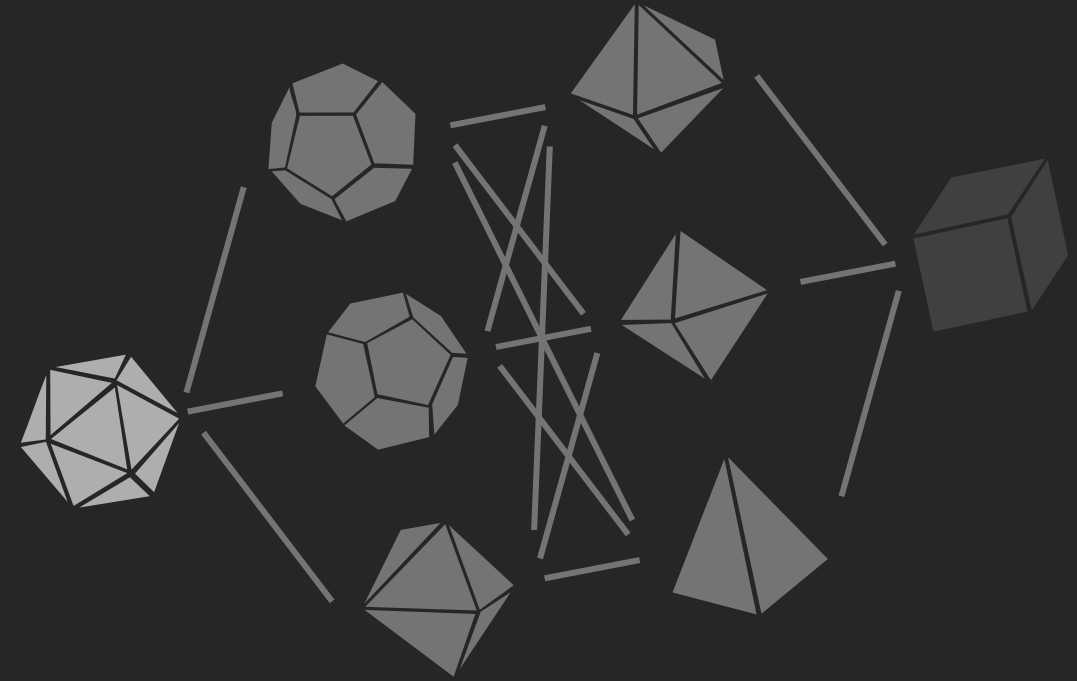


# ► CaVE

**A Cone-Aligned Approach for  
Fast Predict-then-Optimize**



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# Introduction

CaVE (Cone-aligned Vector Estimation) is a **Decision-focused Learning / End-to-end Predict-then-optimize** approach for **Binary Linear Programs** (BLPs).



## End-to-End:

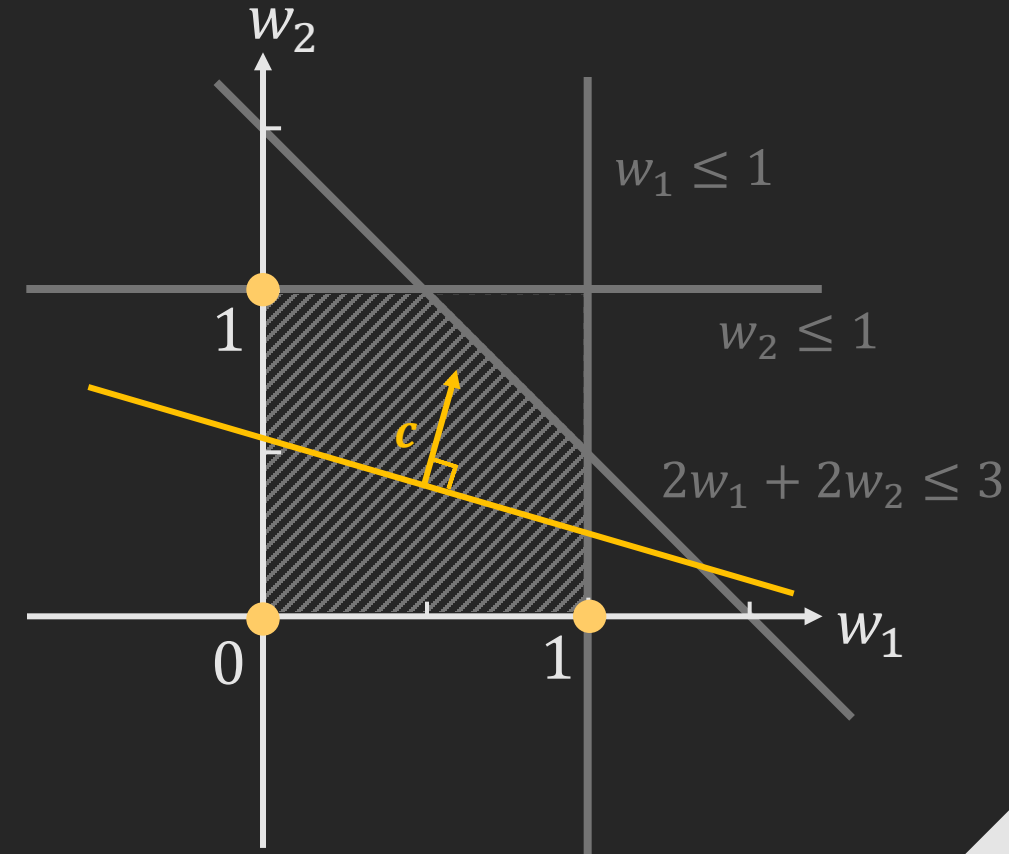
The loss function of CaVE focuses on decision quality.

## Efficiency:

The algorithm of CaVE utilizes non-negative least squares (NNLSs) instead of solving BLPs.

# Notation

$$\min_{\underbrace{w}_{\text{Decision Variables}}} \left\{ \overbrace{c^T w}^{\text{Linear Objective Function}} : \underbrace{w \in S}_{\text{Feasible Region}}, \underbrace{w \in \{0,1\}}_{\text{Binary Domain}} \right\}$$



# Predict, then Optimize

$$\min_w \{ \mathbf{c}_1^\top \mathbf{w} : \mathbf{w} \in S \}$$

$$\min_w \{ \mathbf{c}_2^\top \mathbf{w} : \mathbf{w} \in S \}$$

$$\min_w \{ \mathbf{c}_3^\top \mathbf{w} : \mathbf{w} \in S \}$$

$$\vdots$$

# Predict, then Optimize

Unknown  
Coefficients

$$\begin{aligned} \min_w \{ \mathbf{c}_1^\top \mathbf{w} : \mathbf{w} \in S \} \\ \min_w \{ \mathbf{c}_2^\top \mathbf{w} : \mathbf{w} \in S \} \\ \min_w \{ \mathbf{c}_3^\top \mathbf{w} : \mathbf{w} \in S \} \\ \vdots \end{aligned}$$

Identical  
Constraints

# Predict, then Optimize

Unknown  
Coefficients

$$\begin{aligned} \min_w \{ \mathbf{c}_1^\top \mathbf{w} : \mathbf{w} \in S \} \\ \min_w \{ \mathbf{c}_2^\top \mathbf{w} : \mathbf{w} \in S \} \\ \min_w \{ \mathbf{c}_3^\top \mathbf{w} : \mathbf{w} \in S \} \\ \vdots \end{aligned}$$

Identical  
Constraints

Observed Feature Vectors

$\mathbf{x}_1$

$\mathbf{x}_2$

$\mathbf{x}_3$

$\vdots$

# Predict, then Optimize

Unknown  
Coefficients

$$\begin{aligned} \min_w \{ \mathbf{c}_1^\top \mathbf{w} : \mathbf{w} \in S \} \\ \min_w \{ \mathbf{c}_2^\top \mathbf{w} : \mathbf{w} \in S \} \\ \min_w \{ \mathbf{c}_3^\top \mathbf{w} : \mathbf{w} \in S \} \\ \vdots \end{aligned}$$

Identical Constraints

Observed Feature Vectors

$\mathbf{x}_1$

$\mathbf{x}_2$

$\mathbf{x}_3$

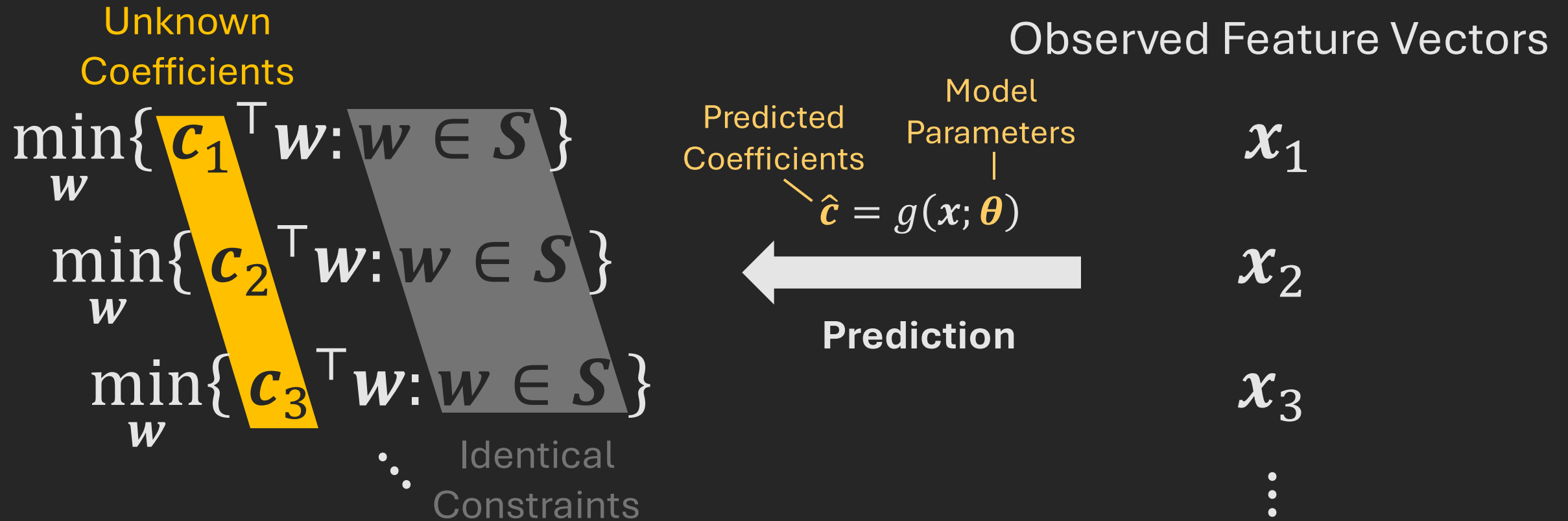
$\vdots$

$$\hat{\mathbf{c}} = g(\mathbf{x}; \theta)$$

Prediction



# Predict, then Optimize



# Examples



❖ Vehicle Routing



❖ Energy Scheduling



❖ Portfolio Optimization

# Examples



❖ Vehicle Routing



❖ Energy Scheduling



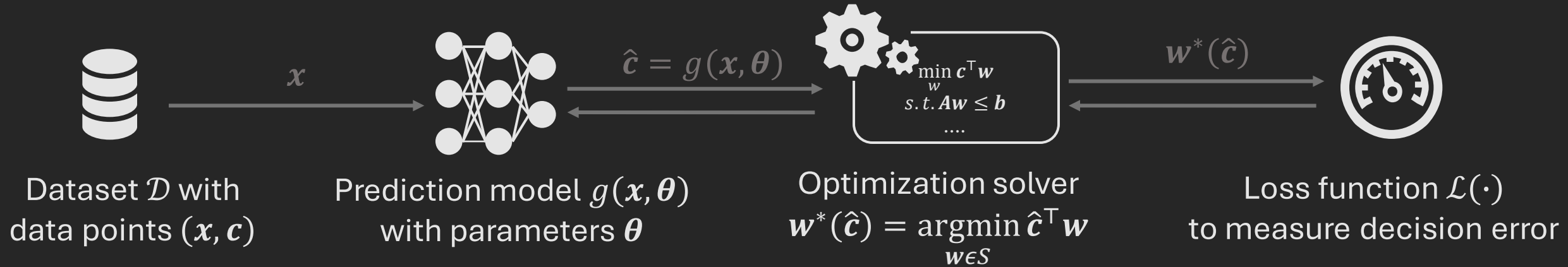
❖ Portfolio Optimization

**? Unknown Costs:** Travel Time, Electricity Prices, Asset Returns

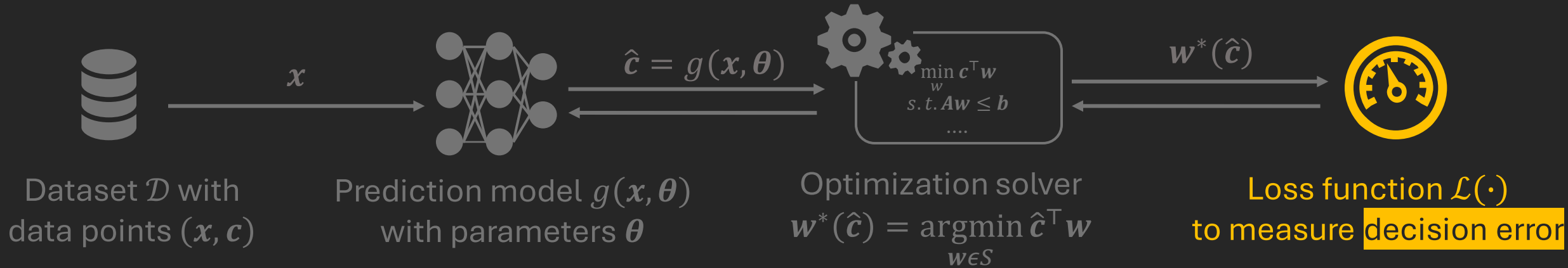


**Observed Features:** Distance, Time, Weather, Financial Factors...

# Decision-focused Learning



# Decision-focused Learning

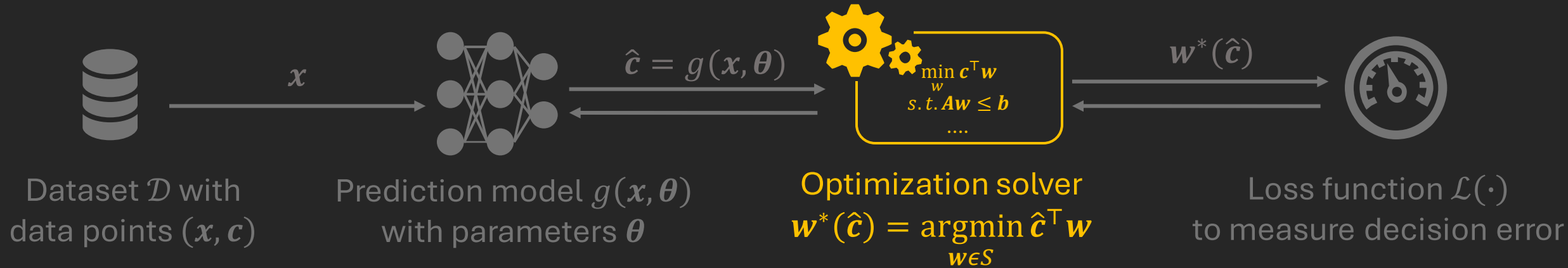


**e.g.**

$$\mathcal{L}_{\text{Regret}}(\hat{c}, c) = c^\top w^*(\hat{c}) - c^\top w^*(c)$$

$$\mathcal{L}_{\text{Square}}(\hat{c}, c) = \frac{1}{2} \|w^*(c) - w^*(\hat{c})\|_2^2$$

# Decision-focused Learning



## Computational Bottleneck:

All state-of-the-art methods require repeated solving during the iteration.

**1,000 Dataset Instances**

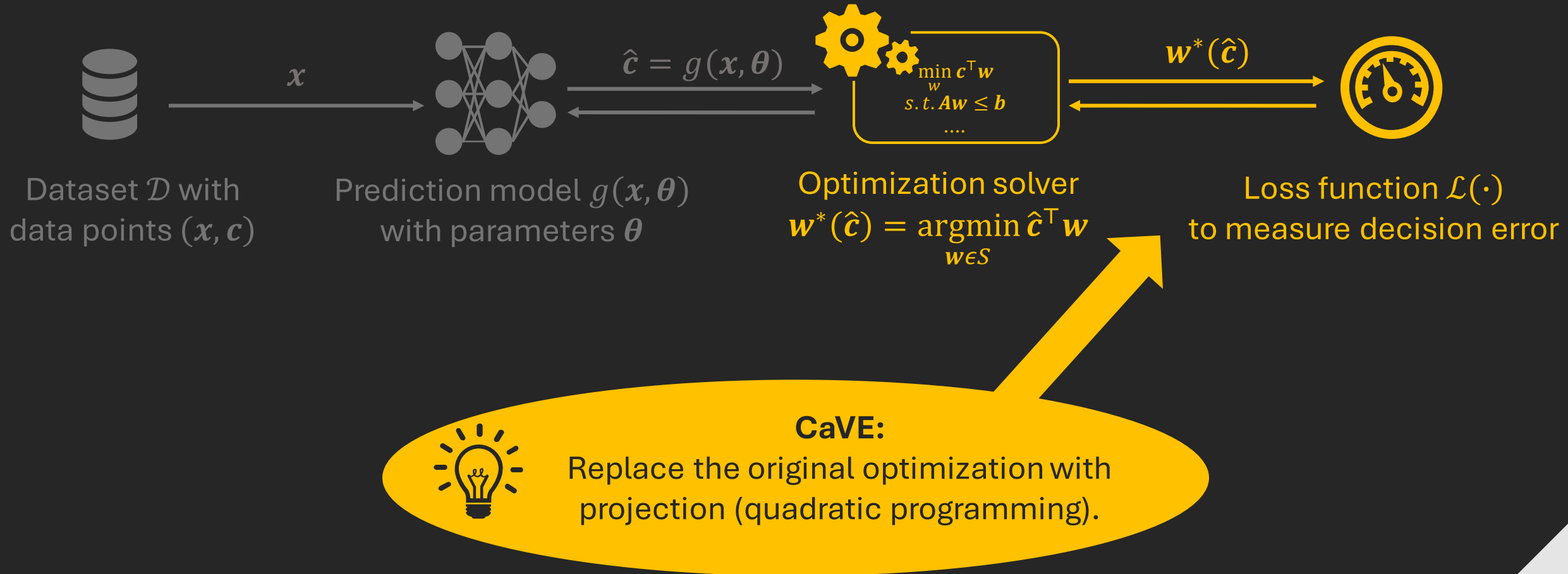
×

**20 Training Epochs**

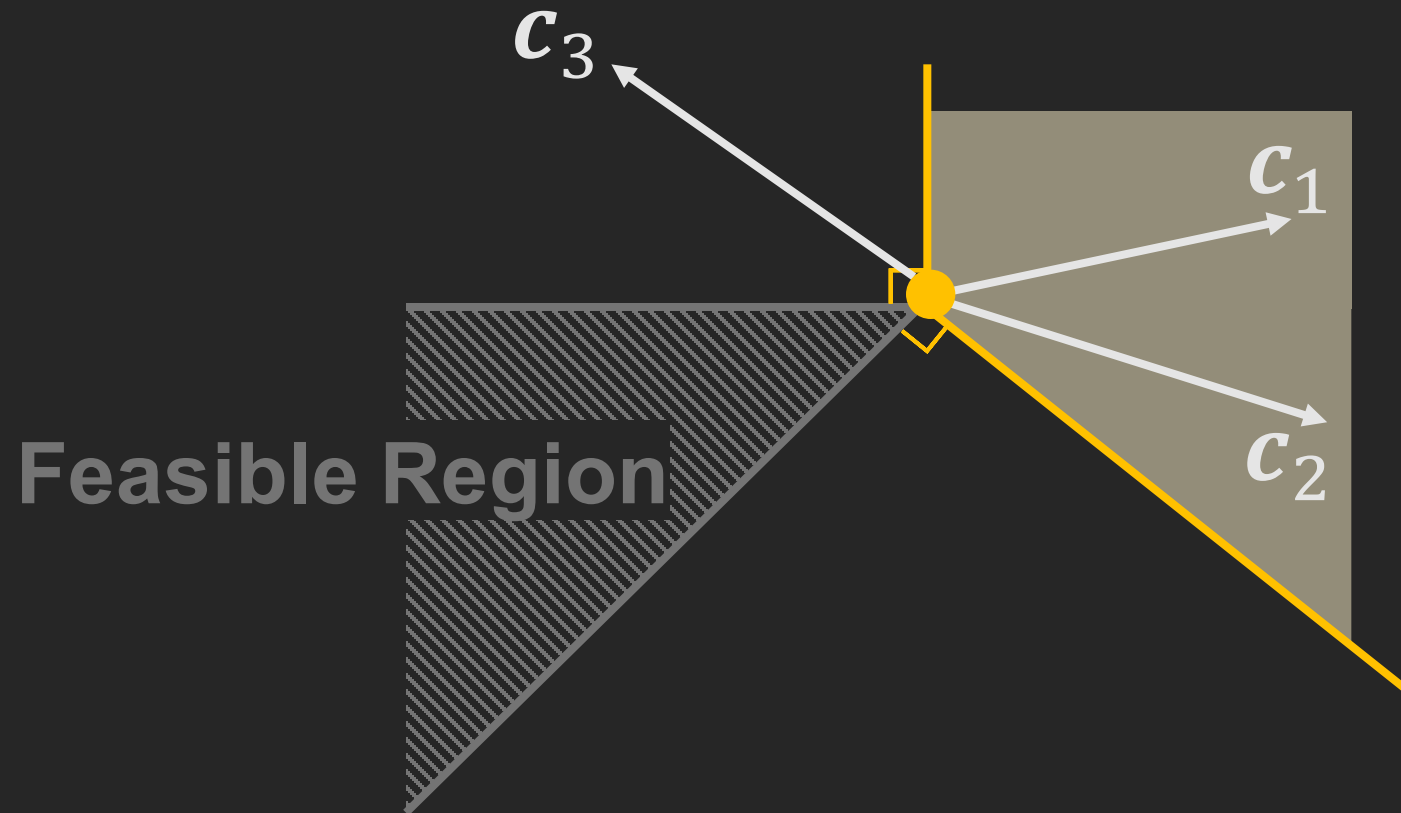
⇓

**≥20,000 Solving!!!**

# Decision-focused Learning



# Geometric Intuition

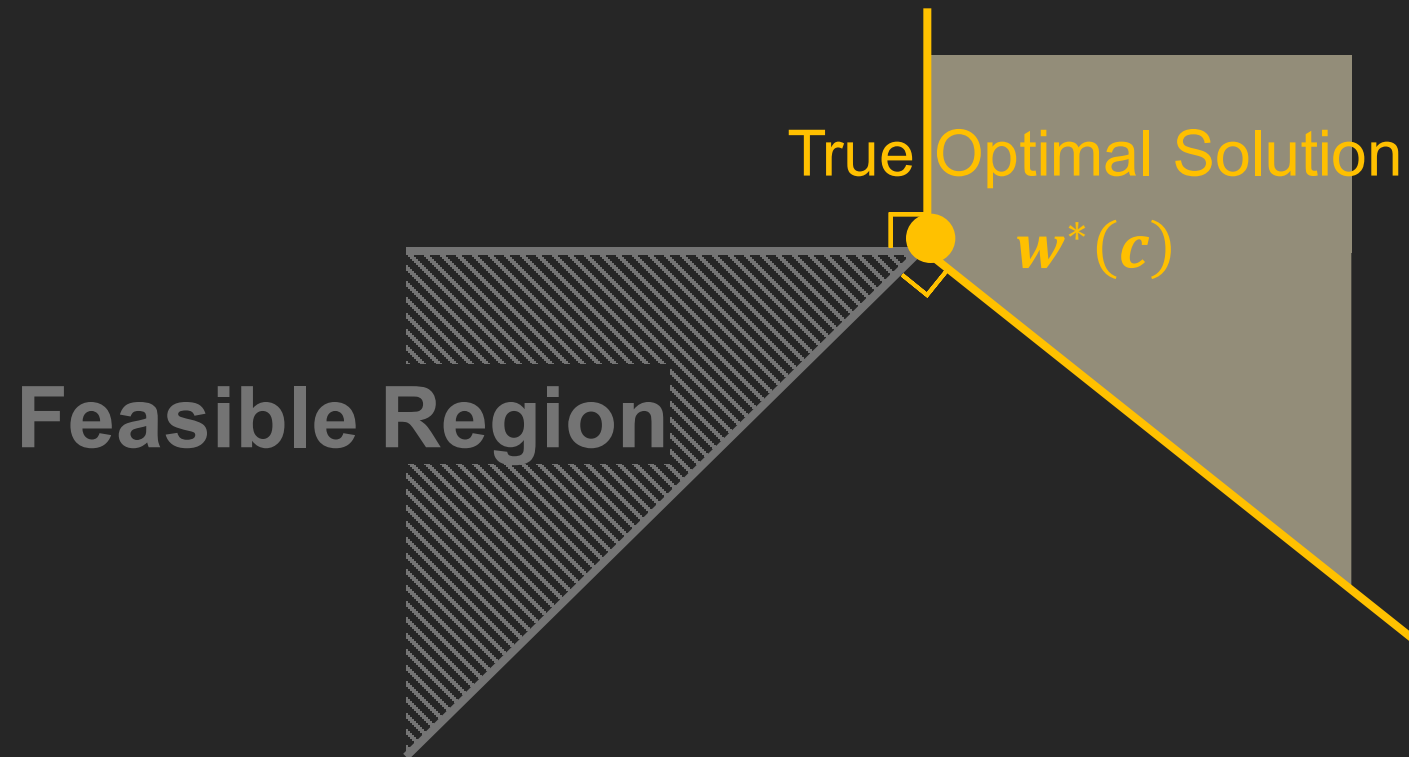


For LP, cost vectors in the same **normal cone** have the same optimal solutions.

$$w^*(c_1) = w^*(c_2) \neq w^*(c_3)$$

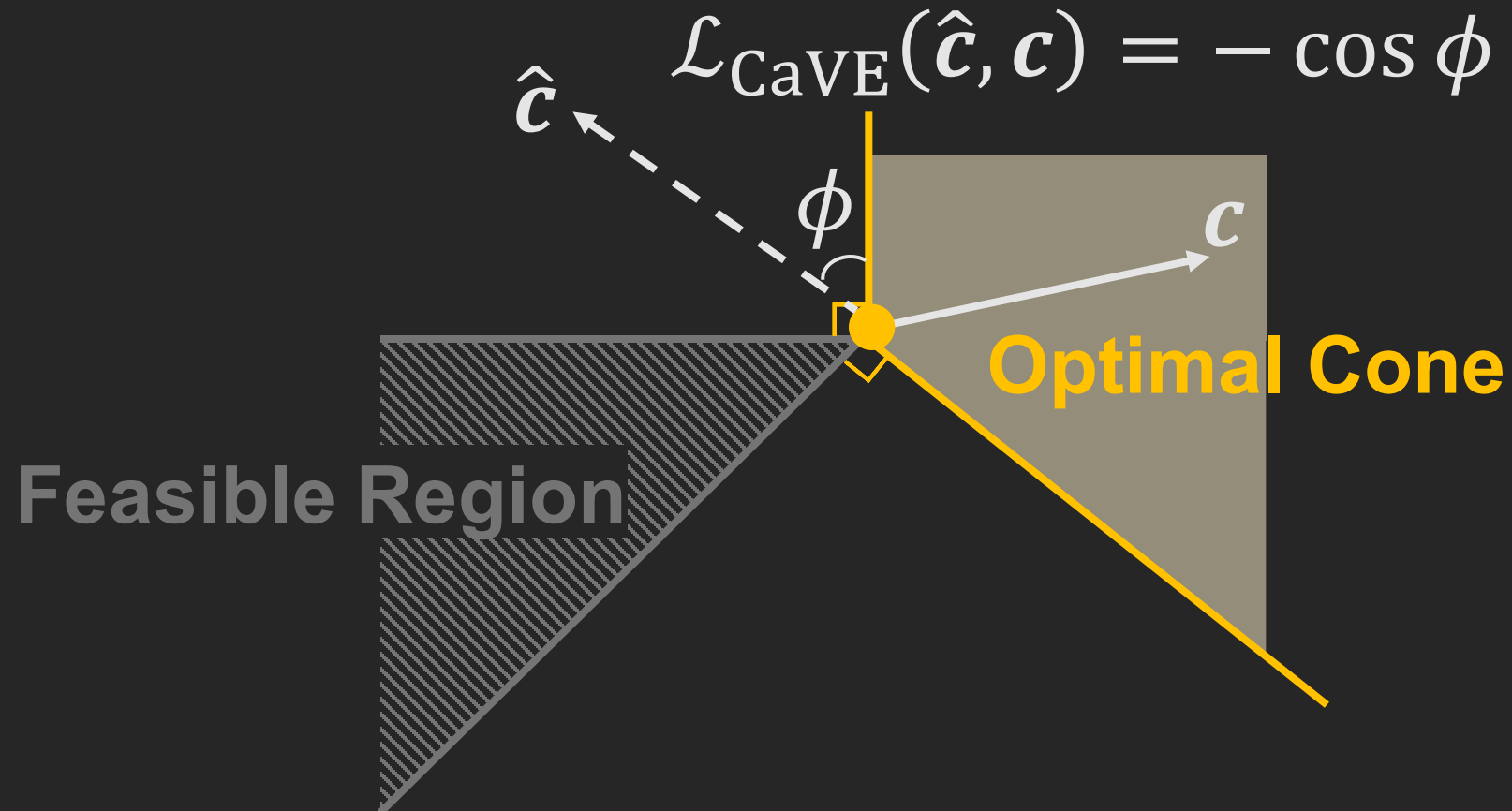


# Geometric Intuition



For LP, **optimal cone**  $\mathcal{C}^*(c)$  is defined as the normal cone at the true optimal solution  $w^*(c)$ .

# Similarity Loss Function



The loss of CaVE employs cosine similarity to minimize the **angle  $\phi$**  between the **predicted costs  $\hat{c}$**  and **optimal subcone** (a subset of **optimal cone**).

# Similarity Loss Function

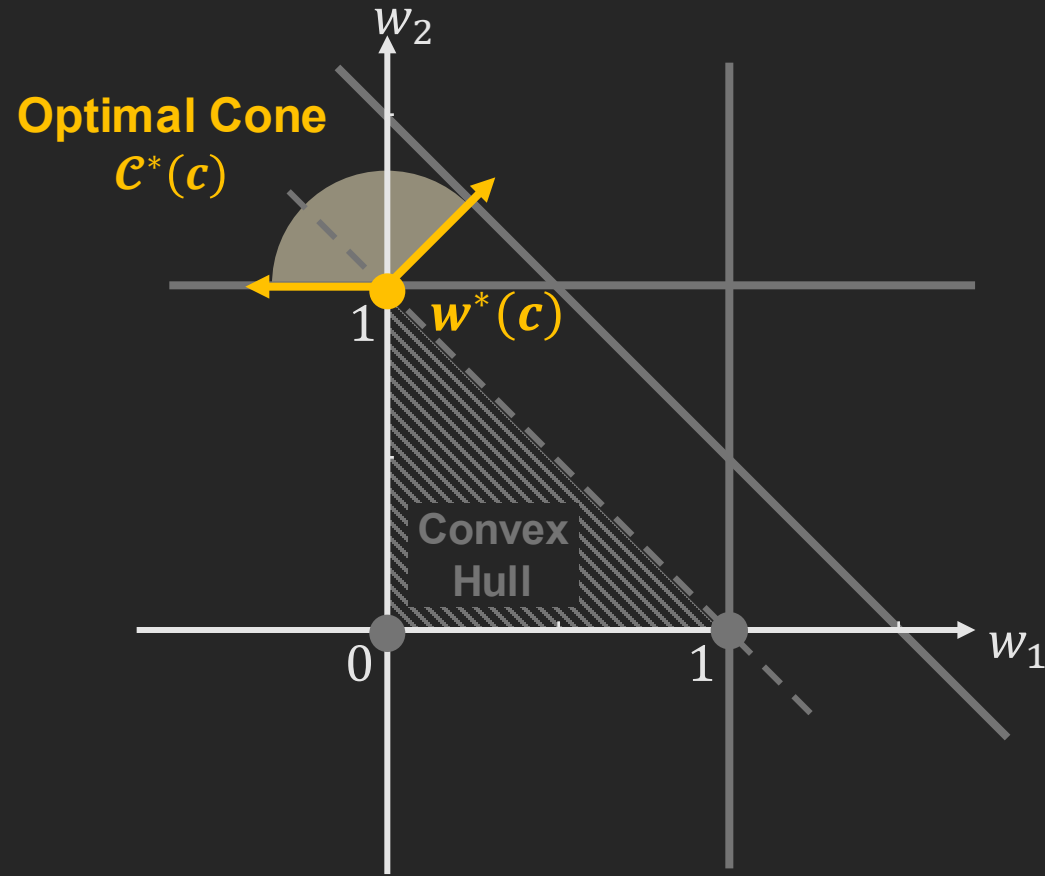

$$\mathcal{L}_{\text{CaVE}}(\hat{c}, c) = -\cos \phi$$

## IMPORTANT QUESTIONS:

- What is the optimal subcone?
- How to obtain the angle  $\phi$ ?

The loss of CaVE employs cosine similarity to minimize the **angle  $\phi$**  between the **predicted costs  $\hat{c}$**  and **optimal subcone** (a subset of **optimal cone**).

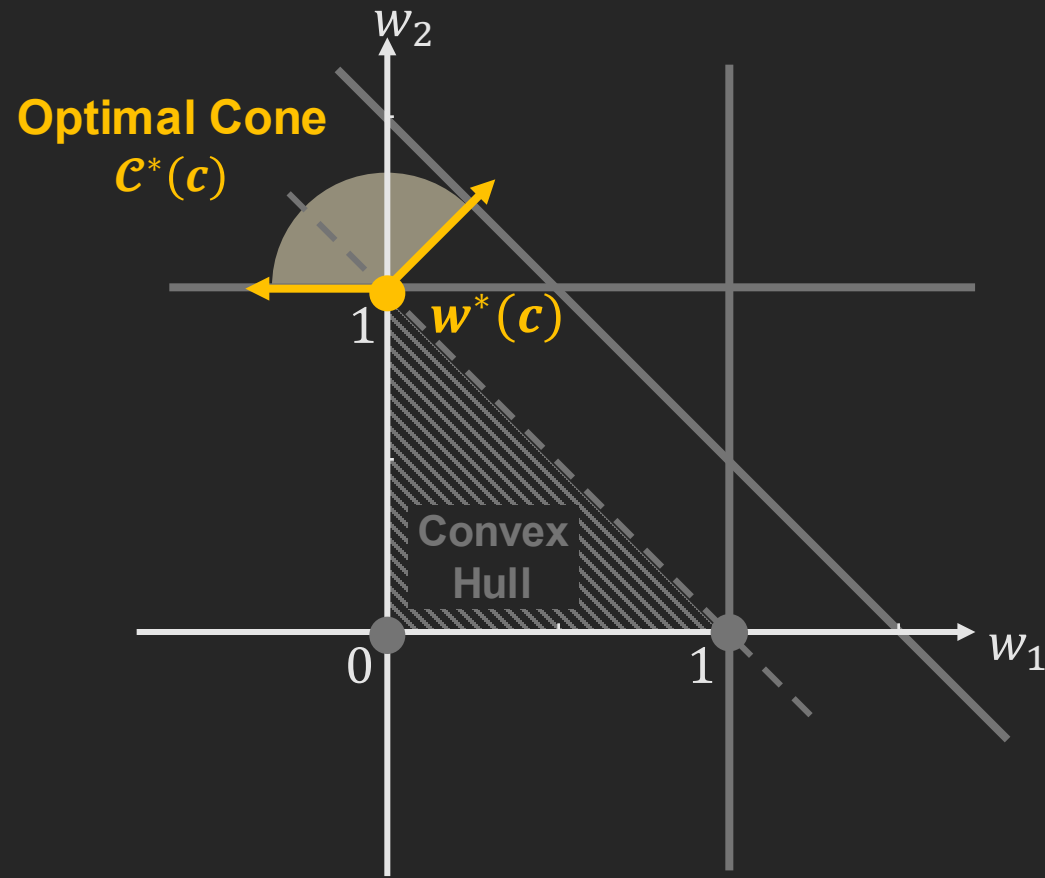
# Optimal Subcone



Binary Linear Program

1. For ILP, the normal cone to the convex hull at the optimal solution  $w^*(c)$  is defined as **optimal cone**  $\mathcal{C}^*(c)$ .

# Optimal Subcone



Binary Linear Program

1. For ILP, the normal cone to the convex hull at the optimal solution  $w^*(c)$  is defined as **optimal cone**  $\mathcal{C}^*(c)$ .
2.  $\forall c' \in \mathcal{C}^*(c), w^*(c') = w^*(c)$ . Cost vectors yield the same optimal solution if and only if they are in the same optimal cone.

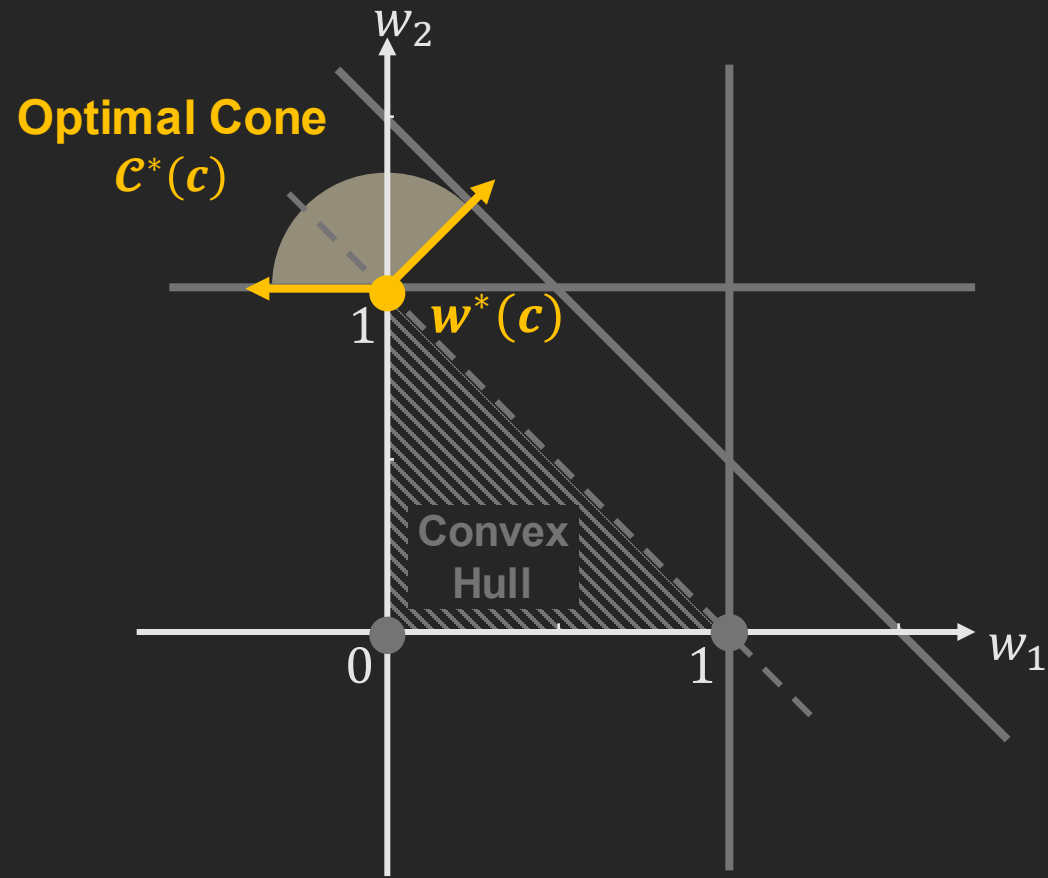
# Optimal Subcone



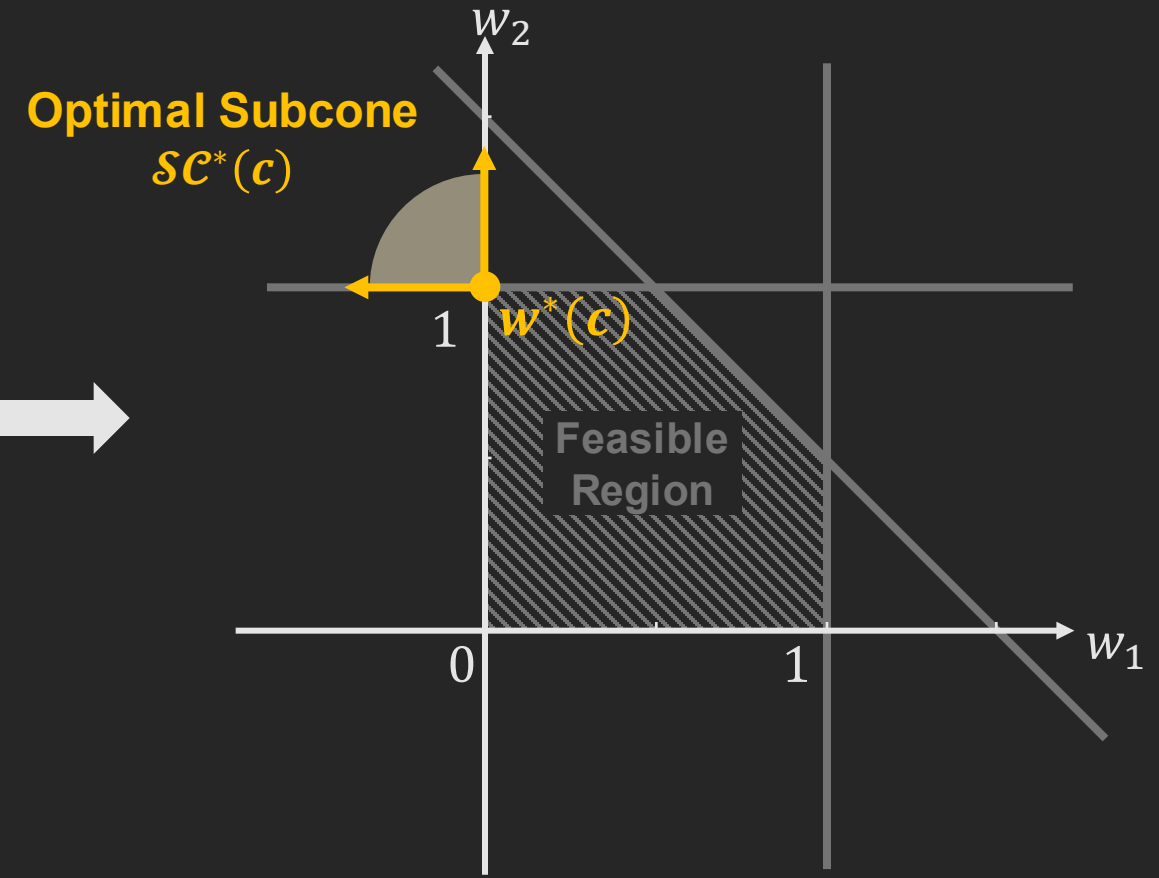
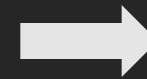
Binary Linear Program

1. For ILP, the normal cone to the convex hull at the optimal solution  $w^*(c)$  is defined as **optimal cone**  $\mathcal{C}^*(c)$ .
2.  $\forall c' \in \mathcal{C}^*(c), w^*(c') = w^*(c)$ . Cost vectors yield the same optimal solution if and only if they are in the same optimal cone.
3. However, for ILP, obtaining the convex hull is **NOT** trivial. e.g., Cutting Plane method...

# Optimal Subcone



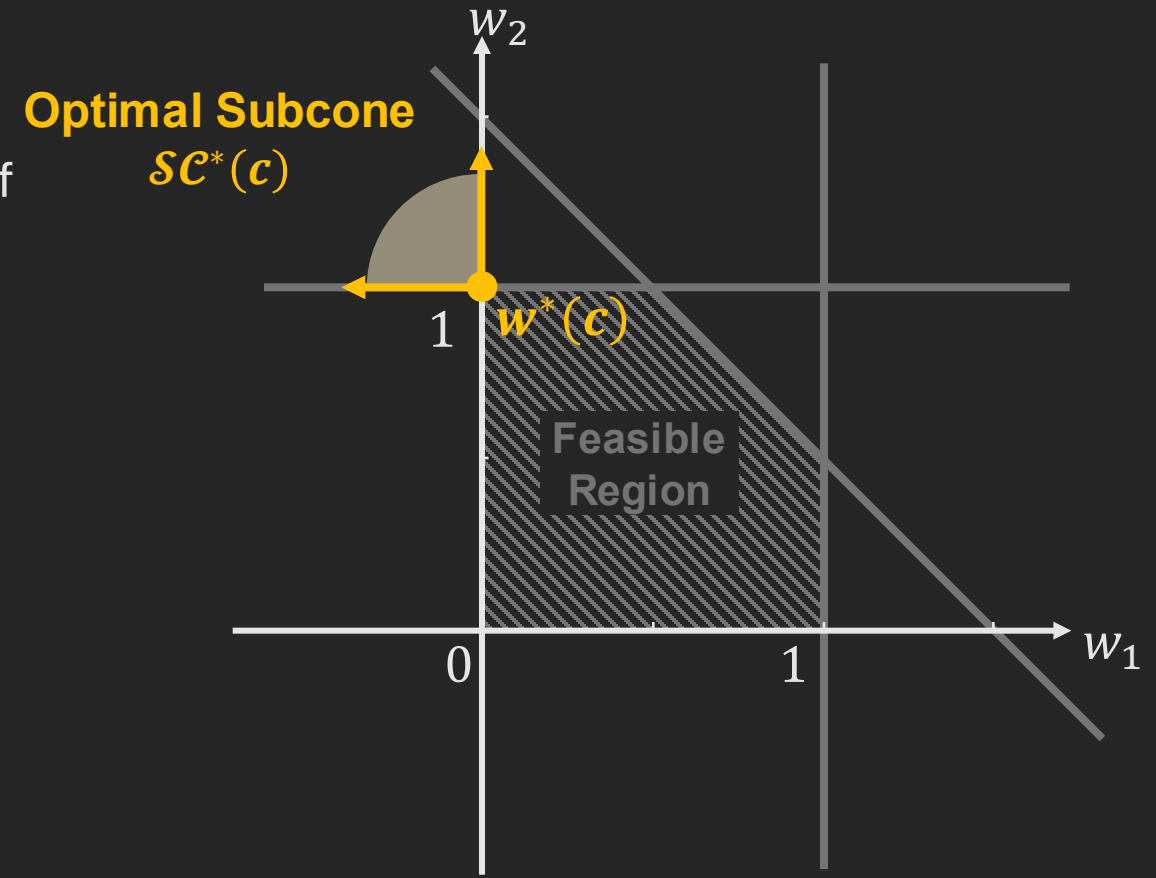
Binary Linear Program



Linear Relaxation

# Optimal Subcone

1. For BLP, the normal cone to the feasible region of linear relaxation at the optimal solution  $w^*(c)$  is defined as **optimal subcone**  $\mathcal{SC}^*(c)$ .



Linear Relaxation

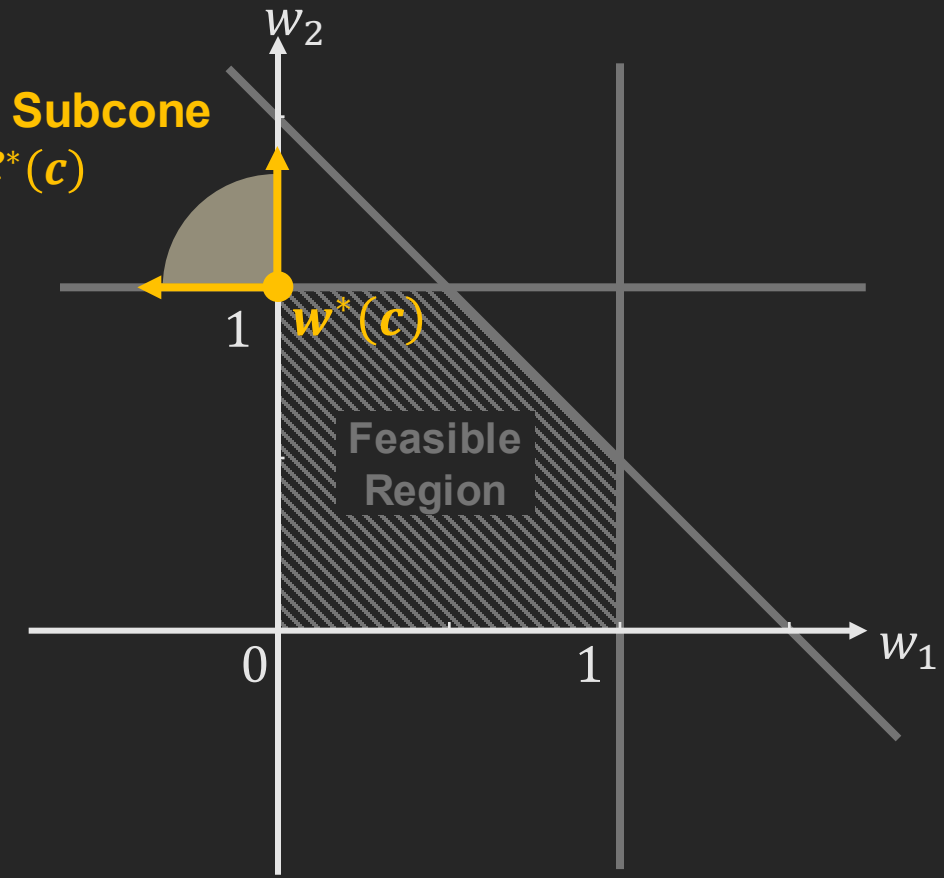


# Optimal Subcone

1. For BLP, the normal cone to the feasible region of linear relaxation at the optimal solution  $w^*(c)$  is defined as **optimal subcone**  $\mathcal{SC}^*(c)$ .

**Note: This does not apply to general ILP!**

**Optimal Subcone**  
 $\mathcal{SC}^*(c)$

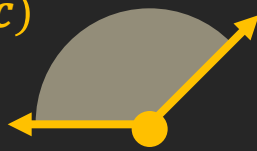


Linear Relaxation

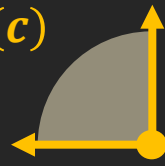
# Optimal Subcone

1. For BLP, the normal cone to the feasible region of linear relaxation at the optimal solution  $w^*(c)$  is defined as **optimal subcone**  $\mathcal{SC}^*(c)$ .
2.  $\mathcal{SC}^*(c) \subseteq \mathcal{C}^*(c)$ . Thus, cost vectors yield the same optimal solution if they are in the same optimal subcone.

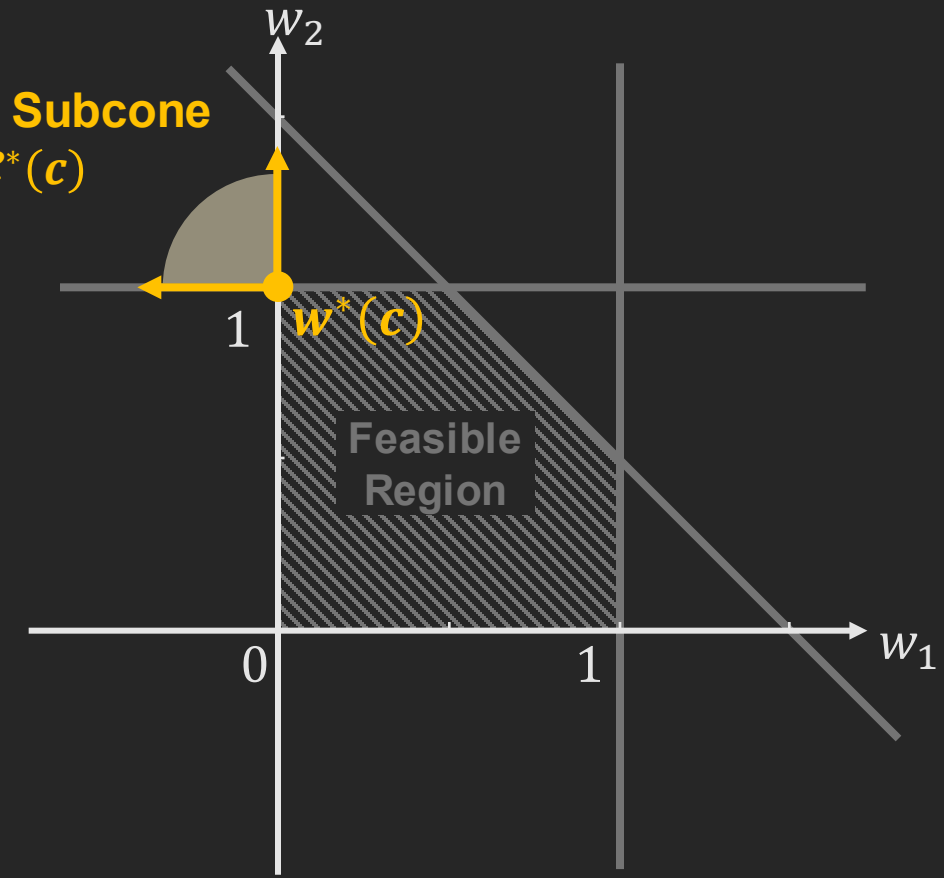
**Optimal Cone**  
 $\mathcal{C}^*(c)$



**Optimal Subcone**  
 $\mathcal{SC}^*(c)$



**Optimal Subcone**  
 $\mathcal{SC}^*(c)$

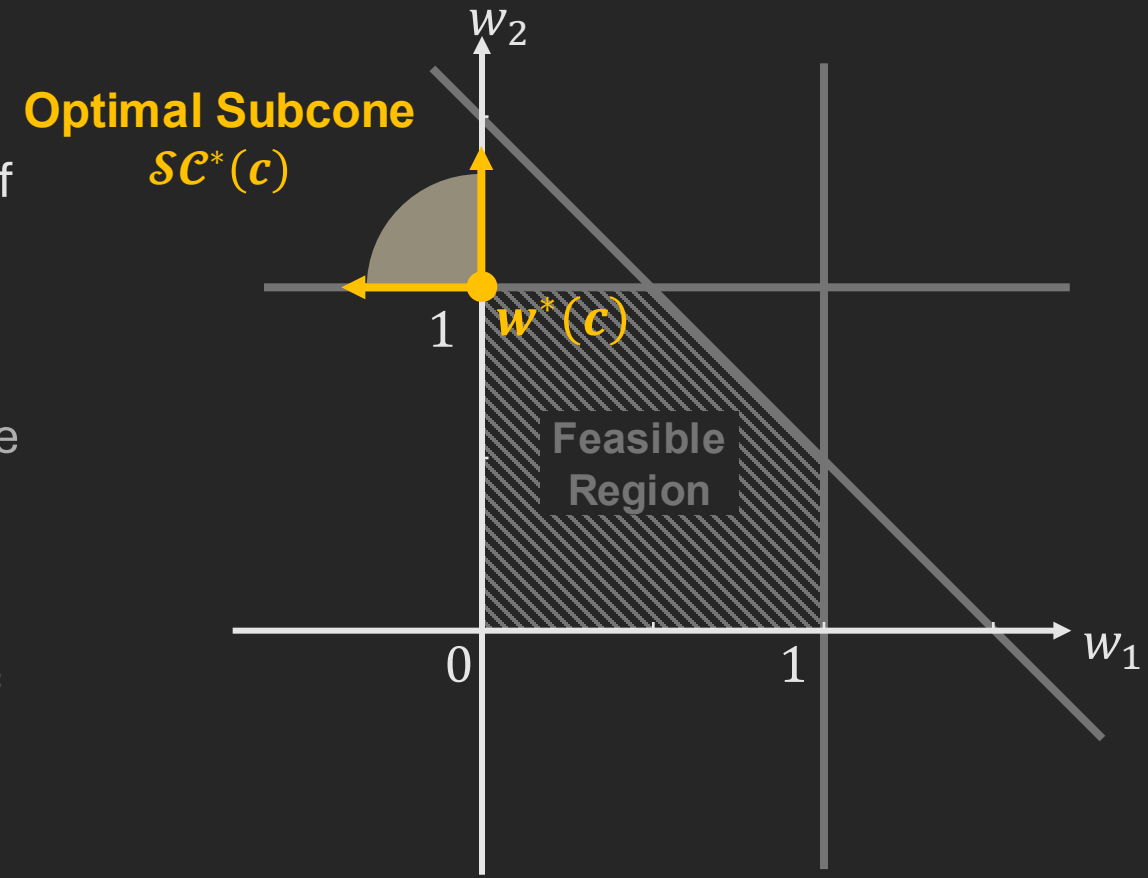


**Linear Relaxation**

# Optimal Subcone

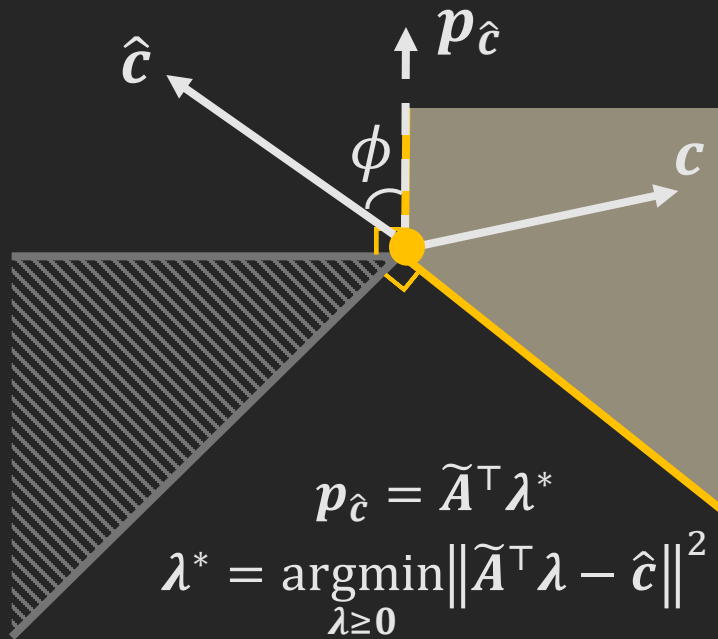
1. For BLP, the normal cone to the feasible region of linear relaxation at the optimal solution  $w^*(c)$  is defined as **optimal subcone**  $\mathcal{SC}^*(c)$ .
2.  $\mathcal{SC}^*(c) \subseteq \mathcal{C}^*(c)$ . Thus, cost vectors yield the same optimal solution if they are in the same optimal subcone.
3. The optimal subcone is the conic combination of **tight constraints**, which is trivial.

$$\tilde{A}(c): \tilde{A}(c)^\top w^*(c) = b$$

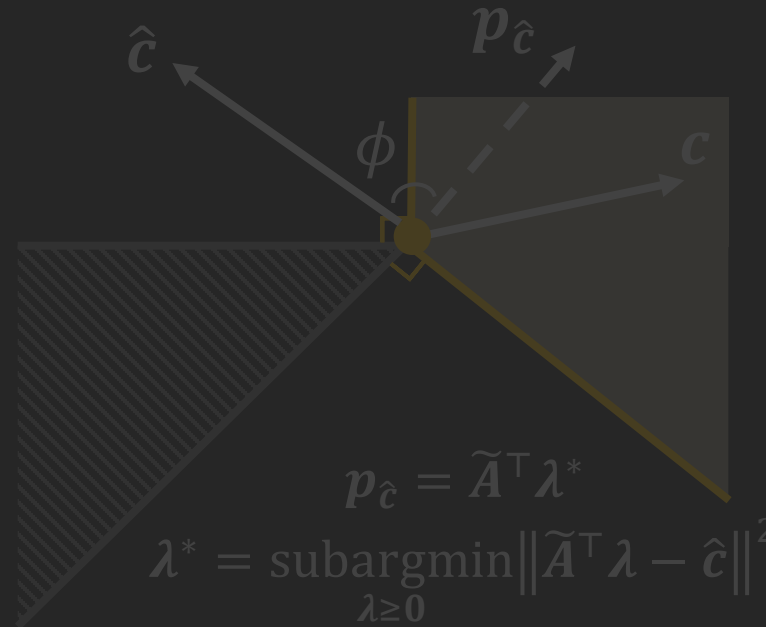


Linear Relaxation

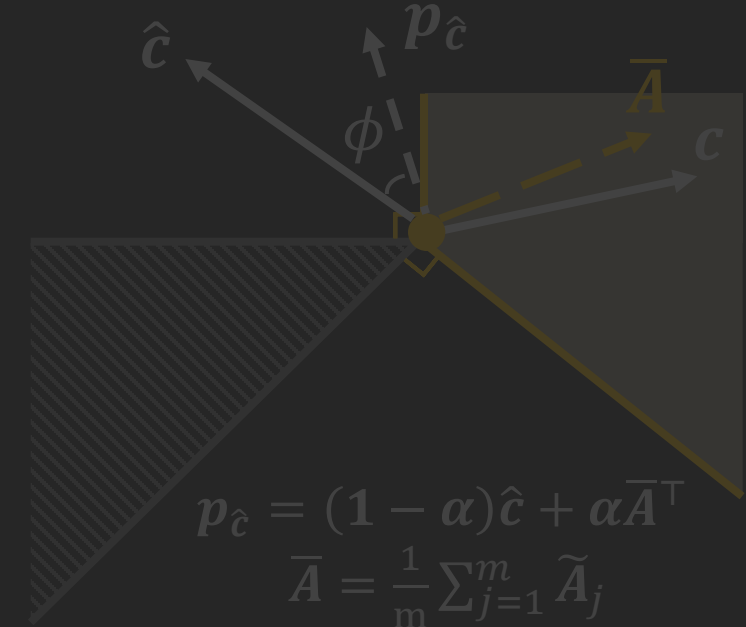
# Projection



Exact Projection



Inner Projection



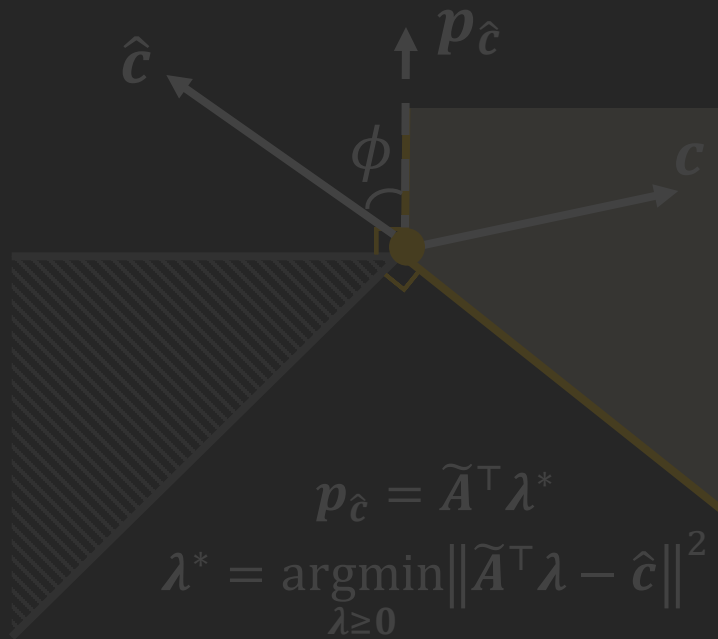
Heuristic Projection



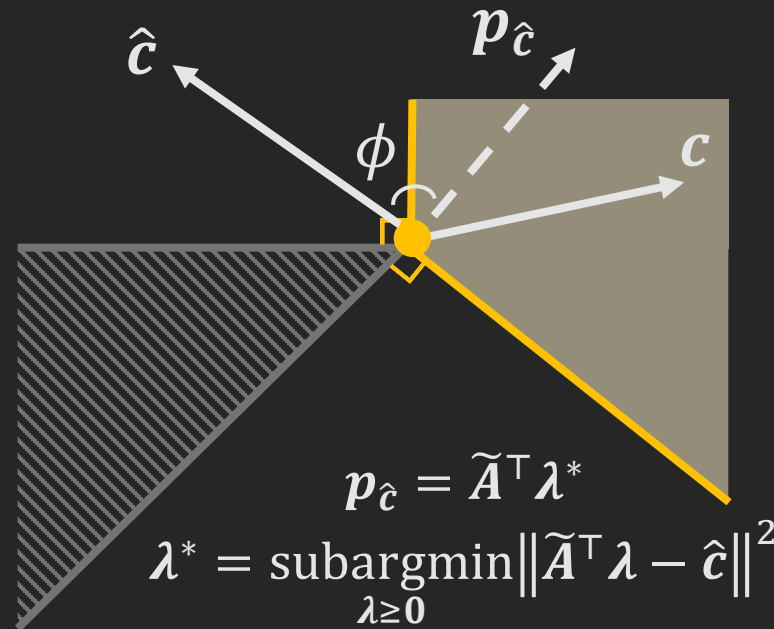
CaVE Exact performs exact projection, wherein the optimal solution of the NNLS is on the surface of the cone.

This approach results in the vanishing gradients as the predicted cost vector close the surface.

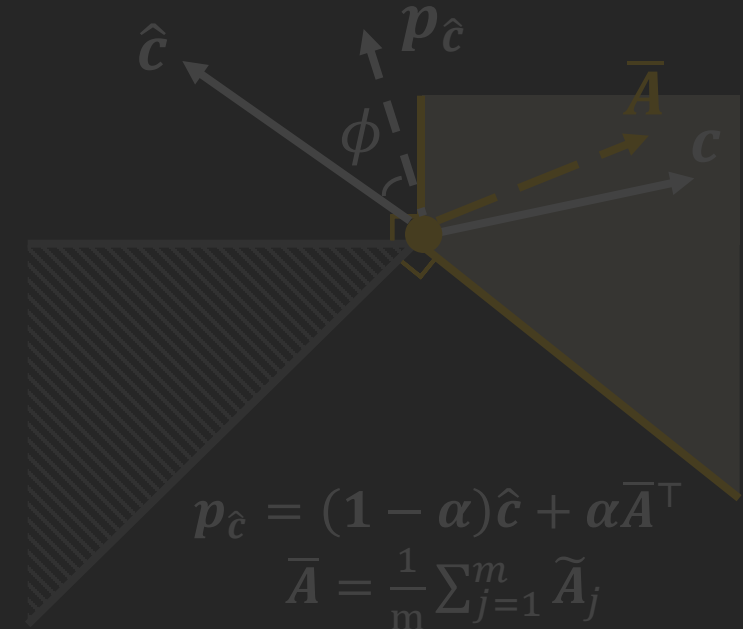
# Projection



Exact Projection



Inner Projection



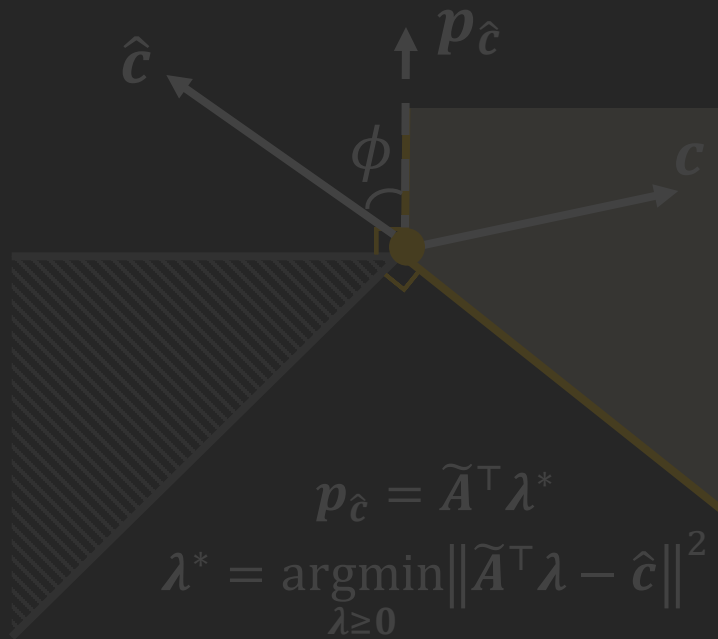
Heuristic Projection



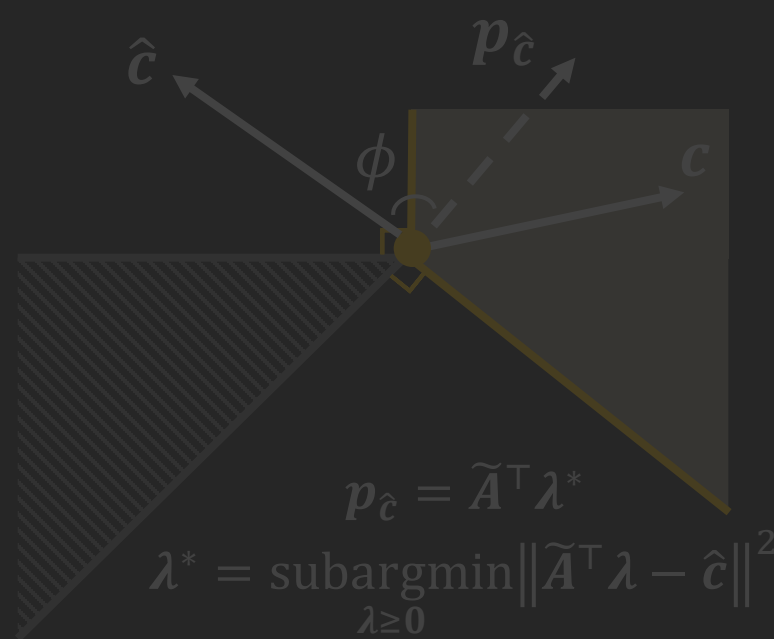
The goal is to obtain a projection of the predicted cost vector that lies inside the subcone. (suboptimal solution)

Since the solver (Clarabel) uses the primal-dual interior point, the feasibility is guaranteed at each iteration. We can simply set the maximum iterations.

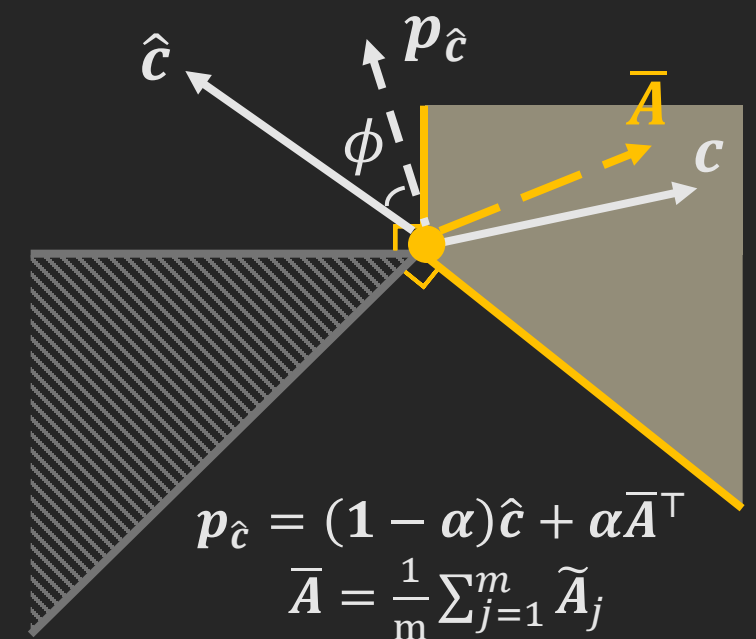
# Projection



Exact Projection



Inner Projection



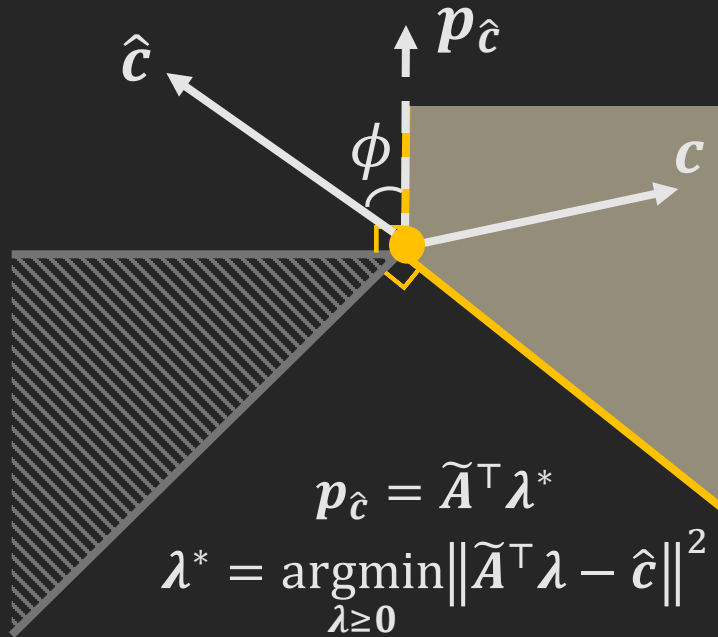
Heuristic Projection



A heuristic projection that does not require solving NNLS and relies on simple operations.

This approach does NOT guarantee feasibility, yet it ensures that the cost vector is pushed towards the cone.

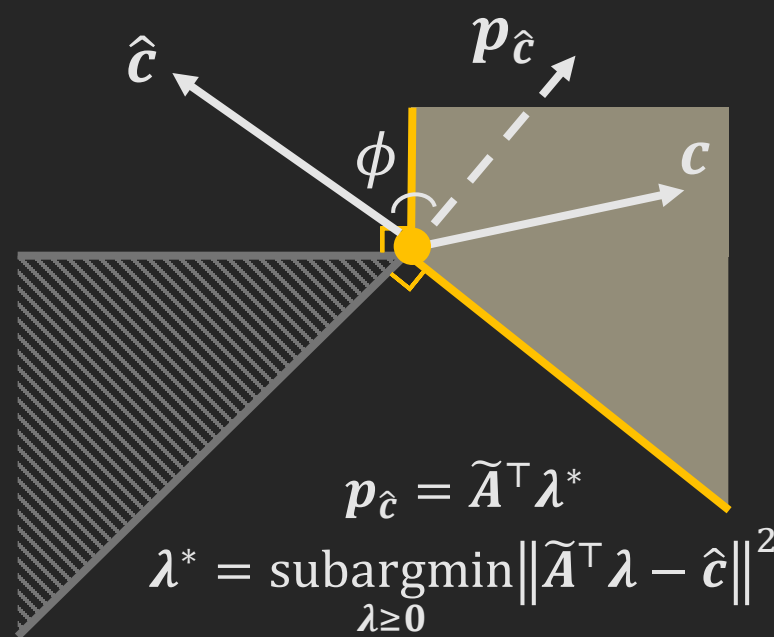
# Projection



Exact Projection



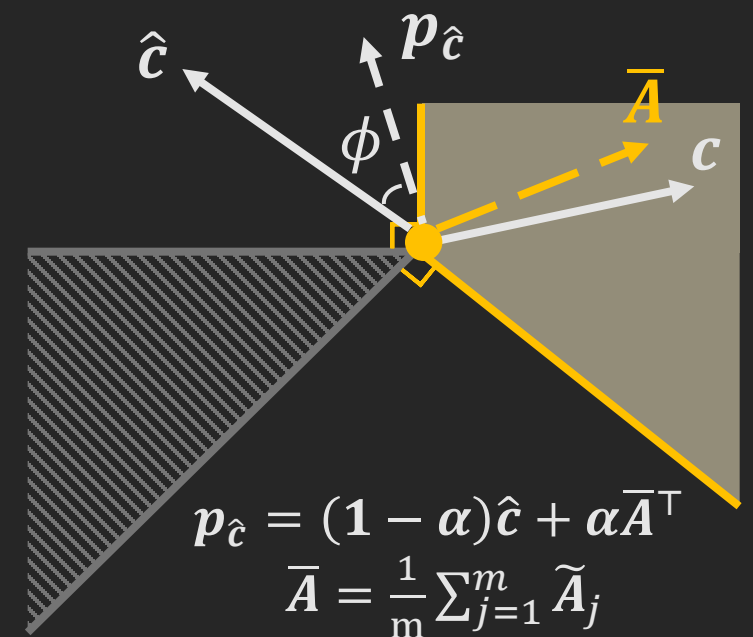
CaVE Exact



Inner Projection



CaVE+



Heuristic Projection



CaVE Hybrid

# Algorithm

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## Algorithm    Cone-aligned Vector Estimation (CaVE)

---

**Require:** Pairs of feature vectors and binding constraints  $\{(\mathbf{x}^i, \tilde{\mathbf{A}}^i)\}_{i=1}^n$  for  $n$  training instances; learning rate  $\alpha > 0$

```
1: Initialize model parameters  $\theta$ 
2: for each training epoch do
3:   for each batch of training samples  $(\mathbf{x}, \tilde{\mathbf{A}})$  do
4:     Predict cost coefficient  $\hat{\mathbf{c}} \leftarrow g(\mathbf{x}, \theta)$ 
5:     Compute projection  $\mathbf{p}_{\hat{\mathbf{c}}}$  with quadratic program
6:     Compute cosine similarity loss  $\mathcal{L}_{\text{CaVE}}(\hat{\mathbf{c}}, \tilde{\mathbf{A}})$ 
7:     Compute the gradient  $\nabla_{\theta} \mathcal{L}_{\text{CaVE}}(\hat{\mathbf{c}}, \tilde{\mathbf{A}})$  with backpropagation
8:     Update ML model parameters  $\theta \leftarrow \theta - \alpha \nabla_{\theta} \mathcal{L}_{\text{CaVE}}(\hat{\mathbf{c}}, \tilde{\mathbf{A}})$ 
9:   end for
10: end for
11: return  $g(\cdot, \theta)$ 
```

---



# Experiments - SP5

## Shortest Path on 5 × 5 Grid

Average Test **Normalized Regret** (%) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
Deg 4	8.82 ± 1.15	10.73 ± 1.54	8.39 ± 0.95	8.35 ± 0.88	7.79 ± 1.00	<b>7.68</b> ± 0.99	11.34 ± 1.11
Deg 6	12.58 ± 2.14	11.30 ± 1.30	8.89 ± 0.90	8.84 ± 1.00	<b>7.72</b> ± 1.11	7.86 ± 0.96	13.78 ± 1.58

Average **Training Time** (Sec) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
Deg 4	1.52 ± 0.14	4.64 ± 0.09	4.89 ± 0.12	<b>2.57</b> ± 0.19	17.64 ± 0.12	18.52 ± 0.31	4.50 ± 0.48
Deg 6	1.38 ± 0.13	3.52 ± 0.11	3.72 ± 0.14	<b>2.39</b> ± 0.19	18.68 ± 0.40	17.78 ± 0.13	4.38 ± 0.42

# Experiments - SP5

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**Synthetic Ground Truth Generation:**

The degree of the polynomial from features to costs increases the complexity.  
 The higher the degree, the greater the difficulty for learning.

$$c \propto (ax + b)^{\text{Deg}} + \epsilon$$

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
Deg 4	1.52 ± 0.14	4.64 ± 0.09	4.89 ± 0.12	<b>2.57</b> ± 0.19	17.64 ± 0.12	18.52 ± 0.31	4.50 ± 0.48
Deg 6	1.38 ± 0.13	3.52 ± 0.11	3.72 ± 0.14	<b>2.39</b> ± 0.19	18.68 ± 0.40	17.78 ± 0.13	4.38 ± 0.42

# Experiments - TSP20

## Traveling Salesperson with 20 Nodes

Average Test **Normalized Regret** (%) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	12.12 ± 0.89	7.35 ± 0.40	6.20 ± 0.24	7.69 ± 0.33	<b>5.95</b> ± 0.16	6.56 ± 0.21	12.21 ± 0.88
<b>Deg 6</b>	21.32 ± 1.81	8.01 ± 0.45	<b>6.97</b> ± 0.37	9.52 ± 0.64	7.48 ± 0.36	7.41 ± 0.37	14.31 ± 0.40

Average **Training Time** (Sec) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	1.52 ± 0.10	113.56 ± 3.16	107.15 ± 3.80	27.06 ± 2.17	175.23 ± 4.95	220.21 ± 24.20	<b>25.92</b> ± 4.23
<b>Deg 6</b>	1.53 ± 0.19	158.66 ± 9.65	102.19 ± 10.38	30.17 ± 2.62	185.13 ± 7.44	185.02 ± 5.09	<b>25.48</b> ± 3.66

# Experiments - TSP50

## Traveling Salesperson with 50 Nodes

Average Test **Normalized Regret** (%) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	28.16 ± 1.08	15.19 ± 0.65	7.69 ± 0.22	9.59 ± 0.44	<b>7.57</b> ± 0.20	8.03 ± 0.23	14.31 ± 0.40
<b>Deg 6</b>	52.61 ± 2.36	23.25 ± 2.41	<b>8.57</b> ± 0.38	11.28 ± 0.80	10.26 ± 0.46	9.00 ± 0.52	17.12 ± 0.48

Average **Training Time** (Sec) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	1.55 ± 0.18	611.47 ± 23.52	518.07 ± 51.89	196.96 ± 35.92	1220.68 ± 85.39	1328.99 ± 28.87	<b>151.80</b> ± 24.21
<b>Deg 6</b>	1.16 ± 0.13	502.71 ± 16.03	573.87 ± 20.19	253.93 ± 27.67	1191.29 ± 42.63	1456.21 ± 34.18	<b>155.95</b> ± 24.46

# Experiments - CVRP20

## Capacity Vehicle Routing with 20 Nodes

Average Test **Normalized Regret** (%) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	10.10 ± 0.64	9.26 ± 1.56	6.44 ± 0.24	7.92 ± 0.52	<b>5.94</b> ± 0.25	6.32 ± 0.28	15.77 ± 0.96
<b>Deg 6</b>	19.50 ± 1.22	11.64 ± 0.25	<b>7.94</b> ± 0.54	11.44 ± 1.14	8.75 ± 0.28	8.09 ± 0.57	18.96 ± 1.01

Average **Training Time** (Sec) with Standard Deviation

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	1.65 ± 0.48	213.56 ± 42.36	153.56 ± 11.08	<b>44.52</b> ± 6.27	7020.11 ± 1043.05	3773.31 ± 288.84	583.56 ± 170.67
<b>Deg 6</b>	1.54 ± 0.25	208.95 ± 12.90	127.94 ± 13.84	<b>51.83</b> ± 8.78	2204.83 ± 99.86	6197.84 ± 288.63	470.20 ± 84.46

# Experiments - CVRP30

## Capacity Vehicle Routing with 30 Nodes

Test **Normalized Regret** (%)

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	19.72	12.54	<b>9.13</b>	9.99	N/A		18.28

**Training Time** (Sec)

Methods	2-Stage	CaVE-E	CaVE+	CaVE-H	SPO+	PFYL	NCE
<b>Deg 4</b>	9.27	331.73	287.77	<b>132.62</b>	≥100h		884.95

Due to the scale of the problem, we did not repeat our experimental evaluation with random seeds.

# Experiments - Relaxation

## Traveling Salesperson with 50 Nodes

Average Test **Normalized Regret** (%)

Methods	CaVE+	SPO+ Rel	PFYL Rel
Deg 4	<b>7.69</b> ± 0.22	10.17 ± 0.23	11.11 ± 0.33
Deg 6	<b>8.57</b> ± 0.38	13.14 ± 0.46	13.38 ± 0.58

Average **Training Time** (Sec)

Methods	CaVE+	SPO+ Rel	PFYL Rel
Deg 4	518.07 ± 51.89	<b>386.06</b> ± 9.69	536.67 ± 4.94
Deg 6	573.87 ± 20.19	636.99 ± 3.04	<b>510.37</b> ± 3.46

## Capacity Vehicle Routing with 20 Nodes

Average Test **Normalized Regret** (%)

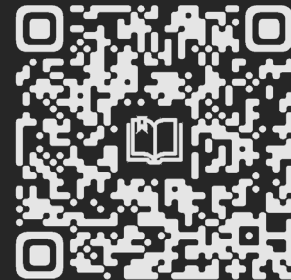
Methods	CaVE+	SPO+ Rel	PFYL Rel
Deg 4	<b>6.44</b> ± 0.24	8.03 ± 0.38	17.07 ± 0.63
Deg 6	<b>7.94</b> ± 0.54	15.73 ± 0.39	19.19 ± 1.66

Average **Training Time** (Sec)

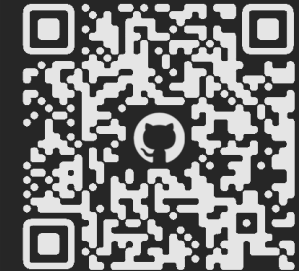
Methods	CaVE+	SPO+ Rel	PFYL Rel
Deg 4	153.56 ± 11.08	78.95 ± 0.73	<b>78.80</b> ± 1.19
Deg 6	127.94 ± 13.84	<b>78.74</b> ± 3.82	81.80 ± 0.86

# Thank You

Paper



GitHub



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