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Extreme Value Theory: Review and Illustration

In Partial Fulfilment of the Requirements for the Degree of Bachelor of Science

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At last, we are grateful to the University of Khartoum and the Faculty of Mathematical Sciences for providing us with the opportunity to learn and grow even in the face of adversity.

May this project serve as a small contribution to the academic and scientific progress of our beloved country Sudan.

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CONTENTS

INTRODUCTION

1. Subject of the Project

What if? A powerful question that puts into consideration what is not always considered. Thought the not considered can have a significant impact, it might still not be considered just because it is ‘not too usual’ or ‘extreme’.

Extreme events are known by their unusual or outlier nature. Looking at a distribution, extreme events (values) will be seen at the tails of this distribution, they represent the most uncommon outcomes within this distribution. Hence, the Extreme Value Theory (EVT) involves the study of maxima or minima of a series of independent and identically distributed random variables.

Researchers have become more concerned with events occurring under extreme conditions because extreme never means never, even though they occur infrequently they often have significant impact or consequences. EVT is widely used in risk management in finance and insurance, environmental sciences, health and engineering. As a review, this project argues that EVT is a useful measure since it provides more appropriate distributions to fit extreme events. As an illustration, this project sheds the light into one of the most important applications of EVT, which is risk measurement. It discusses how to measure risk in stock markets, mainly using Value at Risk (VAR) method.

The statistical theory behind the Extreme Values is quit recent. One dimensional extreme value theory was developed by M.Frechet (1927), R. Fisher and L. Tippet (1928), and R. von Mises (1936), and culminated in the work of B. Gnedenko (1943). The statistical theory was intiated by J. Pickands III (1975).

2. Objectives and Importance of the Project

In statistics, handling extreme values is crucial because they can significantly affect estimates. Therefore, it’s essential to analyze data series for estimation before addressing extreme values, often by using Extreme Value Theory insuring the importance of this project.

The project has the following objectives:

- Review the theory behind extreme values by highlighting these rare events and how statisticians analyse these outliers.
- Showing how Extreme value theory provides insights into the distribution, behaviours of the outliers and how it helps to predict and estimate extreme events of the study.
- Moreover, the project focuses on illustrating the EVT through stock markets and risk measurement. It explores the relationship between the theory and risk measurements and how crucial the theory is in stock markets.

- Finally, the project reviews Value at Risk (VaR) in deep details starting from the importance of it through the methods of measurements all the way to limitations and applications.

3. Sections of the Project

This project consists mainly of two sections. The first one is the review section which focuses on providing a deep understanding of the theory concepts together with its applications. The second section is the illustration part. It aids the first section by providing evidence of how the EVT is effective and illustrating it in relationship with stock markets and risk measurement in addition to (VaR).

1. Literature Review

1.1 Historical background of The Theory

1.1.1 Introduction

The extreme value theory is a blend of an enormous variety of applications involving natural phenomena such as rainfall, floods, wind gusts, Air pollution, corrosion etc.

The founders of the calculus of probabilities were too occupied with the general behavior of statistical masses to be interested in the extremes. However, as early as 1709 Nicolaus Bernoulli considers an actuarial problem: n men of equal age die within t years. What is the mean duration of life of the last survivor? He reduces this question to the following: n points lie at random on a straight line of length t. Then he calculates the mean largest distance from the origin.

The first researches pertaining to the theory of largest values started from the normal distribution.

The early papers by Fuller (1914) and Griffith (1920) on the subject were specialized both in fields of applications and in methods of mathematical Analysis.

1.1.2 Development of the General Theory

A systematic development of the general theory may be regarded as having started with the paper by von Bortkiewicz (1922) that dealt with the distribution of range in random samples from a normal distribution. The importance of the paper by Bortkiewicz is inherent by the fact that the concept of distribution of largest value was clearly introduced in it for the first time. In the very next year von Mises (1923) evaluated the expected value of this distribution, and Dodd (1923) calculated its median, also discussing some non-normal parent distributions.

Of more direct relevance is a paper by Frechet (1927) in which asymptotic distributions of largest values are considered. In the following year Fisher and Tippett (1928) published results of an independent inquiry into the same problem. While Frechet (1927) had identified one possible limit distribution for the largest order statistic, Fisher and Tippett (1928) showed that extreme limit distributions can only be one of three types.

Tippett (1925) had earlier studied the exact cumulative distribution function and moments of the largest order statistic and of the sample range from a normal population.

Von Mises (1936) presented some simple and useful sufficient conditions for the weak convergence of the largest order statistic to each of the three types of limit distributions given earlier by Fisher and Tippett (1928).

In 1943, Gnedenko presented a rigorous foundation for the extreme value theory and provided necessary and sufficient conditions for the weak convergence of the extreme order statistics. Mejzler (1949), Marcus and Pinsky (1969) (unaware of Mejzler's result) and de Haan (1970) (1971) refined the work of Gnedenko.

An important but much neglected work of Juncosa (1949) extends Gnedenko's results to the case of not necessarily identically distributed independent random variables. Although of strong theoretical interest, Juncosa's results do not seem to have much practical utility. The fact that asymptotic distributions of a very general nature can occur does not furnish much guidance for practical applications.

1.1.3 Practical Developments of EVT

The theoretical developments of the 1920s and mid 1930s were followed in the late 1930s and 1940s by a number of papers dealing with practical applications of extreme value statistics in distributions of human lifetimes, radioactive emissions [Gumbel (1937a,b), strength of materials [Weibull (1939)], flood analysis [Gumbel (1941, 1944, 1945, 1949a), Rantz and Riggs (1949)], seismic analysis [Nordquist (1945)], and rainfall analysis [Potter (1949)] to mention a few examples. From the application point of view, Gumbel made several significant contributions to the extreme value analysis; most of them are detailed in his book length account of statistics of extremes [Gumbel (1958)].

Gumbel was the first to call the attention of engineers and statisticians to possible applications of the formal "extreme-value" theory to certain distributions which had previously been treated empirically. The first type of problem treated in this manner in the USA had to do with meteorological phenomena - annual flood flows, precipitation maxima, etc. This occurred in 1941.

In essence, all the statistical models proposed in the study of fracture take as a starting point Griffith's theory, which states that the difference between the calculated strengths of materials and those actually observed resides in the fact that there exist flaws in the body which weaken it.

The first writer to realize the connection between specimen strength and distribution of extreme values seems to be F. T. Peirce (1926) of the British Cotton Industry Association. The application of essentially the same ideas to the study of the strength of materials was carried out by the well-known Swedish physicist and engineer, W. Weibull (1939).

The Russian physicists, F'renkel and Kontorova (1943), were the next to study these problems. Another important neglected early publication related to extreme value analysis of the distribution of feasible strengths of rubbers is due to S. Kase (1953).

1.1.4 The Theory over the years

More studies developed the extreme value theory over the years, In the 1970s, 1980s and 1990s, the foundations were laid for an extreme value theory of dependent sequences.

Pioneering work was done by R. Leadbetter, H.Rootz_en, S. Resnick, J. Husler, T. de Olivera, R.A. Davis, T. Hsing, and many others.

The book by Leadbetter, Lindgren and Rootz_en from 1983 was the first one that treated the extremes of stationary sequences; it essentially solved the problem for Gaussian sequences in a rather complete way. They discussed extremal clusters and how to describe them in a quantitative way. Shortly after, in 1987, S.I. Resnick published his important book Extreme Values, Regular Variation, and Point Processes. The focus of this book is on the relationship between the weak convergence of the point processes of the exceedances in a sample and the distributional convergence of the maxima and upper order statistics. He also provided a rigorous extreme value theory for a multivariate iid sequence.

Since the 1970s and 1980s the number of people interested in extreme value problems has increased by far. In 1998, H.Rootz_en founded the journal Extremes which, by now, has become a successful specialized journal.

1.1.5 Applications of EVT

Extreme value theory had many applications over the years it has been used to model the risk of many extreme, rare events, such as the 1755 Lisbon earthquake.

Rainfall data from Florida was provided by Nadarajah (2003). The data analyzed consisted of annual maximum daily rainfall for the years from 1901 to 2003 for the following fourteen locations in West Central Florida: Clermont, Brooksville, Orlando, Bartow, Avon Park, Arcadia, Kissimmee, Inverness, Plant City, Tarpon Springs, Tampa Intl Airport, St Leo, Gainesville and Ocala.

Little has been published on extreme values of New Zealand wind data. Two references that we are aware of are: De Lisle (1965) and Revfeim and Hessell (1984), the former provided an analysis of maximum wind gusts for thirty three stations in New Zealand while the latter suggested an alternative extreme value model with ‘physically meaningful parameters’ and illustrated its use for wind gust data from four stations around New Zealand.

Withers and Nadarajah (2003) conducted one of the first studies which explored New Zealand wind data for trends. Trends in wind data could be caused by a variety of reasons such as climate change, multidecadal natural climate fluctuations, site movement, site exposure and changes in observational procedure. The data consisted of annual maximum of the daily wind run in km for the following twenty locations in New Zealand: Rarotonga Airport, Leigh, Woodhill Forest, Auckland at Oratia, Waihi, Hamilton at Ruakura, Whatawhata, Rotoehu Forest, Gisborne at Manutuke, Wanganui at Cooks Garden, Palmerston North, Pahiatua at Mangamutu, Wallaceville, Wellington at Kelburn Grassmere Salt Works, Lincoln, Mt Cook at The Hermitage, Winchmore, Waimate and Winton.

Flooding in Taiwan is a common phenomenon and often it causes considerable damage. For example, on August of 1997 heavy floods and destructive landslides in northern Taiwan caused millions of dollars in damage and economic losses, and killed over 40 people.

Agriculture has been the hardest hit sector. In spite of this, there has been no work concerning extreme values of flood data from Taiwan. As a matter of fact, there has been little concerned with extreme values of any kind of climate data from Taiwan–the only piece of work that we are aware of is Yim et al. (1999) where extreme value distributions are applied to study the wind climate at Taichung harbor.

Nadarajah and Shiau (2005) provided the first application of extreme value distributions to flood data from Taiwan. The daily stream flow data from the Pachang River located in southern Taiwan was used. Thirty-nine yearly daily stream flow records, from 1961 to 1999, were employed to investigate the extreme floods. Flood events were defined as daily stream flow exceeding a given threshold.

One of the most famous applications for EVT, in the Netherlands, it is well known that almost 40% of the country is below the sea level. It is tremendously important to secure the country from any possible floods as what happened in 1953. EVT is then needed to answer the important question of how high the dike should be given very small floods probability in a year. By collecting data of the storms for 100 years, and estimating the extreme quantile of the dike height given that the probability of a flood is 0.0001.

While this extensive literature serves as a testimony to the great vitality and applicability of the extreme value distributions and processes, on the other hand unfortunately there are still some deficiencies that have not been addressed, however research is still being conducted to cover the shortcomings and gabs of the previous studies.

1.2 Gaps in the Area of Study

The Extreme Value theory, which explores statistical behaviours of extreme outcomes in datasets, has been crucial to fields ranging from finance to meteorology. Despite its extensive application and development, several significant gaps are present in the area of study that researchers are actively working to address, these gaps include:

Data Limitations: Many extreme value models assume data that are independently and identically distributed, but real-world data often violate these assumptions. For instance, in financial markets, extreme values can exhibit temporal dependence. Studies covering models that carry such assumptions can be considered limited and not enough to close the gap.

Model Robustness: Existing extreme value models may not always handle complex dependencies and non-stationarities well. Advances in robust modeling techniques that account for these factors are still needed.

High-Dimensional Data: While traditional extreme value theory often deals with univariate data, real-world applications increasingly involve high-dimensional datasets. Extending extreme value theory to high-dimensional contexts remains a challenge.

Computational Challenges: The estimation and simulation involved in extreme value theory can be computationally intensive, especially with large datasets or complex models. Hence, improvements in computational methods and algorithms are necessary.

Extreme Value Theory in Machine Learning: Integrating extreme value theory with modern machine learning techniques is an area of growing interest. Research is needed to develop methods that combine the strengths of both fields.

Application to Climate Science: In climate science, extreme events like heatwaves and floods are critical, but accurately modeling these events remains challenging due to the complex and evolving nature of climate data.

Addressing these gaps can improve the accuracy and applicability of extreme value models, making them more useful in diverse fields such as finance, meteorology, engineering, and beyond.

1.3 Conclusion

In this literature review, the historical background of Extreme Value Theory (EVT) is examined by looking into its theoretical and practical developments and how the theory evolved throughout the years along with its applications. Significant gaps in the study of the theory were also identified. This review provides insights into how the theory was developed, the challenges faced by researchers, and the direction of future research.

2. From Normal Distribution to Extreme Value Theory: Addressing Statistical Limitations

2.1 The Normal Distribution

Over many years, statisticians noticed that data from samples or populations often formed very similar patterns, for example, a lot of data were grouped around the middle values with fewer observations at the outside edge of the distribution.

The normal distribution been studied for hundreds of years with many different names. Charles S. Peirce (in 1873) and Francis Galton (in 1877) were among the first mathematicians to use the term “normal”. It is also called Gaussian distribution, named after Carl Gauss (a German mathematician) who first described it.

The normal distribution is a continuous probability distribution that is symmetrical on both sides of the mean, the area under the normal distribution curve represents probability and it sums to one. Most of the continuous data values in a normal distribution tend to cluster around the mean, and the further the value is from the mean, the less probable it becomes. Tails are asymptotic, which means that they approach but never quite meet the horizon. Close approximations to the normal distribution are widely found in nature, especially biology. For example, heights, weights and blood pressure tend to follow this shape of distribution.

The normal distribution is the most important probability distribution in statistics because many continuous data in nature and psychology displays this bell-shaped curve when compiled and graphed, it is also important because many of the most powerful statistical testes require the data to be normal. However normal distribution has some limitations that will be discussed.

2.2 Limitations of the Normal Distribution

In the beginning when statistics was discovered to answer questions related to gambling win chances in the 18th century, normal distribution was a very satisfactory tool. For other various cases where you may be interested in studying the impact of large event for further understanding and future expectation, normal distribution will not do the work! A lot of data can fit under this description, e.g. financial data where you need to study the impact of large financial losses and get its occurrence probability. With the rarity of such events normal distribution overlook it as it doesn’t happen.

A particularly important weakness, in the context of risk models is that real distributions are fat-tailed. When focusing our attention on the tails (extreme values) of the normal distribution, the probability density decreases rapidly. This means that extreme events are less likely to occur according to a normal distribution. However, in many real-world scenarios, extreme events do occur more frequently than predicted by a normal distribution (e.g., financial market crashes, natural disasters).

A number of valuation and risk models assume that the future price of a security is normally distributed. This is clearly false as a normal distribution function has a positive value for any value of the future price, whereas the price of a security cannot fall below zero.

When extreme values are present in the data, fitting a normal distribution can lead to inaccurate parameter estimates. This can affect statistical inference, such as hypothesis testing or confidence interval estimation, which relies on the assumption of normality.

In fields of risk management, it relies on accurate estimation of the probability of extreme events. If these events are underestimated because of the normality assumption, we may not adequately prepare for or protect against such events, leading to potentially severe consequences.

2.3 Overcoming Normal Distribution Constraints with Extreme Value Theory

The main objective of Extreme Value Theory (EVT) is to offer a structured approach for modelling and predicting infrequent events that have a low likelihood of happening but can have substantial consequences.

(EVT) was built in way to fix the shortage of the normal distribution by focusing more on these extreme events. It provides a more accurate framework for analyzing and predicting them since the normal distribution assumptions can underestimate their probability. (EVT) came to underline these events, and model their distribution.

It focuses on the tails of distributions where events are rare but significant. This was not the case with the normal distribution, since it always assumes light tails.

Even though the normal distribution can sometimes marginalize extreme events, EVT categorizes these events into three types based on the distribution of maxima (or minima) of sample data: Type I (Gumbel), Type II (Frechet), and Type III (Weibull), providing a framework to choose the appropriate model for a given dataset.

The quantitative risk analysis used to rely, until recently, on classical probabilistic modelling where only average events were taken into account. Thus the evaluation of “normal” risks was more comfortable because it could be easily predicted and so insured.

The catastrophes of the beginning of this century, natural or financial proved that it is nowadays crucial to take also extreme events into account; indeed, although it concerns events that occur almost never (very small probability), the magnitude of such events is such that their consequences are dramatic.

When (EVT) was studied, it solved this dilemma. The theory has various financial implications, like setting appropriate capital reserves, risk management, and portfolio optimization. However, it also helps identify and manage extreme risks like insurance estimation, market movement, and financial crisis. Moreover, it has also helped in tail risk assessment, enhancing decision-making processes, and improving risk models used in the financial industry. Hence, extreme evaluation theory has applications in finance, where it is used to estimate the value at risk (VaR) and expected shortfall (ES) of financial portfolios.

3. Basic Concepts of Extreme Value Theory

3.1 Overview

Extreme Value Theory (EVT) is a powerful tool that provides the best statistical estimate of the tail behaviour of a distribution by quantifying the stochastic process at unusually large or small levels. EVT usually requires estimation of the probability of severe shocks that are more extreme than any other that has been previously observed. Our interest is to find possible limit distributions for (say) sample maxima of independent and identically distributed random variables.

3.2 Extreme Value Theory

Suppose $X_1, X_2, \ldots$ are independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F. Let M_n = max $\{X_1, \ldots, X_n\}$ denote the maximum of the first n random variables and let $w(F)$= sup $\{x: F(x) < 1\}$ denote the upper end point of F. Since

$$\Pr(M_n \leq x) = \Pr(X_1 \leq x, \ldots, X_n \leq x) = F^n(x),$$

M_n converges almost surely to $w(F)$ whether it is finite or infinite. The limit theory in univariate extremes seeks norming constants $a_n > 0$, b_n and a non-degenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \rightarrow G(x)$$

as $n \rightarrow \infty$, i.e.

$$\lim_{n \to \infty} F^n(a_n x + b_n) = G(x)$$

If this holds for suitable choices of a_n and b_n then we say that G is an extreme value cdf and F is in the domain of attraction of G, written as $F \in D(G)$. We say further that two extreme value cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some a > 0, b and all x.

3.3 Extremes Types and their Relevance with EVT

Extreme Value Theory (EVT) deals with the statistical modelling of extreme events, focusing on the distribution of extreme values rather than the bulk of the data. There are two main types of extremes in EVT:

Type I extremes (minimums): These are the smallest values observed within a dataset or a series of data points. Understanding the minimum extremes is important for many fields such as risk management, where it helps in assessing the likelihood of the least favourable outcomes. For example, in finance, the minimum extreme might represent the worst-case scenario in terms of losses. EVT helps in modelling these rare events accurately, which is essential for setting appropriate safety margins and risk assessments.

Type II extremes (maximums): They are largest values observed within a dataset or a series of data. They are relevant in fields such as meteorology, hydrology, and environmental science, where the focus is often on predicting and managing the occurrence of extreme events like floods hurricanes, or heat waves. EVT helps in understanding the probability distribution of these extreme values, which are important in designing infrastructure, planning disaster responses, and implementing mitigation strategies.

Overall both types of extremes in EVT (Minimums and Maximums) are essential for understanding and managing risks associated with extreme events in various domains, from finance and insurance to environmental science and engineering.

3.4 Types of Extreme Value Theory Distributions

EVT provides mathematical frameworks and distributions to model extreme events. Here are the main distributions of EVT;

3.4.1 Generalized Extreme Value (GEV) Distribution:

The GEV distribution is a family of continuous probability distributions developed within extreme value theory. Extreme value theory provides the statistical framework to make inferences about the probability of very rare or extreme events.

The GEV has three parameters: location (μ), scale (σ), and shape (ξ). Location specifies the shift of the distribution relative to the standard GEV; scale is the spread of the distribution; and shape characterizes the heaviness of the distribution's tails, (see Figure 1 for examples of changing μ, σ, and ξ compared to the standard GEV). The standard GEV has parameters $\mu = 0$, $\sigma = 1$, and ξ varies depending on the type of GEV. The canonical GEV probability density (PDF) and cumulative distribution functions (CDF) are:

$$PDF = \frac{1}{\sigma}\left[t(x)^{\xi+1}e^{t(x)}\right]$$

$$CDF = e^{-t(x)}$$

Where

$$t(x) = \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}$$

There are three types of the GEV, differentiated based on the GEV's shape parameter, ξ:

1. **Gumbel Distribution (Type I) with ξ = 0:** Used to model the distribution of the maximum (or minimum) of a number of samples of various distributions.
   It includes the normal, exponential, gamma and lognormal distributions where only the lognormal distribution has a moderately heavy tail.The PDF for this distribution are as below:
$$f(x;\mu,\sigma) = \frac{1}{\sigma} \times exp\left[-\frac{x-\mu}{\sigma} - exp\left(-\frac{x-\mu}{\sigma}\right)\right]$$
   The CDF can then be given as follows:
$$F(x;\mu,\sigma) = exp\left(-exp\left(-\frac{x-\mu}{\sigma}\right)\right).$$

2. **Fréchet Distribution (Type II) with ξ > 0:** Applied to distributions with heavy tails, where extreme events are more likely than predicted by an exponential distribution.and these include well known fat tailed distributions such as the Pareto, Cauchy, Student-t and mixture distributions.It includes the following PDF:
$$f(x;\alpha,\sigma,\mu) = \frac{\alpha}{\sigma}\left(\frac{\sigma}{x-\mu}\right)^{(\alpha+1)} \exp\left(-\left(\frac{\sigma}{x-\mu}\right)^{\alpha}\right).$$
   CDF is as follows:
$$F(x;\alpha,\sigma,\mu) = \exp\left(-\left(\frac{\sigma}{x-\mu}\right)^{\alpha}\right).$$

3. **Weibull Distribution (Type III) with ξ < 0:** These are short tailed distributions with finite lower bounds and include distributions such as uniform and beta distributions.The PDF for this distribution are as below:
$$f(x;\alpha,\sigma,\mu) = \frac{\alpha}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{(\alpha-1)} \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right)$$
   CDF is as follows:
$$F(x;\alpha,\sigma,\mu) = 1 - \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right).$$

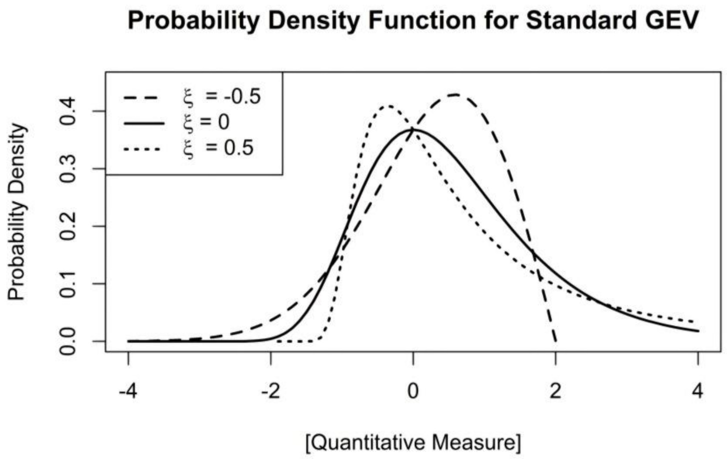


Figure 1: Probability density of the standard GEV with μ = 0, σ = 1, and different shape parameter ξ values (indicated through line type).

3.4.2 Generalized Pareto Distribution (GPD):

The Generalized Pareto distribution is used to model the tail distribution of a random variable above a certain threshold u. It is characterized by three parameters: threshold u, scale σ, and shape ξ. The pdf of the GPD is given by:

$$g(x; u, \sigma, \xi) = \frac{1}{\sigma}\left(1 + \frac{\xi(x-u)}{\sigma}\right)^{-1/\xi - 1}, \qquad x \geq u$$

where:

u is the threshold parameter,

σ >0 is the scale parameter,

ξ is the shape parameter.

4. Methodologies in Extreme Value Theory

4.1 Modeling of the Extreme Value Theory(EVT):

There are two approaches to identify and model the extrema of a random process: the block-maxima approach where the extrema follow a generalized extreme value distribution (BM-GEV), and the peak-over-threshold approach that fits the extrema in a generalized Pareto distribution (POT-GPD).

4.1.1 Block Maxima Method

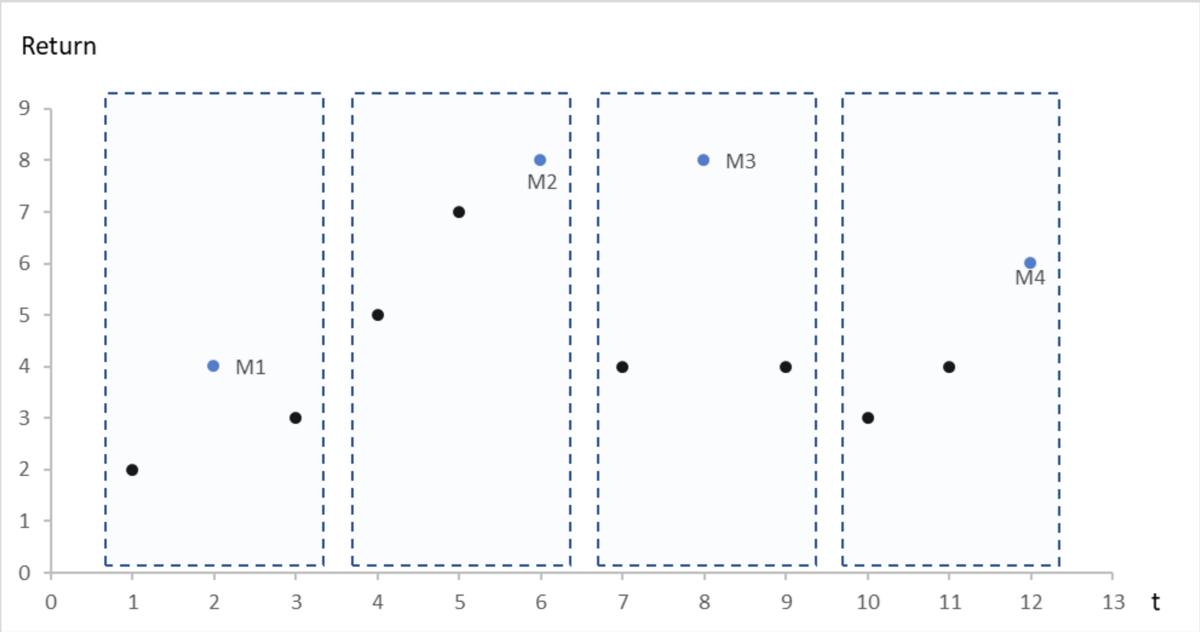
Observing a set of iid observations: $x_1, x_2, \ldots$. the first of the models for estimating extreme values, is the block maxima approach. Here a block size is chosen, depending on the size of the dataset, and within each block the k biggest observations are chosen as extremes. This has the potential limitation that in a single period there can be more extremes than chosen, and in others there can be chosen extremes that in reality are not really extremes.

Under the assumption of independent observations, a series of independent stochastic variables X1,X2... are observed. These observations are blocked appropriately to the chosen block size n, so the block maxima of the series is Mn,1,...,Mn,m, that can then be fitted to the GEV distribution.

One of the great challenges when using the block maxima methods, concerns choice of block size. With block maxima methods, the amount of datapoints are dramatically reduced, as the block size increases.

The fact that the amount of data points are reduced, poses the challenge of reasonably estimating the distribution of the maxima, as estimation becomes less feasible, as for e.g. yearly maxima for a ten year old equity would result in having to estimate the distribution based on ten data points.

Figure2.Illustration of the Block-Maxima approach



Source: Computation by Shengya Zheng

4.1.2 Peak-Over-Threshold (POT) Method:

Threshold models take another approach to finding maxima than the explained in the previous section.Instead of grouping observations on blocks, and only categorizing the block maxima as extreme, threshold models use a chosen threshold, and categorize every observation above (or below for minima), as being extreme.To quantify this, the probability functions is presented as being the conditional probability:

Mn defines to the same as in the previous section, hence the Pr$\{M_n \leq z\ \}$ ≈ G(z)

Where:

$$G(z) = \Pr\{x > u + y\ \ / X > u\} = \frac{1 - F(u + y)}{1 - F(u)}, y > 0$$

The pdf of the GEV , for some given σ and μ > 0, and ξ.

From this the distribution function of our conditional excess term (X − u|X > u), is approximated as:

$$H(y) = 1 - \left(1 + \frac{\xi y}{\sigma}\right)^{-1/\xi}$$

defined on {y : y > 0 and (1+ξy/σ˜) > 0}where σ˜ = σ+ξ(u −μ).

Above equation defines the family of distributions known as the generalized Pareto family. The connection between block maxima, and the generalized Pareto family is so, that if block maxima is approximated by a distribution G, the threshold excess will have an analogous approximating distribution of the generalized Pareto family.

The associated values of the GEV distribution of block maxima will uniquely determine the parameters of the generalized Pareto distribution (hence GPD) of the threshold excess, especially but not limited to the value of the parameter ξ, as this will be identical in the two distributions.

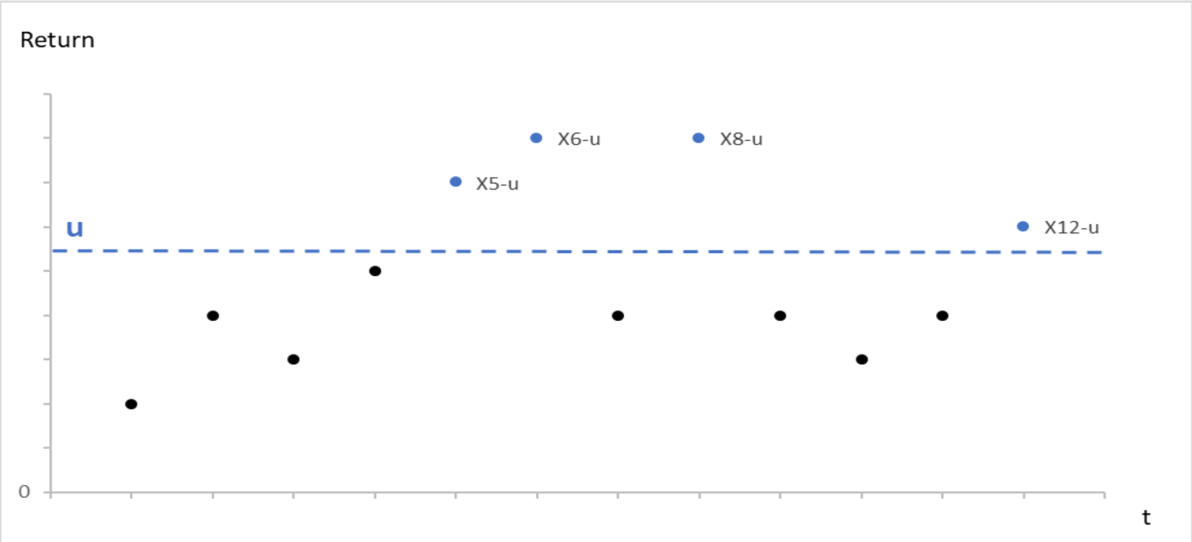
Choice of threshold:

Taking a sequence of iid datapoints $x_1, \dots, x_n$ the datapoints exceeding the threshold u will be identified as the extremes. These extremes are labeled x(1),...,x(k), hence the threshold exceedances can be defined as $y_j = x(j) - u$ for j = 1,...,k. The distribution of these exceedances can be approximated by a form of the generalized Pareto family.

As with the block maxima approach, there is a payoff between bias and variance, as the choice of threshold can be chosen too small, leading to large bias as the asymptotic rules will be violated as there will be too many observations being identified as extremes. On the other hand, a threshold chosen too large, will likely lead to large variance, since the amount of possible extremes will be too small to reasonably estimate the model.

Available for choosing the optimal threshold, there are two methods which offers different approaches. One approach searches to explore how stable, the parameter estimates in the model are, whereas the other approach is more pragmatic, trying out different threshold levels.

Figure3.Illustration of the Peak-Over-Threshold approach



Source: Computation by Shengya Zheng

Choosing Between Methods:

The choice often depends on the characteristics of the data. If extreme events occur infrequently and irregularly, POT method might be more suitable. If extremes are expected to occur more regularly in distinct periods, block maxima method could be preferred. You should also consider the specific application and what types of extreme events are of interest. Each method has strengths in different contexts, such as flood frequency estimation (block maxima) versus modelling high financial losses (POT). Another thing is to assess whether the assumptions of stationary, independence, and distributional form are reasonable for the dataset and application.

4.2 Fitting The Extreme Value Theory Distributions

Fitting Extreme Value Theory (EVT) distributions involves methods tailored to model the extreme values of a distribution, typically focusing on the tail behaviour. Here are some methods commonly used for fitting EVT distributions:

Maximum Likelihood Estimation (MLE): It is used to estimate parameters of the GEV or GPD distributions. In GEV, parameters include location, scale, and shape. While in GPD, parameters include threshold (u), scale, and shape.

Probability Weighted Moments (PWM): It provides estimators for the parameters of GEV distribution. It also involves higher-order moments and is less commonly used compared to MLE.

L-Moments: It is an alternative to traditional moments, used for parameter estimation in GEV distribution based on linear combinations of order statistics, providing robust estimators.

Bayesian Methods: It utilizes Bayesian inference to estimate parameters, incorporating prior distributions and updating with observed data. Particularly useful for incorporating expert knowledge and uncertainty.

Goodness-of-Fit Tests: Assess the fit of GEV or GPD distributions to the data. It includes tests like the Kolmogorov-Smirnov test, Anderson-Darling test, or likelihood ratio test.

Quantile Plotting: Compares empirical and theoretical quantiles to visually assess the fit of GEV or GPD distributions.

Note that choosing the appropriate method depends on the characteristics of the data, such as sample size, the presence of thresholds, and the desired robustness of estimates. MLE is commonly used due to its asymptotic properties, while POT method is effective for modelling extreme values above a threshold. Overall, a combination of methods like MLE, POT, and goodness-of-fit tests often provides a comprehensive approach to fitting EVT distribution.

4.3 Applications of Extreme Value Theory

As mentioned before, EVT is a powerful tool in various fields, providing insights into the behaviour of rare and extreme events. It is widely used in finance, economics, earth sciences, public health and engineering. Applications of (EVT) can include but not limited to:

Modelling Financial Extremes: EVT is crucial in modelling the tail risks of financial returns, such as stock market crashes, extreme losses, or gains. It helps in understanding the overall behaviour of extreme market movements.

Value at Risk (VaR): EVT is used to estimate VaR, which quantifies the maximum potential loss in the value of a portfolio over a given period for a specified confidence interval. This will be discussed in more details later on.

Insurance: EVT is applied to model extreme events like natural disasters, catastrophic losses, and large claims. It assists in setting appropriate premiums and capital reserves.

Climate and Weather Extremes: EVT is well-known to be used in modelling extreme weather events such as floods, heatwaves, and droughts. It helps in understanding the frequency and severity of these events under different climate scenarios.

Structural Reliability: EVT is applied in structural engineering to assess the probability of extreme loads and stresses that structures might encounter, such as extreme wind speeds or seismic events.

River Flow Extremes: EVT models extreme variations in river flows, aiding in water resource management and planning.

Cost-Benefit Analysis: EVT is used to evaluate the economic impacts of extreme events, helping policymakers and businesses in making informed decisions about investments in infrastructure and risk mitigation.

Epidemiology: EVT can be applied to model extreme cases of disease outbreaks or other health-related events, helping in planning and response strategies.

4.4 Extreme Value Theory and Stock Market

4.4.1 Introduction

Extreme stock price movements are of great concern to both investors and the entire economy, financial institutions and investors use EVT to estimate the tail risks associated with their investments. By understanding the distribution of extreme events, investors can better manage their portfolios and hedge against unexpected losses, they may adjust their portfolios based on EVT-derived insights to enhance risk-adjusted returns. For investors, a single negative return, or a combination of several smaller returns, can possible wipe out so much capital that the firm or portfolio becomes illiquid or insolvent. If enough investors experience this loss, it could shock the entire economy.

4.4.2 Value of EVT in Stock Markets

There has been a lot of recent interest regarding the increasing volatility of stock prices. One of the difficulties in examining stock price data is that there is no consensus regarding the correct shape of the distribution function generating the data. An advantage with extreme value theory is that no detailed knowledge of this distribution function is required to apply the asymptotic theory. We focus on the tail of the distribution. Extreme value theory allows us to estimate a tail index, which we use to derive bounds on the returns for very low probabilities on an excess (we will discuss this part in details later on). Such information is useful in evaluating the volatility of stock prices.

In essence Extreme value analysis is a valuable tool in finance by examining stock price movements, it provides a probabilistic framework that complements traditional financial models and enhances the understanding of tail risk, thereby aiding in better decision-making and risk management practices, and it can be more efficient than the usual variance in measuring risk.

4.4.3 Tail Dependences

Estimation of tail dependence between financial assets plays a vital role in various aspects of financial risk modelling including portfolio theory and hedging amongst others. Extreme Value Theory (EVT) that provides well established methods for univariate and multivariate tail distributions which are useful for forecasting financial risk or modelling the tail dependence of risky assets.

Measuring extreme tail dependence of different financial markets or assets is one of the most important subjects in financial risk modelling and plays a vital role in quantification of codependent risk. It has become evident from the recent GFC that extreme shocks in one financial market for a given country may affect other financial markets around the globe. Hence the modelling of extremal dependence and the exploration of whether the tails of stock markets are asymptotically dependent becomes a research question of great interest.

The simplest measure of dependence, the Pearson correlation coefficient is only useful in detecting linear dependence between return series in financial applications. Because it is based on deviations from the mean of the distribution it assigns equal weights to all the instances in a random variable and hence is not a suitable measure for extreme tail dependence. To study tail dependence one of the most sophisticated statistical tools of interest is EVT which provides well established methods for tail based distributions and hence provides better univariate or multivariate tail estimates.

4.4.4 EVT in Financial Applications through the years

The contribution of EVT based techniques in financial applications has increased in recent years. Not only has univariate EVT been implemented in the quantification of extreme risk measures, but there are also now applications of multivariate EVT in the modelling of tail dependence and other financial risk related research. Multivariate extreme value methods for stock market return series have been evaluated in Longin and Solnik (2001), Longin (2000), Bouyé (2002), for VaR and portfolio risk modelling. Cole, Heffernan and Tawn (2000) outline the dependence measures for extremal events.

Poon, Rockinger and Tawn (2003) implemented bivariate EVT based dependence measures to investigate the asymptotic dependence in various international stock markets and how it relates to the presence of heteroskedasticity in financial return series. Fernández (2003) implemented EVT methods for the quantification of VaR and dependence in world markets. Chen, Giles and Feng (2010) study the Chinese stock market and its dependence with other major international stock markets. Schich (2004) studies dependence in European stock markets for large price changes with multivariate EVT methods. Hilal, Poon and Tawn (2011) use the methods of Heffernan and Tawn (2004) to model extremal dependence between financial time series and demonstrate how it can be effectively used in hedging. Dupuis and Jones (2006) illustrate the use of multivariate EVT in actuarial applications.

Poon, Rockinger and Tawn (2003, 2004) found in their study that stock markets do not show statistically significant asymptotic tail dependence for return series filtered for heteroskedasticity. We conduct the empirical investigation for the left tail of the return series data, emphasizing on the losses in financial markets. The asymptotic dependence between these markets is investigated using simple EVT measures and daily log return series. GARCH based filters are used as a filter for heteroskedasticity in the return series to test the filtered return series for tail dependence and asymptotic tail dependence.

5. Risk Measurement in Stock Markets

5.1 Introduction

Risk is an unavoidable part of life. Everyone, from individuals to large organizations, faces the risk of some kind daily. Many types of risks exist, including financial, operational, strategic, and reputational risks.

5.1.1 What is a Risk?

A risk is any activity or investment that presents a potential for gain but also contains the possibility of loss. Risk can be associated with various aspects of life, including business, financial investments, and even personal decisions. In general, risks are divided into two broad categories: economic activities (such as stock market investments) and physical activities (such as driving).

Depending on the type of risk involved, managing or minimizing the associated losses may be possible through careful planning and preparation. As such, understanding what risk is – and how to mitigate it – can help people make better decisions when dealing with uncertain situations.

5.2 Importance of Risk Measurement

Measuring risk in stock markets is crucial for many reasons. Risk measurement is used to make informed decisions about what to buy, hold and sell. It is also essential for portfolio diversification, by understanding the risk profiles of different assets; investors can balance risk and returns.

Measuring risk can also help investors anticipate potential losses and implement strategies to protect their investments. While allowing them to compare the performance of different investments relative to the level of risk they carry. This helps in assessing whether an investment is providing adequate returns for the risk taken.

On a larger scale, understanding and managing risk contributes to the stability of the financial markets. Regulators, financial institutions, and investors all rely on risk assessments to prevent systemic risks that could lead to market crashes or financial crises. It also helps in understanding investor behavior and market sentiment.

5.3 Types of Risks

Systematic risk is defined as the inherent risk that affects the market, not just one sector of the market. This type of risk is uncontrollable by any company and is usually derived from macroeconomic factors. Systematic risk can't be diversified away since it affects the entire market.

Systematic risk is measured by a stock’s beta. A company's beta is the company's risk sensitivity to the market as a whole. A beta of 1 means the stock moves with the market. A beta over 1 means the stock is more volatile than the market. A beta of less than 1 means the stock is less volatile than the market. An investor can calculate his expected return by multiplying the return of the market by the stock's beta.

The most common types of systematic risk are market risk, interest rate risk, purchasing power risk, and exchange risk.

Unsystematic risk represents the firm-specific or industry-specific risk that can be eliminated through diversification. Diversification is an investment strategy to lower risk by investing in uncorrelated or negatively correlated assets. Unsystematic risk is calculated by taking the square root of the variance of stock one minus the product of the beta of stock one squared and the variance of stock two.

The most common types of unsystematic risk are business risk, financial risk, operational risk, strategic risk, and legal and regulatory risk.

5.4 Common Risk Measurement Tools

Understanding and measuring risk is crucial in various fields, from finance and insurance to healthcare and environmental management. Risk measurement involves quantifying the potential for adverse outcomes and uncertainties that could affect objectives or investments. It encompasses a range of methodologies designed to assess the likelihood and impact of different risk scenarios.

In finance, for instance, risk measurement might involve statistical analyses to evaluate the volatility of investment returns or the probability of default on loans. In healthcare, it might include assessing the likelihood of adverse patient outcomes or the effectiveness of preventive measures. Environmental risk measurement could involve evaluating the potential impacts of climate change or natural disasters on ecosystems and communities.

Next we will discuss some of the common methods of risk measurement.

5.4.1 Standard Deviation (volatility):

Standard deviation is a statistical measurement that is often used in finance, particularly in investing. When applied to the annual rate of return of an investment, it can provide information on that investment's historical volatility. This means that it shows how much the price of that investment has fluctuated over time.it can also be used to predict performance trends. It is one of the key fundamental risk measures that analysts, portfolio managers, and advisors use. It is calculated by taking the square root of a value derived from comparing data points to a collective mean of a population.

Standard deviation is widely used in business for risk management. It helps businesses quantify and manage various types of risks. By calculating the standard deviation of certain outcomes, businesses can assess the volatility or uncertainty associated with how they operates.

An alternative to the standard deviation is the semi-deviation, a measurement tool that only assesses part of an investment's risk profile. The semi-deviation is calculated similarly to the standard deviation but can be used to specifically look at only the downside or risk of loss potential of an investment as only half the distribution curve is determined.

5.4.2 Sharpe Ratio:

The Sharpe ratio describes how much excess return you receive for each additional unit of risk you assume. A higher ratio implies a higher investment return compared to the amount of risk of the investment.

The Sharpe ratio measures investment performance by considering associated risks. To calculate the Sharpe ratio, the risk-free rate of return is removed from the overall expected return of an investment. The remaining return is then divided by the associated investment's standard deviation. The result is a ratio that compares the return specific to an investment with the associated level of volatility an investor is required to assume for holding the investment. The Sharpe ratio serves as an indicator of whether an investment's return is worth the associated risk.

One variation of the Sharpe ratio is the sortino ratio which removes the effects of upward price movements on standard deviation to focus on the distribution of returns that are below the target or required return. The Sortino ratio also removes the risk-free rate of return in the numerator of the formula.

The Sharpe ratio is most useful when evaluating differing options. This measurement allows investors to easily understand which companies or industries generate higher returns for any given level of risk.

Another variation of the Sharpe ratio is the Treynor Ratio which integrates a portfolio's beta with the rest of the market. Beta is a measure of an investment's volatility compared to the general market. The goal of the Treynor ratio is to determine whether an investor is being compensated fairly for taking additional risk above the market. The Treynor ratio formula is calculated by dividing the investment's beta from the return of the portfolio less the risk-free rate.

5.4.3 Beta:

Beta used in finance to denote the volatility or systematic risk of a security or portfolio compared to the market. It measures the scale of the volatility of an individual stock compared to the systematic risk of the entire market. Beta represents the slope of the line through a regression of data points. It is calculated by dividing the covariance of the excess returns of an investment and the market by the variance of the excess market returns over the risk-free rate.

Beta effectively describes the activity of a security's returns as it responds to swings in the market. It measures the amount of systematic risk an individual security or sector has relative to the entire stock market. The market is always the beta benchmark an investment is compared to, and the market always has a beta of one.

If a security's beta is equal to one, the security has exactly the same volatility profile as the broad market. A security with a beta greater than one means it is more volatile than the market. A security with a beta less than one means it is less volatile than the market.

Beta is most useful when comparing an investment against the broad market.

5.4.4 Value at Risk (VaR):

Value at risk (VaR) is a statistic that quantifies the extent of possible financial losses within a firm, portfolio, or position over a specific time frame. This metric is most commonly used by investment and commercial banks to determine the extent and probabilities of potential losses in their institutional portfolios.

VaR modeling determines the potential for loss in the entity being assessed and the probability that the defined loss will occur. One measures VaR by assessing the amount of potential loss, the probability of occurrence for the amount of loss, and the time frame.

We will cover this method in details in another chapter.

5.4.5 Stress Testing:

Stress testing is a computer simulation technique used to test the resilience of institutions and investment portfolios against possible future financial situations. Such testing is customarily used by the financial industry to help gauge investment risk and the adequacy of assets and help evaluate internal processes and controls.

Types of Stress Testing:

Stress testing involves running simulations to identify hidden vulnerabilities.

Historical Stress Testing: in a historical scenario, the business or asset class, portfolio, or individual investment is run through a simulation based on a previous crisis.

Hypothetical Stress Testing: it is generally more specific, often focusing on how a particular company might weather a particular crisis.

Simulated Stress Testing: as for the methodology for stress tests, Monte Carlo simulation is one of the most widely known. This type of stress testing can be used for modeling probabilities of various outcomes given specific variables.

However, stress tests have three main attractions. First and foremost, they can give us a lot of information about what we stand to lose in bad states and, indeed, stress testing is explicitly designed to give us information about losses in bad states. The information provided by stress testing is a natural complement to that provided by probabilistic risk measures, most particularly VaR. Second, stress test results can be very useful in risk management decision making in setting capital requirements and position limits, and so on. Finally, stress tests can highlight weaknesses in our risk management systems. Stress testing is essential for sound risk measurement and management.

5.4.6 Conditional Value at Risk (CVaR):

Conditional Value at Risk (CVaR) is another risk measurement used to assess the tail risk of an investment. Used as an extension to the VaR, the CVaR assesses the likelihood, with a certain degree of confidence, that there will be a break in the VaR. It seeks to assess what happens to investment beyond its maximum loss threshold. This measurement is more sensitive to events that happen at the tail end of a distribution.

Generally, if an investment has shown stability over time, then the value at risk may be sufficient for risk management in a portfolio containing that investment. However, the less stable the investment, the greater the chance that VaR will not give a full picture of the risks, as it is indifferent to anything beyond its own threshold.

Conditional Value at Risk (CVaR) attempts to address the shortcomings of the VaR model, which is a statistical technique used to measure the level of financial risk within a firm or an investment portfolio over a specific time frame. While VaR represents a worst-case loss associated with a probability and a time horizon, CVaR is the expected loss if that worst-case threshold is ever crossed. CVaR, in other words, quantifies the expected losses that occur beyond the VaR breakpoint.

5.4.7 Monte Carlo simulation:

A Monte Carlo simulation is a way to model the probability of different outcomes in a process that cannot easily be predicted due to the intervention of random variables. It is a technique used to understand the impact of risk and uncertainty. Monte Carlo simulations can be applied to a range of problems in many fields, including investing, business, physics, and engineering. It is also referred to as a multiple probability simulation.

When faced with significant uncertainty in making a forecast or estimate, some methods replace the uncertain variable with a single average number. The Monte Carlo simulation instead uses multiple values and then averages the results.

Monte Carlo simulations have a vast array of applications in fields that are plagued by random variables, notably business and investing. They are used to estimate the probability of cost overruns in large projects and the likelihood that an asset price will move in a certain way.

The method acknowledges an issue for any simulation technique: The probability of varying outcomes cannot be firmly pinpointed because of random variable interference. Therefore, a Monte Carlo simulation focuses on constantly repeating random samples.

A Monte Carlo simulation takes the variable that has uncertainty and assigns it a random value. The model is then run, and a result is provided. This process is repeated again and again while assigning many different values to the variable in question. Once the simulation is complete, the results are averaged to arrive at an estimate.

However, this method will be discussed later as a VaR Calculation method.

5.5 Limitations of Risk Measurements

In the stock market, risk measurement faces specific limitations, including:

1. Sensitivity analysis is based on management assumptions since it mostly uses historical data. Logically, therefore, wrong assumptions lead to inaccurate forecasts. In addition, Risk models often assume that returns follow a normal distribution, which can underestimate the probability of extreme events (fat tails). This can lead to underestimation of potential losses.

2. It ignores the correlation between variables because sensitivity analysis estimates an outcome by considering individual variables. However, in the real market, some variables may be interrelated. For example, sensitivity analysis considers the impact of factors such as inflation and fluctuation of the market interest on the bond price, on an independent basis. It does not consider the possibility of an interrelation between inflation and the market interest rate.

3. Model Risk measures are also susceptible to manipulation, as the choice of parameters and models can influence them. These models may not capture all relevant factors or interactions, leading to potential inaccuracies.

4. Standard risk measures often ignore liquidity risk, which is the risk that an asset cannot be bought or sold quickly enough to prevent a loss. During periods of market stress, liquidity can dry up, leading to larger-than-expected losses.

4. It cannot tell how much an investor can expect to lose should a tail event occur. Instead, it can only provide information on potential losses if the tail event does not occur. This could have undesirable consequences.

5. Risk measurement may not account for behavioral biases of investors, which can lead to deviations from expected risk models. For example, panic selling or herd behavior can exacerbate losses.

6. Financial markets are interconnected, and risk models often fail to capture the complex interplay between different asset classes, sectors, and global markets, which can lead to incomplete risk assessments.

5.6 Pickands–Balkema–De Haan theorem

5.6.1 Introduction

The Pickands–Balkema–De Haan theorem is a fundamental result in extreme value theory which focuses on the behavior of the tail ends of probability distributions and gives the asymptotic tail distribution of a random variable, when its true distribution is unknown. It is often called the second theorem in extreme value theory. Unlike the Fisher–Tippett–Gnedenko theorem, which concerns with the maximum of a sample, the Pickands–Balkema–De Haan theorem describes the values above a threshold.

5.6.2 The Theorem

Let X be a random variable with a cumulative distribution function (CDF) $F(x)$. For a high threshold u, consider the conditional distribution of exceedances:

$$F_u(y) = P(X - u \leq y \mid X > u), y \geq 0.$$

The theorem states that, as $u \to x_F$ (the upper endpoint of F), if $F_u(y)$ converges, then it converges to a Generalized Pareto Distribution (GPD):

$$G(y; \xi, \beta) = \begin{cases} 1 - \left(1 + \xi \dfrac{y}{\beta}\right)^{-\frac{1}{\xi}}, & \xi \neq 0 \\ 1 - \exp\left(-\dfrac{y}{\beta}\right), & \xi = 0 \end{cases}$$

For $y \geq 0 \; when \; \xi \geq 0 \; and \; 0 \leq y \leq -\frac{\beta}{\xi} \; when \; \xi < 0$.

Where:

ξ: the heaviness of the tail.

$\beta > 0$: scale parameter.

The theorem implies that for a wide range of distributions, the behavior of the excesses over high thresholds can be approximated by a GPD, regardless of the underlying distribution F. This universality makes the theorem a powerful tool in fields like risk management, finance, and environmental sciences.

5.7 Pickands–Balkema–De Haan and Risk Measurement

While risk measurement often focuses on the tail risk which is the risk of extreme outcomes beyond a certain threshold the Pickands–Balkema–De Haan theorem directly addresses this by modeling the tail behavior. The GPD provides a flexible and accurate way to model the distribution of extreme events. Also the theorem estimate ξ and β, practitioners can quantify the probability and impact of rare events as in stock markets.

The key aspect of applying the theorem is choosing an appropriate threshold u.The GPD is effective for modeling exceedances above this threshold, making it crucial in risk modeling. The theorem also helps estimate quantities like Value-at-Risk (VaR) and Conditional VaR (CVaR), which are widely used in finance and stock markets to measure the potential for large losses.

6. Value at Risk

6.1 Introduction

What Is Value at Risk (VaR)?

Value at risk (VaR) is a statistic that quantifies the extent of possible financial losses within a firm, portfolio, or position over a specific time frame. This metric is most commonly used by investment and commercial banks to determine the extent and probabilities of potential losses in their institutional portfolios.

VaR modeling determines the potential for loss in the entity being assessed and the probability that the defined loss will occur. One measures VaR by assessing the amount of potential loss, the probability of occurrence for the amount of loss, and the time frame.

6.2 Importance of Value at Risk

VaR operates on the principle that financial markets can experience fluctuations and that there is always an associated level of risk involved in investments. By quantifying this risk, VaR offers investors and risk managers a powerful metric for evaluating and comparing different investment options. It provides a standardized approach to assess the potential downside, facilitating the identification of riskier investments or portfolios that require additional risk mitigation strategies.

It also serves as a valuable tool for assessing the risk profile of an investment portfolio. By evaluating the potential downside, risk managers can identify and analyze areas of vulnerability. This allows for the formulation and implementation of risk mitigation strategies, such as diversification or hedging, to protect against adverse market conditions. Furthermore, the ability to quantify risks through VaR helps investors set risk tolerance levels and allocate resources accordingly.

The use of VaR as a decision-making tool is becoming increasingly prevalent within the financial industry. By considering VaR estimates, investment managers can evaluate different strategies and make informed choices based on their risk appetite.

One of the primary advantages of VaR is its ability to quantify market risk accurately. By analyzing historical data, market fluctuations, and portfolio compositions, VaR calculations provide risk managers with a clear understanding of potential losses during normal market conditions. This enables them to proactively anticipate and address potential risks, minimizing adverse impacts on their investment portfolios.

Optimizing investment portfolios is a crucial objective for any investor or financial institution. VaR can play a pivotal role in achieving this objective by providing valuable insights into portfolio performance and potential downside risks. Risk managers can use VaR calculations to identify the optimal balance between risk and return, strategically allocating assets across different investment avenues. By curating diverse portfolios with favorable risk-reward profiles, investors can maximize returns while minimizing exposure to potential losses.

6.3 Methods for VaR Measurement

We will discuss several widely used methods to calculate VaR. Typically this is not a very precise type of calculation. There is a need to rely on many different parameters, and each one has a small error.

We will discuss the two basic types of methods: parametric and non-parametric. Parametric methods will include the variance-covariance approach and some analytical methods. The non-parametric model includes historical simulation and the Monte-Carlo approach. All VaR measurement approaches use a similar scheme:

a. Selection of basic parameters (time horizon, confidence level, time of measurement)

b. Selection of relevant market factors

c. Risk mapping

d. VaR calculation

For step (a) we define the relevant parameters according to our goals and resources. The next two steps, (b) and (c), assume some kind of model, either just a set of relevant factors or a completely specified pricing model. In any case the relatively small set of relevant parameters should be defined, and some method for portfolio valuation based on this set should be established. Step (d) includes the calculation itself. This step can be very time consuming, especially when Monte-Carlo methods are used. There are numerous techniques for speeding the calculations.

The following are the different types of techniques to calculate VaR.

6.3.1 Historical simulations

The historical method simply reorganizes actual historical returns, putting them in order from worst to best. It then assumes that history will repeat itself, from a risk perspective.

This is probably the simplest non-parametric method. There is no assumption of a complex structure of the markets. Instead we observe the historical behavior of our current portfolio over the last few years. We begin by measuring the daily percentage changes in the market parameters. Then we apply these changes to our current portfolio and measure the corresponding profits and losses. The most useful version of this approach is when the risk mapping procedure defines the price of the whole portfolio as a deterministic function of the market parameters $P(p)$. Here P is the pricing function and p is the vector of all relevant market parameters. Then today's (day t) price is $P(p_t)$. The market parameters at some day j were p_j and on day j+1 the parameters were p_{j+1}. Then we can model the possible changes in today's parameters in the following ways. We can use the relative change, where each market parameter is multiplied by the ratio of the same parameter at day $j+1$ and day j. Another approach is when we add to today's value the difference between the values at day $j+1$ and day j for each parameter. The multiplicative method is applicable when the volatility increases with the level of the parameter. This method is useful for stock indexes, exchange rates, etc. The additive approach assumes that the volatility is level independent. For example, for the additive approach we would take as a possible price tomorrow $p(p_t + (p_{j+1} - p_j))$. More complex combinations of both methods can be used as well. For example when modeling exchange rates in a band or interest rates.

A typical problem with this approach is that there is not enough data. The further we go into the past for data, the less relevant this information is to today's market. This is not a simple trade-off. On the one hand, we would like to have more data in order to observe the rare events, especially the heavy losses. On the other hand, we do not want to build our current risk estimates on very old market data. Let's assume that we have agreed to take the last five years of data for our VaR estimate. If there was a big loss on a particular day, then exactly five years later the big market jump will not appear in the set of data we use. This will lead to a jump in our VaR estimate from one day to the next. This demonstrates that the results are not stable when using the historical simulations approach.

One important situation in which the historical simulation approach can not be used is for technical trading strategies developed on the basis of historical data. Technical trading strategies are generally conceived on the basis of historical data, and produce the best results on this data (i.e. most of big losses are a posteriori excluded). In such a case, one can certainly not use the data which was already used for calibration as the data set for the VaR estimate.

6.3.2 Variance covariance

This is a parametric method, based on the assumption that the returns are normally distributed. Historical data is used to measure the major parameters: means, standard deviations, correlations. The overall distribution of the market parameters is constructed from this data. Using the risk mapping technique, the distribution of the profits and losses over the time horizon (typically one day) can be found. When the market value of the portfolio is a linear function of the underlying parameters, the distribution of the profits is normal as well.

The strong side of this approach is that it is flexible, simple and widely used. It also enables the addition of specific scenarios and enables the analysis of the sensitivity of the results with respect to the parameters. However, it relies heavily on the important assumption that all of the major market parameters are normally distributed. Therefore when a significant portion of the portfolio is not linear this method can not be used directly.

An important property of the variance covariance matrix is that it must be positive definite. However, because of the very high degree of correlation, even a small error in the data can easily lead to loss of this property.

A second difficulty is related to the fixed rolling time window. Any big market move creates an abrupt change in all major parameters when the window moves.

Overall, the variance-covariance method assumes that stock returns are normally distributed and requires an estimate of only two factors, an expected return, and a standard deviation, allowing for a normal distribution curve.

It is similar to the historical method except it uses a familiar curve instead of actual data.

6.3.3 Monte Carlo simulations

This is another non parametric method. It is probably one of the most popular methods among sophisticated users. It does not assume a specific form of the distributions.

The first step is to identify the important market factors. Next, one should build a joint distribution of these factors based on one of the following: historical data; data implicitly implied by observed prices; data based on specific economic scenarios. Finally, the simulation is performed, typically with a large number of scenarios. The profit and losses at the end of the period are measured for each scenario.

This method has several important advantages. First, it does not assume a specific model and can be easily adjusted to economic forecasts. The results can be improved by taking a larger number of simulated scenarios. Options and other nonlinear instruments can be easily included in a portfolio. In addition, one can track path-dependence because the whole market process is simulated rather than the final result alone.

One important disadvantage is very slow convergence. Any Monte Carlo type simulation converges to the true value as $\frac{1}{\sqrt{N}}$, where N is the total number of simulated trajectories. This means that in order to increase the precision by a factor of 10 one must perform 100 times more simulations. This problem is the most serious disadvantage of this method. However in many cases there are well developed techniques of variance reduction.

When using the Monte Carlo simulations one should generate a large number of trajectories for all of the major parameters and then price every single instrument along these trajectories. The pricing of all of the instruments is very time consuming, however it can be reduced significantly when one divides all of the instruments into similar groups. This method of variance reduction is based on the knowledge that the instruments that we want to group have similar risk characteristics.

An additional problem with Monte Carlo simulations is that one needs to know the joint distribution of many market parameters. When there are more than 3-4 important parameters it is not easy to clean all the data and to build this multidimensional distribution.

An advantage of the Monte Carlo method is that it allows the use of the preliminary results of all of the methods mentioned above. The historical simulations can give a first approximation to the distribution functions. The variance covariance shows which connections between variables are important and which can be neglected.

In addition, one can easily perform stress testing on the Monte Carlo simulation or perform a more detailed analysis of a specific set of scenarios, including dynamic strategies, such as prepayments or partial recoveries.

However, A Monte Carlo Simulation refers to any method that randomly generates trials, but by itself does not tell us anything about the underlying methodology.

For most users, a Monte Carlo simulation amounts to a "black box" generator of random, probabilistic outcomes. This technique uses computational models to simulate projected returns over hundreds or thousands of possible iterations.

6.4 Value At Risk (VAR) Limitations

Value at Risk is a widely used as a risk management tool, popular especially with banks and big financial institutions. There are valid reasons for its popularity – using VAR has several advantages. But for using Value At Risk for effective risk management without unwillingly encouraging a future financial disaster, it is crucial to know the limitations of Value At Risk.

False sense of security: Looking at risk exposure in terms of Value At Risk can be very misleading. Many people think of VAR as "the most I can lose", especially when it is calculated with the confidence parameter set to 99%. Even when you understand the true meaning of VAR on a conscious level, subconsciously the 99% confidence may lull you into a false sense of security.

Unfortunately, in reality 99% is very far from 100% and here's where the limitations of VAR and their incomplete understanding can be fatal.

VAR does not measure worst case loss: 99% percent VAR really means that in 1% of cases (that would be 2-3 trading days in a year with daily VAR) the loss is expected to be greater than the VAR amount. Value At Risk does not say anything about the size of losses within this 1% of trading days and by no means does it say anything about the maximum possible loss.

The worst case loss might be only a few percent higher than the VAR, but it could also be high enough to liquidate your company. Some of those "2-3 trading days per year" could be those with terrorist attacks, big bank bankruptcy, and similar extraordinary high impact events.

You simply don't know your maximum possible loss by looking only at VAR. It is the single most important and most frequently ignored limitation of Value At Risk.

Besides this false sense of security problem, there are other (perhaps less frequently discussed but still valid) limitations of Value At Risk.

Difficult to calculate for large portfolios: When you're calculating Value At Risk of a portfolio, you need to measure or estimate not only the return and volatility of individual assets, but also the correlations between them. With growing number and diversity of positions in the portfolio, the difficulty (and cost) of this task grows exponentially.

VAR is not additive: The fact that correlations between individual risk factors enter the VAR calculation is also the reason why Value At Risk is not simply additive. The VAR of a portfolio containing assets A and B does not equal the sum of VAR of asset A and VAR of asset B.

Only as good as the inputs and assumptions: As with other quantitative tools in finance, the result and the usefulness of VAR is only as good as your inputs. A common mistake with using the classical variance-covariance Value At Risk method is assuming normal distribution of returns for assets and portfolios with non-normal skewness or excess kurtosis. Using unrealistic return distributions as inputs can lead to underestimating the real risk with VAR.

Different VAR methods lead to different results: There are several alternative and very different approaches which all eventually lead to a number called Value At Risk: there is the classical variance-covariance parametric VAR, but also the Historical VAR method, or the Monte Carlo VAR approach (the latter two are more flexible with return distributions, but they have other limitations). Having a wide range of choices is useful, as different approaches are suitable for different types of situations. However, different approaches can also lead to very different results with the same portfolio, so the representativeness of VAR can be questioned.

So many problems... Should we still use VAR?

Having read about several serious limitations of Value At Risk and you might be thinking of never using VAR (again). However, Value At Risk can be useful, as long as you keep its weaknesses in mind and don't take VAR for something it isn't.

Value At Risk should be one little piece in the risk management process and it must be complemented with other tools, especially those taking care of that 1% worst case area, which VAR virtually ignores. As long as you don't let VAR become a false sense of security, it can be very helpful.

6.5 Application of VaR in Risk Management

Here's how VaR is applied in risk management

Portfolio Risk Assessment: Investment banks and financial institutions use VaR to assess the risk of their entire portfolio.It helps them understand the cumulative risks from different trading desks and departments.By analyzing VaR data, institutions can determine if they have sufficient capital reserves or need to reduce concentrated holdings.

Risk Control and Decision-Making: VaR allows risk managers to set risk limits and monitor deviations. It informs decisions related to hedging, asset allocation, and risk-taking

Performance Measurement: VaR is used to evaluate the performance of risk managers and traders. Comparing actual losses with VaR predictions helps assess how well risk management strategies are working.

Stress Testing: Although VaR is typically based on historical data, it can be complemented with stress testing to evaluate potential losses under extreme market conditions. This helps in understanding the limits of VaR and preparing for rare, high-impact events.

Capital Allocation: Financial institutions use VaR to determine the amount of capital needed to cover potential losses. Regulatory frameworks like Basel III require banks to hold capital reserves proportional to their VaR, ensuring they can absorb shocks.

Regulatory Compliance: Regulators often require financial institutions to report VaR figures.It ensures transparency and accountability.

Conclusion

This research provides a comprehensive review of the Extreme Value Theory (EVT) showing its important role in modelling and understanding extreme events.

At first, it explores the historical development of the theory then it shows the significant importance of the theory by highlighting the statistical limitations of the normal distribution.

The theoretical framework of this research introduces and describes the theory in details. The study then illustrates how EVT is indispensable in fields like finance, engineering, and others. Through the detailed examination of methodologies such as the Block Maxima and Peak-Over-Threshold approaches, the research sheds the light on the effectiveness of EVT in predicting rare but impactful events.

The research discusses that the integration of EVT in risk measurement, particularly in financial markets through measures like Value at Risk (VaR), underscores its significance in enhancing decision-making and mitigating potential risks.

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