = a\left(\frac{\partial a}{\partial y}\right) \tag{6}$$

Order of a Partial Differential Equation:

The highest partial derivative that occurs in a partial differential equation is referred to as the order of the partial differential equation. The first order equations are (1), (3), (4), and (6), the second order equation (5), and the third order equation

(2).**Degree of a Partial Differential Equation:**

A partial differential equation's degree is determined by the highest order derivative that occurs in it after it has been rationalized, or rendered free of radicals and fractions in terms of derivatives. The first-degree equations are (1), (2), (3), and (4), whereas the second-degree equations are (5) and (6).

Linear and non-linear a Partial Differential Equation:

Definitions. If the dependant variable and all of its partial derivatives only appear in the first degree and are not multiplied, a partial differential equation is said to be linear. The term "non-linear partial differential equation" refers to a partial differential equation that is not linear. Equations (1) and (4) in Art are linear, whereas equations (2), (3), (5), and (6) are nonlinear.

First order partial differential equations are divided into linear, semi-linear, quasi-linear, and non-linear equations, with examples of each type:

f(x,y,z,p,q)=0 is a first order linear equation. is said to as linear if it is linear in p, q, and z, or if an equation has the following form.$P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$. For examples, $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$ these two partial differential equations are both linear first order.

Semi-linear equation: If the given equation has the form f(x,y,z,p,q)=0, then the first order partial differential equation is said to be semi-linear since the coefficients of p and q are just functions of x and y.

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

For examples,
$xyp + x^2yq = x^2y^2z^2$ and $yp + xq = (x^2z^2/y^2)$ are both first order semi-linear partial differential equations.

Quasi-linear equation: The partial differential equation of first order f(x,y,z,p,q)=0 If the given equation has the form P(x,y,z)p+Q(x,y,z)q=R, then it is a quasi-linear equation since it is linear in p and q. (x,y,z). Examples of this

$x^2zp + y^2zp = xy$ and $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ are first order quasi-linear partial differential equations.

Non-linear equation: The partial differential equation of first order f(x,y,z,p,q)=0 is referred to as a non-linear equation and does not fall under the first three categories. Examples of, $p^2 + q^2 = 1, pq = z \ and \ x^2p^2 + y^2q^2 = z^2$ are all non-linear partial differential equation.[1]

CHAPTER 2
Solution Technique of First-Order Differential Equations

2.1 Separable Equations

2.1.1 General Solution

The solution to the differential equation of first order with separable components.

$$A(p)\,dp + B(q)dq = 0 \tag{2.1}$$

Is $$\int A(p)dp + \int B(q)dq = c \tag{2.2}$$

where c is a freely chosen constant.

It's possible that it's impossible to assess the integrals from Equation 2.2 practically. Numerical methods are applied in this situation to arrive at a close solution. The algebraic solution of y in terms of x may not be feasible, even if the integrations in 2.2 that are suggested can be carried out. The solution is left implicit in the situation.

2.1.2 Solutions to the Initial-Value Problem

The solution to the initial-value problem

$$A\,(p)dp + B(q)dq = 0;\ y(p_0)=q_0 \tag{2.3}$$

can be acquired, as per usual, by first solving the differential equation with Equation 2.2 and then immediately applying the initial condition to determine the value of c.

As an alternative, the answer to Equation 2.3 can be found by

$$\int_{xo}^{x} A(p)dp + \int_{yo}^{y} B(q)dq = 0 \tag{2.4}$$

p and q are integration variables here..[3]

2.1.3 Solved Problems

Solved Problem 1.Solve $\frac{dy}{dx} = \frac{x^2+2}{y}$

Solution: There is a differential form that can be used to rewrite this equation.

$$(x^2 + 2)dx - ydy = 0$$

Which is separable with $A(x) = x^2 + 2$ and B(y) = -y. Its solution is

$$\int(x^2 + 2)dx - \int ydy = c$$

Or

$$\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = c$$

We arrive at the solution in implicit form as after solving for y.

$$y^2 = \frac{2}{3}x^3 = 4x + k$$

using $k = -2c$. The two solutions are obtained by explicitly solving for y. $\quad y = \sqrt{\frac{2}{3}x^3 = 4x + k}$

and $y = -\sqrt{\frac{2}{3}x^3 = 4x + k}$

Which is the required solution.

Solved Problem 2. Solve $\tan x\, dy = \cot y\, dx$

Solution: $\tan x\, dy = \cot y\, dx$

$\Rightarrow \tan y\, dy = \cot x\, dx$ (Multiply both sides by $\cot x \tan y$)

$\Rightarrow \int \tan y\, dy = \int \cot x\, dx$

$\Rightarrow \int \tan y\, dy = \int \cot x\, dx$

$\Rightarrow \ln \sec y = \ln \sin x + \ln c$

$\Rightarrow \ln \sec y = \ln c \sin x$

$\Rightarrow \sec y = c \sin x$

$\Rightarrow \sec y \csc y = c$

Which is the required solution.

Solved Problem 3. Solve $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

Solution: $\quad \frac{\mathbf{dy}}{\mathbf{dx}} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

$$\Rightarrow dy + \frac{\sqrt{\mathbf{1-y^2}}}{\sqrt{\mathbf{1-x^2}}}dx = 0$$

$$\Rightarrow \frac{dx}{\sqrt{\mathbf{1-x^2}}} + \frac{dy}{\sqrt{\mathbf{1-y^2}}} = 0$$

$$\Rightarrow \int \frac{dx}{\sqrt{\mathbf{1-x^2}}} + \int \frac{dy}{\sqrt{\mathbf{1-y^2}}} = 0$$
$$\Rightarrow \sin^{-1} x + \sin^{-1} y = c$$
Which is the required solution.

2.2 Homogeneous Equations

The homogeneous differential equation

$$\frac{dy}{dx} = f\,(x, y) \tag{2.5}$$

bearing the property f(tx, ty) = f(x, y) may be changed into a separable equation by putting

2.2.1 DIFFERENTIAL EQUATIONS

Let
$$y = xv \tag{2.6}$$
Along with its equivalent derivative
$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{2.7}$$
Equation 2.5's necessary solution is acquired by back substituting into the resulting equation in the variables v and x, which is then solved as a separable differential equation.

As an alternative, the differential equation can be changed to read as follows to find the solution to problem 2.5.
$$\frac{dy}{dx} = \frac{1}{f(x,y)} \tag{2.8}$$

and then substituting
$$x = yu \tag{2.9}$$

and the corresponding derivative
$$\frac{dy}{dx} = u + y\frac{du}{dy} \tag{2.10}$$
2.8 into the equation. The differential equation that results from simplification will have separable variables (this time, u and y). Typically, it doesn't matter which type of solution is employed. The replacements 2.6 or 2.9 can, on occasion, both be clearly superior to one another. In these situations, the better substitute is typically obvious from the differential equation's form. [3]

2.2.2 Solved Problems

Solved Problem 1. Solve $y' = \frac{y+x}{x}$

Solution: This differential equation can't be divided into two parts. Instead, it has the form.

$$y' = f\,(x, y), \text{ with}$$

$$f(x,y) = \frac{y+x}{x}$$

Where

$$f(xt, yt) = \frac{yt+xt}{xt} = \frac{t(y+x)}{xt} = \frac{y+x}{x} = f(x,y)$$

It is uniform, thus. Equations 2.6 and 2.7 are substituted into the formula, and the result is

$$v + x\frac{dv}{dx} = \frac{vx+x}{x}$$

This, when reduced using algebra, becomes

$$x\frac{dv}{dx} = 1 \text{ or } \frac{1}{x}dv - dx = 0$$

This final equation can be separated, and its solution is

$$\int \frac{1}{x}dv - \int dx = 0$$

Which is the result of the evaluation v=ln[x]-c or v=ln |kx|, where c=ln| k| is set, and where it is observed that

$$\ln[x] + \ln[k] = \ln[xk]$$

The solution of the previous differential equation is finally obtained by re-inserting v = y/ x into 4.11. y = x ln | kx .

Solved Problem 2. Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution: This differential equation cannot be separated. The form is there instead. $\frac{dy}{dx}$ = f (x, y),with

$$f(x,y) = \frac{y}{x} + \tan\frac{y}{x}$$

Let y= vx ; $\qquad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (1)$

$$(1) \Rightarrow\; v + x\frac{dv}{dx} = v + \tan v$$

$$\Rightarrow \cot v\, dv = \frac{dx}{x} \qquad (2)$$

Integrating both side of (2), we get

$$\ln \sin v = \ln x + \ln c$$
$$\Rightarrow \sin v = cx$$
$$\Rightarrow \sin \frac{y}{x} = cx$$
Which is the required solution.

Solved Problem 3. Solve $\frac{dy}{dx} + \frac{2y}{x} = \frac{y^3}{x^3}$

Solution: This differential equation is not separable. Instead it has the form $\frac{dy}{dx}$ = f (x, y), with

$$f(x, y) + \frac{2y}{x} = \frac{y^3}{x^3}$$

Let y= vx ; $\frac{d}{dx}(vx) + 2y = y^3 \quad (1)$

$$(1) \Rightarrow v + x\,\frac{d}{dx}(v) + 2v = v^3$$
$$\Rightarrow x\,\frac{d}{dx}(v) = v^3 - 3v$$
$$\Rightarrow \frac{1}{v(v^2-3)}dv = \frac{dx}{x}$$
$$\Rightarrow \frac{1}{3}\left(\frac{v}{v^2-3} - \frac{1}{v}\right)dv = \frac{dx}{x} \quad (2)$$

Integrating both side of (2), we get

$$\frac{1}{3}\left[\frac{1}{2}\ln(v^2 - 3)\right] - \ln v = \ln x + \frac{1}{6}\ln c$$
$$\Rightarrow \{\ln(v^2 - 3)\ 2\ln v\} = 6\ \ln x + \ln c$$
$$\Rightarrow \ln\left(\frac{v^2-3}{v^2}\right) = \ln cx^6$$
$$\Rightarrow \frac{v^2-3}{v^2} = cx^6$$
$$\Rightarrow v^2 - 3 = cv^2x^6$$
$$\Rightarrow \left(\frac{y}{x}\right)^2 - 3 = c\left(\frac{y}{x}\right)^2 x^6$$
$$\Rightarrow y^2 - 3x^2 = cx^6y^2$$

Which is the required solution.

2.3 Exact Equations

2.3.1 Defining Properties

A differential equation

$$M(a, b)da + N(a, b)db = 0 \qquad (2.11)$$

If g(a, b) has a function such that, then the statement is accurate.

$$dg(a, b) = M(a, b)da + N(a, b)db \qquad (2.12)$$

Test for accuracy: Equation 2.12 is accurate if and only if M(a,b) and N(a,b) are continuous functions having continuous first partial derivatives on a certain x-y rectangle.

$$\frac{M(a,b)}{b} = \frac{N(a,b)}{a} \qquad (2.13)$$

Method of Solution

If Equation 2.11 is an exact solution, then first solve the equations.

$$\frac{\partial g(a,b)}{\partial a} = M(a, b) \qquad (2.14)$$

$$\frac{\partial g(a,b)}{\partial b} = N(a, b) \qquad (2.15)$$

for g(a, b). The solution to 2.11 is then given implicitly by

$$g(a, b) = c \qquad (2.16)$$

where c denotes any random constant. From Equations 2.14 and 2.15, equation 2.16 follows immediately. We get dg(a, b(a)) = 0 when 2.14 is changed to 2.15. By integrating this equation, we arrive at, which suggests 2.16 (remember that 0 can be written as 0 da).

Integrating Factors

Equation 2.11 is generally not accurate. By carefully multiplying, 2.12 can occasionally be converted into an accurate differential equation. If the equation calls for an integrating factor of 2.16, it is a function I(a, b).

DIFFERENTIAL EQUATIONS

$$I(a, b)[M(a, b)da + N(a, b)db] = 0 \qquad (2.17)$$

is exact. By resolving the precise differential equation that 2.17 defines, 2.16 can be solved. Table 2.1 and the following conditions show some of the more common integrating factors:

If , $\frac{1}{N}\left(\frac{\partial M}{\partial b} - \frac{\partial N}{\partial a}\right) \equiv g\,(a)$ a function of x alone, then

$$I(a, b) = e^{\int g(a)da} \qquad (2.18)$$

If , $\frac{1}{M}\left(\frac{\partial M}{\partial b}-\frac{\partial N}{\partial a}\right) \equiv h(y)$a function of y alone, then

$$I(a,b) = e^{-\int h(b)db} \tag{2.19}$$

If M = bf(ab) and N = ag(ab), then

$$I(a,b) = \frac{1}{aM-bN} \tag{2.20}$$

In general, it can be challenging to identify integrating elements. Finding an integrating factor is unlikely to be successful if a differential equation does not have one of the previous forms, and it is suggested to use different methods of solution instead..[3]

2.3.2 Solved Problems

Solved Problem 1. Solve $2xydx + (1 + x^2\ )dy = 0$

Solution: Equation (2.11) with is how this one is written.

$$M(x, y) = 2xy \text{ and } N(x, y) = 1 + x^2.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$ a exact differential equation exists. We have now identified a function g(x, y) that satisfies Equations 2.14 and 2.15 because this equation is exact. When M(x,y) = 2xy is substituted into 2.14, we get $\frac{\partial g}{\partial x} = 2xy$ With this equation′s two sides to be integrated with regard to x, we determine

$$\int \frac{\partial g}{\partial x} dx = \int 2xydx$$

Or

$$g(x, y) = x^2y + h(y) \tag{1}$$

The constant of integration (with regard to x) can rely on y when integrating with respect to x, as should be noted. Now, we establish h. (y). When we differentiate (1) with relation to y, we get

$$\frac{\partial g}{\partial y} = x^2 + h'(y)$$

Substituting this equation along with
N(x, y) = 1 + x^2 into 3.15, we have

$$x^2 + h'^{(y)} = 1 + x^2 \text{ or } h'(y) = 1$$

When we integrate this final equation with consideration to y, we get h(y) = y + c_1 (c_1 = constant). Adding this expression to (1) results in

$$g(x, y) = x^2y + y + c_1$$

The differential equation's solution, which is given by 2.16, is written as

$$g(x, y) = c, \text{ is } \; x^2y + y = c_2 \quad (c_2 = c - c_1)$$

Solving for y explicitly, we obtain the solution as

$$y = c_2/(x^2 + 1).$$

Solved Problem 2. Solve If the differential equation is true, determine it.

$$y\,dx - x dy = 0 \text{ is exact.}$$

Solution: Equation 2.11 with is the solution for this equation M (x, y) = y and N (x, y)= −x. Here

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = -1$$

Which are not equal; hence, the differential equation is not exact.

Solved Problem 3.Solve Evaluate whether or not the differential equation's $-1/x^2$is an integrating factor. y dx – x dy = 0.

Solution: The differential equation's lack of accuracy was shown in Problem 2.

Multiplying it by $-1/x^2$, we obtain

$$\frac{-1}{x^2}(y\,dx - x\,dy) = 0$$

$$\text{Or} \qquad \frac{-y}{x^2}\,dx + \frac{1}{x}dy = 0 \qquad (1)$$

Equation 2.17 has the form of Equation 2.11 with M (x, y) = $-y/x^2$and N (x, y) = 1/x. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2}\right) = \frac{-1}{x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{\partial N}{\partial X}$$

Since (1) is exact, it follows that $-1/x^2$serves as an integrating factor for the initial differential equation.

Solved Problem 4. Solve y dx – x dy = 0.

Solution: We may rewrite the given differential equation as based on the findings of Solved Problem 3:

$$\frac{y\,dx - x\,dy}{x^2} = 0$$

This is exact. The steps in Equations 2.11 through 2.16 can be used to solve Equation (1).

Alternatively, we see that (1) can be expressed as d(y/x) = 0 from Table 2.1. As a result, y = xc or y = x/c is the solution by direct integration.

Table 2.1

Group of terms	Integrating factor $I(x, y)$	Exact differential $dy(x, y)$
$y\,dx - x\,dy$	$-\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
$y\,dx - x\,dy$	$\frac{1}{y^2}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
$y\,dx - x\,dy$	$-\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$
$y\,dx - x\,dy$	$-\frac{1}{x^2 + y^2}$	$\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\arctan\frac{y}{x}\right)$
$y\,dx + x\,dy$	$\frac{1}{xy}$	$\frac{y\,dx + x\,dy}{xy} = d(\ln xy)$
$y\,dx + x\,dy$	$\frac{1}{(xy)^n}, \quad n > 1$	$\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$
$y\,dy + x\,dx$	$\frac{1}{x^2 + y^2}$	$\frac{y\,dy + x\,dx}{x^2 + y^2} = d\left[\frac{1}{2}\ln(x^2 + y^2)\right]$
$y\,dy + x\,dx$	$\frac{1}{(x^2 + y^2)^n}, \quad n > 1$	$\frac{y\,dy + x\,dx}{(x^2 + y^2)^n} = d\left[\frac{-1}{2(n-1)(x^2 + y^2)^{n-1}}\right]$
$ay\,dx + bx\,dy$ (a, b constants)	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$

2.4 Linear Equations

2.4.1 Method of Solution

This is the format of a first-order linear differential equation.

$$y' + p(a)y = q(a) \qquad (2.21)$$

Equation 4.22 has an integrating factor of

$$I(a) = e^{\int p(y)da} \qquad (2.22)$$

This is independent of y and only dependent upon a. Equation is formed when 2.21's both sides are multiplied by I(a).

$$I(a)y' + p(a)I(a)y = I(a)q(a) \qquad (2.23)$$

is exact. By using the previous mentioned method, this problem can be resolved.

To make things easier, rewrite 2.22 as

$$\frac{d(yI)}{da} = Iq(a)$$

DIFFERENTIAL EQUATIONS

Solve the resulting equation for y by integrating both sides of the final equation with respect to x. Equation 2.21 has a general solution, which is

$$y = \frac{\int I(a)\, q\,(a)\, da + c}{I(a)}$$

Where c is the constant of integration.[3]

2.4.2 Solved Problems

Solved Problem 1. Solve $y' + (\frac{4}{x})y = x^4$

Solution: Equation 2.21 is the format of the differential equation. $p(x) = 4 / x$ and $q(x) = x^4$, and is linear. Here

$$\int p(x)dx = \int \frac{4}{x}dx = 4\ln x = \ln x^4$$

So 2.22 becomes

$$I(x) = e^{\int p(y)dx}$$
$$= e^{\ln x^4} = x^4 \qquad (1)$$

Once the differential equation is multiplied by the integrating factor specified by (1), the result is obtained.

$$x^4y' + 4x^3y = x^8$$

$$\text{or } \frac{d}{dx}(x^4y) = x^8$$

By integrating both sides of this last equation with relation to x, we obtain at $x^4y = \frac{1}{9}x^9 + c$

$$\text{or } y = \frac{c}{x^4} + \frac{1}{9}x^5$$

Which is the required solution.

Solved Problem 2. Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solution : $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\Rightarrow \frac{dy}{dx} + y\sec^2 x = \sec^2 x \tan x$$

Equation 2.21 is the solution of the differential equation. p(x) = $\sec^2 x$and q(x) = $\sec^2 x \tan x$, and is linear. Here

$$\int p(x)dx = \int \sec^2 x\, dx$$

So the I. F. : $I(x) = e^{\int p(y)dx}$

$$= e^{\int \sec^2 x dx}$$

$$=e^{\tan x}$$

So the required general solution is :

$$ye^{\int p(x)dx} = \int q(x)\, e^{\int p(x)dx}dx$$

$$\Rightarrow ye^{\tan x} = \int \sec^2 x \tan x\, e^{\tan x}dx + c$$

$$= \int \tan x\ e^{\tan x}d(\tan x) + c$$

$$= \tan x\, e^{\tan x}\text{-}e^{\tan x} + c$$

$$\Rightarrow\ y = e^{-\tan x}(\tan x\, e^{\tan x} - e^{\tan x}) + ce^{-\tan x}$$

$$= \tan x - 1 + \ ce^{-\tan x}$$

Which is the required solution.

2.5 Bernoulli Equations

2.5.1 Defining Properties

The structure of a differential equation for Bernoulli is

$$y' = p(x)y = q(x)y^n \tag{2.24}$$

A real number, n, is present. It was replaced

$$z = y^{1-n} \tag{2.25}$$

Transforms 2.24 into a linear differential equation in the unknown function z(x).[3]

2.5.2 Solved Problems

Solved Problem 1. Solve $y' + xy = xy^2$

Solution: This equation is nonlinear. However, it is a Bernoulli differential equation with n = 2 and the variables p(x) = q(x) = x. The change suggested by 2.25 is implemented, specifically

$z = y^{1-2} = y^{-1}$, from which follow

$$y = \frac{1}{z} \text{ and } y' = -\frac{z'}{z^2}$$

We obtain the differential equation by substituting these equations into it.

$$-\frac{z'}{z^2} + \frac{x}{z} = \frac{x}{z^2} \text{ or } z' - xz = -x$$

The unknown function z is linear in the last equation (x). Equation 2.21's shape, with z in place of y, and
$p(x) = q(x) = -x$. The integrating factor is

$$I(x) = e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$$

Multiplying the differential equation by I(x), we obtain

$$e^{-\frac{x^2}{2}}\frac{dz}{dx} - xe^{-\frac{x^2}{2}}z = -xe^{-\frac{x^2}{2}}$$

$$\text{or } \frac{d}{dx}\left(ze^{-\frac{x^2}{2}}\right) = -xe^{-\frac{x^2}{2}}$$

Upon integrating both sides of this last equation, we have

$$e^{-\frac{x^2}{2}}z = e^{-\frac{x^2}{2}} + c$$

Where upon

$$z(x) = ce^{-\frac{x^2}{2}} + 1$$

The solution of the original differential equation is then

$$y = \frac{1}{ce^{-\frac{x^2}{2}} + 1}$$

Solved Problem 2.Solve $xy - \frac{dy}{dx} = y^3e^{-x^2}$

Solution : $xy - \frac{dy}{dx} = y^3e^{-x^2}$

$\Rightarrow -\frac{dy}{dx} + xy = y^3e^{-x^2}$

$\Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{xy}{y^3} = e^{-x^2}$ (divided by y^3)

$\Rightarrow \frac{1}{2}\frac{dy^{-2}}{dx} + xy^{-2} = e^{-x^2}$ [$\because dy^m = my^{m-1}dy$]

$$\Rightarrow \frac{dy^{-2}}{dx} + 2xy^{-2} = 2e^{-x^2}$$

Which is a one order linier differential equation of y^{-2}

Now I.F $= e^{\int 2xdx} = e^{x^2}$
So the required general solution is :

$$y^{-2}e^{x^2} = \int 2\ e^{-x^2}e^{x^2}dx$$

$$= 2\int dx = 2x + c$$

$$\Rightarrow \quad e^{x^2} = y^2(2x + c)$$

Which is the required solution.

CHAPTER 3
Applications of First-Order Differential Equations

3.1 Growth and Decay Problem

3.1.1 Law of natural growth or natural decay

Quantities grow or degrade at a rate proportional to their magnitude in many natural wonders. It would seem logical to predict that the rate of growth, denoted by y_0(t), is proportional to the population, denoted by y(t), that is, y_0 (t) = ky(t), for some constant k, if y = y(t) represents the number of individuals in a population of animals or bacteria at time t. In ideal circumstances, what really occurs can be fairly accurately anticipated by the mathematical model provided by the equation y_0(t) = ky(t) (unlimited environment, adequate nutrition, immunity to disease). Additionally, there are several examples in the fields of economics, chemistry, and nuclear physics.

Generally speaking, if a quantity's value at a given time is expressed as y(t), and if y's rate of change with regard to t is proportionate to its magnitude at any given time is expressed as y(t), then

$$\frac{dy}{dt} = ky \tag{3.1}$$

Equation (3.1), which is frequently referred to as the law of natural growth (if k > 0) or the law of natural decay (if k 0), is a mathematical formula where k is a constant.

As a result, the law of exponential growth and decay can be expressed as

$$y = ce^{kt}; \tag{3.2}$$

The initial value, c, can be determined from the first circumstance. $y(t_0) = y_0$; k is the proportionality constant, which can be discovered from a further condition that may be provided in the problem.

There is an exponential growth if k > 0 and an exponential decrease if k 0.

Let's use a known initial time value, . y(t_0), which equals y_0 as evidence. The differential equation

$$\frac{\mathrm{dy}}{\mathrm{dt}} = ky$$

is a separate differential equation that we are able to solve.

$$\frac{\mathrm{dy}}{\mathrm{dt}} = ky$$

$$\Rightarrow \frac{\mathrm{dy}}{\mathrm{y}} = kdt$$

$$\Rightarrow \int \frac{\mathrm{dy}}{\mathrm{y}} = \int kdt$$

$$\Rightarrow \ln y = kt + c$$

$$\Rightarrow e^{\ln y} = e^{kt+c}$$

$$\Rightarrow y = e^c e^{kt}$$

$$\Rightarrow y = c_1 e^{kt}; c_1 = \pm e^c$$

Using the initial condition:

y(0) = y_0, i.e t_0 = 0, y = y_0

$y_{o=c_1e^0} \Rightarrow y_0 = c_1$

$y = y_0 e^{kt}$

We need further conditions, which may be provided in the problem, to determine the additional constant k..[6][7]

4.1.2 The Malthusian theory of population growth

A single species' population at time t is denoted by N(t) .Followed by the rate of change

$$\frac{dN}{dt} = \text{births - deaths + migration} \qquad (3.3)$$

is a population conservation equation.

The birth and death terms in the most basic model are proportionate to N, and there is no migration. Therefore, (3.3) takes the form

$$\frac{\mathrm{dN}}{\mathrm{dt}} = \mathrm{bN} - \mathrm{dN} \qquad (3.4)$$

Where b and d are positive constants

$$\Rightarrow \frac{dN}{dt} = (b\text{-}d)N$$

$$\Rightarrow \frac{dN}{dt} = (b\text{-}d)dt$$

$$\Rightarrow \int \frac{1}{N} dN = \int (b-d)dt$$

$$\Rightarrow \ln N = (b-d)t + \ln A, \ln A \; is\; a\; constant$$

$$\Rightarrow \ln\left(\frac{N}{A}\right) = (b-d)t$$

$$\Rightarrow \frac{N}{A} = e^{(b-d)t}$$

$$\Rightarrow N = Ae^{(b-d)t}$$

$$\Rightarrow N(t) = Ae^{(b-d)t} \tag{3.5}$$

At t = 0, let the initial population N(0) = No.

Then from (3.5) we have

$$N(0) = Ae^{0} = A$$

$$\Rightarrow A = N(0) = No$$

Putting the value of A in (3) we get,

$$N(t) = Noe^{(b-d)t} \tag{3.6}$$

This is the Malthusian model for population growth.[7][8]

4.1.3 Logistic population growth

A single species' population at time t is denoted by N(t) .Followed by the rate of change

$$\frac{dN}{dt} = \text{births - deaths + migration} \tag{3.7}$$

is a population conservation equation.

The birth and death terms in the most basic model are proportionate to N, and there is no migration. Therefore, (3.7) takes the form

$$\frac{\mathrm{dN}}{\mathrm{dt}} = \mathrm{bN} - \mathrm{dN} \tag{3.8}$$

Where b and d are positive constants

$$\Rightarrow \frac{dN}{dt} = (b\text{-}d)N$$

$$\Rightarrow \frac{dN}{dt} = (b\text{-}d)dt$$

$$\Rightarrow \int \frac{1}{N} dN = \int (b-d)dt$$

$$\Rightarrow \ln N = (b-d)t + \ln A, \ln A \; is\; a\; constant$$

$$\Longrightarrow \ln\left(\frac{N}{A}\right) = (b-d)t$$
$$\Longrightarrow \frac{N}{A} = e^{(b-d)t}$$
$$\Longrightarrow N = Ae^{(b-d)t}$$
$$\Longrightarrow N(t) = Ae^{(b-d)t} \qquad (3.9)$$

At t = 0, let the initial population N(0) = No.
Then from (3.5) we have

$$\mathrm{N(0)}= \mathrm{A}e^{0}=\mathrm{A}$$
$$\Longrightarrow \mathrm{A}= \mathrm{N(0)} = \mathrm{No}$$

Putting the value of A in (3) we get,

$$\mathrm{N(t)}= \mathrm{No}e^{(b-d)t} \qquad (3.10)$$

This is the Malthusian model for population growth
The logistic model is

$$\frac{\mathrm{dN}}{\mathrm{dt}} = \mathrm{rN}\left(1-\frac{\mathrm{N}}{\mathrm{K}}\right) \qquad (3.11)$$

Where r and K are positive constants.

In this model the per capita birth rate is $\mathrm{r}\left(1-\frac{\mathrm{N}}{\mathrm{K}}\right)$ and K is the carrying capacity of the environment.[8][9]

3.1.4 Mathematical problem solve

Solved problem 1. Solve A specific bacterial culture develops at a rate proportional to its size. Calculate how long it will take the culture to grow to be 10 times larger than it was originally if the size doubles in 4 days.

Solution: Let P(t) be the size of the culture after t days.

$$\frac{\mathrm{dp}}{\mathrm{dt}} = kP \qquad (1)$$

Finding the arbitrary constant c will be done using the initial condition P(0) =P_0, and finding the additional constant k will be done using the additional condition P(4) = 2P_0.

: We have, p = ce^{kt} from the initial condition P(0) = P_0 i.e. t = 0, P= P_0 we will have arbitrary constant c, (thus P_0 = ce^{o}) c = P_0 ;

Hence, we have P= P_0 e^{kt}:

Now by using the additional condition P(4) = 2P_0 i.e.

When t = 4, we can determine the new constant k.

$$2P_0 = P_0\, e^{4k}$$

$$\Rightarrow e^{4k} = 2$$

$$\Rightarrow \ln e^{4k} = \ln 2$$

$$\Rightarrow 4k = \ln 2$$

$$\Rightarrow k = \frac{\ln 2}{4} \simeq 0{:}\,173$$

As a result, the time needed for the culture to double in size and back to its original capacity may be calculated from

$$10P_0 = P_0\, e^{o.173t}$$

$$\Rightarrow e^{o.173t} = 10$$
$$\Rightarrow \ln(e^{o.173t}) = \ln 10$$
$$\Rightarrow t = \frac{\ln 10}{0.173}$$
$\Longrightarrow$ t $\simeq$13:31 days.

Solved problem 2. Solve Model the world's population in the second half of the 20th century using the figures of 2560 million in 1950 and 3040 million in 1960. Suppose that the growth rate is inversely proportional to the population. The proportional growth rate is k, what is it? Use the model to calculate the global population in 2020 and estimate its amount in 1993.

Solution: In millions, we express the population p(t).

We've got

$$\frac{\mathrm{dp}}{\mathrm{dt}} = \mathrm{kp}$$
$$\Rightarrow \mathbf{p} = \mathbf{ce^{kt}}$$

and here is the initial condition.

$$p(t_0) = p_0$$
$$\Rightarrow p(0) = 2560$$

thus, we can find the arbitrary constant c

$$p = ce^{kt}$$
$$\Rightarrow p(0) = ce^{0}$$
$$\Rightarrow\ 2560 = c:$$

In order to determine the additional constant k (the relative growth rate), we will now apply the additional condition. p(10) = 3040.

$$\Longrightarrow p = ce^{kt}$$
$$\Rightarrow\ 3040=2560e^{10k}$$
$$\Rightarrow e^{10k} = \frac{3040}{2560}$$
$$\Rightarrow \ln\left(e^{10k}\right) = \ln\left(\frac{3040}{2560}\right)$$
$$\Rightarrow 10k=\ln 1.1875$$
$$\Rightarrow k = \frac{\ln 1.1875}{10} \simeq 0.01785.$$

The model's relative growth rate is approximately 1.7% yearly.

$$p(t) = 2560e^{0.01785t}:$$

Using the model, we determine what the world's population was in 1993.

$$p(t) = 2560e^{0.01785t}$$
$$\Rightarrow p(45) = 2560e^{0.01785(45)} \simeq 5360 \text{ milion}$$

The model predicts that the population in 2020 will be

$$p(70) = 2560e^{0.01785(70)} \simeq 5360 \text{ milion}$$

3.2 Temperature Problems

3.2.1 Newton's law of cooling:

The time rate of change of a body's temperature is proportional to the temperature differential between the body and its surrounding medium, according to Newton's law of cooling, which is also generally relevant to heating.

Let T and T_m stand for the body's and the girding medium's respective temperatures, respectively.
In addition, Newton's law of cooling can be expressed as: where: $\frac{dT}{dt}$ is the time rate of change of the body's temperature, and

$$\frac{dT}{dt} = -k(T - T_m), \text{ or as}$$

$$\frac{dT}{dt} + kt = kT_m \tag{3.12}$$

where k is a positive proportionality constant. The negative sign is necessary in Newton's law after k is selected as positive in order to make $\frac{dT}{dt}$ negative in a cooling system when T is more than Tm and positive in a heating process when T is less than T_m.[3]

3.2.2 Mathematical problem solve

Solved problem 1. When placed in a room where the temperature is 30°C, a glass of hot water that is initially 80°C in temperature. The water's temperature drops to 70°C after one minute. What temperature will there be in three minutes? When does the water reach 40 °C?

Solution: We have the Newton's Law of Cooling is given from

$$\frac{dT}{dt} = -k(T - T_m), \text{ or as}$$

$$\frac{dT}{dt} + kt = kT_m$$

Additionally, we have the initial condition T(0) = 80°C, the ambient temperature Ts = 30°C, and the additional condition T(1) = 70°C.

$$T = T_m + ce^{kt}$$

$$\Rightarrow T(0) = 30 + ce^0$$

$$\Rightarrow 80 = 30 + c$$

$$\Rightarrow c = 50$$

Thus

$$T = 30 + 50e^{kt}$$

Now, using the additional condition, we will calculate the additional constant k.

$$T(1) = 70°C$$
$$T(1) = 30 + 50e^{k}$$
$$\Rightarrow 70 = 30 + 50e^{k}$$
$$\Rightarrow e^{k} = \frac{40}{50}$$
$$\Rightarrow \ln\left(e^{k}\right) = \ln\left(\frac{40}{50}\right)$$
$$\Rightarrow k = \ln(0.8)$$

As a result, to determine the water's temperature at any given time,

$$T(t) = T_{m} + ce^{\ln(0.8)t}$$

As a result, the water's temperature is known when t =3.

$$T(3) = 30 + ce^{3\ln(0.8)}$$
$$\Rightarrow T(3) = 30 + 25.6 = 55.6°C$$

Finding the time at which the water cools to 40°C is the next step.

$$T = 30 + 50e^{kt}$$
$$\Rightarrow 40 = 30 + 50e^{kt}$$
$$\Rightarrow e^{\ln(0.8)t = \frac{10}{50}}$$
$$\Rightarrow \ln\left(e^{\ln(0.8)t}\right) = \ln(.2)$$
$$\Rightarrow \ln(.8)t = \ln 0.2$$
$$\Rightarrow t = \frac{\ln 0.2}{\ln 0.8} \simeq 7.2 \text{ mins}$$

3.3 Electrical Circuits

The initial equation defining the electromotive force (short e.m.f.) E (in volts) and the resistance R (in ohms) and inductor L (in henries) in a simple RL circuit is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{R}{L} \qquad (3.13)$$

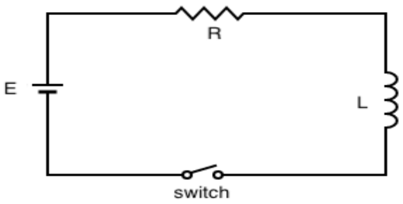
The equation defining the quantum of electrical charge q (in coulombs) on the capacitor for an RC circuit consisting of a resistance, a capacitance C (in farads), an e.m.f., and no inductance is

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R} \qquad (3.14)$$

Q and I are related in the following way:

$$I=\frac{dQ}{dt} \tag{3.15}$$

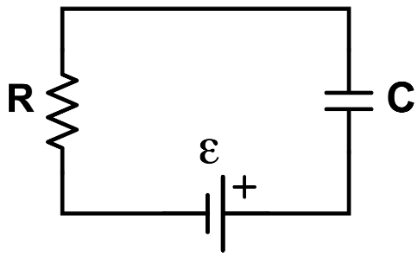
3.3.1 LR series circuit:

	$L\frac{di}{dt}+Ri=E(t)$ **i(t) =current** **L= Inductor** **R= Resistance** **E(t)= Impressed voltage**

Kirchhoff's second law states that the amount of voltage dissipated by an inductor ($L\frac{di}{dt}$) and a resistor (iR) in a series circuit consisting just of these two components is equal to the circuit's induced voltage E(t).

3.3.2 RC series circuit :

The term "RC Series Circuit" refers to a circuit in which a pure capacitor of capacitance C farads is connected in series with pure resistance R ohms. When a sinusoidal voltage is provided, current I flows through the circuit's capacitance (C) and resistance (R).

	$R\frac{dq}{dt}+\frac{1}{c}q=E(t)$ **$q(t)$= charge** **C= capacitance** **R= Resistance**

[4][3]

3.3.3 Solved Problems

Solved problem 1. An LR-series circuit with an inductance of 0.1 Henry and a resistance of 50 ohms is subjected to a 30-volt electromotive force. Seek out the current i(t) if. Calculate t to be the current.

Solution: LR-series circuit DE is

$$L\frac{di}{dt} + Ri = E(t) \tag{1}$$

Given that, Inductor L=0.1 henry the resistance R=50 ohms and E(t)=30 volts.

We have to find the current i(t) at any time t for the initial value i(0)=0.

For this, we need to solve IVP

$$0.1\frac{di}{dt} + 50i = 30; i(0) = 0 \tag{2}$$

S.F. of the DE in (2):

$$\frac{di}{dt} + 5001i = 300 \tag{3}$$

Multiply (3) I.F $e^{\int 500dt}$=e^{500t}

$$\frac{di}{dt}e^{500t} + 500e^{500t}i = 300e^{500t}$$

$$\Rightarrow \frac{d}{dt}[i.e^{500t}] = 300e^{500t}$$

$$\Rightarrow \int d[i.e^{500t}] = \int 300e^{500t}\,dt$$

$$\Rightarrow\ i.e^{500t} = \frac{3}{5}e^{500t}\text{+c}$$

$$\Rightarrow i(t) = \frac{3}{5} + ce^{-500t} \tag{4}$$

Using the i.c. $i(0) = \frac{3}{5} + ce^{0}$

$$\Rightarrow c = -\frac{3}{5}$$

$(4) \Rightarrow\ i(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}$ and as t→∞;

i$(t) \to \frac{3}{5}$.

Solved problem 2: An electromotive force of 100 volts is supplied to an RC-series circuit with a 200 ohm resistance and a 10^{-4}F capacitance. If q(0) = 50, determine the charge q(t) on the capacitor. Discover the most recent I (t).

Solution: We have a 100-volt electromotive force in a RC circuit in series with a Resistance R=200 ohms and a capacitance 10^{-4}farad, and then we have the differential equation for Charge(q) as

$$R\frac{dq}{dt}+\frac{1}{c}q = E(t) \qquad (1)$$

and we have to obtain the charge q(t) on the capacitor if we have the initial charge as q(0)=0 as the following:

Since we have R=200ohms,C=10^{-4}farad and E=100 volt, then we have the differential equation shown in (1) as

$$200\frac{dq}{dt}+\frac{1}{10^{-4}}q = 100$$
$$\Rightarrow \frac{dq}{dt}+50q = 0.5$$
$$\Rightarrow \frac{dq}{dt} = 0.5-50q \qquad (2)$$

It is possible to separate the variables in this equation since it is a first order linear and separable differential equation.

$$\frac{dq}{0.5-50q} = dt$$
$$\Rightarrow \int\frac{dq}{0.5-50q} = \int dt$$
$$\Rightarrow -\frac{1}{50}\int\frac{-50\,dq}{0.5-50q} = \int dt$$
$$\Rightarrow -\frac{1}{50}\, ln(0.5-50q) = t+c_1$$
$$\Rightarrow ln(0.5-50q) = -50t+c_2$$
$$\Rightarrow e^{ln(0.5-50q)} = e^{(-50t+c_2)}$$

$\Rightarrow$ 0.5-50q $= e^{c_2} e^{(-50t)}$
$\Rightarrow$ 0.5-50q $= ce^{(-50t)}$
$\Rightarrow 50q = 0.5 - ce^{(-50t)}$

Then we have

$$q(t) = \frac{1}{100} - ke^{(-50t)}$$

(3) After that, to find the value of constant k,we have to apply this point of condition (q,t)= (0 C , 0 min) into equation (3) as

$$q(0) = \frac{1}{100} - ke^{0}$$

Then we have

$$k = \frac{1}{100}$$

After that, substitute with the value of k into equation (3), then we have

$$q(t) = \frac{1}{100} - \frac{1}{100} e^{(-50t)} \quad (4)$$

is the charge on the capacitor of RC series circuit at time t.

Also, we have to obtain the current of this circuit as using equation (4) as

$$i(t) = \frac{dq}{dt}$$

$$= \frac{d\left(\frac{1}{100} - \frac{1}{100} e^{(-50t)}\right)}{dt}$$

$$i(t) = \frac{1}{2} e^{(-50t)}$$

Charge is: $q(t) = \frac{1}{100} - \frac{1}{100} e^{(-50t)}$

Charge is : $i(t) = \frac{1}{2} e^{(-50t)}$

3.4 Orthogonal Trajectories

Let's say we have a family of curves with the formula

F(x, y, and c) = 0. (3.16)

as well as an other family of curves denoted by

G(x, y, k) = 0; (3.17)

So that the tangents of the curves are perpendicular at every point where a curve from the family F(x, y, c) and a curve from the family G(x, y, k) = 0 intersect.
Hence, two families of curves always intersect perpendicularly.

3.4.1 How to Find Orthogonal Trajectories

To determine the family's orthogonal trajectories

$$F(x; y; c) = 0; \qquad (3.18)$$

Step 1: Implicitly differentiate (3) with respect to x to obtain a relation of the type (3)

$$g\left(x, y, \frac{dy}{dx}, c\right); \qquad (3.19)$$

Step 2: Remove the parameter c from (3) and (4) to produce the differential equation.

$$F\left(x, y, \frac{dy}{dx}, c\right); \qquad (3.20)$$

Corresponding to the first family (3);

Step 3: Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ by in (5) to obtain the differential equation

$$H\left(x, y, \frac{dy}{dx}\right); \qquad (3.21)$$

On the basis of the orthogonal trajectories (as depicted in the picture below), Step 4: The general solution of equation (6) results in the necessary orthogonal trajectories.

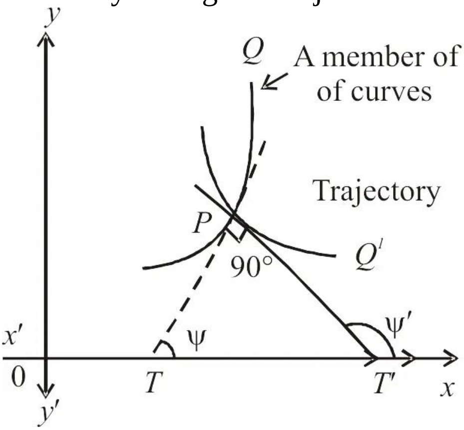


Figure : orthogonal trajectories [5]

3.4.2 Solved Problems

Solved problem 1: Discover the orthogonal trajectories that a family of straight lines takes through the origin.

Solution: It is stated by that the family of lines passing through the origin is

$$y = kx \tag{1}$$

We take the preceding four actions to get the orthogonal trajectories:

Step 1: Apply implicit differentiation to (1) with regard to x to get

$$\frac{dy}{dx} = k; \tag{2}$$

Step 2: Remove the parameter k from (1) and (2) to obtain the differential equation.

$$\frac{dy}{dx} = \frac{y}{x}; \tag{3}$$

This gives the differential equation of the family (1).

Step 3: Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ by in (3) to obtain the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}; \tag{4}$$

Step 4: Solving differential equation (4), we obtain

$$x^2 + y^2 = c \tag{5}$$

The family of straight lines' orthogonal trajectories through the origin are thus given by (11). A family of circles with the center at the origin is represented by the number 11.

Solved problem 2 : Find the orthogonal trajectories of the family

$$cx^2 - y^2 = 1; \tag{1}$$

Solution: We take the preceding four actions to get the orthogonal trajectories:

Step 1: Apply implicit differentiation to (1) with regard to x to get

$$2cx - 2y\frac{dy}{dx} = 0; \tag{2}$$

Step 2: Remove the parameter c from (1) and (2) to obtain the differential equation.

$$c = \frac{1+y^2}{x^2};$$

Thus, we obtain the differential equation

$$\frac{dy}{dx} = \frac{1+y^2}{xy}; \tag{3}$$

This gives the differential equation of the family (1).

Step 3: Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ by in (3) to obtain the differential equation

$$\frac{dy}{dx} = \frac{-xy}{1+y^2}; \tag{4}$$

Step 4: When we use the method of variable separation to solve differential equation (4), we get

$$\int \frac{1+y^2}{y} = -\int x\,dx$$

$$\Rightarrow \ln y + \left(\frac{y^2}{2}\right) = \left(-\frac{x^2}{2}\right) + c_1$$

$$\Rightarrow 2\ln y + y^2 + x^2 = c_{1;} \qquad (5)$$

As a result, the necessary equation for orthogonal trajectories is defined by (4).

CONCLUSION

Ordinary differential equations are frequently utilized in mechanics, astronomy, physics, and a number of chemistry and biology problems. They have a wide range of applications and are an effective tool in the study of a variety of problems in the natural sciences and technology.

Differential equations are widely used in engineering and scientific applications. It appears in a wide range of engineering applications, such as electromagnetic theory, signal processing, computational fluid dynamics, etc. The usual approaches for solving these equations are analytical or numerical. Since many differential equations that arise in real-world applications cannot be solved analytically, we can claim that there is no analytical solution for certain differential equations. There are certain numerical techniques in the literature that can be used for these kinds of issues.

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