PHYS 310 Work and Results

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1 Interpolation

1.1 Lagrange Interpolation

1.1.1 Manual Function

The first attempt to code a second-degree Lagrange Interpolation of the data sets modeling

$$y = x^2$$

is by utilizing a self-coded function:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.interpolate import lagrange

# Starting with the manual Interpolation

x2=np.array([1, 3, 5])
y2=np.array([0.8, 10, 23.5])

x_lagrange=np.arange(0,10,0.1)

def lagrange_interpol(x,x2, y2):
        P1 = (y2[0]*(x-x2[1])*(x-x2[2]))/((x2[0]-x2[1])*(x2[0]-x2[2]))
        P2 = (y2[1]*(x-x2[0])*(x-x2[2]))/((x2[1]-x2[0])*(x2[1]-x2[2]))
        P3 = (y2[2]*(x-x2[0])*(x-x2[1]))/((x2[2]-x2[0])*(x2[2]-x2[1]))
```

```
        y = P1 + P2 + P3
        return y


x1= np.arange(0,10, 0.1)
y1 = np.array([])
for i in x1:
        y1 = np.append(y1, lagrange_interpol(i,x2,y2))


plt.plot(x2[0],y2[0], 'bo',label='P1')
plt.plot(x2[1],y2[1], 'mo',label='P2')
plt.plot(x2[2],y2[2], 'ko',label='P3')
plt.plot(x1,y1, color='r')
plt.grid()
plt.legend()
plt.title('Lagrange Interpolation of y=x^2')
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```

The outputted Graph recovers the data points and the result of the interpolation:

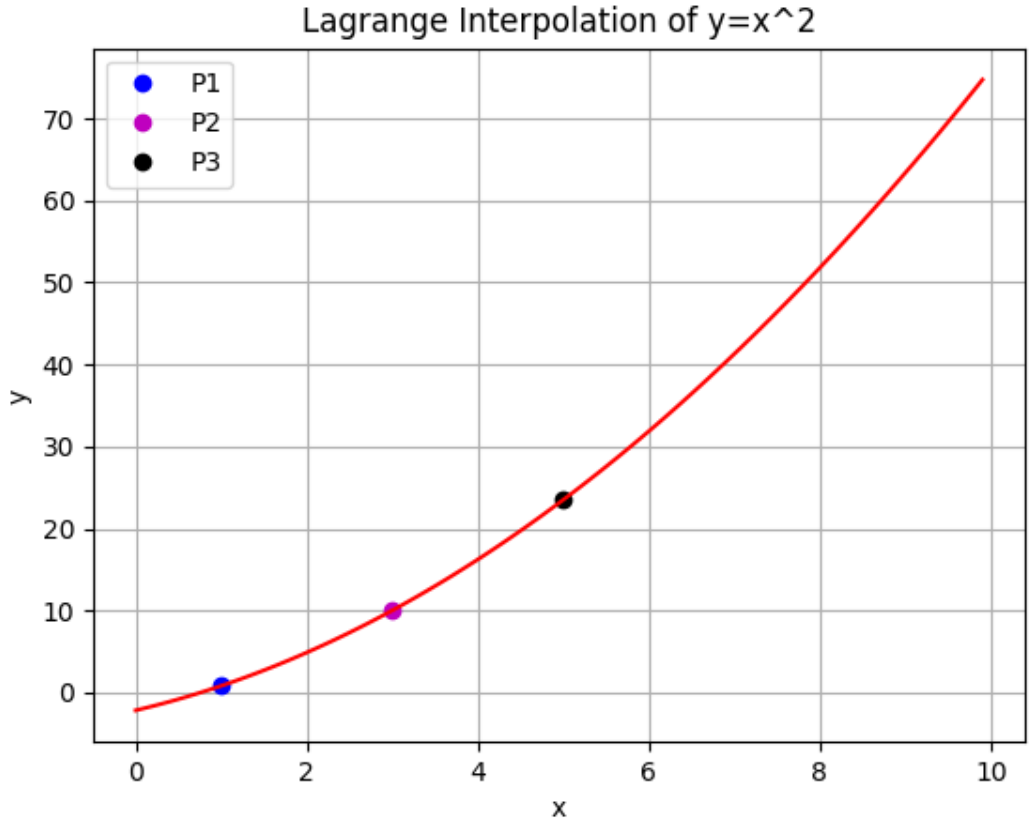

Figure 1: Lagrange Interpolation using a Manual Function

1.2 Atkins Method

Attempting to Model

$$y = \frac{1}{1+x}$$

using both the Atkins Method of Polynomials and the built in function in Scipy library.

```
        import numpy as np
        import matplotlib.pyplot as plt
        from scipy.interpolate import lagrange
# Newton/Atkins Method And Quadratic Splines
def Atkins_method(x,x1,y1):
        p12 = (x-x1[1])/(x1[0] - x1[1]) * y1[0] + (x-x1[0])/(x1[1]-x1[0]) * y1[1]
```

```
        p23 = (x-x1[2])/(x1[1] - x1[2]) * y1[1] + (x-x1[1])/(x1[2]-x1[1]) * y1[2]
        p34 = (x-x1[3])/(x1[2] - x1[3]) * y1[2] + (x-x1[2])/(x1[3]-x1[2]) * y1[3]
        p123 = (x-x1[2])/(x1[0] - x1[2]) * p12 + (x-x1[0])/(x1[2]-x1[0]) * p23
        p234 = (x-x1[3])/(x1[1] - x1[3]) * p23 + (x-x1[1])/(x1[3]-x1[1]) * p34
        p1234 = (x-x1[3])/(x1[0] - x1[3]) * p123 + (x-x1[0])/(x1[3]-x1[0]) * p234
        return p1234

# Modeling equation f(x) = 1/1+x

x_f=np.array([1, 2, 3, 4])
y_f=np.array([0.45, 0.35, 0.23, 0.18])

x_r=np.linspace(np.min(x_f),np.max(x_f),50)
y_quadratic = interp1d(x_f, y_f, kind='quadratic')
y_r=np.array([])
for i in x_r:
        y_r = np.append(y_r,Atkins_method(i,x_f,y_f))
y_real=1/(1+x_r)
plt.figure()
plt.plot(x_f[0],y_f[0], 'bo',label='P1')
plt.plot(x_f[1],y_f[1], 'mo',label='P2')
plt.plot(x_f[2],y_f[2], 'ko',label='P3')
plt.plot(x_f[3],y_f[3], 'ro',label='P4')
plt.plot(x_r,y_r, color='b',label='Atkins Interpolation')
plt.plot(x_r,y_real,color='k',label='Real Function')
plt.plot(x_r,y_quadratic(x_r),color='m',label='Quadratic Splines')
plt.grid()
```

```
plt.legend()
plt.title('Atkins Method for Interpolating y=1/1+x')
plt.xlabel('x')
plt.ylabel('y')
plt.show()

# Scipy Python Library
f_lag= lagrange(x2,y2)
plt.figure
plt.plot(x1,f_lag(x1), 'g' , x2, y2 , 'mo')
plt.grid()
plt.legend()
plt.title('Lagrange Interpolation of y=x^2')
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```

Here one can see when the function doesn't follow an apparent trend or when the data points are not too far off and don't cover a wide range the estimated interpolation is very far from the actual function being modeled. This is seen below.

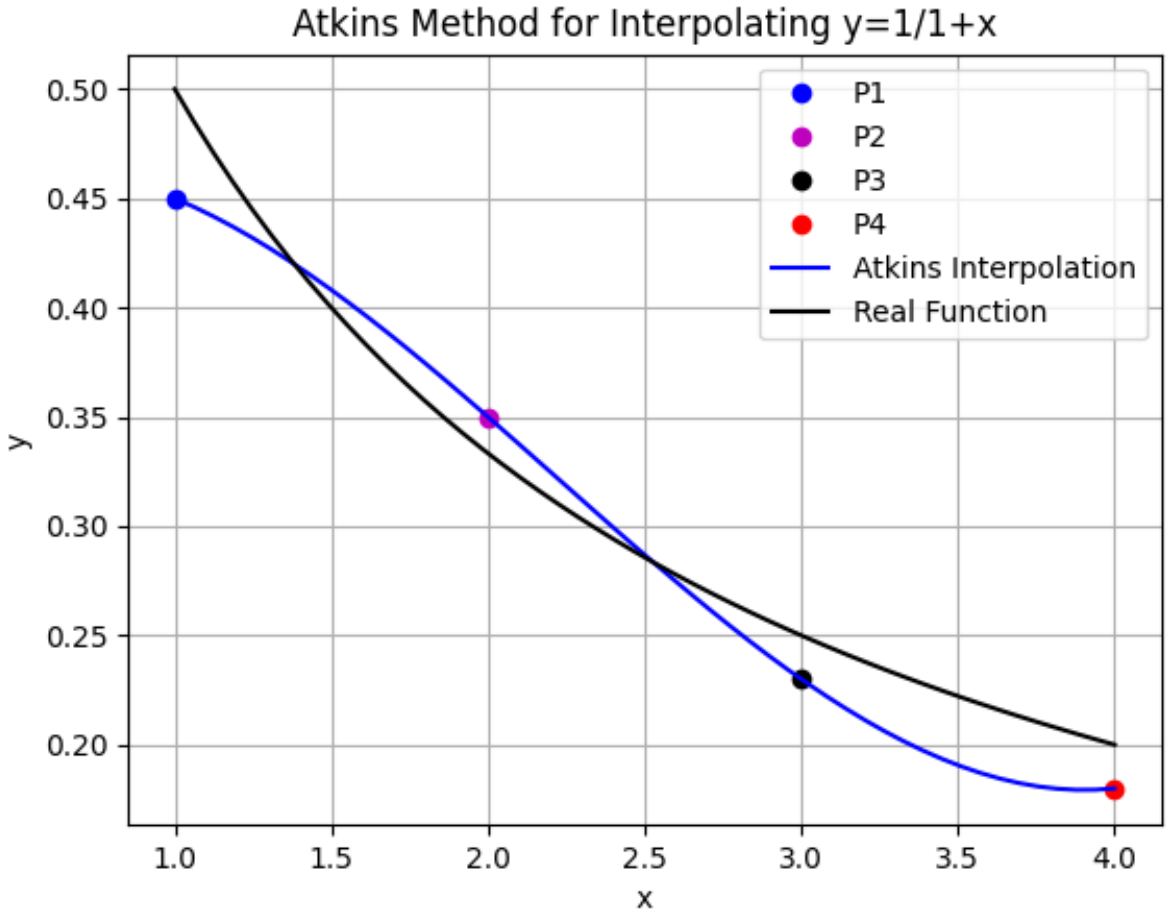

Figure 2: Bad Interpolation for $y = \frac{1}{1+x}$

1.3 Quadratic Splines

Using the built in function scipy.interpole from SciPy, one can produce better results using Quadratic Splines.

```
from scipy.interpolate import interp1d

y_quadratic = interp1d(x_f, y_f, kind='quadratic')
```

This will output a quadratic splines estimate between the data points and is a more accurate estimate than Lagrange interpolation as the figure below shows.

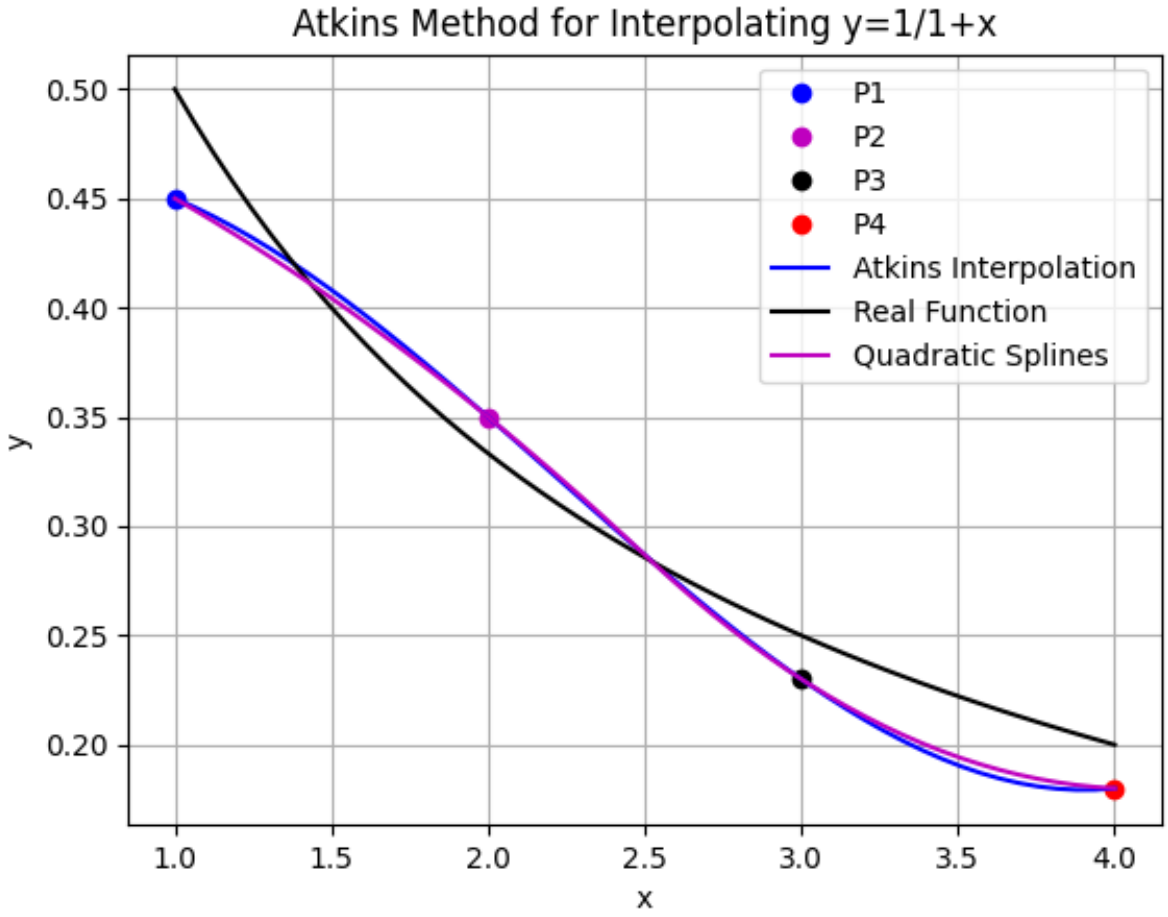

Figure 3: Quadratic Splines and Lagrange Interpolation for $y = \frac{1}{1+x}$

Additionally, below is the code used for the function lagrange in scipy.interpole that automatically outputs the Lagrange Interpolation of the the Data point. Below is the code and the output for Data Sets modeling $y = \frac{1}{1+x}$.

```
# Scipy Python Library

f_lag= lagrange(x_f,y_f)

plt.figure

plt.plot(x_r,f_lag(x1), 'g' , x_f, y_f , 'mo')

plt.plot(x_r,y_real,color='k')

plt.grid()

plt.legend()
```

```
plt.title('Lagrange Interpolation of y=x^2')
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```

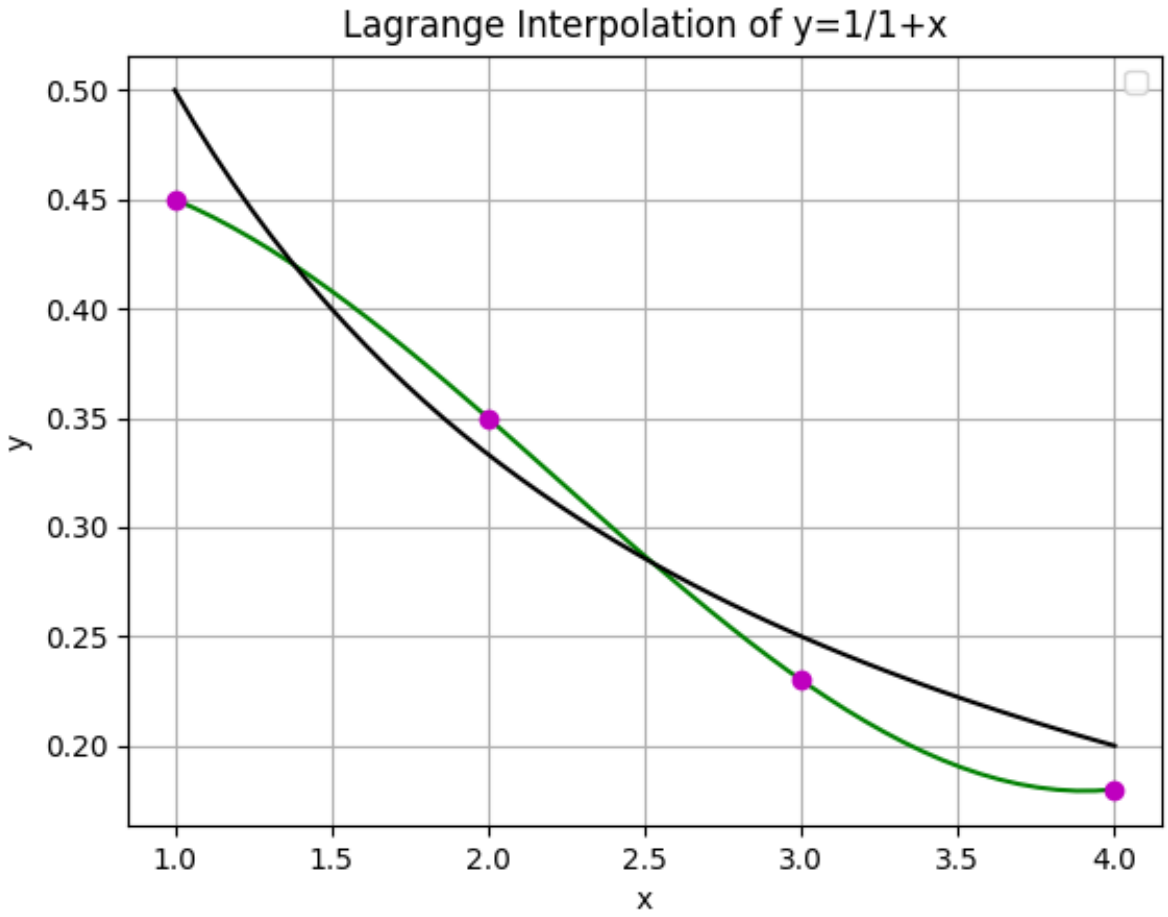

Figure 4: Lagrange Interpolation Using the Built-in function in SciPy Library for $y = \frac{1}{1+x}$

The output is identically the same as the ones above. The only difference that will help the interpolation is introducing more points in different ranges to have a better estimate towards the trend of the points.

2 Fitting

2.1 Least Square Approximation

To get the line of best-fit for

$$y = 2x + 3$$

The following code is executed. The two methods being used, are the manual method were the coefficients of the linear linear to describe the fit $y = mx + b$ are calculated using the sums and mean of the data point. The other method is to write up the data points in matrix form and solve for the coefficients. The last is the built in function, optimize.curve_fit in scipy.

```
import numpy as np
from scipy import optimize
import matplotlib.pyplot as plt


# generating data of equation y = 2x + 3 with errors
x= np.linspace(0,20,50)
n=len(x)
y= 2*x + 3 + 10* np.random.random(n)


# Physically getting the least squares through the sums equations


xbar = np.mean(x)
ybar = np.mean(y)


num = 0
den = 0
for i in range(n):
        num += (x[i] - xbar) * (y[i] - ybar)
```

```
        den += (x[i] - xbar) ** 2
m = num/den # slope
b = ybar - (m * xbar) # y-intercept
print(m, b)
y_reg = m*x + b
plt.plot(x,y,'r*',label='Data Point')
plt.plot(x,y_reg,color='b',label='Linear Regression')
plt.title('Least Square Approximation of y = 2x + 3')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()

# Writing it in Matrix formation
A= np.vstack([x,np.ones(len(x))]).T
y = y[:,None] # changing y into a column vector
alpha = np.dot((np.dot(np.linalg.inv(np.dot(A.T,A)),A.T)),y) # Getting
print(alpha)
plt.figure()
plt.plot(x,y,'r*',label='Data Point')
plt.plot(x,alpha[0]*x + alpha[1],color='b',label='Linear Regression')
plt.legend()
plt.grid()
plt.show()

# Using optimize.curve_fit from scipy
```

```
def func(x,m,b):
        y=m*x + b
        return y

beta = optimize.curve_fit(func,xdata=x,ydata=y)[0]
print(beta)
```

The following code outputs the same value for the coefficients:

$$m = 2.07772003$$

$$b = 7.21019203$$

And the following graph describing the fit:

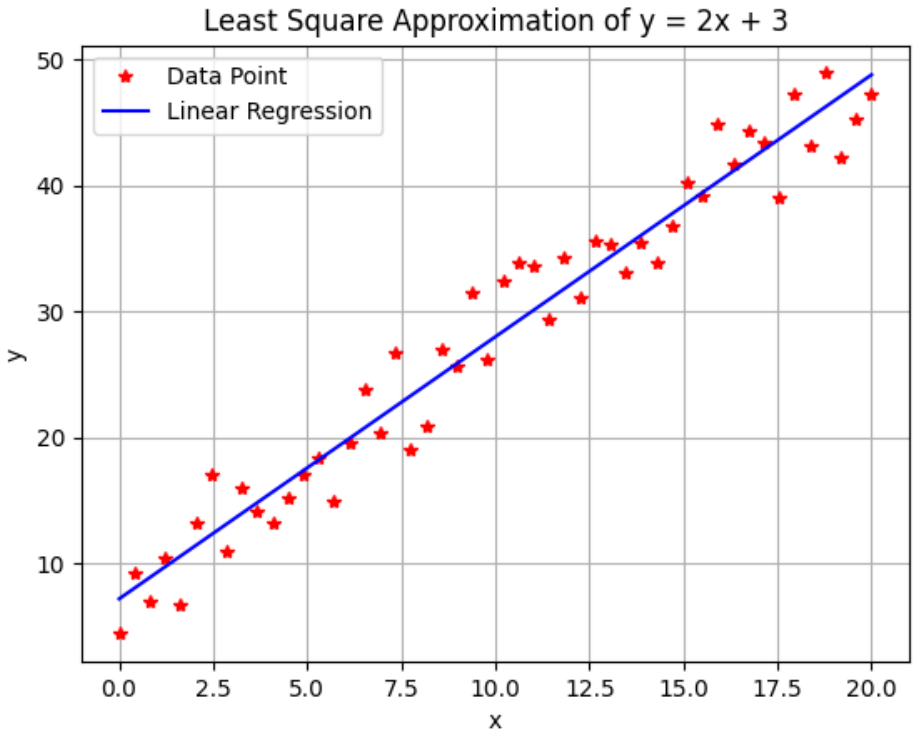

Figure 5: Linear Regression using Least Square Method

3 Integration Methods

Calculating the integral of

$$f(x) = \sin(x)$$

using the Trapezoidal Rule, Simpson's Rule, and the built in packages in Scipy.

3.1 Trapezoidal Rule

The code written will emulate this formula, where x_0 and x_n are the boundary points:

$$I_{Trapizoidal} = \frac{h}{2}(f(0) + f(n)) + h\sum_{i=1}^{n-1} f(x_n)$$

The code that will implement the rule:

```
import numpy as np
from scipy.integrate import quad
import matplotlib.pyplot as plt


x1 = 0
x2 = np.pi
n=100
h= (x2 - x1) / (n-1 )
x = np.linspace(x1,x2,n)
func = np.sin(x) #+ np.random.random(n)


# Trapizoidal Rule


I_trapizoid = (h/2) * (func[0] + func[n-1]) + np.sum(func[1:n-1]) * h
print(I_trapizoid)
```

```
plt.plot(x,func)

plt.show()
```

3.2 Simpson's Rule

The Simpson's rule will emulate the following relation:

$$I_{simpson} = \frac{h}{3}(f(x_0) + f(n)) + \frac{2h}{3}\sum_{i=1}^{n/2-1} f(x_{2i}) \frac{4h}{3}\sum_{i=1}^{n/2} f(x_{2i-1})$$

The code that will implement the rule:

```
# Simpson's Rule


I_simpson = (h/3) * (func[0] + func[n-1]) + (2*h/3) * (np.sum(func[:n
    ↪ -2:2])) + (4*h/3) * (np.sum(func[1:n-1:2]))
print(I_simpson)


# Python integrator


def f(x):
        return np.sin(x)


I_package = quad(f,x1,x2)
print(I_package)
```

Both methods output the following results: 1.9998321638939927, 1.9991609434038167, and 2.0.

The theoretical answer is 2 so the methods are successful.

4 Root Finding Method

I intend to find the root of:

$$f(x) = \sin(x)$$

using the Bisection Method and Newton-Raphson method.

4.1 Bisection method

Using initial boundaries of $\frac{\pi}{4}$ and $\frac{3\pi}{2}$

The code that will implement this method is:

```
import numpy as np


def bisection_method(f,x1,x2,e):
        if f(x1)*f(x2) >=0:
                return print('No root exists in these boundaries')
        x1_n=x1
        x2_n=x2
        while (x2_n - x1_n) < e:
                mid = (x1_n + x2_n) / 2
                f_mid = f(mid)
                if f(x1_n) * f_mid < 0:
                        x2_n = mid
                elif f(x2_n) * f_mid < 0:
                        x1_n = mid
                elif f_mid == 0:
                        return print('Exact solution found: ',mid)
                else:
                        return print('Method Failed')
```

```
        return print((x1_n + x2_n) / 2)


        bisection_method(f,np.pi/2,1.5*np.pi,0.001)
```

The root finding method outputs a root of: 3.141592653589793

Which is correct

4.2 Newton-Raphson

The initial guess for the Newton-Raphson method is: $\frac{\pi}{4}$

```
        def f(x):
                return np.sin(x)
        def df(x):
                return np.cos(x)


        def Newton_Raphson(f,df,x,e):
                h = f(x) / df(x)
                while abs(h) > e:
                        h=f(x) / df(x)
                        x = x - h
                print('The root found is: ',x)


        Newton_Raphson(f,df,np.pi/4,0.0001)
```

It outputs: 1.6543612251060553e-24, which is almost zero which is the correct root.

5 Differential Equations

To examine the validity of different method of solving differential Equations, we will examine the Euler Method, Range-Kutta 2 and 4 methods for solving a simple exponential growth function:

$$\frac{dy}{dt} = f(t, y) = y$$

With the solution

$$y = \exp^t$$

5.1 Euler Method

The Euler Method work with iteration as such:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

The python code that will be used to examine the method:

```
import numpy as np
import scipy as sp
import math
import matplotlib.pyplot as plt


# Euler Method
# Defining an ODE with exponential growth dy/dt = y


def f(t,y):
        return y


def euler(f,y0,t): # where f is the function, y0 is the initial condition
```

```
    ↪ , and t is the time interval
        y=np.zeros(len(t))
        y[0] = y0
        h = t[1] - t[0]
        for i in range(0,len(t) - 1):
        y[i+1] = y[i] + h*f(t[i],y[i])
        return y

y0=1
t=np.linspace(0,2,10)

y_soln= euler(f,y0,t)
y1= np.exp(t)
plt.plot(t, y_soln,label='Euler Estimate')
plt.plot(t,y1,label='Actual Solution')
plt.legend()
plt.xlabel('t')
plt.ylabel('y')
plt.grid()
plt.title('Graphing ODE Estimate of dy/dt=y usig the euler mnethod')
plt.show()
```

Taking the time interval from 0 to 2 seconds and only taking 10 points, the outcome is the following:

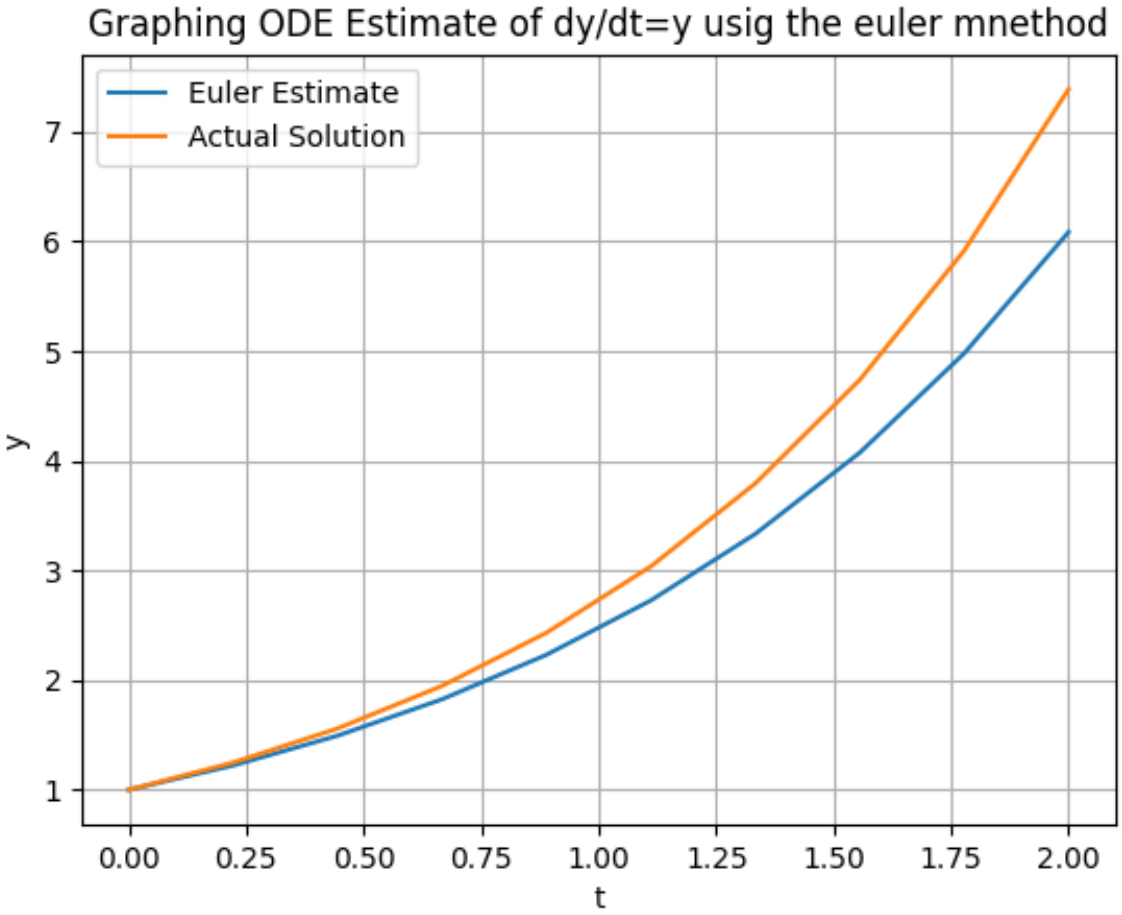

Figure 6: Euler Method for solving $\dot{y} = y$ with wide time step

This will compute an error of ≈ 3.34. If we take 100 points then the outcome would be closer to the estimate.

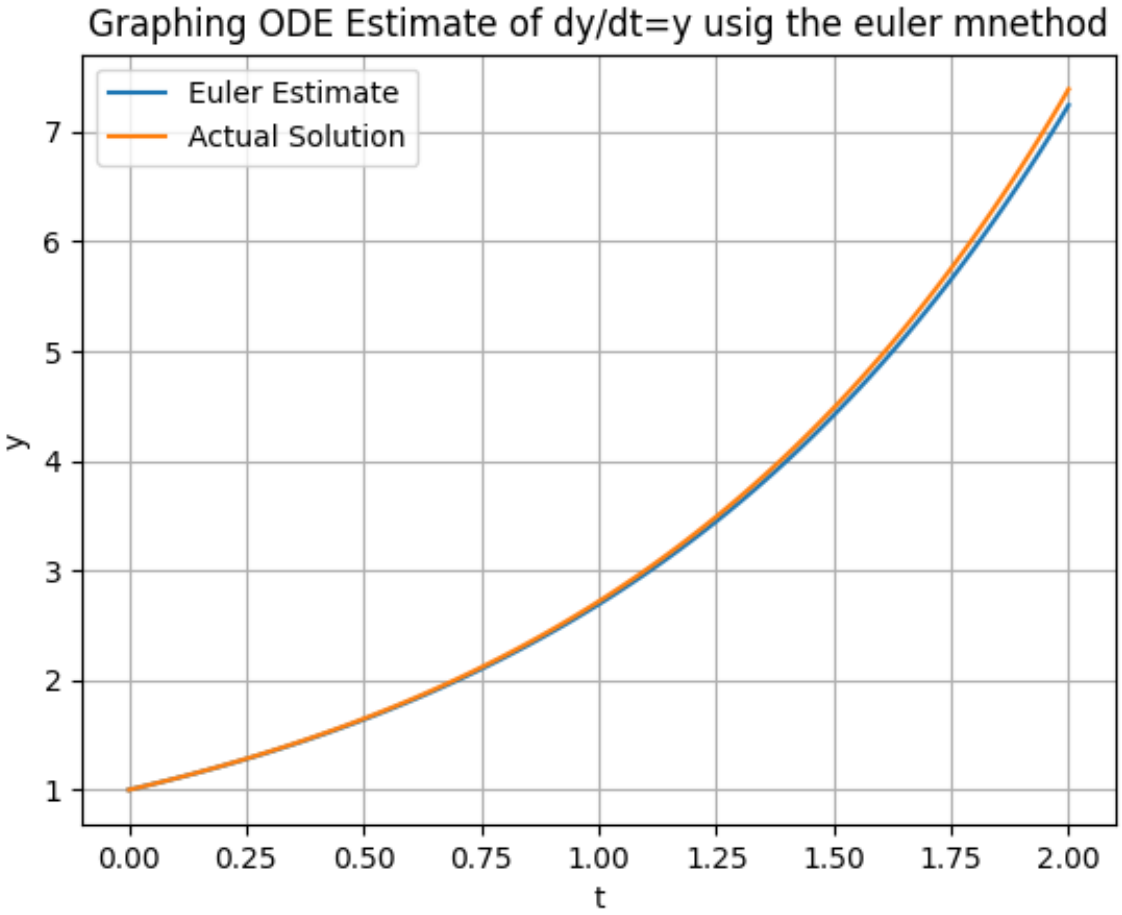

Figure 7: Euler Method for solving $\dot{y} = y$ with narrow time step

This will give an error of ≈ 0.34.

5.2 Range-Kutta 2 and 4 methods

The code that will undertake the Range-Kutta 2 and 4 methods:

```
# Range-Kutta 2 Method
def RK2(f,y0,t):
        y=np.zeros(len(t))
        y[0] = y0
        h= t[1] - t[0]
```

```
        for i in range(0,len(t) - 1):
            k1 = h * f(t[i],y[i])
            k2 = h * f(t[i] + 0.5*h,y[i] + 0.5*k1)
            y[i+1] = y[i] + k2
        return y


# Range-Kutta 4 Method
def RK4(f,y0,t):
        y=np.zeros(len(t))
        y[0] = y0
        h= t[1] - t[0]
        for i in range(0,len(t)-1):
            k1=h*f(t[i],y[i])
            k2=h*f(t[i]+0.5*h,y[i]+0.5*k1)
            k3=h*f(t[i]+0.5*h,y[i]+0.5*k2)
            k4=h*f(t[i]+h,y[i]+k3)
            y[i+1] = y[i] + (1/6)*(k1+2*k2+2*k3+k4)
        return y
```

To show the variation of all three methods on trying to graph the solution of the above example

$$\frac{dy}{dt} = y$$

The following graph includes the solution predicted by all three methods using only 20 points.

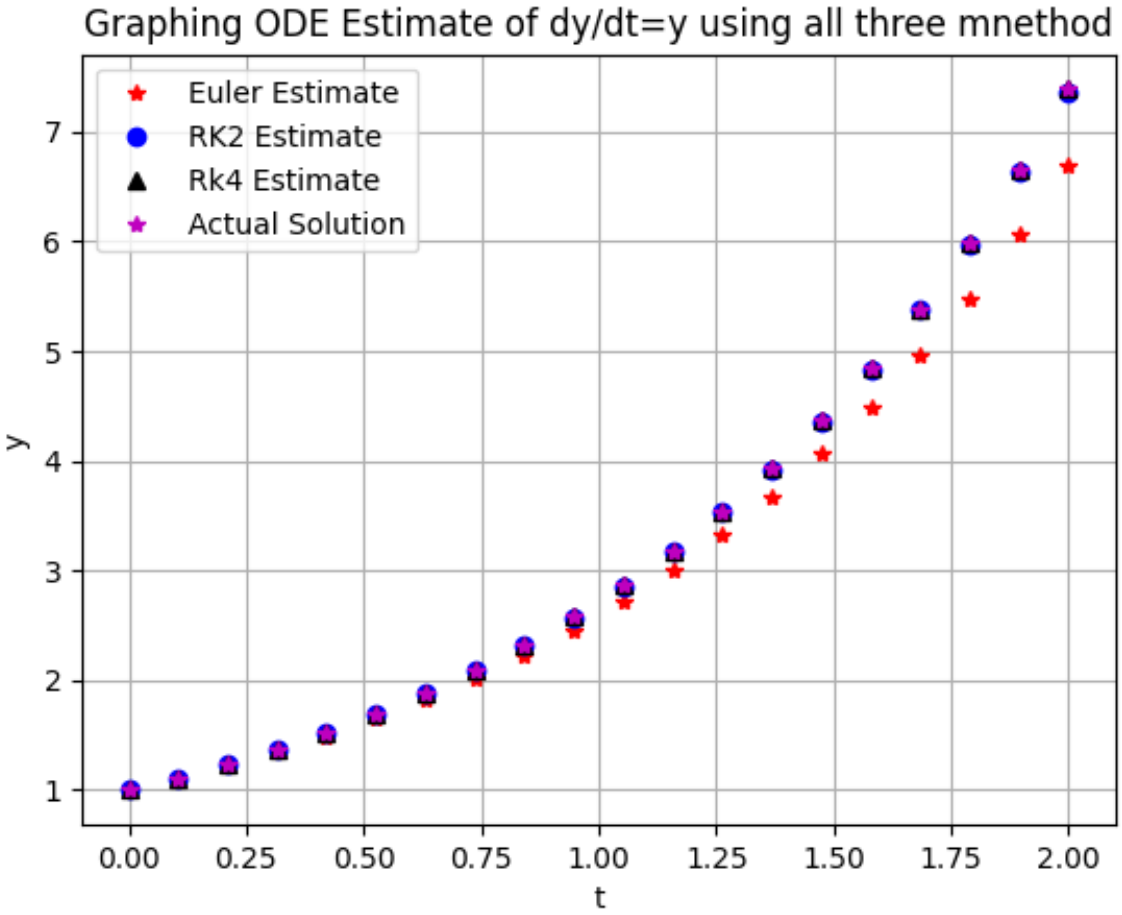

Figure 8: All Methods for solving $\dot{y} = y$ Attempted

The following is the error on each perspective method:

- Euler Method: 1.6909867190479142
- RK2 Method: 0.0022106615370554774
- RK4 Method: 6.684782034258244e-10

Which proves that each method has respectively less error and is a better method to be used when solving complicated ODEs.

RK4 will be used to solve the example of a forced pendulum.

5.3 Simple Harmonic Oscillator Solution

The following is the code for solving the example of a forced pendulum with the following differential equations:

$$\dot{y} = v$$

$$v = f\cos(wt) - w^2 y$$

The code ventures through all three methods and the universal code for each of them.

```
# Simple Harmonic Oscillator
# Euler Method
t=np.linspace(0,10,1000)
y=np.zeros(len(t))
v=np.zeros(len(t))
h=t[1]-t[0]
y[0]=10
v[0]=-2
for i in range(0,len(t)-1):
        y[i+1] = y[i] + h*v[i]
        v[i+1] = v[i] + h*(10*np.cos(2*np.pi*t[i]))
plt.figure()
plt.plot(t,y)
plt.show()
plt.figure()
plt.plot(t,v)
plt.show()


# RK 2 Method
# y' = v
```

```
# v' = f cos(wt) - w^2y
f=3
w=2
def F(y,v,t):
        return f*np.cos(w*t) - (w**2)*y
t=np.linspace(0,10,100)
ypoints=[]
vpoints=[]
h=t[1]-t[0]
y=2
v=2
for i in range(0,len(t)):
        ypoints.append(y)
        vpoints.append(v)

        m1 = h*v
        k1 = h*F(y, v, t[i])

        m2 = h*(v + 0.5*k1)
        k2 = h*F(y+0.5*m1, v+0.5*k1, t[i]+0.5*h)

        y += m2
        v += k2

plt.plot(t,ypoints)
plt.show()
plt.figure
```

```
plt.plot(t,vpoints)
plt.show()


# RK 4 Method
# y' = v
# v' = f cos(wt) - w^2y
f=3
w=2
def F(y,v,t):
        return f*np.cos(w*t) - (w**2)*y
t=np.linspace(0,10,100)
ypoints=[]
vpoints=[]
h=t[1]-t[0]
y=2
v=2
for i in range(0,len(t)):
        ypoints.append(y)
        vpoints.append(v)


        m1 = h*v
        k1 = h*F(y, v, t[i])


        m2 = h*(v + 0.5*k1)
        k2 = h*F(y+0.5*m1, v+0.5*k1, t[i]+0.5*h)


        m3 = h*(v + 0.5*k2)
```

```
            k3 = h*F(y+0.5*m2, v+0.5*k2, t[i]+0.5*h)


            m4 = h*(v + k3)
            k4 = h*F(y+m3, v+k3, t[i]+h)


            y += (m1 + 2*m2 + 2*m3 + m4)/6
            v += (k1 + 2*k2 + 2*k3 + k4)/6


    plt.plot(t,ypoints)
    plt.show()
    plt.figure
    plt.plot(t,vpoints)
    plt.show()
```

Showing the results of RK4 method with initial conditions set as:

$$y0 = 2$$

$$v0 = 2$$

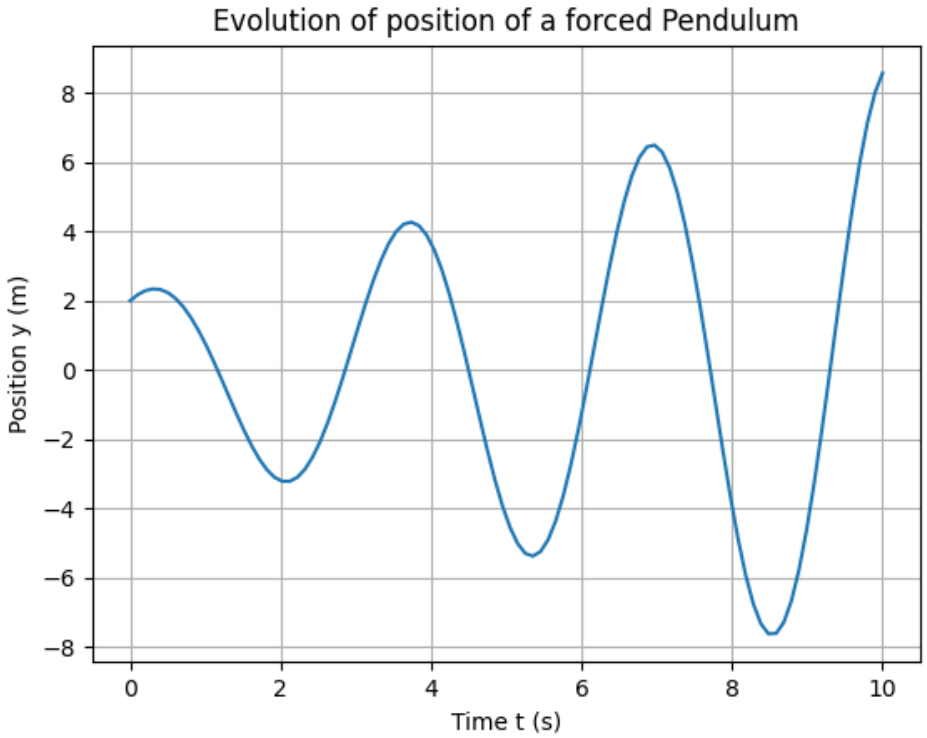

Figure 9: RK4 solution for position

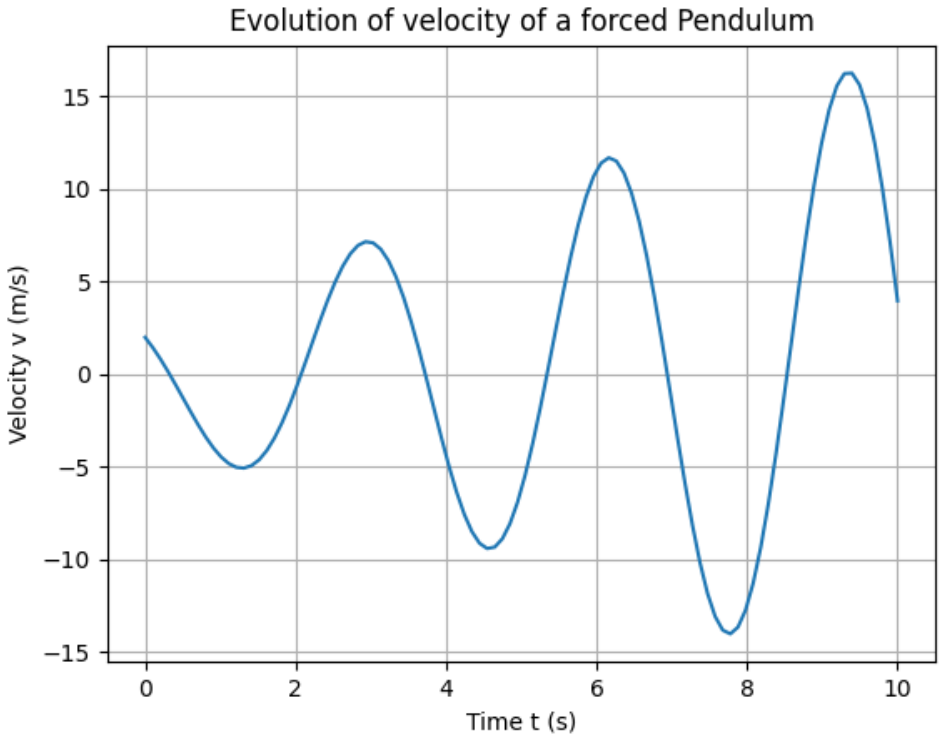

Figure 10: RK4 solution for velocity