Euler method and freefall

Computing exercise 3

The aims of this investigation were to use the Euler method to numerically solve the differential equation for an object in freefall being acted on by a variable drag force to confirm the statistics of Felix Baumgartner's skydive. Felix Baumgartner jumped from a helium balloon at a height of 39045m, he fell for 4 minutes and 19 seconds and reached a maximum speed of 373 ms^{-1}.

Background

Euler's method can be used to solve ordinary differential equations in the form,

$$\frac{dy}{dt} = f(y, t), \tag{1}$$

from some initial conditions. Euler's method works by calculating the gradient of the solution at some initial point and then assume that the gradient is constant over some small step, Δt. The gradient is then used to approximate the value of the solution at the point t+ Δt, this value is then used to calculate the gradient of the new position and the process is repeated. The smaller you make Δt the more accurate the numerical solution becomes where in the limit as Δt goes to 0 the numerical solution becomes exact. This process can be summarised by

$$y_{n+1} = y_n + \Delta t \cdot f(y_n, t_n), \tag{2}$$

$$t_{n+1} = t_n + \Delta t. \tag{3}$$

The equation of motion for a freefalling object is a second order ordinary differential equation, as the acceleration of the object varies. To solve a second order ODE using the Euler method the second order ODE can be separate into 2 first order ODEs, one for velocity and the other for position. The first order ODE's are,

$$\frac{dv_y}{dt} = -g - \frac{k}{m}|v_y|v_y, \tag{4}$$

$$\frac{dy}{dt} = v_y. \tag{5}$$

Where v_y is the velocity, m is mass, g is the acceleration due to gravity and k is a constant. The value of k is given by,

$$k = \frac{c_d \rho_0 A}{2}, \tag{6}$$

where c_d is the drag coefficient (which for a person is between 1 and 1.3), ρ_0 is the air density which is about 1.2 kg m^{-3} at ambient temperature and pressure and A is the cross-sectional area of the object in freefall (which for a skydiver is about 0.75 m^3). Using equation 2 the numerical solutions of velocity and position can be written as,

$$v_{y,n+1} = v_{y,n} - \Delta t\left(g + \frac{k}{m}\left|v_{y,n}\right|v_{y,n}\right), \tag{7}$$

$$y_{n+1} = y_n + \Delta t \cdot v_{y,n}\,. \tag{8}$$

The time at any point is given by equation 3.

Part A): Euler method

The task for part A was to create a program to numerically calculate y(t) and v(t) for a free-falling object. A while loop was used where each iteration a value for the velocity and position was calculated using equation 7 and 8. The loop stops when the object reaches the ground, i.e. y=0. The object was given an initial velocity of 0 ms^{-1} and an initial position of 1 km. The object was chosen to be a skydiver so, the drag coefficient was chosen to be equal to 1 and the cross-sectional area was chosen to be $0.75m^2$. Equation 7 requires a value for the acceleration due to gravity, a value for g was calculated for each iteration using newtons law of gravity.

$$g = \frac{GM_e}{(R_e + y)^2}, \tag{9}$$

where G is the gravitational constant, M_e is the mass of the earth, R_e is the average radius of the earth and y is the position of the skydiver above the ground.

Once the values of y(t) and v(t) had been calculated they were plotted generating the following graphs:

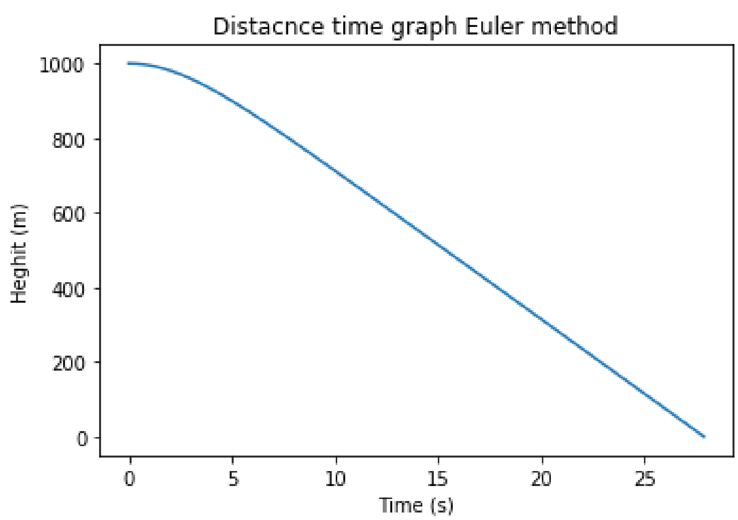

Figure 1 Predicted position against time graph for a skydiver in free fall. Calculated using the Euler method, with a time step of 0.0001s and an initial height of 1000m.

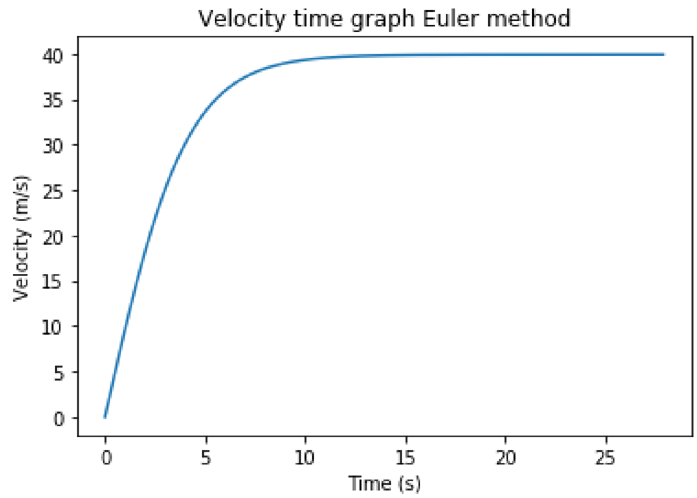

Figure 2 Predicted magnitude of velocity against time graph for a skydiver in free fall. Calculated using the same method and parameters as figure 1.

The skydiver's velocity accelerates until it reaches a terminal velocity at around 40m/s. This behavior is expected and is easier to explain when you look at a plot of the 2 forces acting on the skydiver.

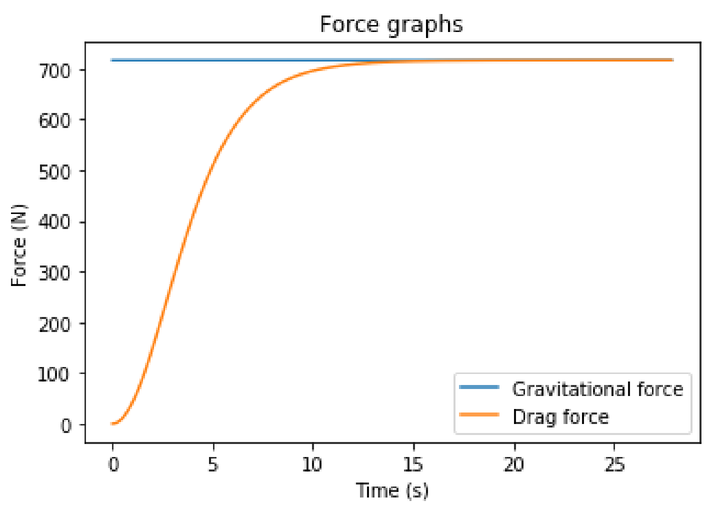

Figure 3 Plot of the magnitude of the gravitational and drag forces acting on a skydiver over time. Calculated using the same parameters as figure 1/2.

Initially the gravitational force dominates, and the skydiver accelerates, however, as the velocity of the skydiver increases so does the magnitude of the drag force. Eventually the drag force is equal in magnitude to the gravitational force and the two forces cancel causing the skydiver to stop accelerating.

Part B): Analytical model

Equations 4 and 5 have analytical solutions given by

$$y = y_0 - \frac{m}{2k} ln\left[cosh^2\left(t\sqrt{\frac{kg}{m}}\right)\right], \qquad (10)$$

$$v_y = -\sqrt{\frac{mg}{k}} tanh\left(t\sqrt{\frac{kg}{m}}\right), \tag{11}$$

Where y_0 is the initial position of the skydiver. The task in part b was to compare the behaviour of the analytical solution to the numerical solutions. To do this the difference between the analytical and numerical solutions were plotted for different time steps.

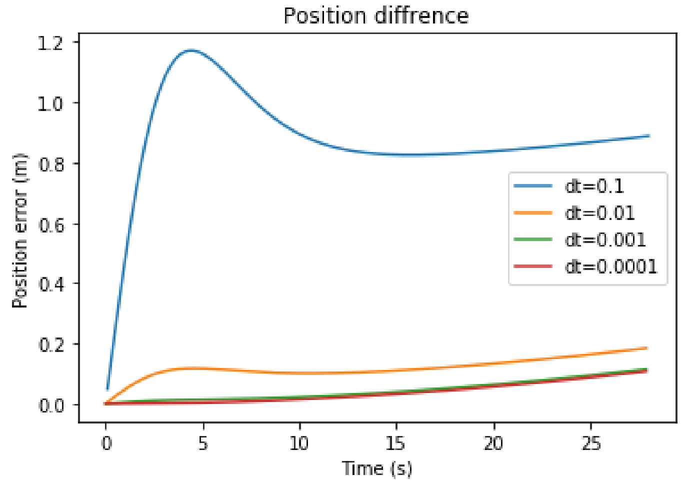

Figure 4 The difference between the analytical and numerical solutions for position plotted against time for various time steps. An initial height of 1 km was used to produce these plots.

By decreasing the size of the time step the errors at each point reduce, especially the errors in the region between 0 and 5 seconds where the skydiver is still accelerating. For bigger time steps the Euler method calculations seem to overshoot in this region. This is because the Euler method works by calculating the gradient of the solution and assumes that it is constant. Therefor in the regions where there is the greatest rate of change of the gradient (greatest acceleration) the errors will be largest. As the time step is reduced however the errors in this region seem to smooth out. A similar pattern emerges when the error on the velocities are plotted for different time steps.

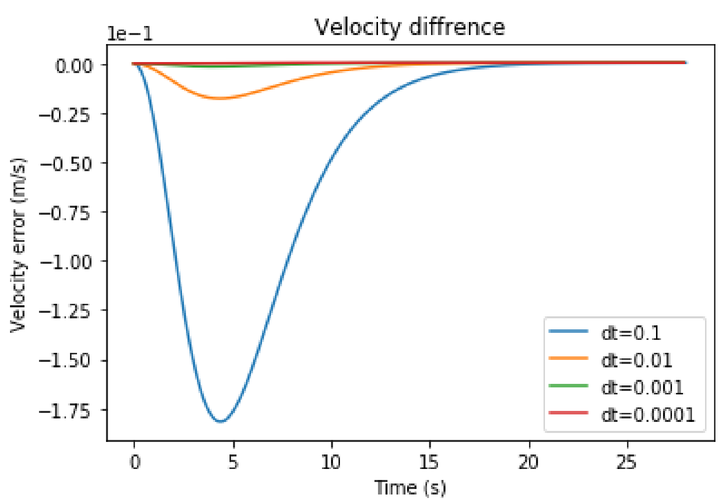

Figure 5 Plots of the difference between the analytical and numerical solutions against time for different time steps. Skydiver fell from a height of 1000m.

Again, as you decrease the time steps the error at each point decreases, the largest error occurs in the region between 0 and 5 seconds when the skydiver is accelerating. As the velocity becomes constant the error on the velocity becomes smaller.

Part C): Modified Euler method

The Euler method can be improved by estimating the gradient at the midpoint, (using the original Euler method) and then using said gradient to calculate the solution at the endpoint. This helps to reduce the overshoot on the Euler method. For part c the modified Euler method was used to calculate the freefall time of a skydiver and to compare the results to the original Euler method. The modified Euler method gives the solution to a first order ODE as,

$$y_{mid} = y_n + \frac{\Delta t}{2} f(y_n, t_n) \ ; \ t_{mid} = t_n + \frac{\Delta t}{2} \tag{12}$$

$$y_{n+1} = y_n + \Delta t f(y_{mid}, t_{mid}). \tag{13}$$

Again, because the differential equation for free fall is a second order ODE it must be split into 2 first order ODE and then solved using the modified Euler method. applying equations 12 and 13 to equation 4 and 5 the numerical solutions to the freefall problem become,

$$v_{mid} = v_n - \frac{c}{2}\left(g + \frac{k}{m}|v_n|v_n\right) , \tag{14}$$

$$y_{n+1} = y_n + \Delta t \cdot v_{mid}, \tag{15}$$

$$v_{n+1} = v_n - v_{mid}\left(g + \frac{k}{m}|v_{mid}|v_{mid}\right) . \tag{16}$$

The modified Euler method was compared to the analytical models.

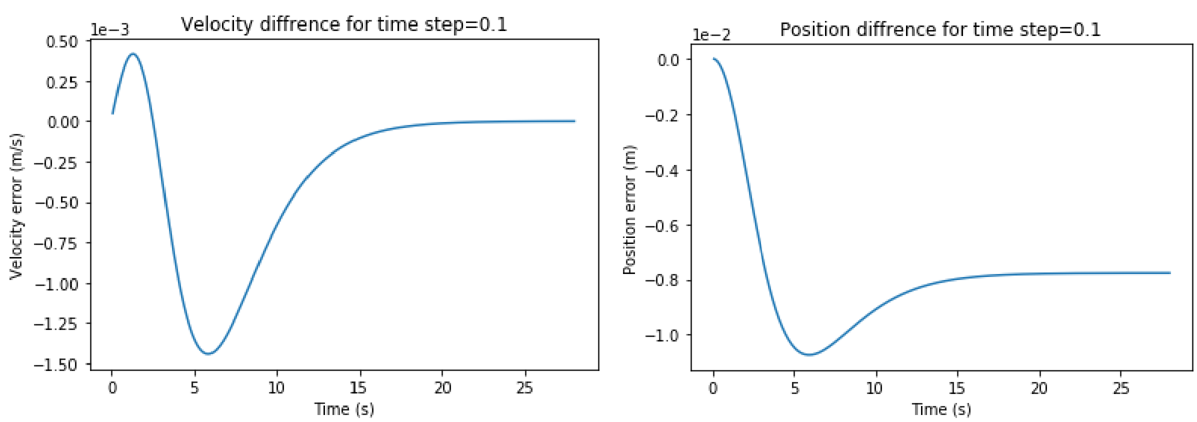

Figure 6 Plots of the difference between the analytical and numerical solutions calculated by the modified Euler method. The height the skydiver fell from was 1000m and the time step was 0.1.

The error on the modified Euler method is smaller at every point then on the original Euler method for a time step of 0.1. A graph was created to directly compare the two but the difference in the relative size of the errors were so great they could not be plotted on the same graph. Even though the modified Euler method has reduced the errors the greatest errors still occur in the region of 0 to 5 seconds. Figure 6 then shows that the modified Euler method more accurately predicts the position and velocity at each point then the original Euler method.

Part d): Variable air density

Equation 6 depends on air density, as an object falls the air density will change. The air density can be modelled as a function of height using,

$$\rho(y) = \rho_0 \exp\left(-\frac{y}{h}\right), \tag{17}$$

where ρ is the air density, ρ_0 is the air density at the ground (1.2kgm^-3), y is the height above the ground and h is the scale height of the atmosphere where an appropriate value of h seems to be 7.64km. By including variable air density, the model becomes more realistic. The program can now be compared to the results of Felix Baumgartner's skydive. Plots were created for y(t) and v(t).

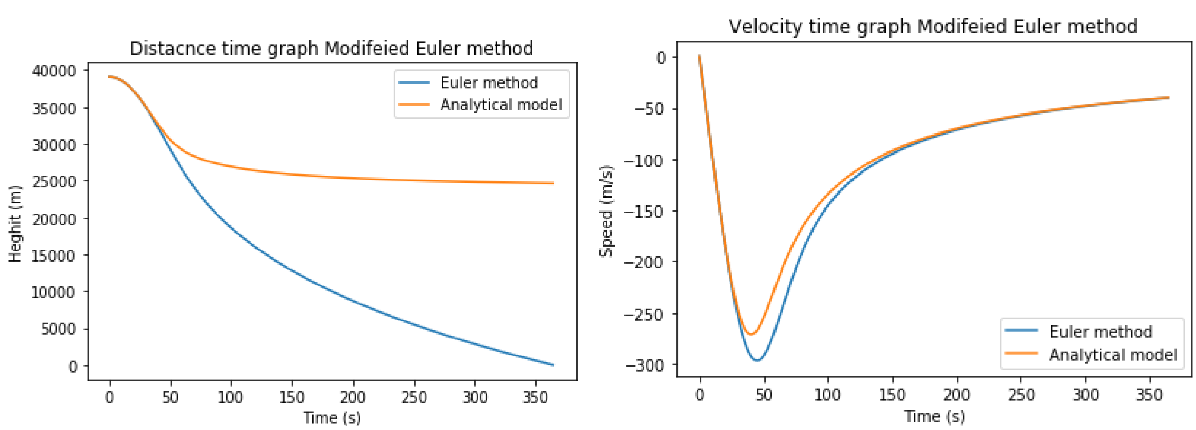

Figure 7 Plots of the distance time and speed time graphs using the modified Euler method and variable air density. The starting position was 39045m, the initial velocity was 0 m/s and the time step used in the calculations was 0.001s. The time taken to fall was 365s with a maximum velocity of -297 m/s.

Whilst the time taken to fall, and the maximum speed are not the same as in Felix's skydive this is not to surprising. The model used is simplified in a few ways that could account for the discrepancy. For the time taken, Felix deployed a parachute which would have slowed his descent increasing the time taken, the model does not take into account effect such as wind resistance which would have also slowed down his descent time and using equation 17 might increase how realistic the model is but there is doubt that it can accurately predict something as complicated as air density at every point along the descent. For the maximum velocity again, the air density model may be oversimplified resulting in a lower maximum velocity and whilst Felix fell he would have most likely been actively trying to reduce his cross-sectional area which may have caused an increased maximum velocity. The more worrying result is the difference between the analytical and numerical solution plots for position. This may be caused by the analytical solution assuming k is constant and not a function of position which would change the solution to the differential equations.

Improvements

The simplest improvement to the freefall model for Felix Baumgartner's skydive would be to find out at what altitude he deployed his parachute and at that altitude in the model increase the cross-sectional area and change the drag coefficient.

The modified Euler method is a two stage Runge-Kutta method, the Runge-Kutta methods are a family of numerical methods used to solve first order ODE's. An improvement to the problem may be to use the 4^{th} order Runge-Kutta method. This would reduce both the local and global truncation error's and would not require time steps as small as the modified Euler method saving computing time.

There are other methods of modelling the drag force, it would be interesting to see how the change if different models where used. In aerodynamics laminar and turbulent flow change the effects and magnitudes of the drag force on an object. Reynolds number gives an indication of which region an object is in depending on its velocity. If at each point Reynolds number was calculated and then used to decide whether turbulent or laminar drag models should be applied, it could make the model more realistic.