Level 5 Laboratory: Computational Physics — Exercise 3

The deadline for this exercise is **Friday 16th March 2018** at 12:30 p.m. A short report and program (*.py) files should be uploaded into Blackboard at the appropriate point in the Second Year Laboratory (PHY2DLM_2017) Course.

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Objectives of the exercise

- To become familiar with some basic tools for solving ordinary differential equations;
- To apply the Euler method to a 1D free fall problem with varying air resistance.

Problem: Free-fall with fixed or varying drag — (20 marks total)

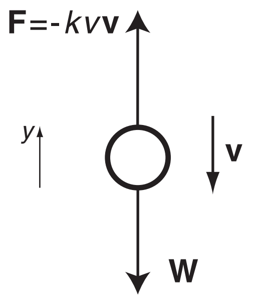
On 14th October 2012, Felix Baumgartner set the world record for falling from a great height. He jumped from a helium balloon at a height of 39045 m, fell for 4 minutes and 19 seconds and reached a maximum speed of 373 m.s^{-1}. In this problem, you will solve the equations of motion for a free-falling object, and use your program to confirm, or otherwise, Felix Baumgartner's statistics.[1]

For a projectile travelling at speed through the air, the air resistance is proportional to the square of the velocity and acts in the opposite direction. i.e. $\mathbf{F} = -kv^2\hat{\mathbf{v}} = -k|\mathbf{v}|\mathbf{v} = -kv\mathbf{v}$. The constant, k, is given by:

$$k = \frac{C_\mathrm{d}\rho_0 A}{2} \tag{1}$$

in which C_d is the drag coefficient (~ 0.47 for a sphere; $\sim 1.0 - 1.3$ for a sky diver or ski jumper), A is the cross sectional area of the projectile and ρ_0 is the air density ($\sim 1.2\,\mathrm{kg\,m^{-3}}$ at ambient temperature and pressure).

In this problem, the acceleration is varying, so Newton's equations of motion produce a second order ODE to solve. As illustrated in lectures, we can do this if we separate it into two first order equations, one for the derivative of the velocity, the other for the position:



$$m\mathbf{a} = \mathbf{W} + \mathbf{F}$$

$$i.e. \qquad m\frac{dv_y}{dt} = -mg - k|v_y|v_y \tag{2}$$

$$\text{and} \quad \frac{dy}{dt} = v_y \tag{3}$$

The y coordinate is taken vertically upwards. Euler's method for solving:

$$\frac{dy}{dt} = f(y, t)$$

is summarised by:

$$y_{n+1} = y_n + \Delta t . f(y_n, t_n) \qquad ; \qquad t_{n+1} = t_n + \Delta t \tag{4}$$

in which we are determining y and t at the $(n+1)$th step from their values at the nth step. Applying this scheme to Eqs. (2) and (3), we obtain:

$$v_{y,n+1} = v_{y,n} - \Delta t\left(g + \frac{k}{m}|v_{y,n}|v_{y,n}\right) \tag{5}$$

$$y_{n+1} = y_n + \Delta t . v_{y,n} \tag{6}$$

$$t_{n+1} = t_n + \Delta t \tag{7}$$

If we provide the initial conditions i.e. y_0 and $v_{y,0}$, we can use the above scheme repeatedly to find y and v_y for all t.

[1] In fact Felix BaumGartner's record stood until 24th October 2014, when Alan Eustace (a senior Google vice president) jumped from 41419 m, reaching a maximum speed of 367 m.s^{-1}. A sonic boom was heard by observers on the ground. However, Eustace's attempt was very low-key compared with Baumgartner, and it is the latter who tends to be remembered.

Attempt the following programming tasks and address the points raised in your report. Incorporate your code in a menu system similar to that used in the previous two exercises:

a) Write a Python program using the Euler method to calculate $y(t)$ and $v_y(t)$ for a free falling body. You will need to provide sensible values for C_d, A and m. Use a starting height of 1 km and zero initial velocity. You will need to specify a condition for ending the simulation i.e. when the body reaches the ground. Plot your results.

b) Verify the correct functioning of your program by comparing with the following analytical prediction:

$$y = y_0 - \frac{m}{2k}\log_e\left[\cosh^2\left(\sqrt{\frac{kg}{m}}.t\right)\right] \tag{8}$$

$$v_y = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}.t\right) \tag{9}$$

N.B. In these equations, you can set $y_0 = 1\,\mathrm{km}$ and calculate y and v_y for *any* t.

Examine the effect on your solution of varying the step size Δt. Investigate how closely your solution tracks the real trajectory for a range of step sizes. What happens to the simulation when Δt is very large? Explore the effect on the motion of varying the ratio k/m.

c) Include an option to use the “modified Euler method” as outlined in the Lecture Notes. Compare the results from the two methods for different values of Δt.

Now you are going to make the problem more realistic. Baumgartner jumped from very high altitude, where the air density is very low, and so the drag factor, k should be replaced with a function $k(y)$. The simplest way to approach this is to make use of the scale height for the atmosphere, h, and model the variation of density as an exponential decay:

$$\rho(y) = \rho_0\exp(-y/h) \tag{10}$$

from which $k(y)$ follows using Eq. (1). An appropriate value of h appears to be 7.64 km (see http://en.wikipedia.org/wiki/Scale_height).

d) Adapt your program to use $\rho(y)$ instead of ρ_0. Test your program using the parameters for Baumgartner’s jump. Plot $y(t)$ and $v_y(t)$ against time. The most interesting feature of the motion is that the downwards speed goes through a maximum that is much greater than the terminal velocities predicted for constant ρ. Do you observe this? There was great interest in whether Baumgartner would break the sound barrier during this stage of his jump. According to your simulation, does he manage it?

e) Investigate the effect of varying the jump parameters on the maximum speed achieved, and the total duration of the jump. You should consider varying the jump height and the quantity $C_\mathrm{d}A/m$.

Submitting your work

You should submit the following to Blackboard:

1. A concise report, in MS Word or pdf format;
2. Final version of your program for the free-fall problem.

As before, please note:

- Only upload your “prog.py" files.
- Blackboard anti-plagiarism software won’t accept a “.py" extension so please rename your file with a “.txt" extension.
- Please also give your programs sensible distinguishing names, including your name or userid e.g. “my_userid_ex3.txt".

If you have any problems submitting your work, please contact Dr. Hanna (s.hanna@bristol.ac.uk) or ask a demonstrator.