Monte Carlo method and random numbers

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INTRODUCTION

Monte Carlo methods are a group of algorithms that use random number generators to approximate numerical solutions to problems. The main objective of this exercise was to generate random numbers in a specific distribution. The random number distributions were then used to investigate two different physics problems. Random number generators usually produce random numbers in a uniform distribution $P(x)$ (the probability of any sample value, x, is the same). Two different methods were used to transform the uniform distribution generated into the non-uniform distributions $P'(x')$ (the probability of any sample value, x', is given by $P'(x)$). The fist method is called inverse transform sampling. Inverse transform sampling takes advantage of the fact that the cumulative distribution function $F(X)$ of any $P'(x')$ gives the probability that a sample value x' is less then an input value X i.e. $F(X) = P(x \leq X)$. The inverse of the cumulative distribution function, $F^{-1}(X)$, takes a probability and returns a sample value. If the probabilities are generated from a uniform random distribution between 0 and 1 then the inverse cumulative distribution function will return a random value distribution proportional to P'(x'). The second method is called rejection sampling. Two random numbers are generated one in the x domain of the required sample distribution and one in the y range. The function that describes the required distribution is then used to calculate a ymax value using the random x value if the random y value is greater then the ymax value then the x value is discarded if not it is stored.

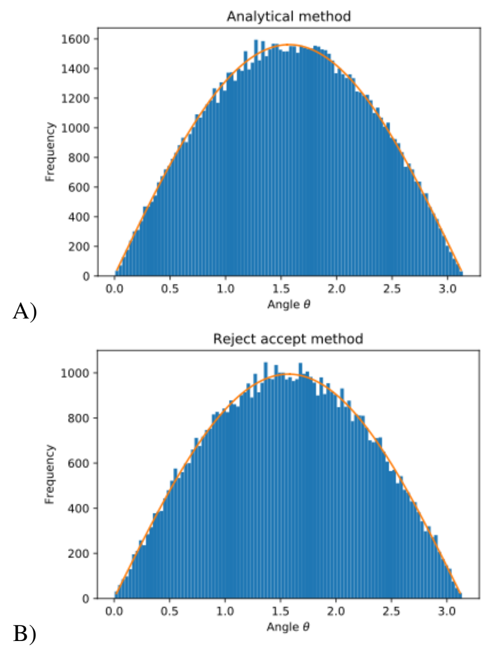

FIG. 1. A and B are random number distributions that are proportional to a sine curve. The reduced chi square value of A and B are 1.06 and 1.1 respectively. Both graphs are fitted to sine curves using the curve fit function from the SciPy library

TASK 1

The first task was to produce a random number distribution that was proportional to sin(θ) over the interval 0 to π using the inverse transform sampling and the rejection sampling methods. To use the inverse transform method the inverse of the cumulative distribution function for the normalised sine curve had to be calculated the result is shown bellow:

$$\theta = arccos(1 - 2u) \qquad 1$$

where θ is the random number in the sine distribution and u is the random number in the uniform distribution in the range. The results of the two methods are shown in figure 1.

Both methods fit the theoretical distribution well with reasonable reduced chi squared values. Originally it was hoped that the reduced chi-squared values could be used to evaluate which method converges to the correct distribution in the fewest number of samples however because of the random nature of the methods the chi squared value fluctuated a lot and the plots did not impart any useful information. On top of this because the curve fit function from the SciPy library was used to work out the theoretical fits which optimizes the fits the reduced chi square value is somewhat meaningless as it is more a measure of the curve fit ability to optimise any distribution. A better method could be to normalise the bin data and then compare that distribution to that of $\frac{1}{2}sin(\theta)$ by working out the residual.

The rejection sampling offers more versatility in terms of the functions that can be evaluated as it doesn't require the integral of the function to be known but it is more computationally expensive as shown by figure 2. Rejection sampling is also less efficient as it discards points. It is for these reason that inverse transform sampling was chosen for the next section.

TASK 2

The second task asked to calculate the distribution of gamma rays on a detector array. The gamma rays are cre-

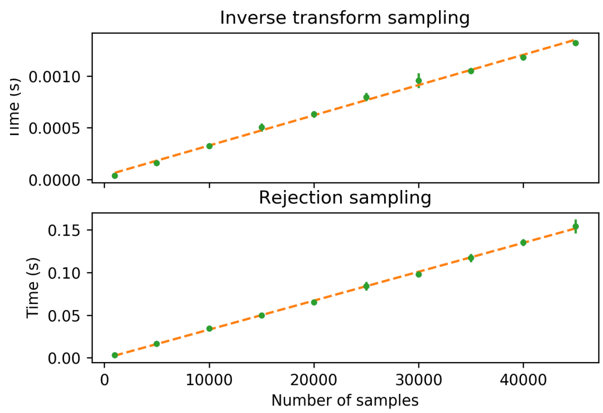

FIG. 2. Plot of the run time of the inverse transform and rejection sampling functions for different numbers of samples. The errors are given by the standard deviation of the run time over ten runs.

ated by a beam of unstable particles traveling at 2000m/s and are moving perpendicular to the detector array. The detector array is placed 2m from the source of the unstable particles. This problem can be split into three parts, calculating the decay positions of each particle, calculating the angular distribution of the gamma rays and then using the previous two results to calculate the position the gamma rays hit the screen. The probability that a particle will decay after some time t is given by

$$P_{decay}(t) = \lambda e^{-\lambda t} \qquad 2$$

where λ is the decay constant of the particle. The inverse transom sampling method can be used to create a distribution of random decay times that is proportional to equation 2. The inverse cumulative distribution function was calculated to be

$$T_{decay} = -\frac{1}{\lambda}\ln(1-u) \qquad 3$$

where T is the decay time and u is a random number produced in a uniform distribution between 0 and 1. From the decay time the decay position can be calculated. Figure 3 is a histogram of the decay positions of the virtual particles.

The next problem was calculating the emission angles of each decay. The gamma rays should be emitted isotropically. At first uniform random number generators were used to generate both the azimuthaual and polar emission angles of the gamma rays. However because of the shrinking of the differential surface element of a sphere at the poles less random number samples need to be taken at the poles then at the equator. The size of a spherical differential surface element is given by

$$dA = \sin(\phi)d\phi d\theta \qquad 4$$

where ϕ is the polar angle and θ is the azimuthaual angle. Equation 4 shows that the surface element does not depend

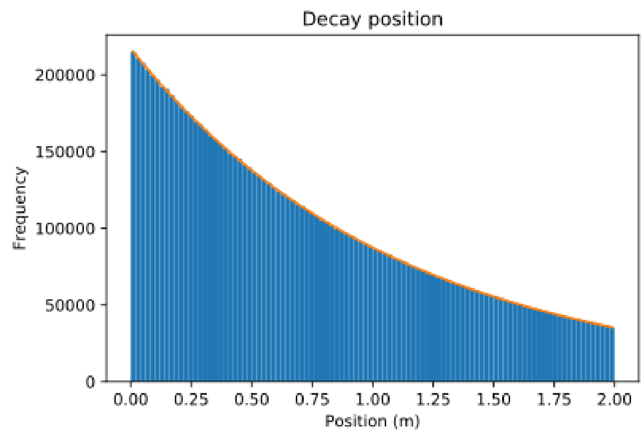

FIG. 3. Frequency of decay positions for 10^7 virtual particles. The random number distribution was generated using the inverse transform sampling method acting on the probability of a particle decaying. The curve was fitted with an exponential function, the reduced chi squared of the plot is 1.1 indicating a good fit.

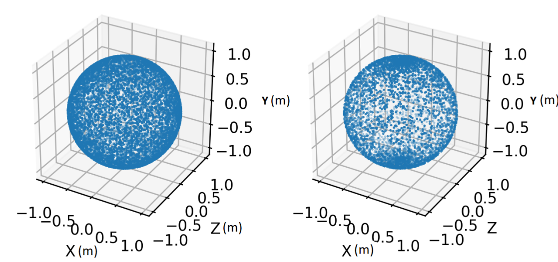

FIG. 4. 3D scatter plots of gamma ray position for a range of different emissions angles. The right hand plot shows the distribution of emission angles when the azimuthaual and polar angles are generated using uniform random number generators. The left hand plot shows an isotropic distribution of emission angles. This is generated by using a sinusoidal random number distribution to sample the polar emission angles.

on the azimuthaual angle and so it can still be sampled using a uniform distribution. The polar angle however has to be sampled using a sinusoidal distribution to offset the scaling of the differential surface element. Figure 4 shows the spherical distribution of the emission angles

Figure 4 shows that when the azimuthaual and polar angles are generated using uniform random number generators there is a sparsity of points around the equator of the sphere and a clumping of points at the poles. This agrees with equation 4. As θ approaches the poles the differential surface element decreases and so there will be a higher density of points for a uniform distribution of polar angles and vise versa for the equator.

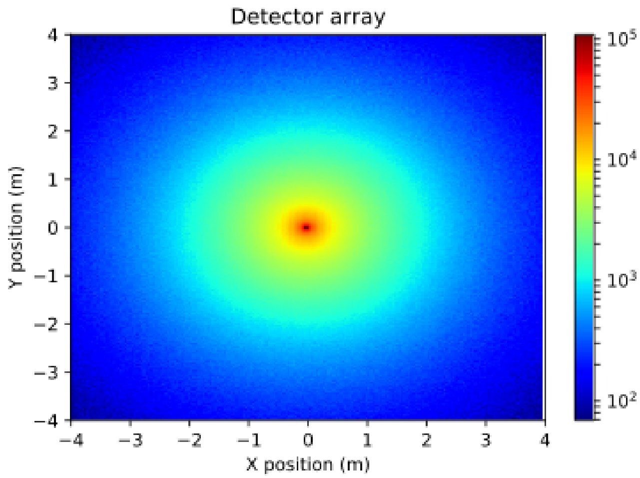

FIG. 5. Intensity plot of the detector array for a 10^8 virtual particles.

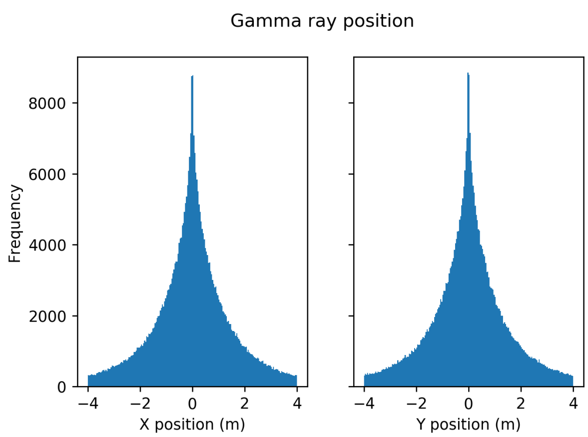

FIG. 6. Plot of the x and y distribution of gamma rays.

The final task was to use the decay position and the emission angle to calculate the incident position of a gamma ray on the detector array. From simple trigonometry the following equations were derived that gave the position of a gamma ray on a 2D plane 2m away from the particle source depending on the emission angle and emission position:

$$y' = \frac{z'}{\sin(\theta)\tan(\phi)} \quad 5$$

$$x' = \frac{z'}{\tan(\theta)} \quad 6$$

where x' and y' are the positions of the gamma ray on the screen, z' is the decay position subtracted by the distance from the source to the detector. Figure 5 shows the detector array intensity plot.

There is not really any statistical test that can be done to evaluate the 'goodness' of the plot however the intensity pattern does look like what is expected. Because of the isotropic

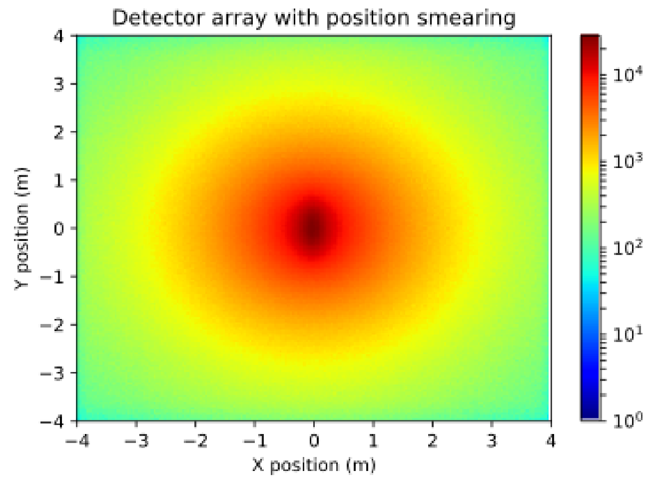

FIG. 7. Resolution limited detector array for 10^8 virtual particles.

distribution of emission angles the plot is expected to be highly symmetric. This is what is observed and is shown more clearly by figure 6. A circular pattern is also observed with the highest intensity point at the center this reflects the projection of the isotropic spheres of radiation being projected on a 2D plane.

Real arrays are resolution limited to approximate this a normally distributed random number generator term was added to the position calculations the resoultion limited array is shown in figure 7. The x and y resolution were different. This breaks the symmetry seen in figure 5.

TASK 3