3 × 1 Product

The traceless $I[ZZZ]J$ is composed of five guys:

$$IZZZJ,\ I(ZZ)\Omega ZJ,\ IZ\Omega(Z,Z)J,\ I\Omega Z(Z,Z)J,\ I(Z,Z)Z\Omega J$$

Let's study the OPE of the first one:

- $$I'Z'Z'Z'J'\ IZJ \sim \frac{\lambda}{(z'-z)^2}\, IZZ\Omega J + \frac{1}{z'-z}\Big(IZZZZJ + \lambda\, \overbrace{\partial(IZZ)\Omega J}^{(I)}\Big)$$

Using the Q-exact relation $\lambda\overleftarrow{\partial} \sim \lambda\overrightarrow{\partial} + (Z,Z)$ we want to turn (I), $\lambda\,\partial(IZZ)\Omega J$, into a total derivative $\kappa\,\partial(IZZ\Omega J)$ plus stuff.
In this case we find:

$$(I) \sim \lambda\frac{3}{4}\partial(IZZ\Omega J) + \frac{1}{4}I(Z,Z)ZZ\Omega J + \frac{2}{4}IZ(Z,Z)Z\Omega J + \frac{3}{4}IZZ(Z,Z)\Omega J$$

Plugging this back into the OPE

$$\boxed{\begin{aligned} I'Z'Z'Z'J'\ IZJ \sim{}& \frac{1}{z'-z}\Big[IZZZZJ + \frac{1}{4}\Big(I(Z,Z)ZZ\Omega J + 2\,IZ(Z,Z)Z\Omega J + 3\,IZZ(Z,Z)\Omega J\Big) \\ &\qquad + \lambda\frac{3}{4}\partial(IZZ\Omega J)\Big] \\ &+ \frac{\lambda}{(z'-z)^2}\, IZZ\Omega J \end{aligned}}$$

Moving on to the OPE's with the other expressions:

- $$\Omega I'(Z',Z')Z'J'\ IZJ \sim \frac{1}{z'-z}\Big\{I\Big[\Omega(Z,Z)ZZ + \frac{1}{4}\Big((Z,Z)(Z,Z)\Omega\Omega + 2\,\overline{Z(Z,Z)Z}\,\Omega\Omega J + 3(Z,Z)(Z,Z)\Omega\Omega\Big)\Big] + \lambda\frac{3}{4}\partial(I(Z,Z)J)\Omega\Omega\Big\} + \frac{\lambda}{(z'-z)^2}\, I(Z,Z)J\Omega\Omega$$

$$\Rightarrow \boxed{\begin{aligned} \Omega I(Z',Z')Z'J'\ IZJ \sim{}& \frac{1}{z'-z}\Big\{I\Big[\Omega(Z,Z)ZZ + \frac{1}{4}\Big(4\,(Z,Z)(Z,Z)\Omega\Omega + 2\,\overline{Z(Z,Z)Z}\,\Omega\Omega J\Big)\Big]J \\ &\qquad + \lambda\frac{3}{4}\partial(I(Z,Z)J)\Omega\Omega\Big\} \\ &+ \frac{\lambda}{(z'-z)^2}\, I(Z,Z)J\Omega\Omega \end{aligned}}$$

- $$I'\,z'(z',z')\,\Omega\,J'\;I\,z\,J \sim \frac{1}{z'-z}\Big[I\,z(z,z)\,\Omega\,z\,J + \frac{1}{4}I(z,z)\,z\,\Omega\,z\,J + \frac{2}{4}I\,z(z,z)\,\Omega\,z + \frac{3}{4}I\,z\,\Omega\,z(z,z)\,J + \lambda\,\frac{3}{4}\,\partial(I\,z\,\Omega\,z\,J)\Big] + \frac{\lambda}{(z'-z)^2}\,I\,z\,\Omega\,z\,J$$

- $$I'\,\Omega\,z'(z',z')\,J'\;I\,z\,J \sim \frac{1}{z'-z}\Big[I\,\Omega\,z(z,z)\,z\,J + \frac{1}{4}I\,\Omega(z,z)\,z\,z\,J + \frac{2}{4}I\,\Omega\,z(z,z)\,z + \frac{3}{4}I\,\Omega\,z\,z(z,z)\,J + \lambda\,\frac{3}{4}\,\partial(I\,\Omega\,z\,z\,J)\Big] + \frac{\lambda}{(z'-z)^2}\,I\,\Omega\,z\,z\,J$$

- $$I'(z',z')\,z'\,\Omega\,J'\;I\,z\,J \sim \frac{1}{z'-z}\Big[I(z,z)\,z\,\Omega\,z\,J + \frac{1}{4}I(z,z)(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)} + \frac{2}{4}I\,\overline{z(z,z)}\,z\,\Omega^{(1)}\Omega\,\Omega^{(2)}J + \frac{3}{4}I(z,z)(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J + \lambda\,\frac{3}{4}I(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J\Big] + \frac{\lambda}{(z'-z)^2}\,I(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J$$

So in the end we have:

$$\begin{aligned}
I'[z'z'z']J'\,I\,z\,J \sim \frac{1}{z'-z}\Big\{ & I\,z\,z\,z\,z\,J + \frac{1}{4}I(z,z)\,z\,z\,\Omega\,J + \frac{2}{4}I\,z(z,z)\,z\,\Omega\,J + \frac{3}{4}I\,z\,z(z,z)\,\Omega\,J \\
& + \frac{2}{3}\Big(+ I\,\Omega(z,z)\,z\,z\,J + \cancel{\tfrac{1}{4}}^{\,1}I(z,z)(z,z)\,\Omega\,\Omega\,J + \frac{2}{4}I\,\overline{z(z,z)}\,z\,\Omega\,\Omega\,J + \cancel{\frac{3}{4}I(z,z)(z,z)\,\Omega\,\Omega\,J}\Big) \\
& + \frac{2}{3}\Big(+ \frac{3}{2}I\,z(z,z)\,\Omega\,z\,J + \frac{1}{4}I(z,z)\,z\,\Omega\,z\,J + \cancel{\frac{2}{4}I\,z(z,z)\,\Omega\,z\,J} + \frac{3}{4}I\,z\,\Omega\,z(z,z)\,J\Big) \\
& + \frac{1}{3}\Big(+ \frac{3}{2}I\,\Omega\,z(z,z)\,z\,J + \frac{1}{4}I\,\Omega(z,z)\,z\,z\,J + \cancel{\frac{2}{4}I\,\Omega\,z(z,z)\,z\,J} + \frac{3}{4}I\,\Omega\,z\,z(z,z)\,J\Big) \\
& + \frac{1}{3}\Big(+ I(z,z)\,z\,\Omega\,z\,J + \cancel{\tfrac{1}{4}}^{\,1}I(z,z)(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J + \frac{2}{4}I\,\overline{z(z,z)}\,z\,\Omega^{(1)}\Omega\,\Omega^{(2)}J + \cancel{\frac{3}{4}I(z,z)(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J}\Big)
\end{aligned}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{(A)}$$

$$+ \frac{3}{4}\lambda\Big[\underbrace{\partial(I\,z\,z\,\Omega\,J) + \frac{2}{3}\partial(I(z,z)\,\Omega\,\Omega\,J) + \frac{2}{3}\partial(I\,z\,\Omega\,z\,J) + \frac{1}{3}\partial(I\,\Omega\,z\,z\,J) + \frac{1}{3}\partial(I(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J)}_{(B)}\Big]\Big\}$$

$$+ \frac{\lambda}{(z'-z)^2}\Big(\underbrace{I\,z\,z\,\Omega\,J + \frac{2}{3}I(z,z)\,\Omega\,\Omega\,J + \frac{2}{3}I\,z\,\Omega\,z\,J + \frac{1}{3}I\,\Omega\,z\,z\,J + \frac{1}{3}I(z,z)\,\Omega^{(1)}\Omega\,\Omega^{(2)}J}_{(C)}\Big)$$

In the expression above, (A), is the traceless meson $I[zzzz]J$, (B) $= \partial(C)$, and (C) can be expressed as a linear combination of traceless mesons:

$$(c) = I\,Z\,Z\,\Omega\,J + \underbrace{\tfrac{2}{3}\,I\,(Z,Z)\,\Omega\,\Omega\,J}_{\frac{1}{2}+\frac{1}{6}} + \tfrac{2}{3}\,I\,Z\,\Omega\,Z\,J + \tfrac{1}{3}\,I\,\Omega\,Z\,Z\,J + \tfrac{1}{3}\,I\,(Z,Z)\,\Omega^{(1)}\,\Omega\,\Omega^{(1)}\,J$$

$$= I\,[Z\,Z]\,\Omega\,J + \tfrac{2}{3}\,I\,[Z|\Omega|Z]\,J + \tfrac{1}{3}\,I\,\Omega\,[Z\,Z]\,J$$

Where $[Z|\Omega|Z]$ means the trace of the Z's has been removed but not the one for Ω.

Integrating over $\oint dz'\, z'^{\,n_1+n_2+n_3}$:

$$u^{(3)}_{n_1 n_2 n_3} \cdot I\,Z\,J = z^{n_1+n_2+n_3}\, I\,[Z\,Z\,Z\,Z]\,J \;+$$

$$\lambda\left(\tfrac{3}{4}\, z^{n_1+n_2+n_3}\,\partial + (n_1+n_2+n_3)\, z^{n_1+n_2+n_3-1}\right)\Big(\tfrac{1}{3}\,\Omega\, I\,[Z\,Z]\,J + \tfrac{2}{3}\, I\,[Z|\Omega|Z]\,J + I\,[Z\,Z]\,\Omega\,J\Big)$$

which leads to the product

$$u^{(3)}_{n_1 n_2 n_3} \cdot u^{(1)}_{n_4} = u^{(4)}_{n_1 n_2 n_3 n_4} + \frac{\lambda}{3}\Big(\overbrace{-\tfrac{3}{4}(n_1+n_2+n_3+n_4) + n_1+n_2+n_3}^{\frac{1}{4}(n_1+n_2+n_3)-\frac{3}{4}n_4}\Big) \cdot$$
$$\cdot\Big(\Omega\, u^{(2)}_{n_1+n_2+n_3+n_4-1} + 2\, u^{(2)(1)}_{n_1+n_2+n_3+n_4-1}\,\Omega\, u^{(2)(2)}_{n_1+n_2+n_3+n_4-1} + 3\, u^{(2)}_{n_1+n_2+n_3+n_4-1}\,\Omega\Big)$$

where $\left(u^{(2)(1)}\,\Omega\, u^{(2)(2)}\right)^{a_1 a_2 a_3 a_4} = u^{a_1 a_4}\,\Omega^{a_2\,3}$.

Substituting $n_i - n_4 = -\varepsilon_{n_i n_4}$

$$\boxed{\begin{aligned} u^{(3)}_{n_1 n_2 n_3} \cdot u^{(1)}_{n_4} &= u^{(4)}_{n_1 n_2 n_3 n_4} \\ &- \frac{\lambda}{12}\Big[\varepsilon_{n_1 n_4}\Big(\Omega\, u^{(2)}_{n_2 n_3} + 2\, u^{(2)(1)}_{n_2 n_3}\,\Omega\, u^{(2)(2)}_{n_2 n_3} + 3\, u^{(2)}_{n_2 n_3}\,\Omega\Big) \\ &\quad \varepsilon_{n_2 n_4}\Big(\Omega\, u^{(2)}_{n_1 n_3} + 2\, u^{(2)(1)}_{n_1 n_3}\,\Omega\, u^{(2)(2)}_{n_1 n_3} + 3\, u^{(2)}_{n_1 n_3}\,\Omega\Big) \\ &\quad \varepsilon_{n_3 n_4}\Big(\Omega\, u^{(2)}_{n_1 n_2} + 2\, u^{(2)(1)}_{n_1 n_2}\,\Omega\, u^{(2)(2)}_{n_1 n_2} + 3\, u^{(2)}_{n_1 n_2}\,\Omega\Big) \end{aligned}}$$