University Of Khartoum
Faculty of Mathematical Science and informatics
Statistics Department

Graduation Project for the year 2024

Data Envelopment Analysis
Vs
Measuring relative Efficiency

Conducted by:

Albasheer Mohammed Abdallah Mostafa (S 16-101)

Mohamed Shihab Hassan Alhag Yousif (CS 16-208)

Osama Elhag Yousif Elhag (CS 16-204)

Supervised by:

Dr. Abdalfattah Abdalmmuti Muhammed

TABLE OF CONTENTS

Introduction 2

(I.1) PROBLEM STATEMENT 2

(I.2) Motivation 2

(I.3) LITERATURE REVIEW 3

Mathematical Model 5

(II.1) Mathematical Model 5

(II.2) fractional programming 6

(II.3) Fractional Programming: A Measure for Relative Efficiency 6

(II.4) Transforming Fractional Programming to a Linear Programming Model 7

Practical Application 9

Conclusion 20

Reference 21

CHAPTER ONE
INTRODUCTION

(I.1) PROBLEM STATEMENT

Data Envelopment Analysis (DEA) application for measuring efficiency in banking

DEA is a non-parametric method used to evaluate the efficiency of decision-making units (DMUs) such as banks. This approach can be applied to assess the relative efficiency of banks by analyzing their inputs (e.g. labor, capital) and outputs (e.g. loans, deposits). Measuring bank efficiency using DEA can provide insights to improve performance and operations.

(I.2) Motivation

i. Handling Multiple Inputs and Outputs:
- Banks utilize a variety of inputs, such as labor, capital, and deposits, to generate multiple outputs, including loans, investments, and other services.
- DEA is well-suited to handle this complex, multiple-input, multiple-output environment, which is common in the banking sector.

ii. The Importance of Efficiency Measurement:
- Measuring the efficiency of banks is crucial for identifying best practices, allocating resources, and improving overall performance.
- DEA provides a robust framework for evaluating the relative efficiency of banks, considering their diverse operations and objectives.

iii. Navigating Competitive Environment:
- The banking industry is highly competitive, with banks facing increasing pressure to improve their efficiency and profitability.
- DEA can be used to benchmark the performance of banks, identify efficient institutions, and provide insights for improving the efficiency of less-efficient banks.

iv. Regulatory Considerations:
- Banks operate in a heavily regulated environment, with regulatory authorities focusing on the financial stability and soundness of the banking system.
- DEA can be used to assess the efficiency of banks, which can be relevant for regulatory purposes and policy decisions.

v. Managerial Implications:
- The efficiency scores and benchmarking information provided by DEA can be valuable for bank managers in making informed decisions about resource allocation, process improvements, and strategic planning.

(I.3) LITERATURE REVIEW

Data Envelopment Analysis (DEA) is a mathematical programming technique that has found a number of practical applications
for measuring the performance of similar units, such as a set of
hospitals, a set of schools, a set of banks, etc. Ramanathan , 2003 [7].

(I.3.1) The development of efficiency measurement

The fundamental efficiency measure used in DEA is the ratio of total outputs to total inputs, as expressed by the simple formula: Efficiency = Output / Input [7]. This basic concept of efficiency measurement was initially applied in the simplest case of a single input and a single output.

The tool of Graphical Descriptive Frontier Analysis was then developed to handle the case of a single output and two inputs, allowing for a graphical representation of the efficiency frontier [7].

Building upon these foundational concepts, the seminal work by Charnes, Cooper, and Rhodes (CCR) in 1978 [3] led to the development of the basic CCR DEA model. This model extended the original efficiency equation by incorporating multiple inputs and outputs, while maintaining the core principle of maximizing the ratio of weighted outputs to weighted inputs Ayadi, 1998. [1]

Further extending the CCR model, Banker, Charnes, and Cooper (BCC) introduced the BCC DEA model in 1984. The BCC model complemented the CCR model by incorporating additional criteria to measure input excesses and output shortfalls Cooper, W. W., Seiford, L. M., & Tone, k. 2006 [4], providing a more comprehensive approach to efficiency evaluation Zhu, 2006. [8]

This evolution from the basic single-input, single-output concept to the more sophisticated CCR and BCC DEA models has enabled the widespread application of DEA in various industries, including the banking sector, where the analysis of multiple inputs and outputs is crucial for assessing the relative efficiency of financial institutions.

(I.3.2) Application To Banking Efficiency

Banking as a financial institution needs to have a good performance. Moreover, the increasingly fierce competition in the banking business requires banks to improve their performance in order to attract investors before investing their funds. One of the indicators used to measure bank performance is efficiency.

Efficiency measurement can provide information about the condition of bank performance that is relevantly needed in decision-making.

Efficiency in banking can be measured by the comparison between the output produced and the input it has. Banking can be said to be efficient compared to its competitors if it can produce a larger output with fixed inputs or produce a fixed amount of output using fewer inputs. Efficient commercial banks can certainly attract investors to invest where efficient banks indicate that banking performance is going well. The term efficiency comes from the field of engineering, which is used to indicate the ratio between the outputs of a system to the input of the system.

According to [8], overall productivity of a bank depends on four components of efficiency classification as shown in Figure 1 and they are:

i. Technical efficiency: Also known as global efficiency, it measures the ability of banks to produce actual outputs with fewer inputs, or less resources used indicates higher efficiency.
ii. Scale efficiency: Refers to the optimal activity volume level whereby inefficiency may arise if goods or services are produced above or below optimal level that resulted in added fixed cost.

iii. Price efficiency: A bank could increase its efficiency if it could purchase the inputs (human capital and material) at lower prices without sacrificing the quality Ong, H. K., et al. 2003 [5]

iv. Allocative efficiency: Measures the optimal mix of several inputs in order to produce products or services, such as banks incorporating automatic teller machines (ATM) and Internet banking for capital-labour tradeoffs to increase efficiency [5].

CHAPTER TWO
MATHEMATICAL MODEL

(II.1)Mathematical Model

Notation

Let:

- K = Number of DMUs
- N = Number of inputs
- M = Number of outputs
- I_{ik} = Value of input i for DMU k
- O_{jk} = Value of output j for DMU k
- x_{ik} = Weight assigned to input i for DMU k
- y_{jk} = Weight assigned to output j for DMU k
- E_k = Efficiency score of DMU k

Efficiency Calculation

The efficiency of DMU k Beasley, n.d [2]. is given by:

Objective Function

$$E_k = \frac{\sum_{j=1}^{M} y_{jk} O_{jk}}{\sum_{i=1}^{N} x_{ik} I_{ik}}$$

Where:

The goal is to maximize the efficiency of each DMU k:

$$\max E_k$$

Constraints

i. Efficiency Bound

The efficiency of every DMU (including k) must not exceed 1:

$$\frac{\sum_{j=1}^{M} y_{jk} O_{jk'}}{\sum_{i=1}^{N} x_{ik} I_{ik'}} \leq 1, \qquad \forall k' = 1,2,\dots,K$$

ii. Non-Negativity

All weights must be non-negative:

$$x_{ik} \geq 0, \qquad \forall i, k$$

$$y_{jk} \geq 0, \qquad \forall j, k$$

I. We need to explain that this program actually provides a measure for relative efficiency.
II. Then show, in detail, how the above FR can transformed perfectly to a linear programming model

(II.2) fractional programming

Fractional programming is an optimization technique used to solve problems where the objective function is a ratio of two functions, typically a ratio of linear functions [7].

(II.3) Fractional Programming: A Measure for Relative Efficiency

By optimizing the ratio above, Fractional programming seeks to find the decision variable values that maximize (or minimize) the relative performance or efficiency.
The ratio captures the relative efficiency, i.e., the output or benefit obtained per unit of input or resource consumed.

(II.4) Transformation Fractional Programing To Linear Programming Model

It is generally difficult to solve fractional programs. If they are converted to simpler formulations, such as the linear programming (LP) formats, then they can be solved easily. The simplest way to convert these fractional programs to linear programs is to normalize either the numerator or the denominator of the fractional programming objective function [2].

- ### Reformulation Steps

Step 1: Maximize the Weighted Output

Instead of maximizing the ratio, the problem is transformed to maximize only the **weighted sum of outputs**:

$$\max \sum_{j=1}^{M} y_{jk} O_{jk}$$

Step 2: Normalize the Input Weights

The weighted sum of inputs is set to 1:

$$\sum_{i=1}^{N} x_{ik} I_{ik} = 1$$

Step 3: Reformulate the Constraints

The original constraint ensuring efficiency for all DMUs is rewritten as:

$$\sum_{j=1}^{M} y_{jk} O_{jk'} \leq \sum_{i=1}^{N} x_{ik} I_{ik'}, \qquad \forall k' = 1,2, \dots, K$$

This ensures that when using the weights of k, no other DMU exceeds an efficiency score of 1.

- **Final Linear Programming Model and general output maximization CCR DEA model can be represented as follows:**

For each DMU $k' = 1,2,\dots,K$, solve the following **LP problem**:

- **Objective Function**

$$\max \sum_{j=1}^{M} y_{jk} O_{jk}$$

- **Subject to Constraints**

Input Normalization

$$\sum_{i=1}^{N} x_{ik} I_{ik} = 1$$

Efficiency Constraints

$$\sum_{j=1}^{M} y_{jk} O_{jk'} \leq \sum_{i=1}^{N} x_{ik} I_{ik'}, \qquad \forall k' = 1,2,\dots,K$$

Non-Negativity

$$x_{ik} \geq 0, \qquad \forall i, k$$

$$y_{jk} \geq 0, \qquad \forall j, k$$

CHAPTER THREE
PRACTICAL APPLICATION

Data Envelopment Analysis (DEA) Project: Efficiency of Bank Branches

Project Specifications

The number of decision making units (DMUs): 5

The number of inputs: 2

The number of outputs: 2

The model: Basic Radial Model

Orientation: Input oriented

Return to Scale: Constant return to scale

- ### Define The Decision-Making Units (DMUs)

  · The DMUs are the individual bank branches.

- ### Inputs and Outputs

  #### Inputs

  1. Number of Staff: Total employees working in the branch.

  2. Total Branch Expenses: All operational costs, including salaries, utilities, and rent.

  #### Outputs

  1. Number of Accounts Opened: Total new customer accounts created within a specific period.

  2. Total Value of Loans Disbursed: The monetary value of loans granted by the branch.

- **Data**

Branch	Staff	Expenses (£'000)	Accounts Opened	Loans Disbursed (£'000)
bank of khartoum Sinja branch	20	250	120	400
bank of khartoum Rabak branch	25	300	150	500
bank of khartoum Gadaref branch	15	200	100	350
bank of khartoum Kosti branch	30	350	180	600
bank of khartoum Al-manaqil branch	22	280	130	450

- **Solving the LP**

Using MCDM Software OnlineOutput software for DEA. (n.d.), we solve the linear program for each branch.

Using linear programming, we calculate the efficiency scores for each branch. The model optimizes weights for inputs and outputs to maximize efficiency while ensuring the efficiency scores of all other branches remain ≤ 1.

- **Efficiency**

The value of efficiency obtained by the defined model is presented in Table 1. In addition to the value of efficiency, its type can also be shown in this table.

Table 1: Efficiency

Branch	Efficiency	Status
bank of khartoum Sinja branch	0.947368421	Inefficient
bank of khartoum Rabak branch	0.97826087	Inefficient
bank of khartoum Gadaref branch	1	Efficient
bank of khartoum Kosti branch	1	Efficient
bank of khartoum Al-manaqil branch	0.922619048	Inefficient

- **Reference Set**

In each linear program of DEA, the solution technique will attempt to make the efficiency of the target unit as large as possible. This search procedure will terminate when either the efficiency of the target unit or the efficiency of one or more other units is equal to one. Therefore, for an inefficient unit at least one other unit has the efficiency equal to 1 with the same weight of the target unit obtained from the solution of the model. These efficient units are known as the peer group for the inefficient unit. Table 2 shows the peers.

Table 2: References

Branch	Peer 1	Peer 2
bank of khartoum Sinja branch	bank of khartoum Gadaref branch	bank of khartoum Kosti branch
bank of khartoum Rabak branch	bank of khartoum Gadaref branch	bank of khartoum Kosti branch
bank of khartoum Gadaref branch	bank of khartoum Gadaref branch	-
bank of khartoum Kosti branch	bank of khartoum Kosti branch	-
bank of khartoum Al-managil branch	bank of khartoum Gadaref branch	bank of khartoum Kosti branch

- **λ (Weights for Peer Units)**

Interpretation:

λ values indicate how an inefficient branch can be represented as a combination of efficient branches. A higher λ for a peer unit means it serves as a stronger reference for improving efficiency.

- **Efficient branches (Gadaref & Kosti) have λ = 1**, meaning they are self-sufficient benchmarks.
- **Inefficient branches (Sinja, Rabak, Al-manaqil) use a weighted mix of Gadaref and Kosti to construct a virtual efficient branch.**
- **Example:** Sinja branch's efficiency is modeled **63.2% on Gadaref and 31.6% on Kosti**, suggesting it should adopt best practices from these peers to improve efficiency.

The values of λ are presented in Table 3.

Table 3: Lambdas

Branch	Sinja	Rabak	Gadaref	Kosti	Al-manaqil
bank of khartoum Sinja branch	0	0	0.632	0.316	0
bank of khartoum Rabak branch	0	0	0.326	0.652	0
bank of khartoum Gadaref branch	0	0	1	0	0
bank of khartoum Kosti branch	0	0	0	1	0
bank of khartoum Al-manaqil branch	0	0	1	0.167	0

- **Weights (values of the variables for the primary model)**

Tables 4 and 5 DEA assigns **weights** to inputs and outputs to measure their contribution to efficiency.

Tables 4 **Input Weights** show how much each input affects efficiency.

- **Higher input weights** indicate a stronger impact on efficiency.
- **Higher output weights** indicate that these outputs contribute more to efficiency.

Table 4: Input Weights

Input Weights show how much each input affects efficiency.

Branch	Input 1	Input 2
bank of khartoum Sinja branch	0.011	0.003
bank of khartoum Rabak branch	0.009	0.003
bank of khartoum Gadaref branch	0.013	0.004
bank of khartoum Kosti branch	0.007	0.002
bank of khartoum Al-manaqil branch	0	0.004

Table 5: Output Weights

Weights show the impact of outputs on efficiency.

Branch	Output 1	Output 2
bank of khartoum Sinja branch	0.008	0
bank of khartoum Rabak branch	0.007	0
bank of khartoum Gadaref branch	0.01	0
bank of khartoum Kosti branch	0.006	0
bank of khartoum Al-manaqil branch	0.003	0.001

- ## Input and Output slacks

Input and Output slacks related to each unit are provided in Tables 6 and 7 respectively.

Table 6: Input Slacks

Slacks measure excess inputs that do not contribute to output efficiency.

Branch	**Input 1**	**Input 2**
bank of Khartoum Sinja branch	0	0
bank of Khartoum Rabak branch	0	0
bank of Khartoum Gadaref branch	0	0
bank of Khartoum Kosti branch	0	0
bank of Khartoum Al-manaqil branch	0.298	0

EX: The Al-manaqil branch has excess staff (0.298), meaning it can reduce staff without affecting efficiency.

Table 7: Output Slacks

Branch	Output 1	Output 2
bank of Khartoum Sinja branch	0	10.526
bank of Khartoum Rabak branch	0	5.435
bank of Khartoum Gadaref branch	0	0
bank of Khartoum Kosti branch	0	0
bank of Khartoum Al-manaqil branch	0	0

EX: Sinja and Rabak branches need to increase loan disbursement to reach efficiency.

- **Target Values**

Tables 8,9 present the actual and target values of each input.

Table 8:

Branch	Input 1	Input 2
bank of khartoum Sinja branch	20	250
bank of khartoum Rabak branch	25	300
bank of khartoum Gadaref branch	15	200
bank of khartoum Kosti branch	30	350
bank of khartoum Al-manaqil branch	22	280

Table 9: Target Inputs

Targets suggest reducing input usage for inefficient branches.

Branch	Input 1	Input 2
bank of khartoum Sinja branch	18.947	236.842
bank of khartoum Rabak branch	24.457	293.478
bank of khartoum Gadaref branch	15	200
bank of khartoum Kosti branch	30	350
bank of khartoum Al-manaqil branch	20	258.333

- Tables 10,11 present the actual and target values of each output.

Table 10: Outputs

Branch	Output 1	Output 2
bank of khartoum Sinja branch	120	400
bank of khartoum Rabak branch	150	500
bank of khartoum Gadaref branch	100	350
bank of khartoum Kosti branch	180	600
bank of khartoum Al-manaqil branch	130	450

Table 11: Target Outputs

Targets suggest increasing outputs to improve efficiency.

Branch	Output 1	Output 2
bank of khartoum Sinja branch	120	410.526
bank of khartoum Rabak branch	150	505.435
bank of khartoum Gadaref branch	100	350
bank of khartoum Kosti branch	180	600
bank of khartoum Al-manaqil branch	130	450

Analysis result:

- **Gadaref and Kosti branches** are **efficient** and serve as models.
- **Sinja, Rabak, and Al-manaqil branches** need to **reduce staff and expenses** while **increasing loan disbursement** to improve efficiency.
- The results guide **bank managers** in optimizing operations for better performance.

Conclusion

First, this study examined the DEA technique-one of the most powerful non-parametric methods to estimate the efficiency of DMUs-and underlined its application, especially in banks. Then, we introduced a practical method for evaluating relative efficiency concerning a transformation from fractional programming to linear programming for computational feasibility. Such a transformation from fractional models provided the means necessary to show their complexity and yielded a well-structured efficiency analysis.

The results obtained through this study thus prove that DEA is an efficient method for the identification of efficient and inefficient bank branches, hence providing valuable information on performance gaps. Efficiency scores will reflect areas where resource allocation, cost management, and operational strategies could be improved. Inefficient branches can benchmark their performance against efficient peers by adopting best practices that will improve productivity and financial sustainability.

All these applications are very broad, extending beyond banking to education, healthcare, and corporate management, among many others, where there is a need for simultaneous analysis of several inputs and outputs.

In conclusion, DEA is an indispensable tool in performance evaluation since it offers a quantitative, objective, and data-driven approach to the measurement of efficiency. From these insights provided by DEA, decision-makers can make strategic improvements, optimize resource utilization, and improve competitive positioning in an increasingly dynamic and complex business environment.

References

1. **Ayadi, F.** (1998). *A DEA Model for the Analysis of Efficiency of Banks in Tunisia*. 1st ed. Tunis, Tunisia: [Publisher Name Not Provided].

2. **Beasley, J. E.** (n.d.) "DEA - Data Envelopment Analysis." Brunel University, https://people.brunel.ac.uk/~mastjjb/jeb/or/dea.html.

3. **Charnes, A., Cooper, W. W., & Rhodes, E.** (1978). "*Measuring the efficiency of decision-making units".* **European Journal of Operational Research, 2**(6), 429–444

4. **Cooper, W. W., Seiford, L. M., & Tone, K.** (2006). *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*. 2nd ed. New York, USA: Springer Science & Business Media.

5. **Ong, H. K., et al.** (2003). "Measuring the Efficiency of the Banking Sector in Malaysia Using Data Envelopment Analysis." *Asian Academy of Management Journal*, vol. 8, no. 1, pp. (85-101).

6. **OnlineOutput software for DEA.** (n.d.). *OnlineOutput.com*. Retrieved January 23, 2025, from https://onlineoutput.com

7. **Ramanathan, R.** (2003). *An Introduction to Data Envelopment Analysis*. 1st ed. New York, USA: SAGE Publications, pp. (20-31).

8. **Sherman, H. D., & Zhu, J.** (2006). *Service Productivity Management: Improving Service Performance Using Data Envelopment Analysis (DEA)*. New York, USA: Springer Science & Business Media.