3 × 1 Product

The traceless $I[ZZZ]J$ is composed of five guys:

$$IZZZJ,\quad I(ZZ)\Omega ZJ,\quad IZ\Omega(Z,Z)J,\quad I\Omega Z(Z,Z)J,\quad I(Z,Z)Z\Omega J$$

Let's study the OPE of the first one:

- $$I'Z'Z'Z'J'\; IZJ \sim \frac{\lambda}{(z'-z)^2}\, IZZ\Omega J + \frac{1}{z'-z}\Big(IZZZZJ + \overbrace{\lambda\,\partial(IZZ)\Omega J}^{(I)}\Big)$$

Using the Q-exact relation $\lambda\overleftarrow{\partial} \sim \lambda\overrightarrow{\partial} + (Z,Z)$ we want to turn (I), $\lambda\,\partial(IZZ)\Omega J$, into a total derivative $k\,\partial(IZZ\Omega J)$ plus stuff.
In this case we find:

$$(I) \sim \lambda\frac{3}{4}\partial(IZZ\Omega J) + \frac{1}{4} I(Z,Z)ZZ\Omega J + \frac{2}{4} IZ(Z,Z)Z\Omega J + \frac{3}{4} IZZ(Z,Z)\Omega J$$

Plugging this back into the OPE

$$\boxed{\begin{aligned} I'Z'Z'Z'J'\; IZJ \sim \frac{1}{z'-z}\Big[& IZZZZJ + \frac{1}{4}\big(I(Z,Z)ZZ\Omega J + 2\, IZ(Z,Z)Z\Omega J + 3\, IZZ(Z,Z)\Omega J\big) \\ & + \lambda\frac{3}{4}\partial(IZZ\Omega J)\Big] \\ & + \frac{\lambda}{(z'-z)^2} IZZ\Omega J \end{aligned}}$$

Moving on to the OPE's with the other expressions:

- $$\begin{aligned} \Omega I'(Z',Z')Z'J'\; IZJ \sim \frac{1}{z'-z}\Big\{ I\Big[& \Omega(Z,Z)ZZ + \frac{1}{4}\big((Z,Z)(Z,Z)\Omega\Omega + 2\,\overline{Z(Z,Z)Z}\,\Omega\Omega J + 3(Z,Z)(Z,Z)\Omega\Omega\big)\Big] \\ & + \lambda\frac{3}{4}\partial(I(Z,Z)J)\Omega\Omega\Big\} \\ & + \frac{\lambda}{(z'-z)^2} I(Z,Z)J\Omega\Omega \end{aligned}$$

$$\Rightarrow\ \boxed{\begin{aligned} \Omega I'(Z',Z')Z'J'\; IZJ \sim \frac{1}{z'-z}\Big\{ I\Big[& \Omega(Z,Z)ZZ + \frac{1}{4}\big(4(Z,Z)(Z,Z)\Omega\Omega + 2\,\overline{Z(Z,Z)Z}\,\Omega\Omega J\big)\Big]J \\ & + \lambda\frac{3}{4}\partial(I(Z,Z)J)\Omega\Omega\Big\} \\ & + \frac{\lambda}{(z'-z)^2} I(Z,Z)J\Omega\Omega \end{aligned}}$$

- $I' Z'(Z',Z') \Omega J' \; I Z J \sim \frac{1}{z'-z}\Big[I Z(Z,Z)\Omega Z J + \frac{1}{4} I(Z,Z) Z \Omega Z J + \frac{2}{4} I Z (Z,Z)\Omega Z + \frac{3}{4} I Z \Omega Z (Z,Z) J + \lambda \frac{3}{4} \partial (I Z \Omega Z J)\Big] + \frac{\lambda}{(z'-z)^2} I Z \Omega Z J$

- $I' \Omega Z'(Z',Z') J' \; I Z J \sim \frac{1}{z'-z}\Big[I \Omega Z(Z,Z) Z J + \frac{1}{4} I \Omega (Z,Z) Z Z J + \frac{2}{4} I \Omega Z (Z,Z) Z + \frac{3}{4} I \Omega Z Z (Z,Z) J + \lambda \frac{3}{4} \partial (I \Omega Z Z J)\Big] + \frac{\lambda}{(z'-z)^2} I \Omega Z Z J$

- $I'(Z',Z') Z' \Omega J' \; I Z J \sim \frac{1}{z'-z}\Big[J(Z,Z) Z \Omega Z J + \frac{1}{4} I(Z,Z)(Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} + \frac{2}{4} I \overline{Z (Z,Z) Z} \Omega^{(1)} \Omega \Omega^{(2)} J + \frac{3}{4} I(Z,Z)(Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J + \lambda \frac{3}{4} I(Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J \Big] + \frac{\lambda}{(z'-z)^2} I(Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J$

So in the end we have:

$$
\begin{aligned}
I'[Z'Z'Z'] J' \; I Z J \sim \frac{1}{z'-z}\Big\{ & I Z Z Z Z J + \frac{1}{4} I (Z,Z) Z Z \Omega J + \frac{2}{4} I Z (Z,Z) Z \Omega J + \frac{3}{4} I Z Z (Z,Z) \Omega J \\
& + \frac{2}{3}\Big(+ I \Omega (Z,Z) Z Z J + \overset{1}{\cancel{\tfrac{1}{4}}} I (Z,Z)(Z,Z) \Omega \Omega J + \frac{2}{4} I \overline{Z (Z,Z) Z} \Omega \Omega J + \cancel{\frac{3}{4} I (Z,Z)(Z,Z) \Omega \Omega J} + \Big) \\
& + \frac{2}{3}\Big(+ \frac{3}{2} I Z (Z,Z) \Omega Z J + \frac{1}{4} I (Z,Z) Z \Omega Z J + \cancel{\frac{2}{4} I Z (Z,Z) \Omega Z J} + \frac{3}{4} I Z \Omega Z (Z,Z) J + \Big) \\
& + \frac{1}{3}\Big(+ \frac{3}{2} I \Omega Z (Z,Z) Z J + \frac{1}{4} I \Omega (Z,Z) Z Z J + \cancel{\frac{2}{4} I \Omega Z (Z,Z) Z J} + \frac{3}{4} I \Omega Z Z (Z,Z) J + \Big) \\
& + \frac{1}{3}\Big(+ J (Z,Z) Z \Omega Z J + \overset{1}{\cancel{\tfrac{1}{4}}} I (Z,Z)(Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J + \frac{2}{4} I \overline{Z (Z,Z) Z} \Omega^{(1)} \Omega \Omega^{(2)} J + \cancel{\frac{3}{4} I (Z,Z)(Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J} \Big) \\
& + \frac{3}{4}\lambda \Big[\partial (I Z Z \Omega J) + \frac{2}{3} \partial (J (Z,Z) \Omega \Omega J) + \frac{2}{3} \partial (I Z \Omega Z J) + \frac{1}{3} \partial (I \Omega Z Z J) + \frac{1}{3} \partial (I (Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J) \Big] \Big\} \\
& + \frac{\lambda}{(z'-z)^2} \Big(\underbrace{I Z Z \Omega J + \frac{2}{3} J (Z,Z) \Omega \Omega J + \frac{2}{3} I Z \Omega Z J + \frac{1}{3} I \Omega Z Z J + \frac{1}{3} I (Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J}_{(I)} \Big)
\end{aligned}
$$

Let's manipulate (I) so it looks like a combination of u's:

$$(I) = I Z Z \Omega J + \underset{\frac{1}{2} + \frac{1}{6}}{\frac{2}{3}} J (Z,Z) \Omega \Omega J + \frac{2}{3} I Z \Omega Z J + \frac{1}{3} I \Omega Z Z J + \frac{1}{3} I (Z,Z) \Omega^{(1)} \Omega \Omega^{(2)} J$$

$$= \boxed{u^{(2)} \Omega + \frac{2}{3} u^{(2)(1)} \Omega u^{(2)(2)} + \frac{1}{3} \Omega u^{(2)}}$$

So finally we find:

$$u^{(3)}_n \cdot IZJ = z^n u^{(4)} + \lambda\left(\tfrac{3}{4} z^n \partial + n z^{n-1}\right)\left(\tfrac{1}{3}\Omega u^{(2)} + \tfrac{2}{3} u^{(2)(1)} \Omega u^{(2)(2)} + u^{(2)}\Omega\right)$$

$$\Longrightarrow \boxed{u^{(3)}_n \cdot u^{(1)}_m = u^{(4)}_{n+m} + \lambda \underbrace{\left(-\tfrac{3}{4}(n+m) + n\right)}_{\left(\frac{1}{4}n - \frac{3}{4}m\right)} \left(\tfrac{1}{3}\Omega u^{(2)}_{n+m-1} + \tfrac{2}{3} u^{(2)(1)}_{n+m-1}\Omega u^{(2)(2)}_{n+m-1} + u^{(2)}_{n+m-1}\Omega\right)}$$