Lecture 2 : SVMs

By : Khalil idrissi

Recap

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Linear regression

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$$w = w - \eta \frac{\partial J(w)}{\partial w}$$

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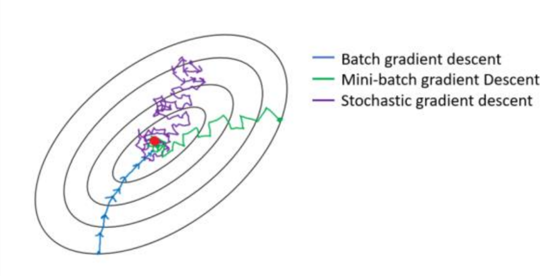

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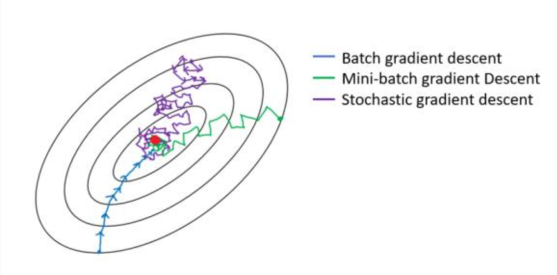

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MLE ,MAP, full bayesian approach

Using GD ,SGD,...

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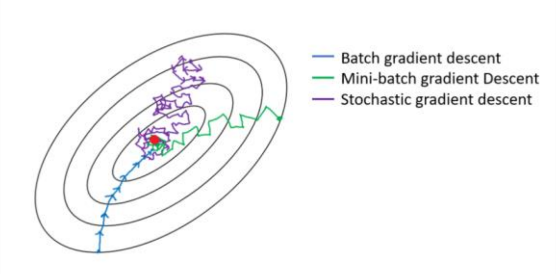

SVMs =Support Vector Machines

From Wikipedia, the free encyclopedia

In machine learning, **support-vector machines** (**SVMs**, also **support-vector networks**[1]) are supervised learning models with associated learning algorithms

Quiz

Quiz

①

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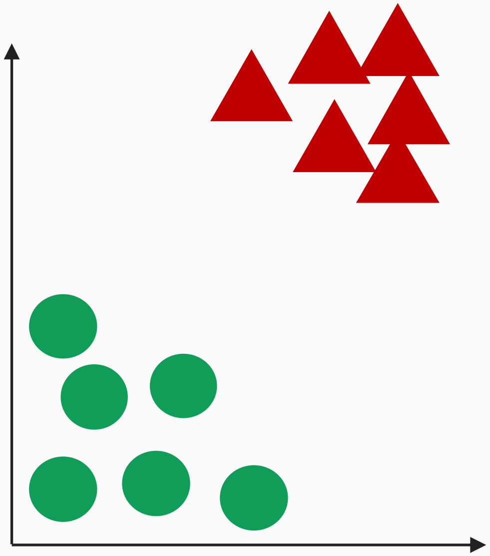

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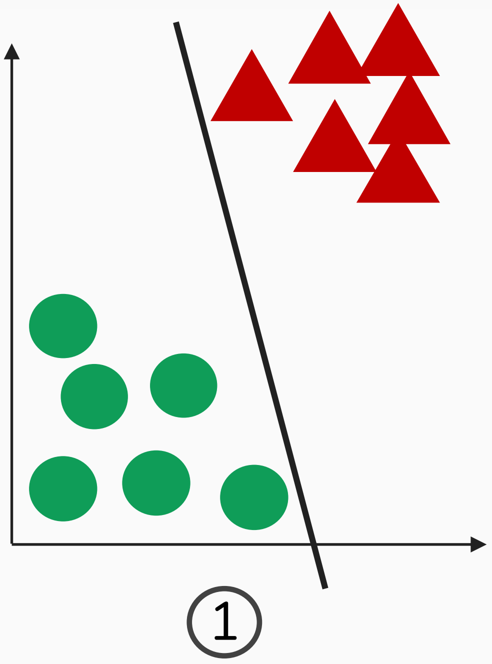

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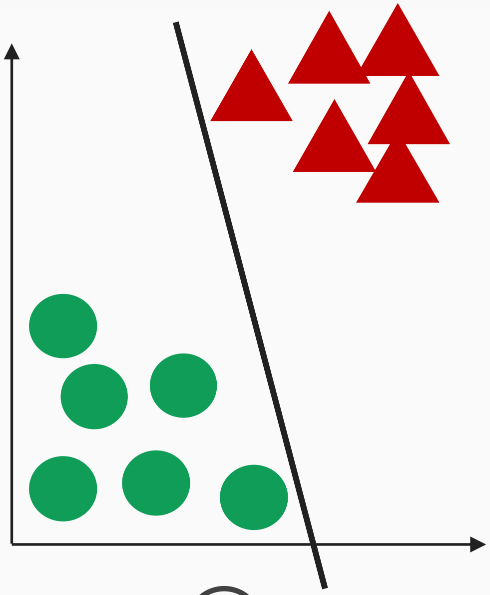

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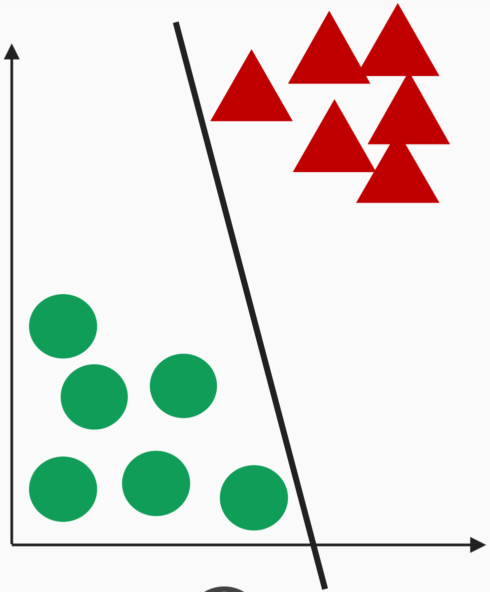
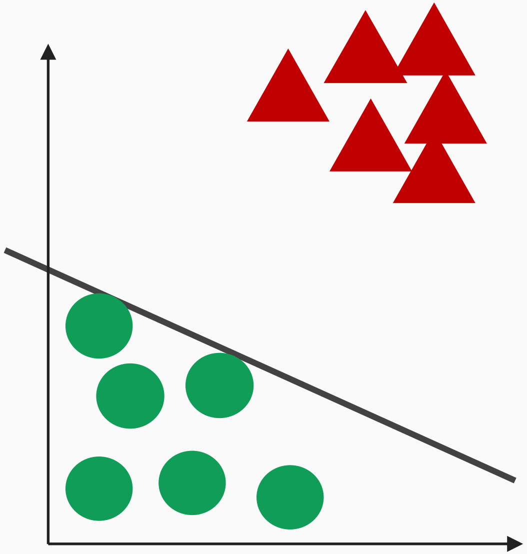


1



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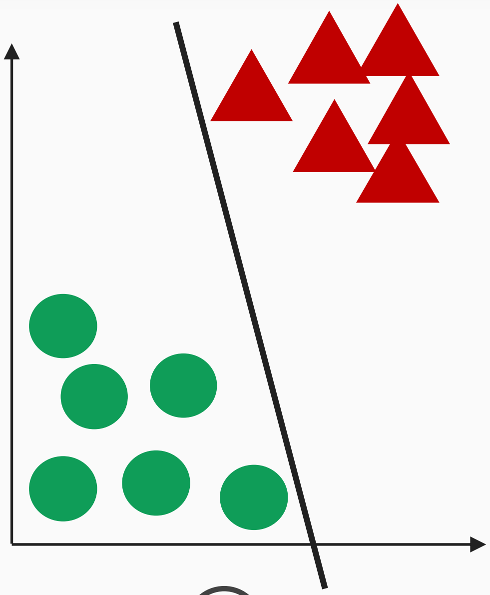

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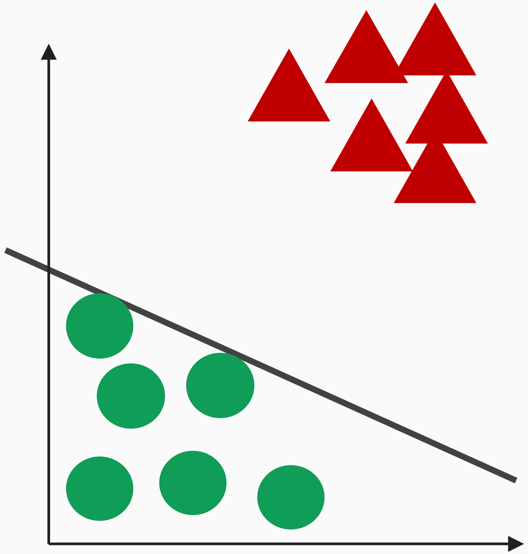

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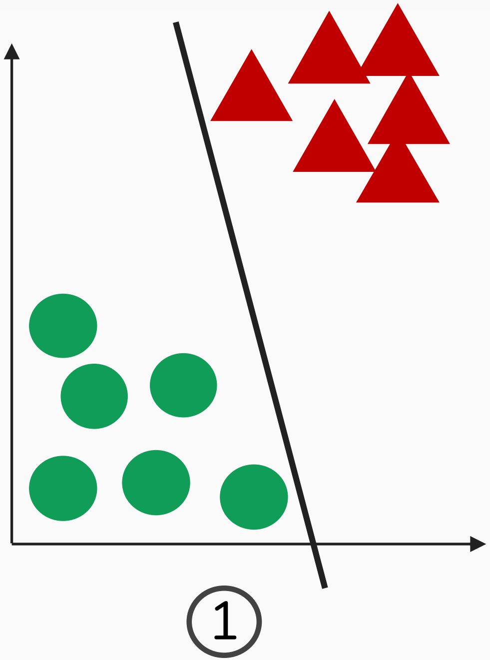

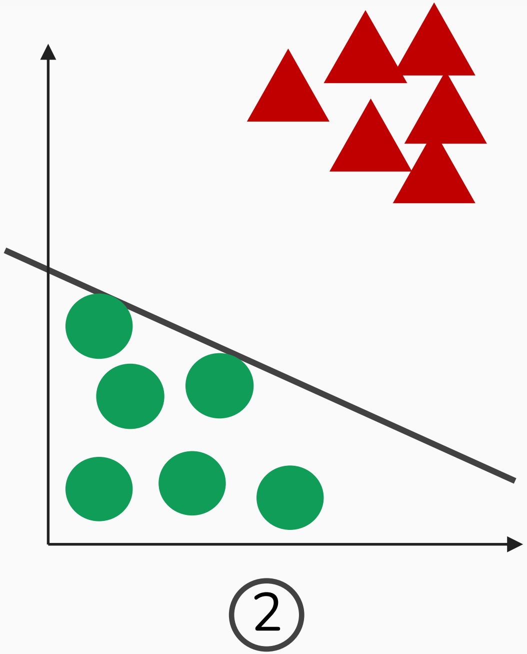

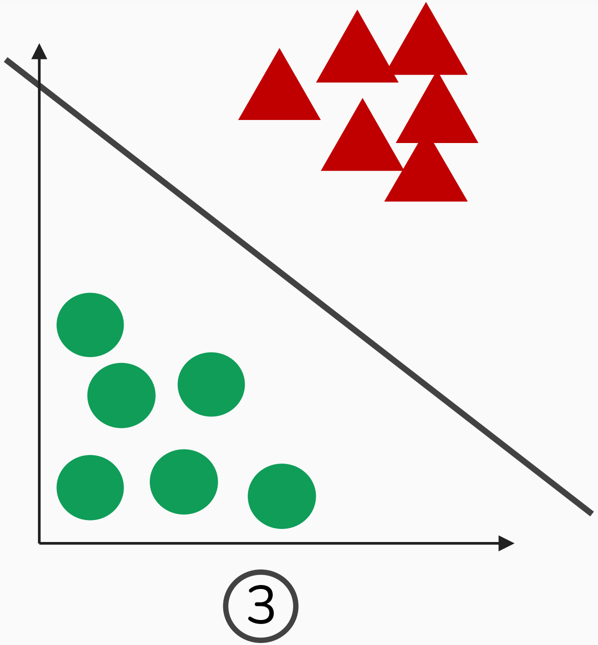

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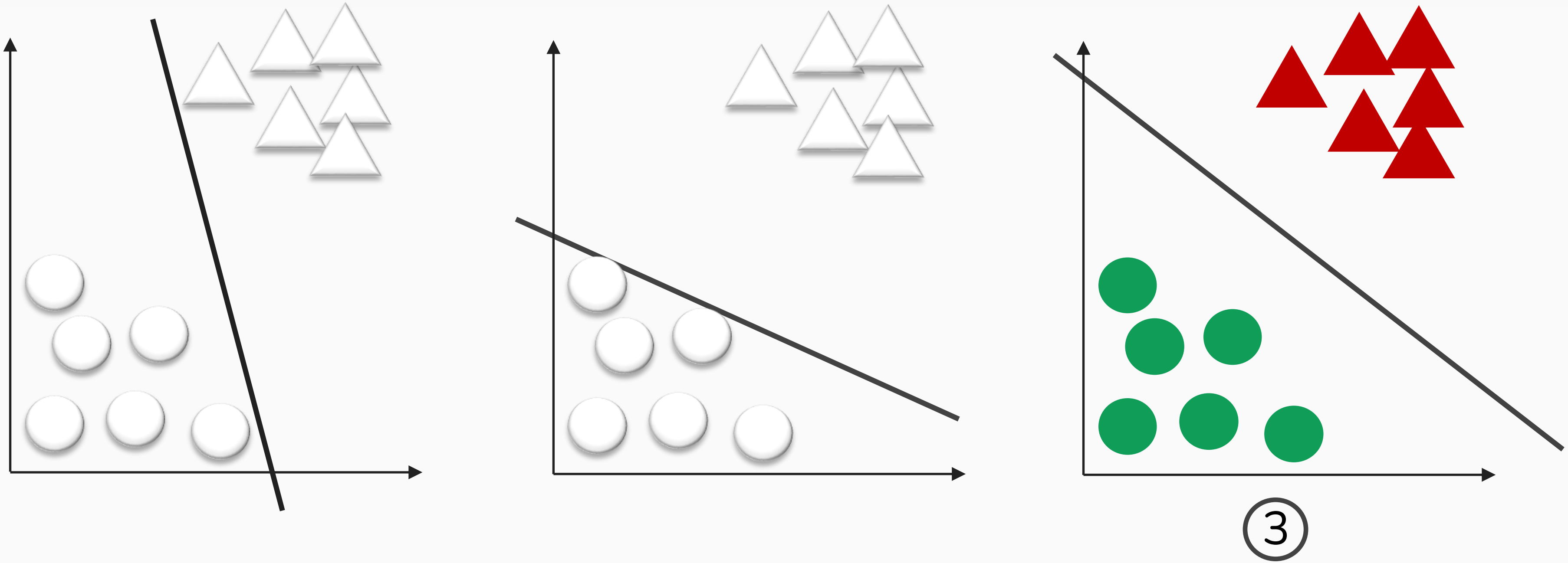

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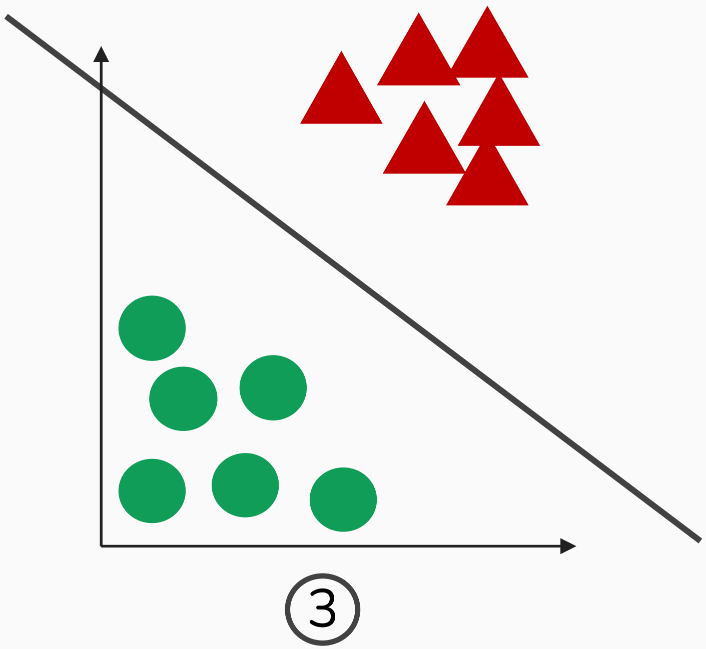

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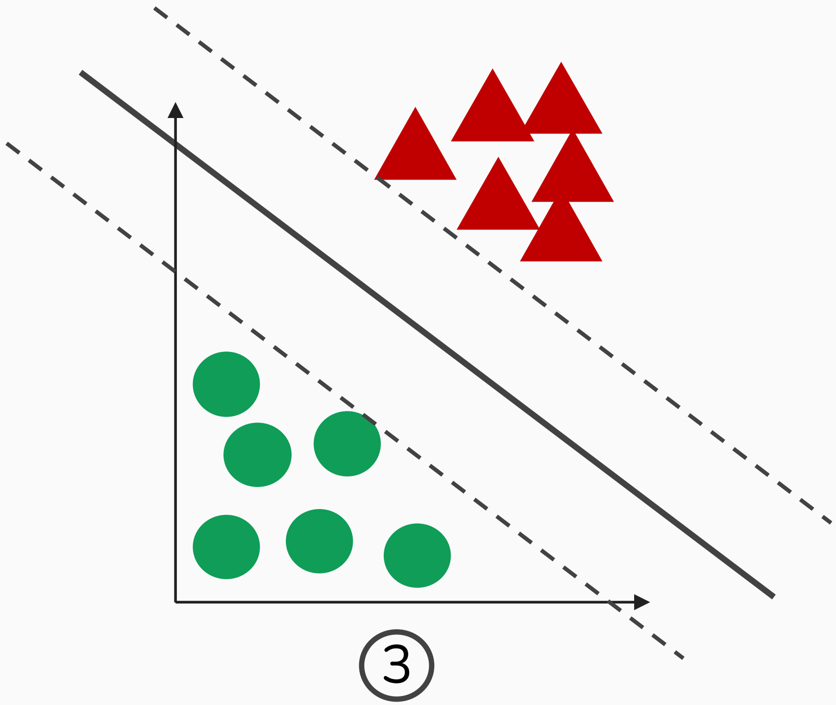
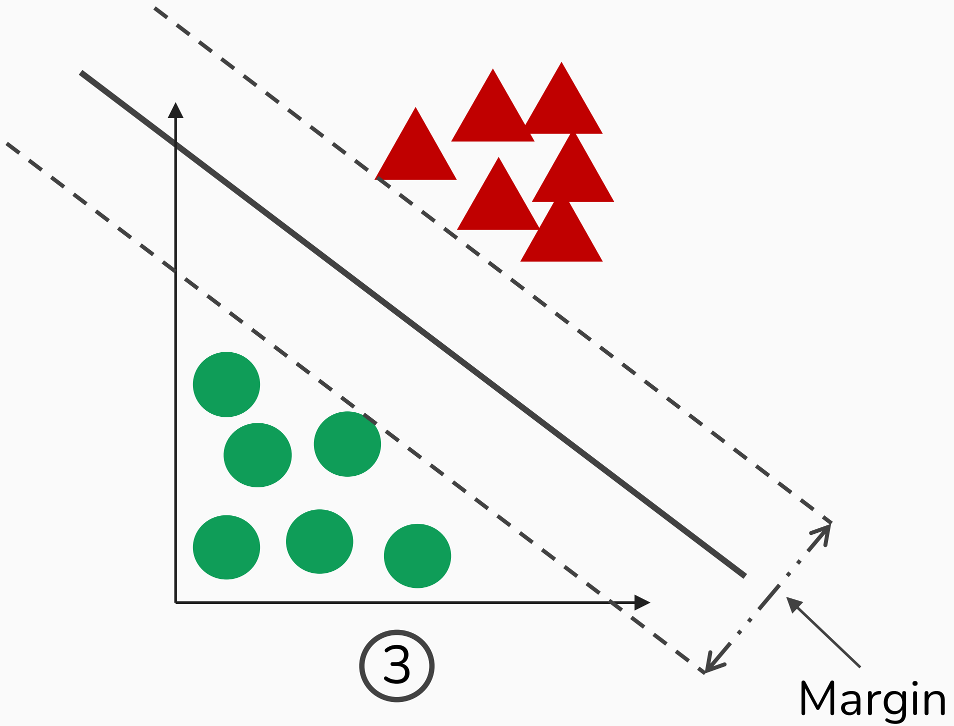


We maximized the margin



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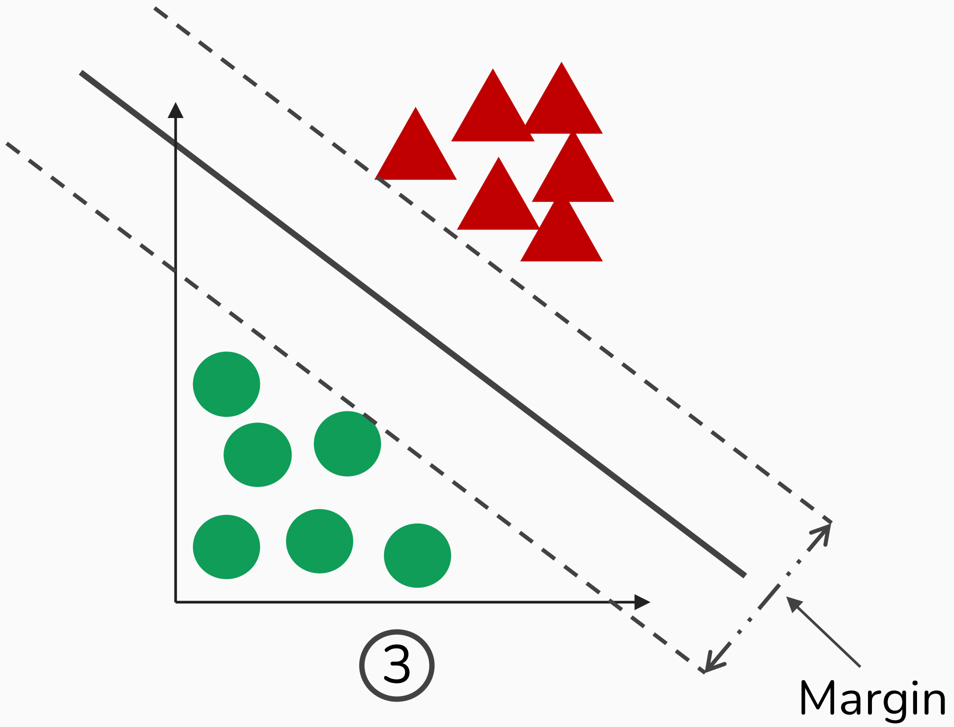
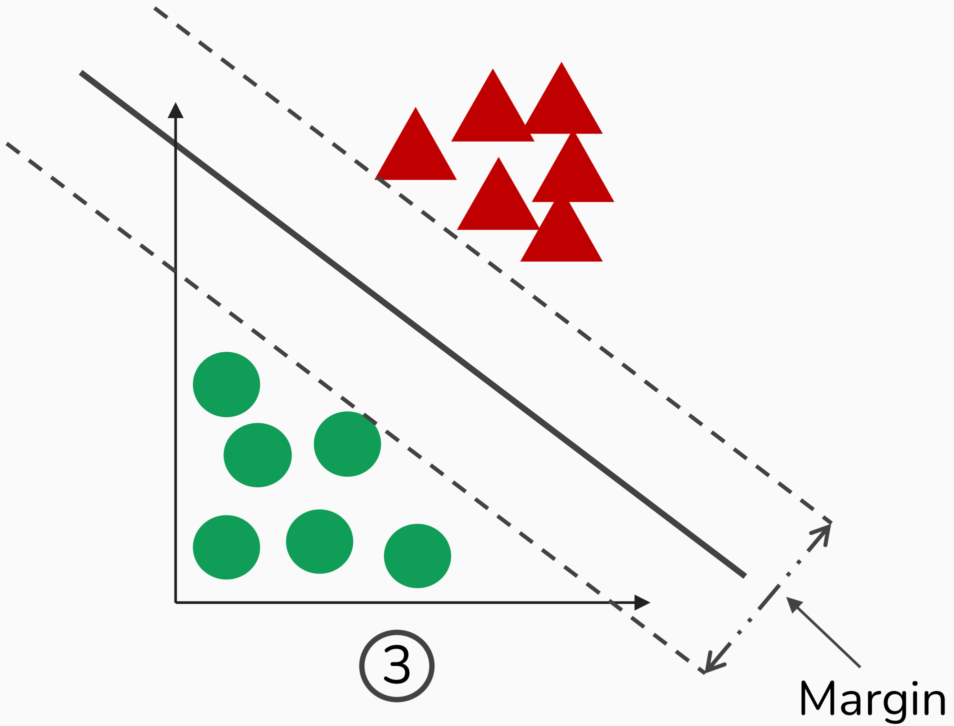
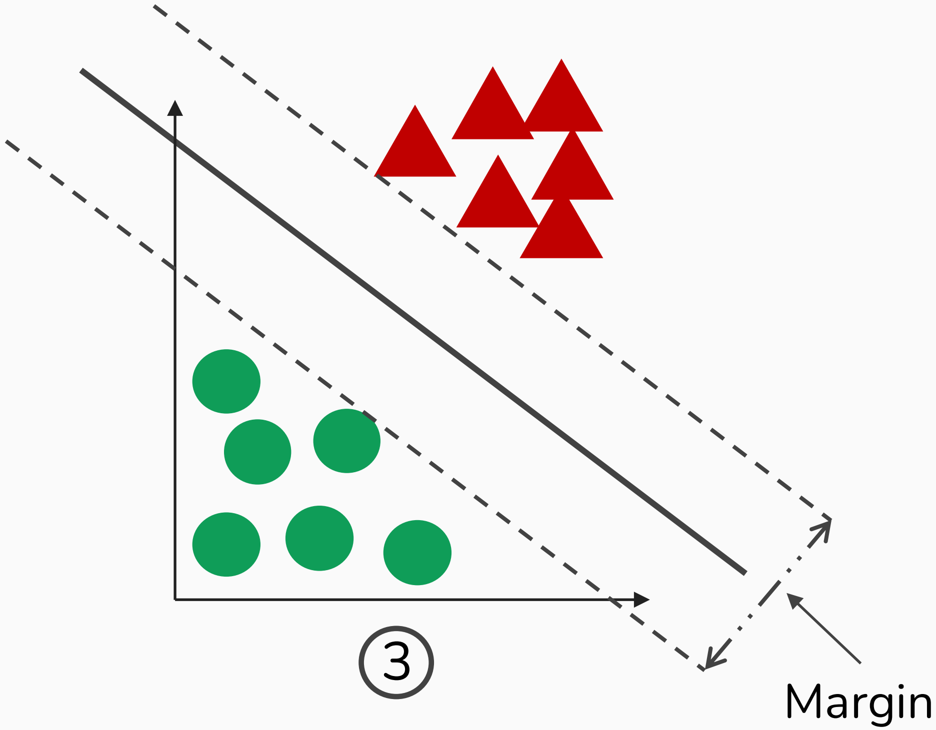
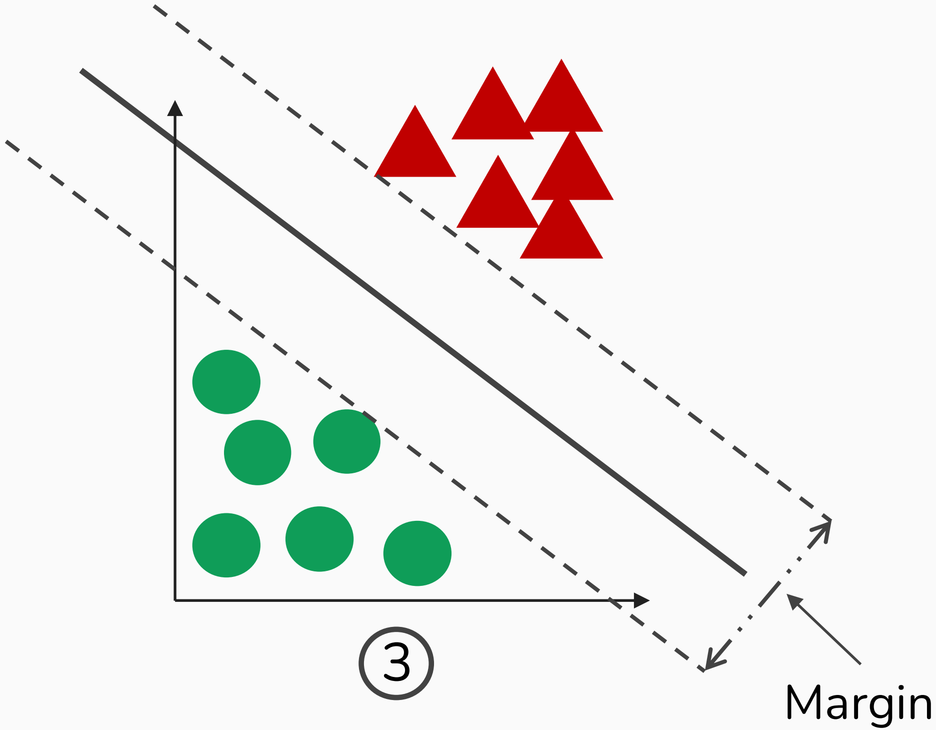


And … This is …. SVMs

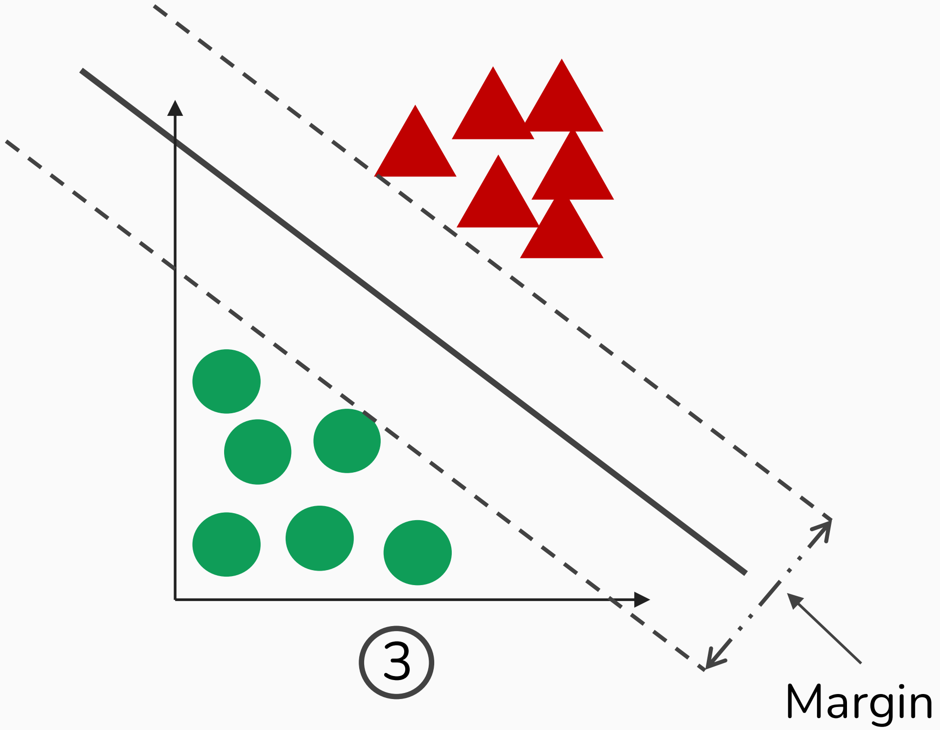



3
Margin

SVMs algorithm learns a linear model (hyperplane)



SVMs algorithm learns a linear model (hyperplane)
SVMs algorithm maximizes the margin around the hyperplane



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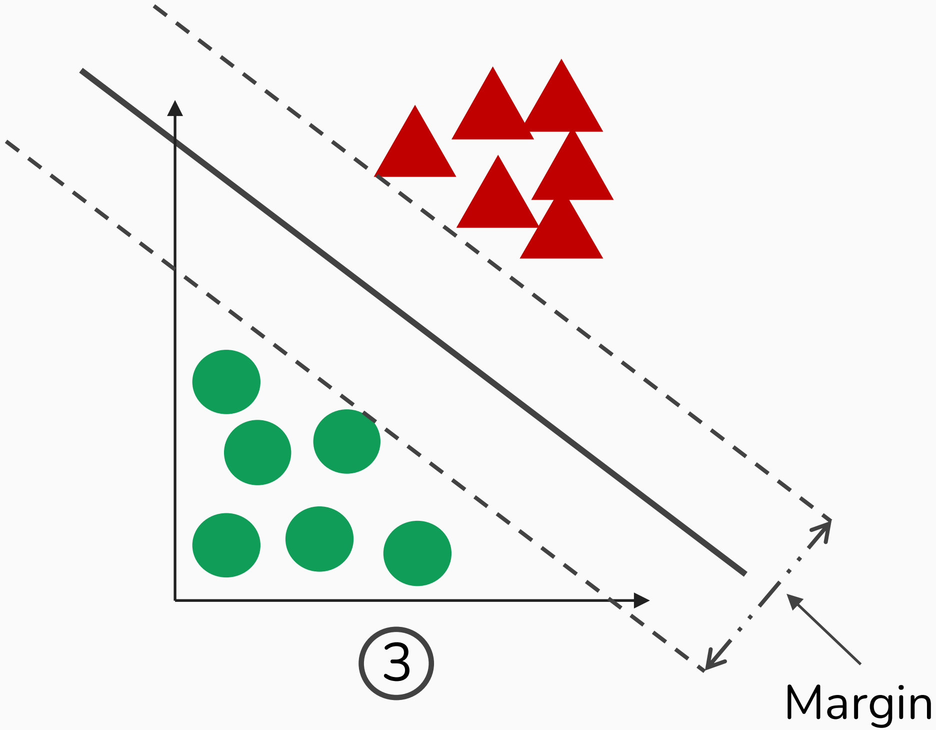
These points are called
support vectors



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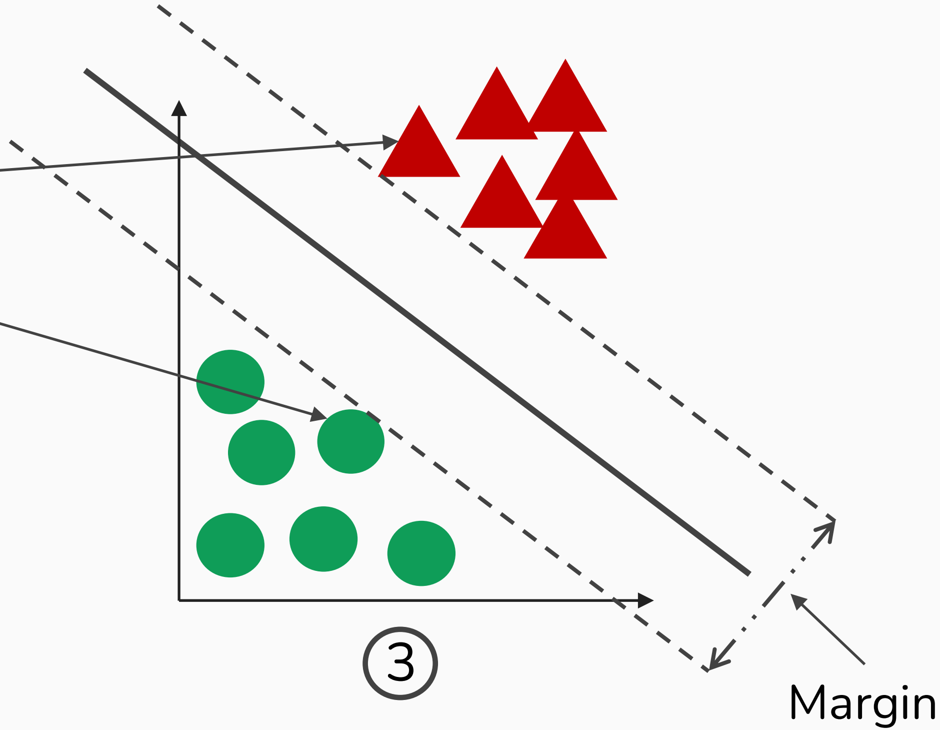

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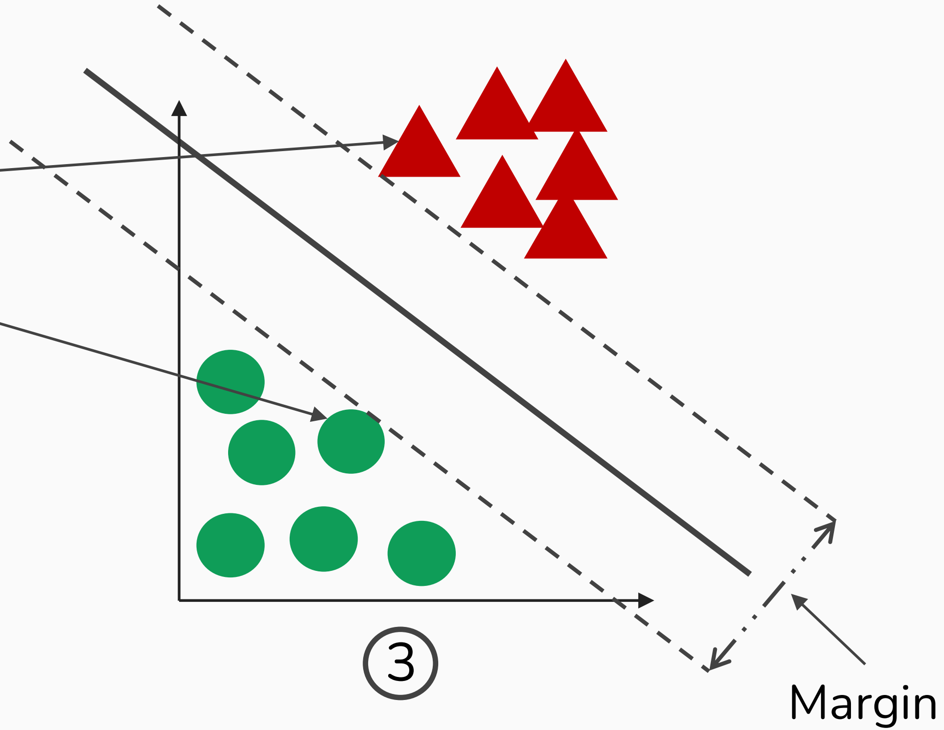
These points are the data points that lie closest to the hyperplane



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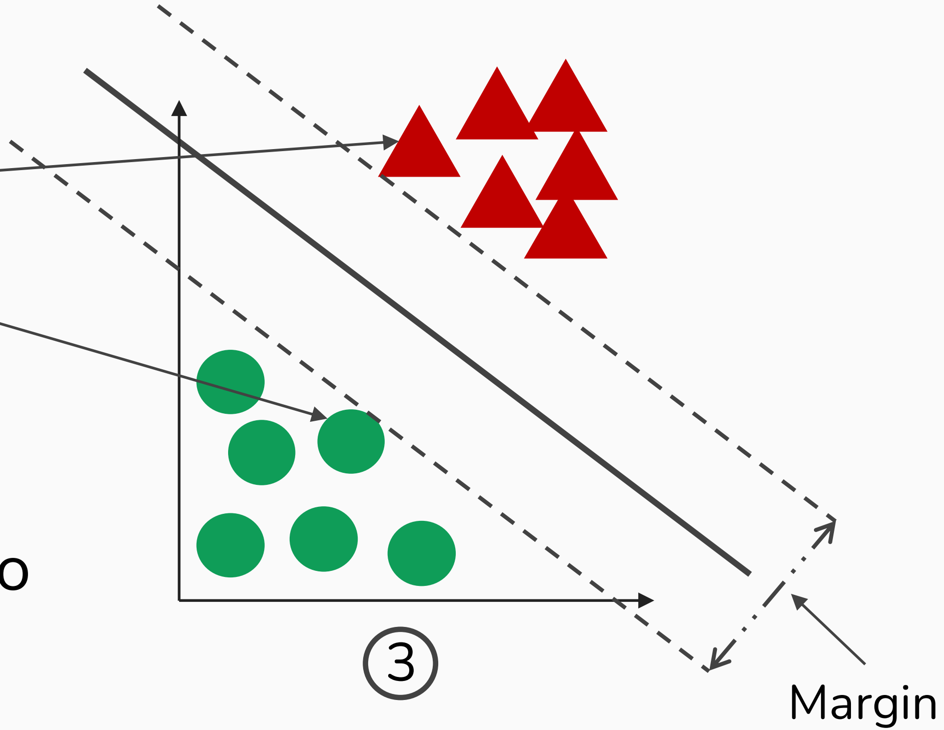
The data points most difficult to classify



2 types of data :

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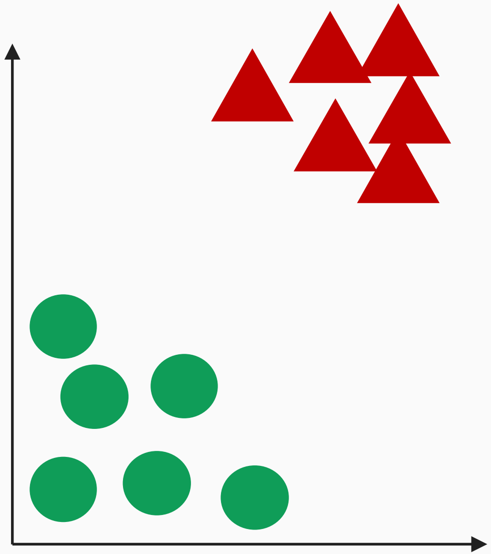
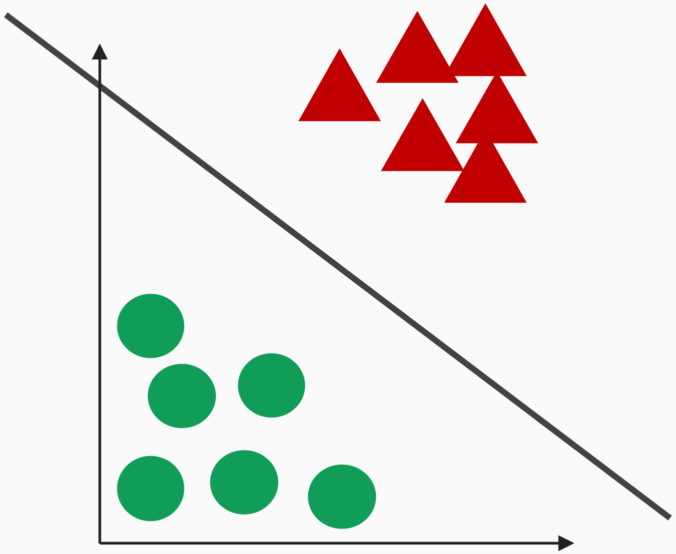


2 types of data :

Linearly separable data



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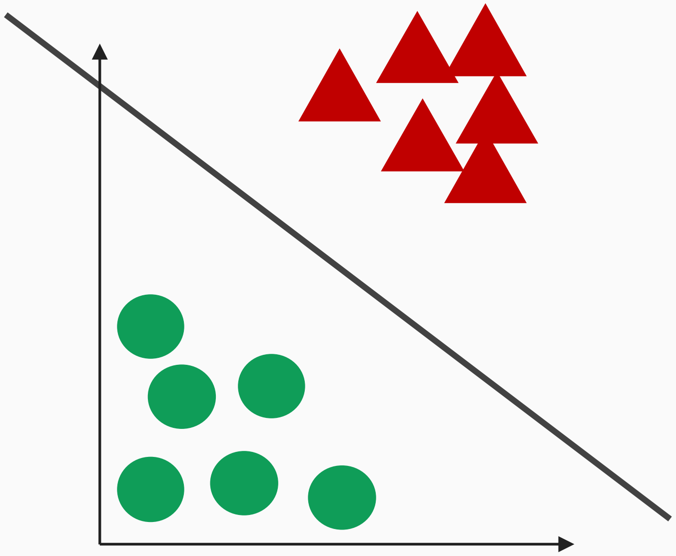

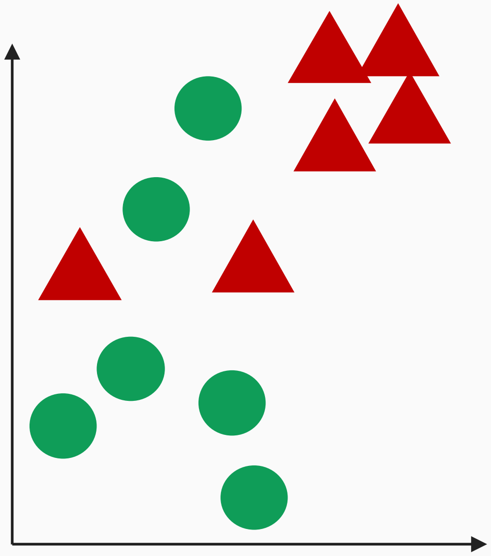

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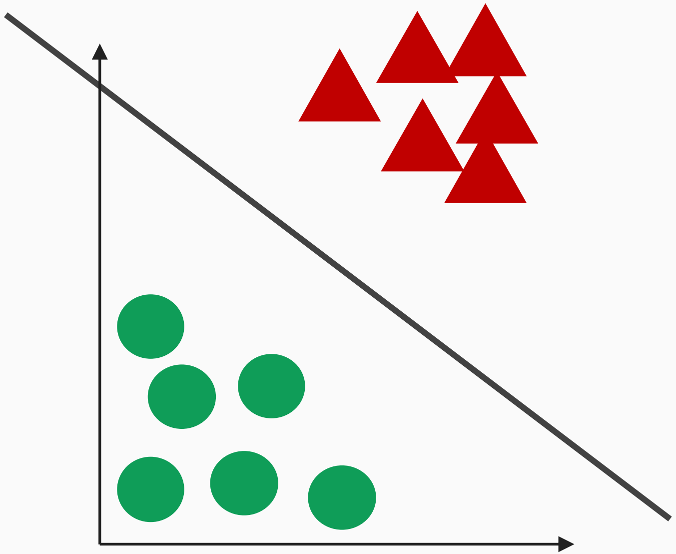
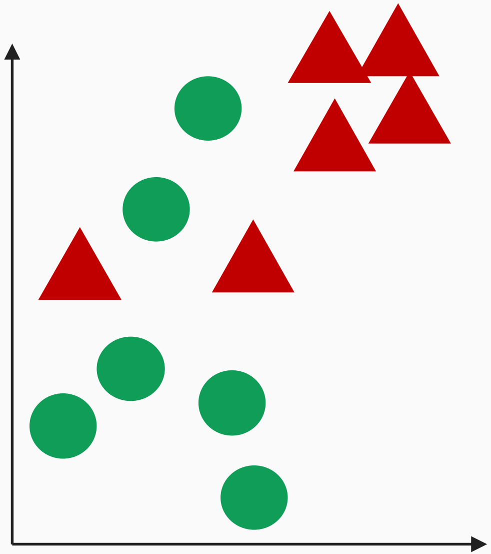


Non linearly separable data



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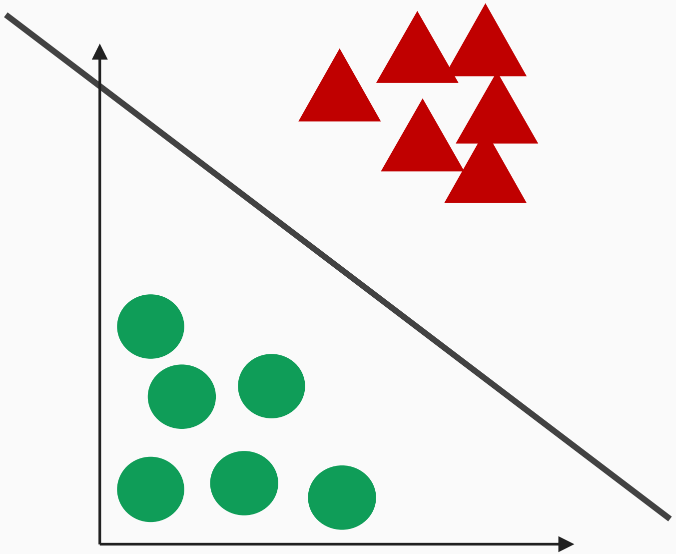

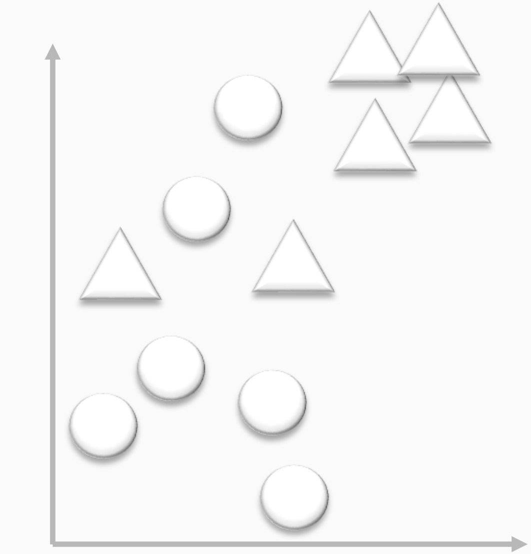

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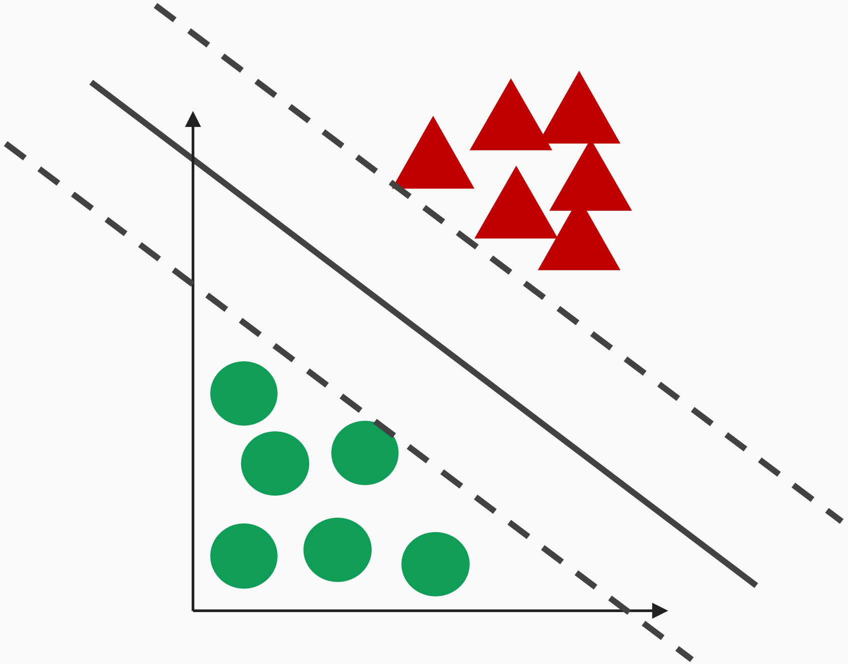
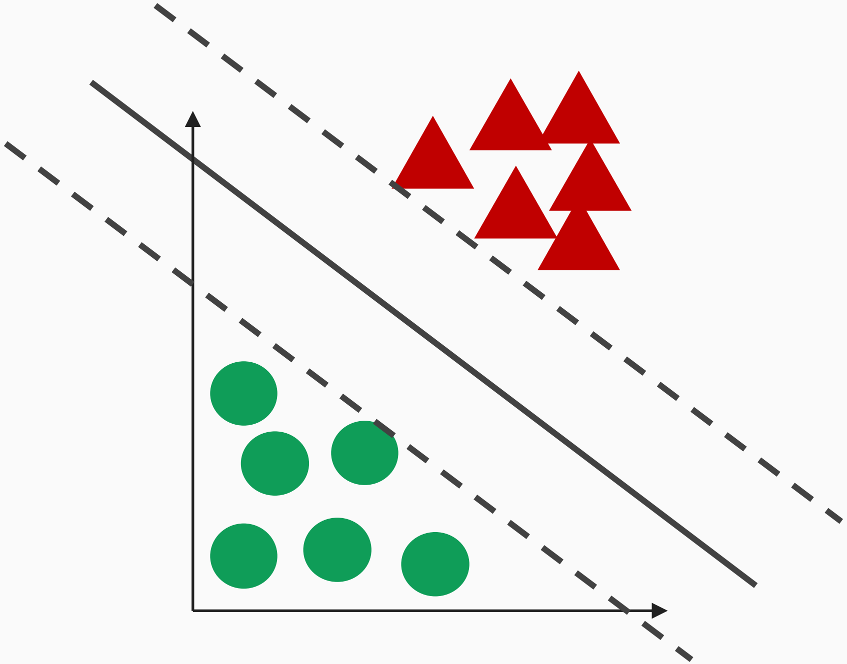


Linearly separable data

Hard margin SVM



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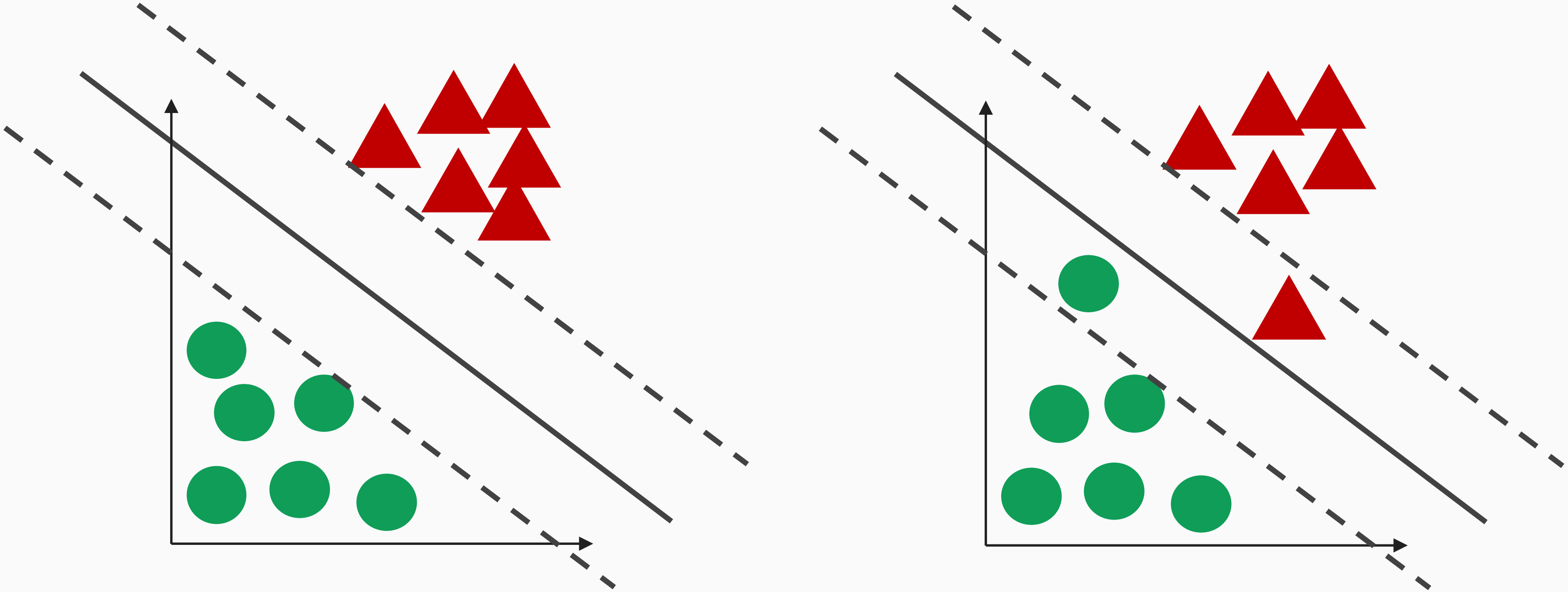

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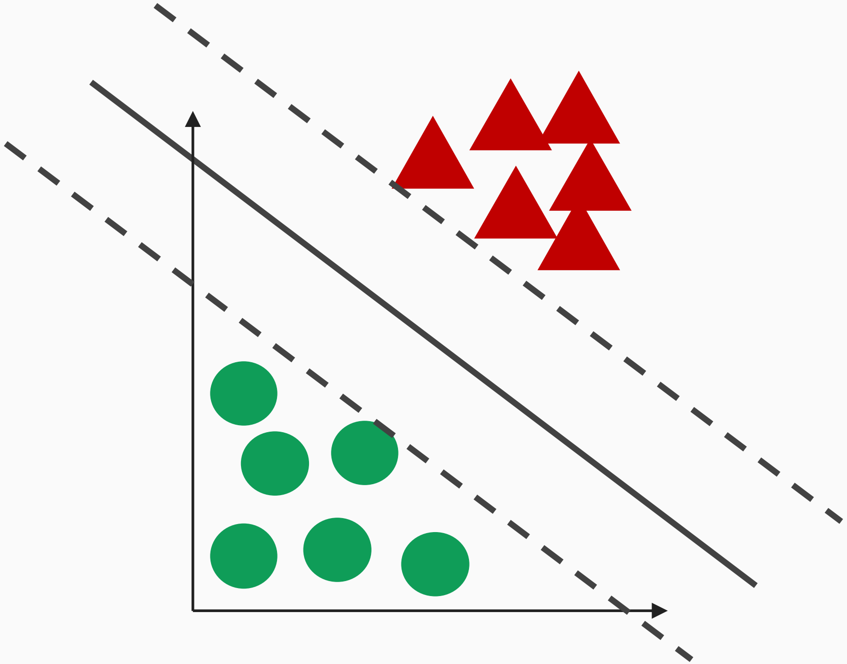
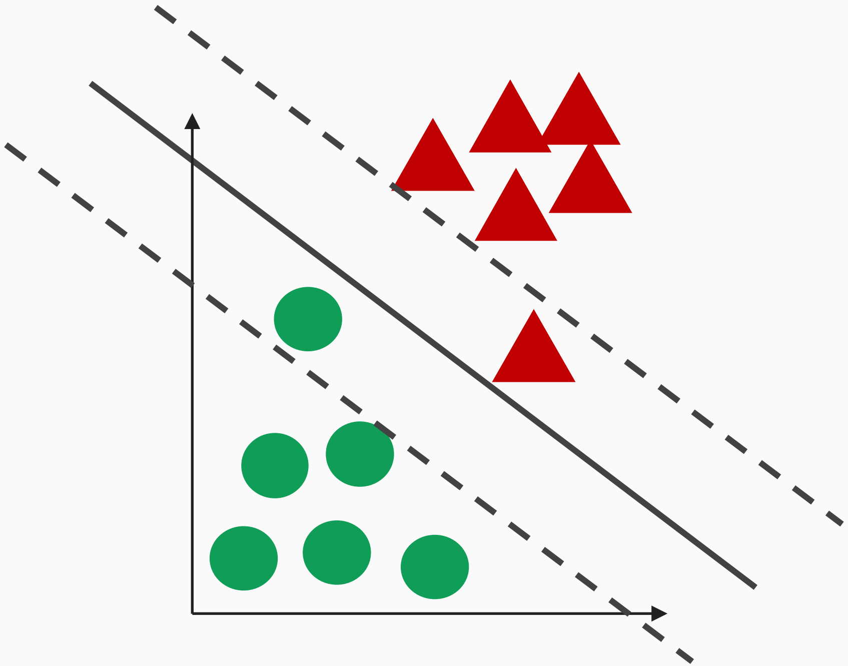


Soft margin SVM



NOTE : hard margin SVM is only applicable when the dataset is linearly separable

WARNING !

The rest of the slides
are math heavy !!

(LINEAR) HARD MARGIN SVMs

Equation of a line : $y = ax + b \Rightarrow y - ax - b = 0$

Equation of an hyperplane: $w^T x = 0 \Leftrightarrow w^T x + b = 0$

$assuming\ that\ x_0 = 1$

First step : prepare the dataset

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$$D = \{(x_i, y_i) \,|\, x_i \in \mathbb{R}^p, y_i \in \{\bullet, \blacktriangle\}\}_{i=1}^{N}$$

Second Step : select 2 hyperplanes that separates the data

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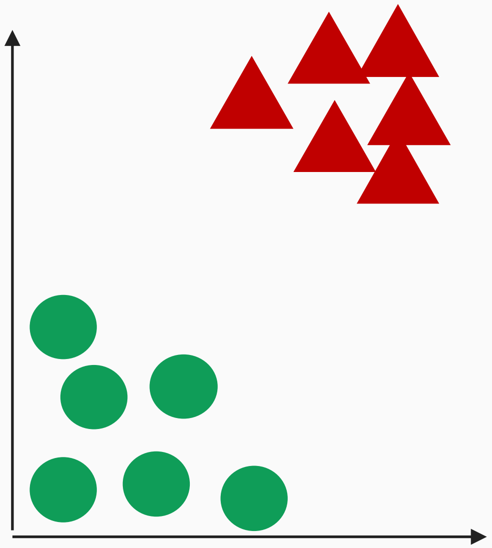

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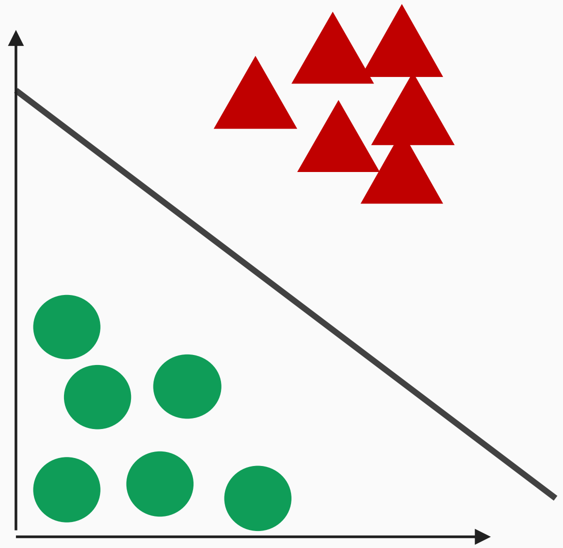

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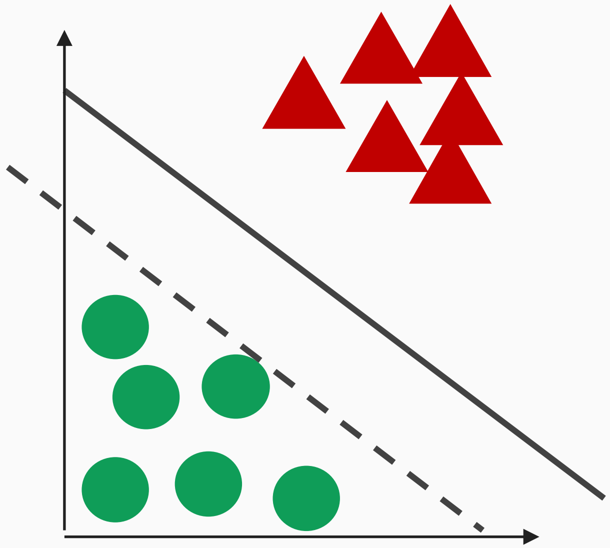

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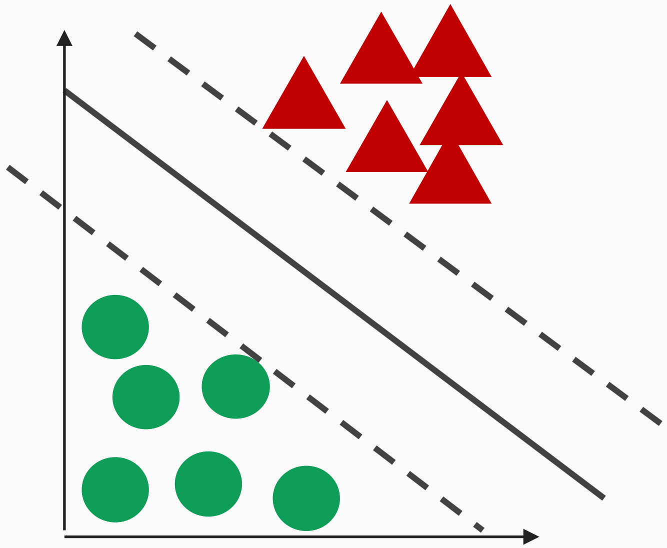

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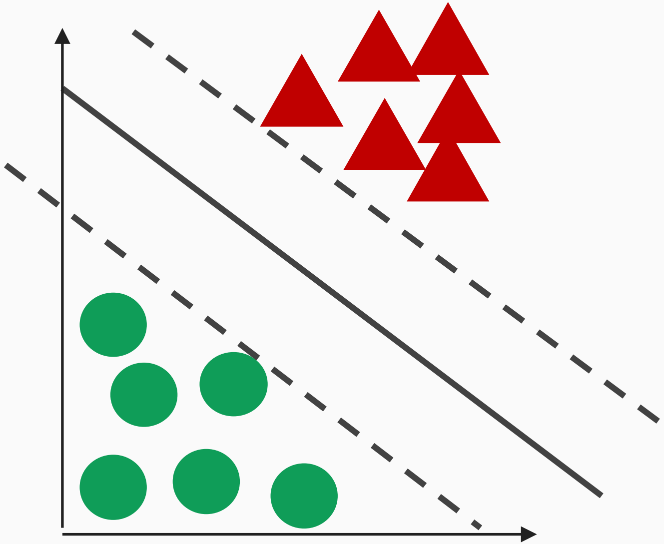

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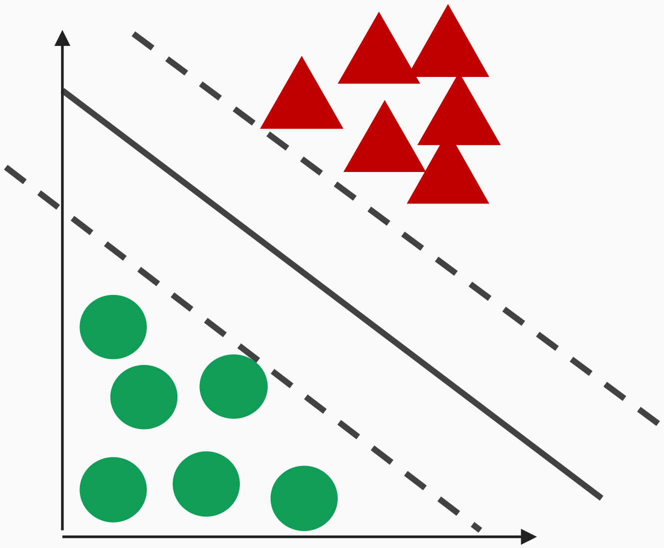

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$$H_0 : w^T x + b = 0$$

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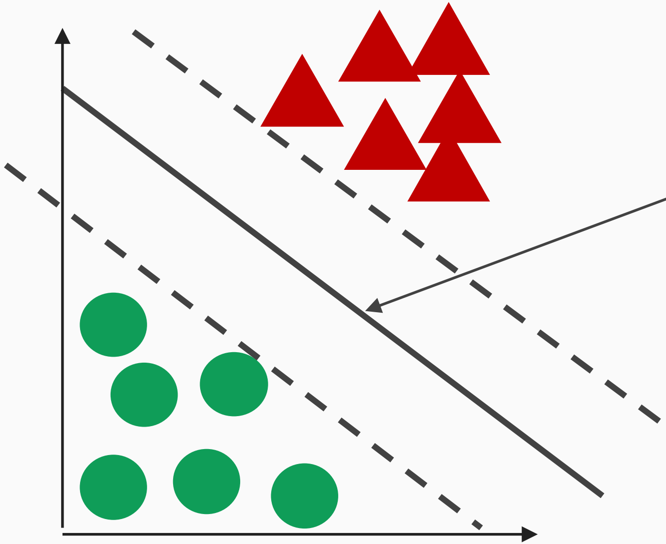

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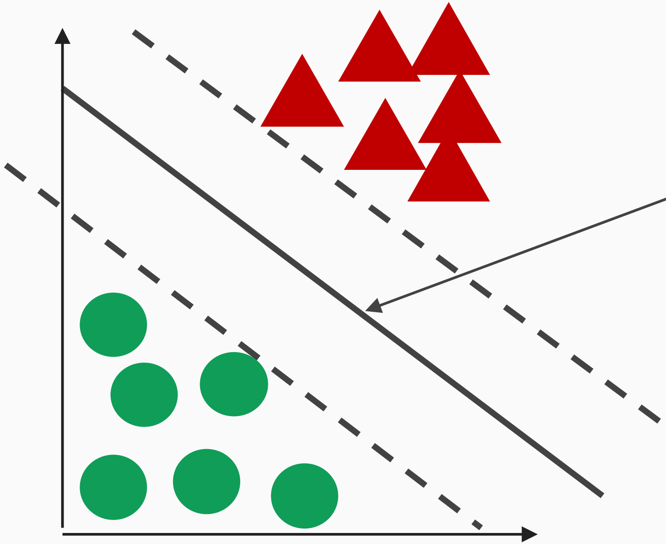

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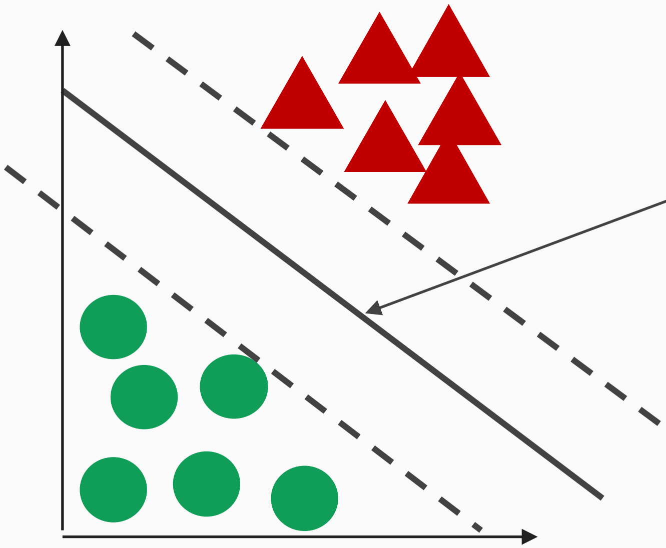

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$$\begin{cases} H_1 : wx + b = \delta \\ H_2 : wx + b = -\delta \end{cases}$$

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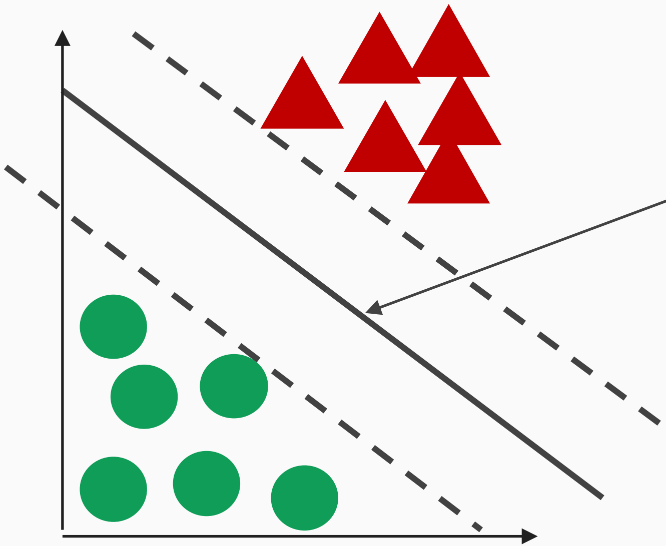

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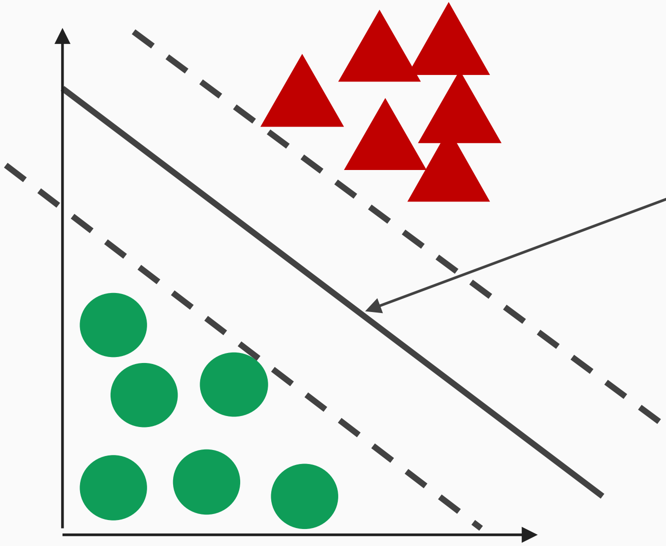

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$$\begin{cases} H_1 : wx + b = 1 \\ H_2 : wx + b = -1 \end{cases}$$

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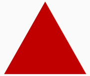
For x_i having the class  :

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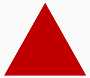
For x_i having the class  : $$wx_i + b \geq 1$$

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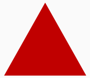
For x_i having the class  : $$wx_i + b \geq 1$$

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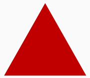
For x_i having the class  : $$wx_i + b \geq 1$$

i goes from 1 to N

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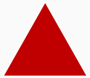
For x_i having the class  : $$wx_i + b \geq 1$$

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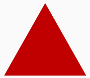
For x_i having the class  : $$wx_i + b \geq 1$$

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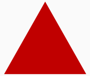
For x_i having the class  : $$wx_i + b \geq 1$$

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REMEMBER : our main goal is to maximize the margin

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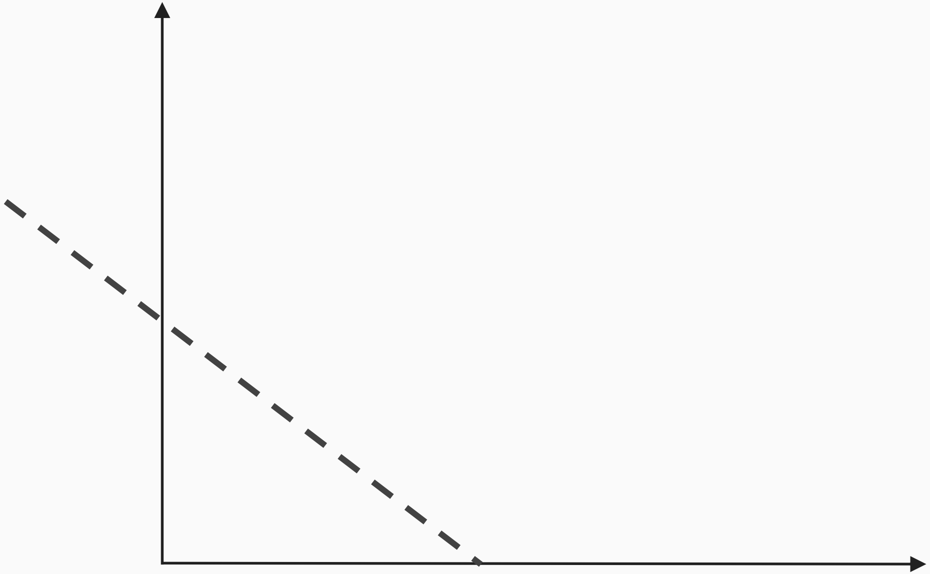

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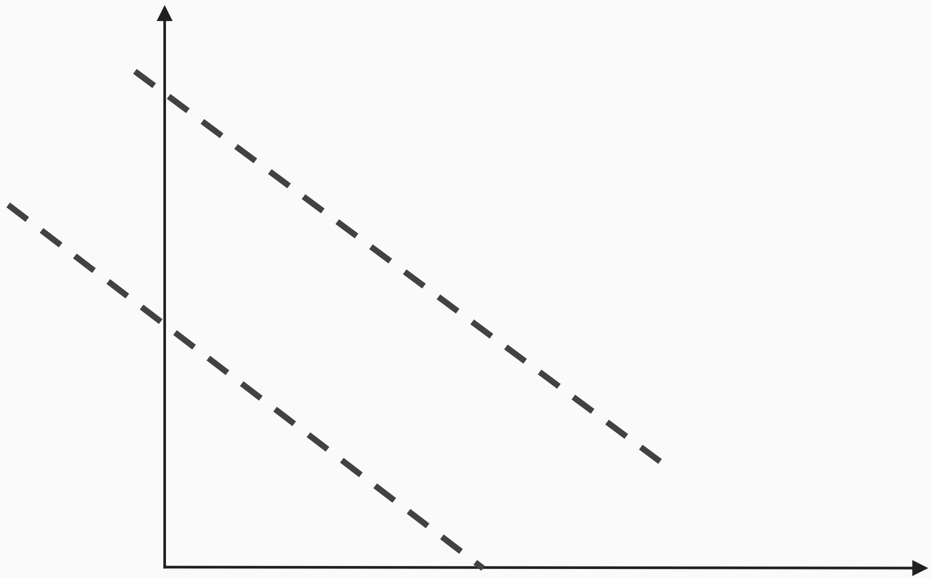

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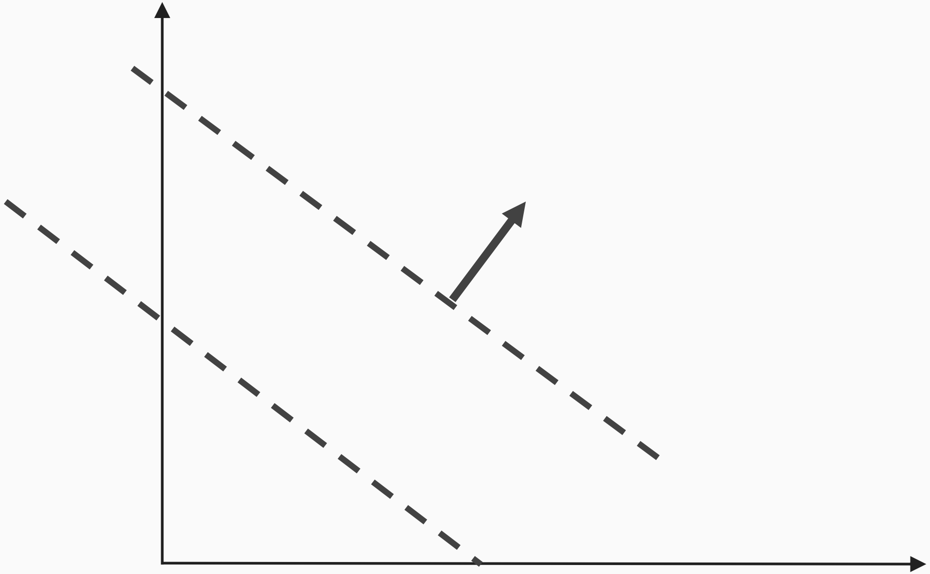

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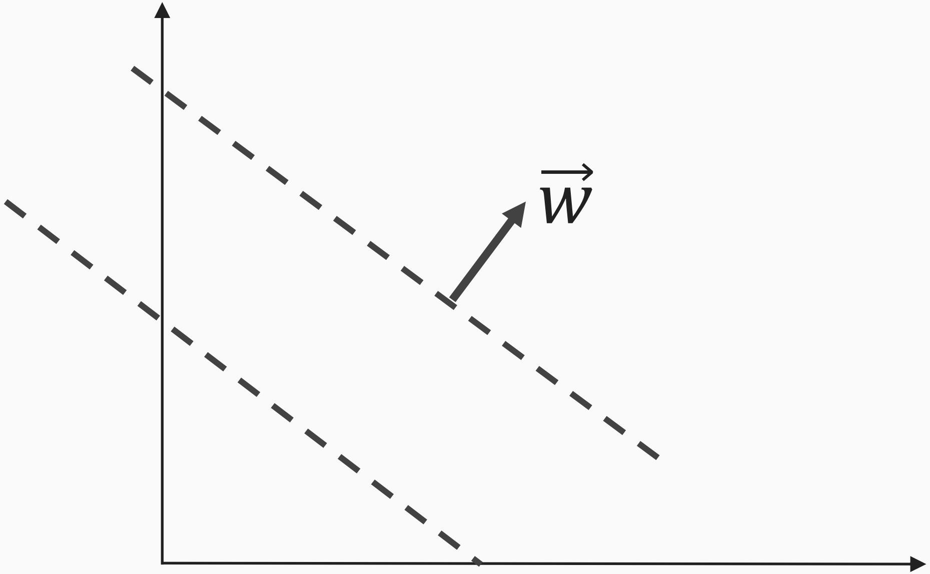

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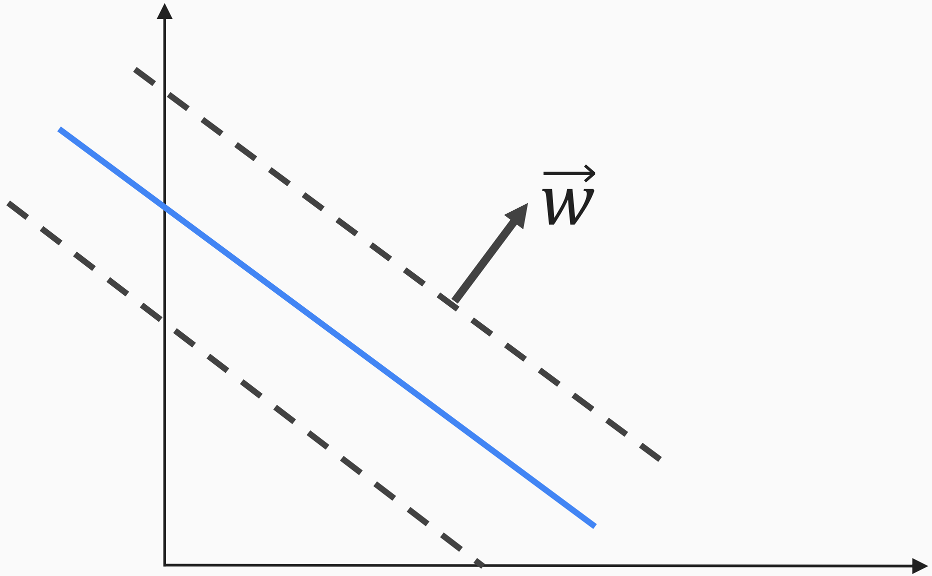

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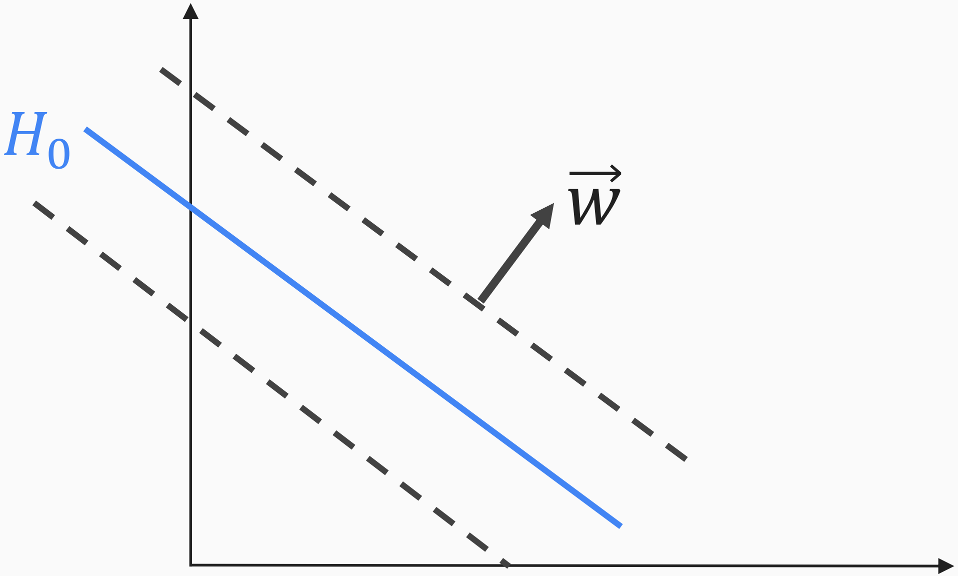

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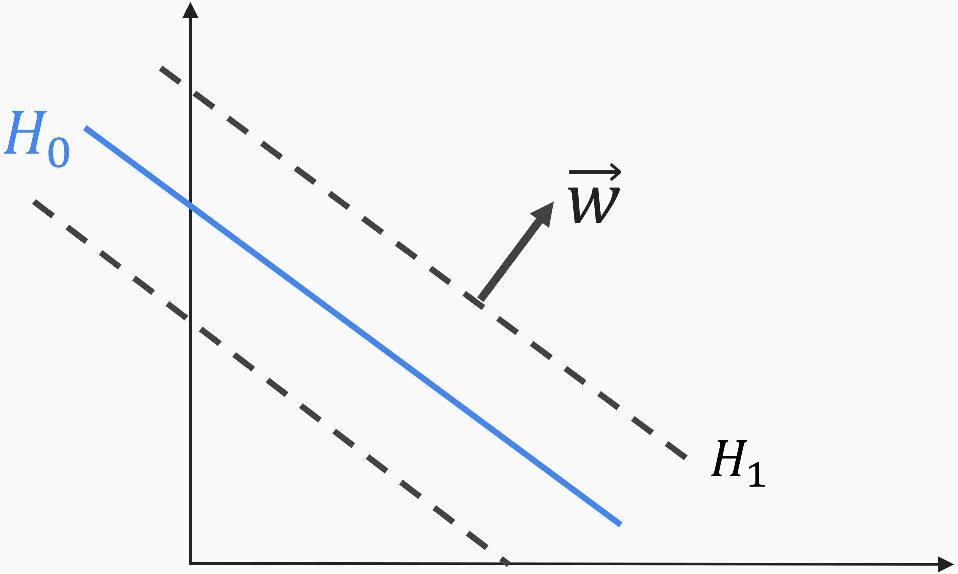

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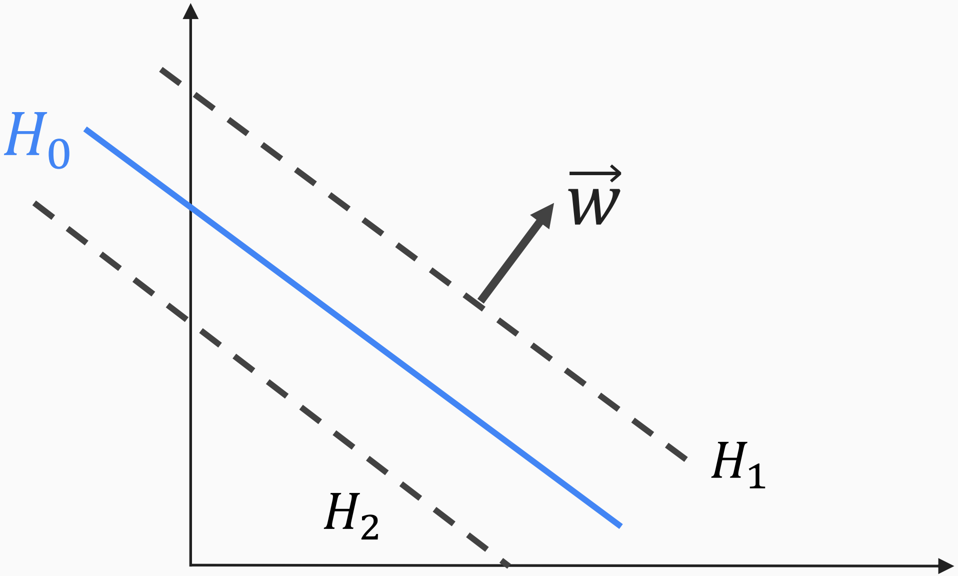

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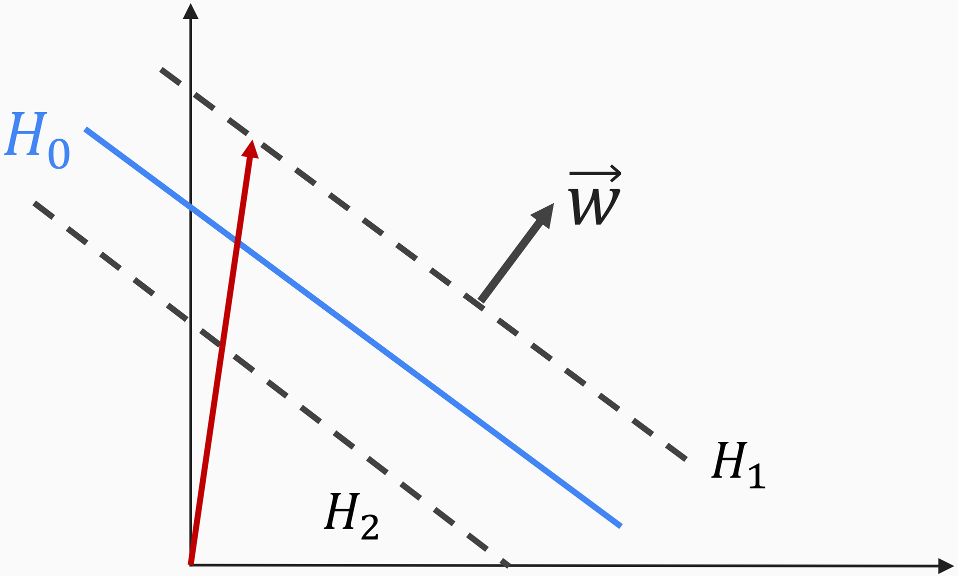

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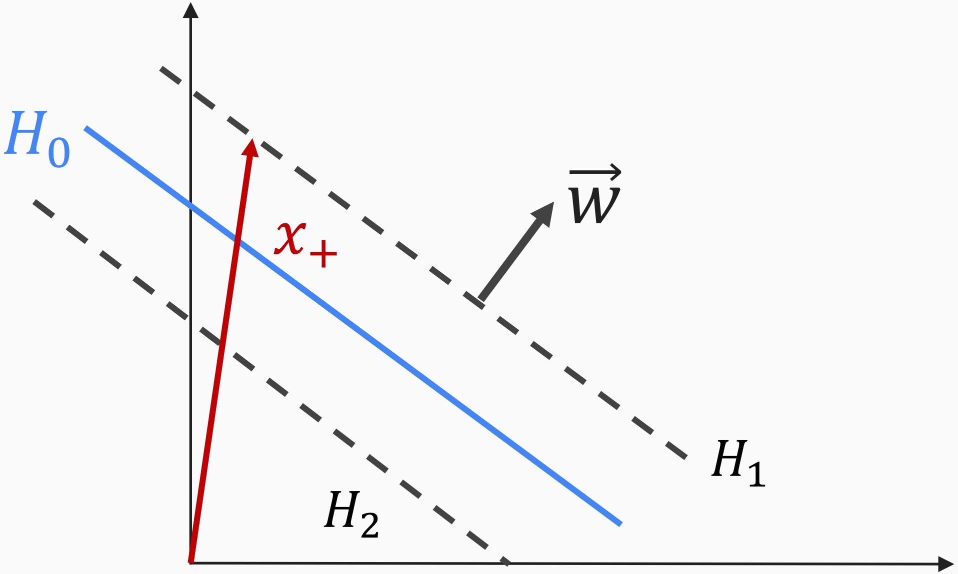

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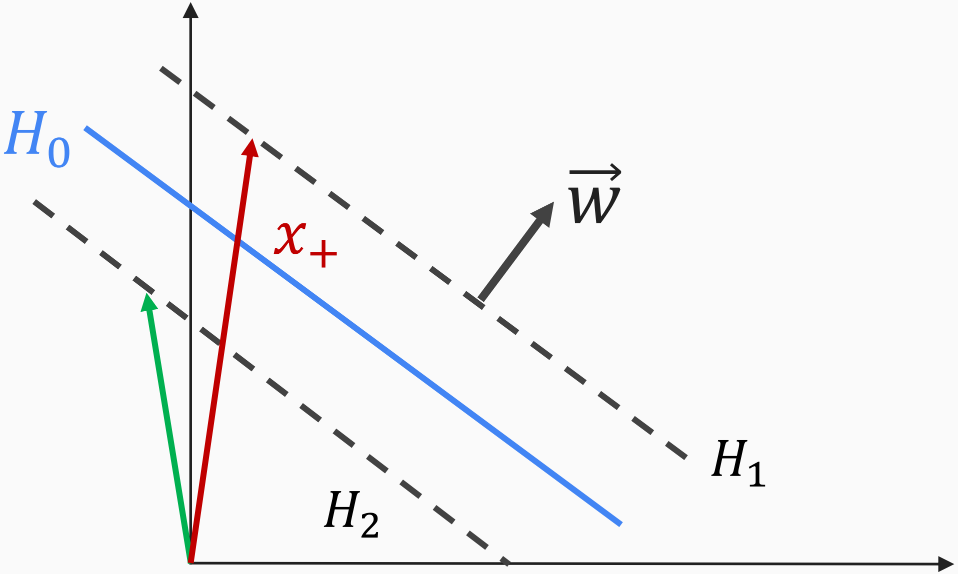

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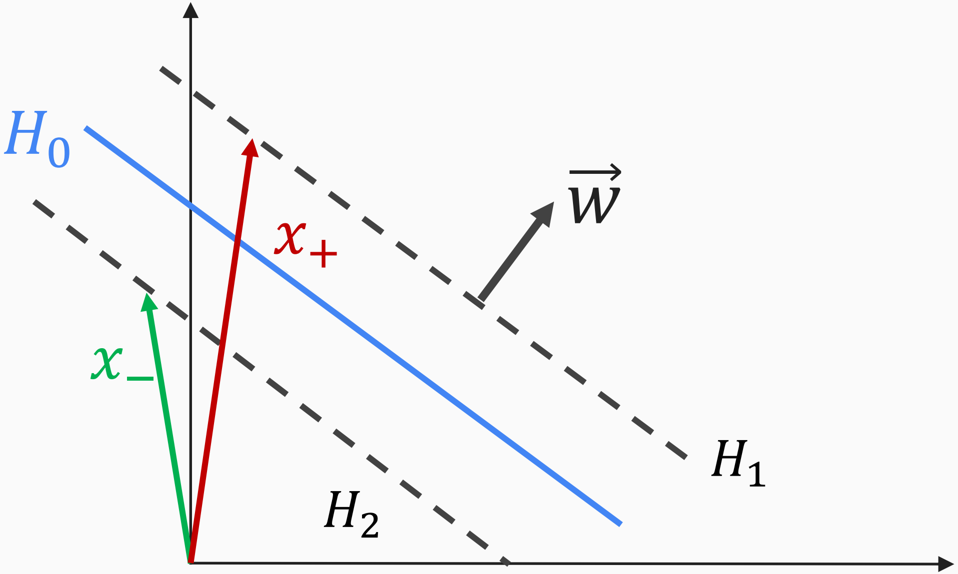

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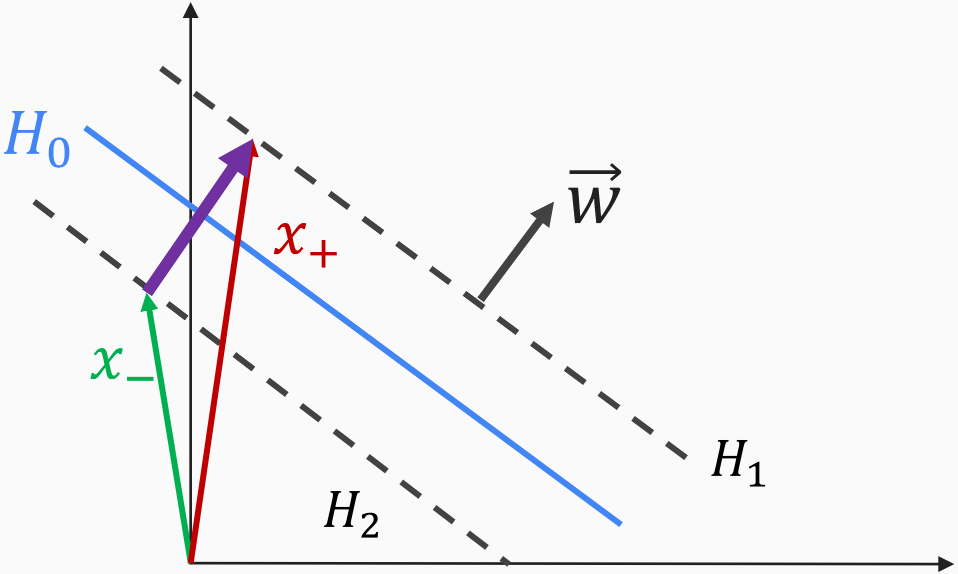

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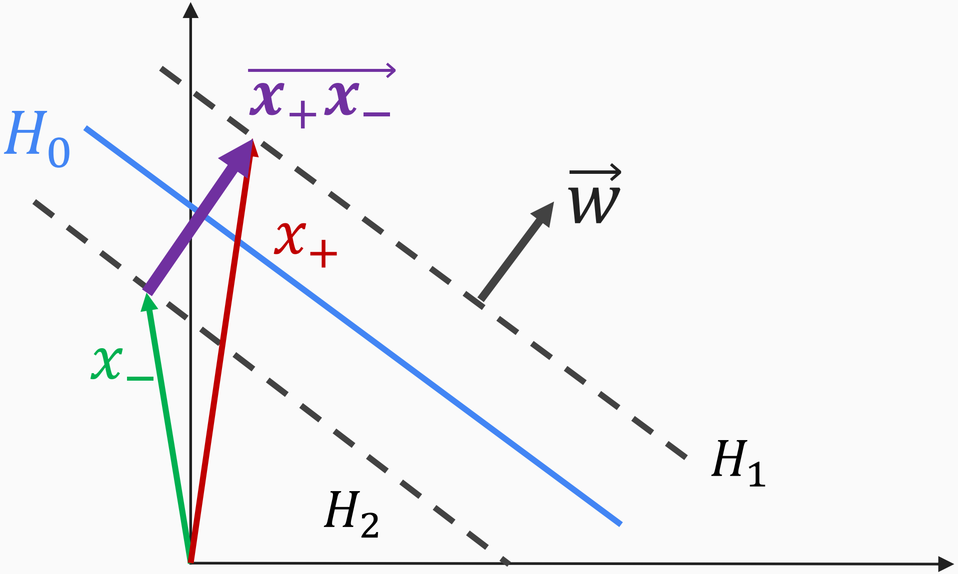

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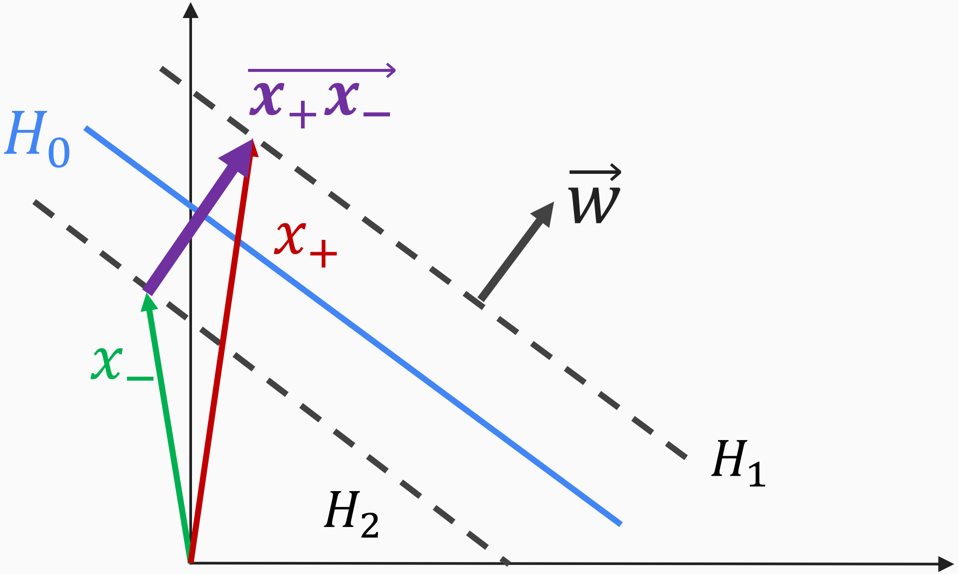

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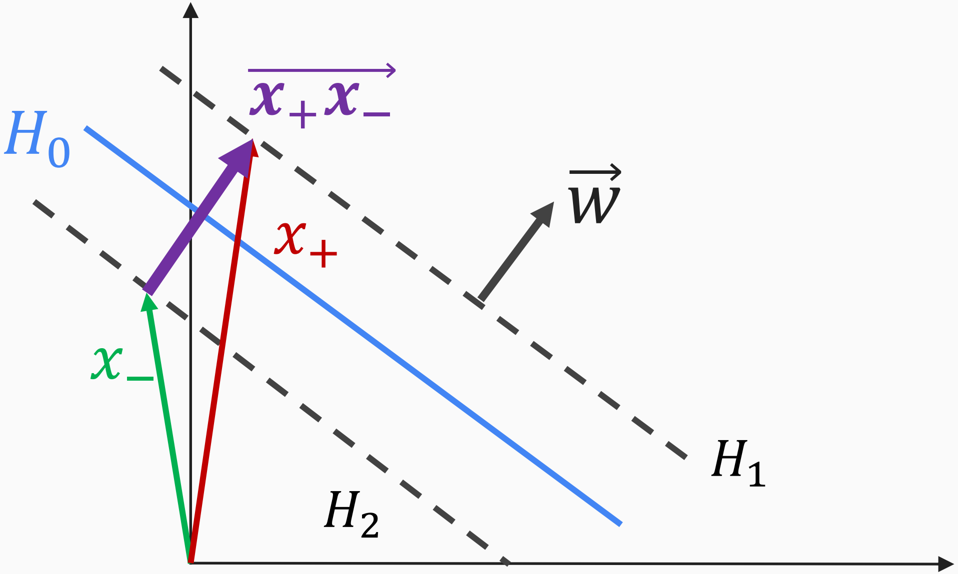

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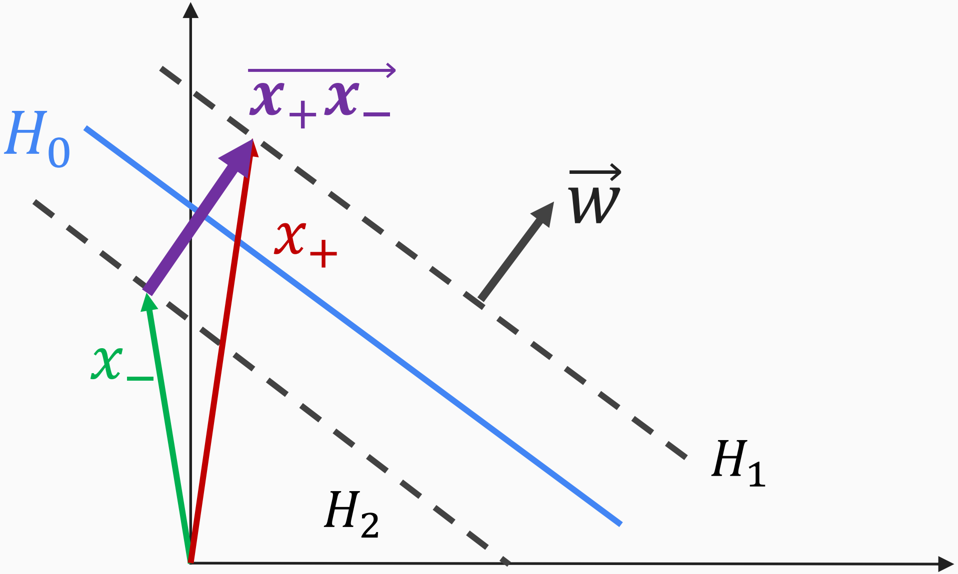

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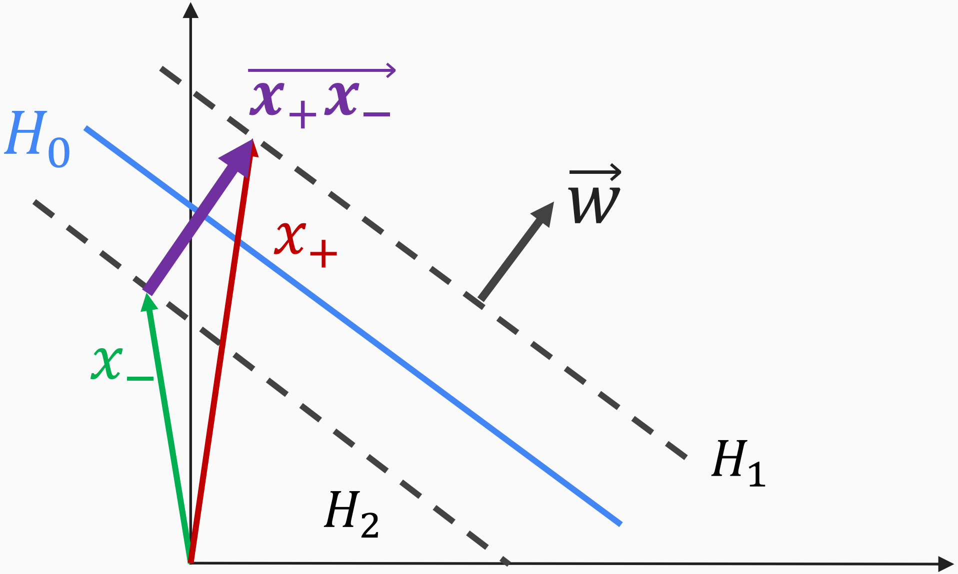

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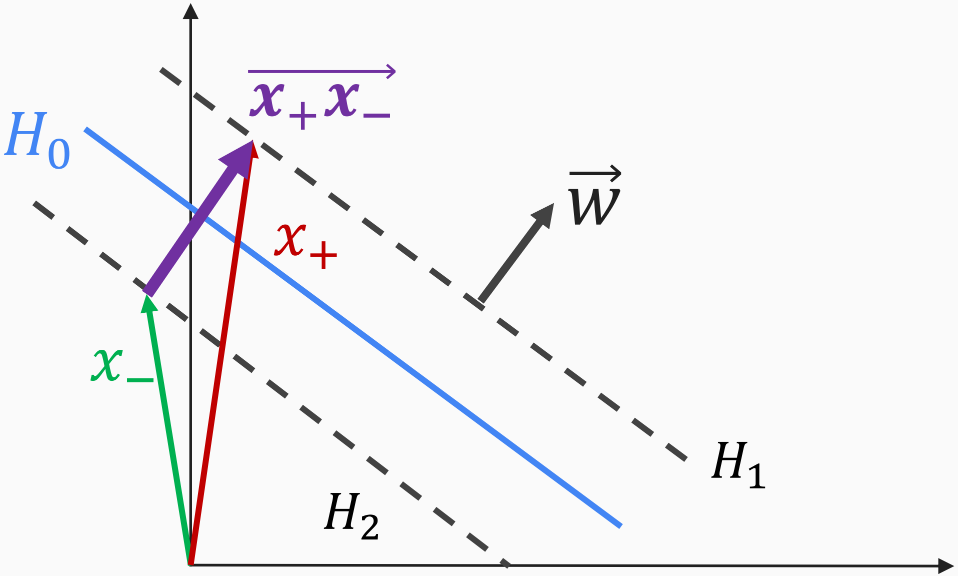

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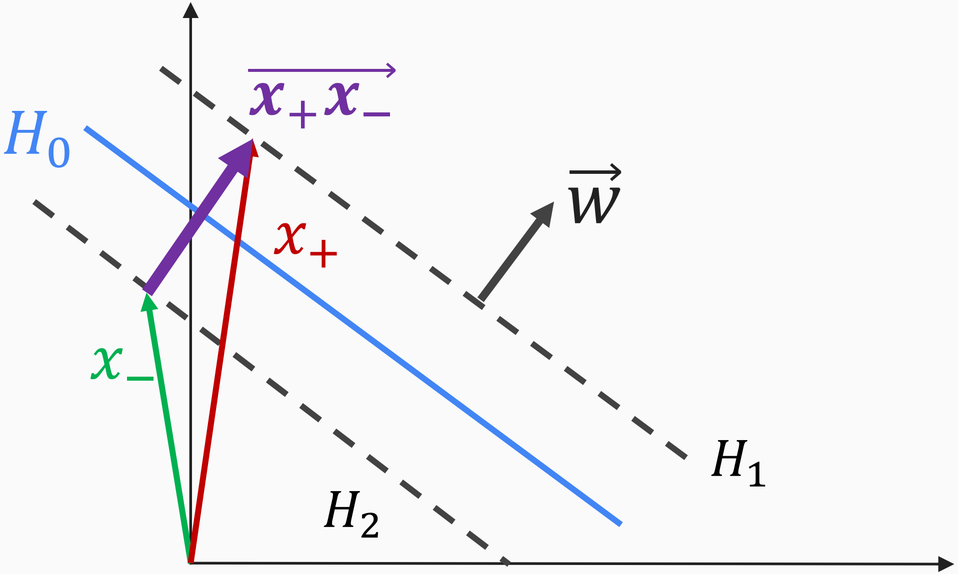

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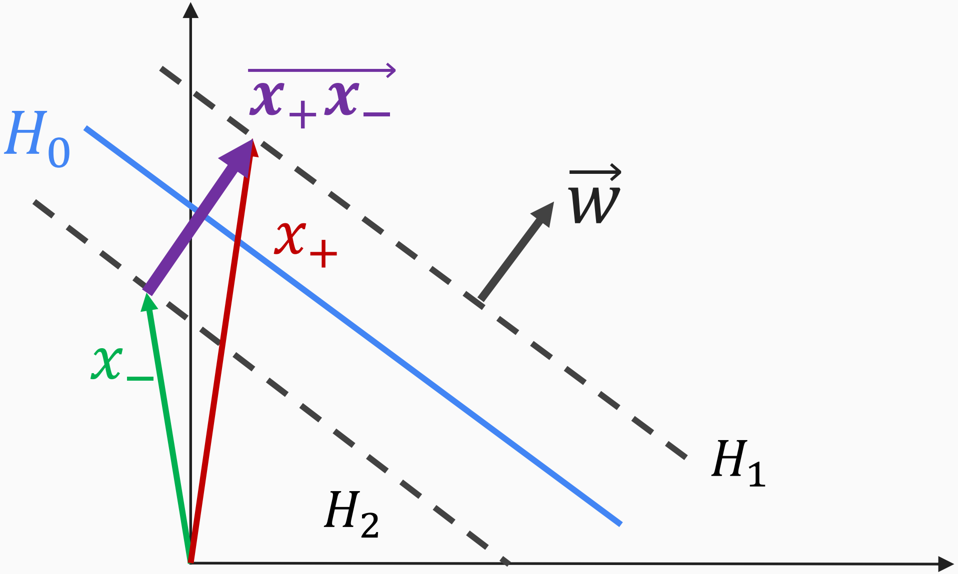

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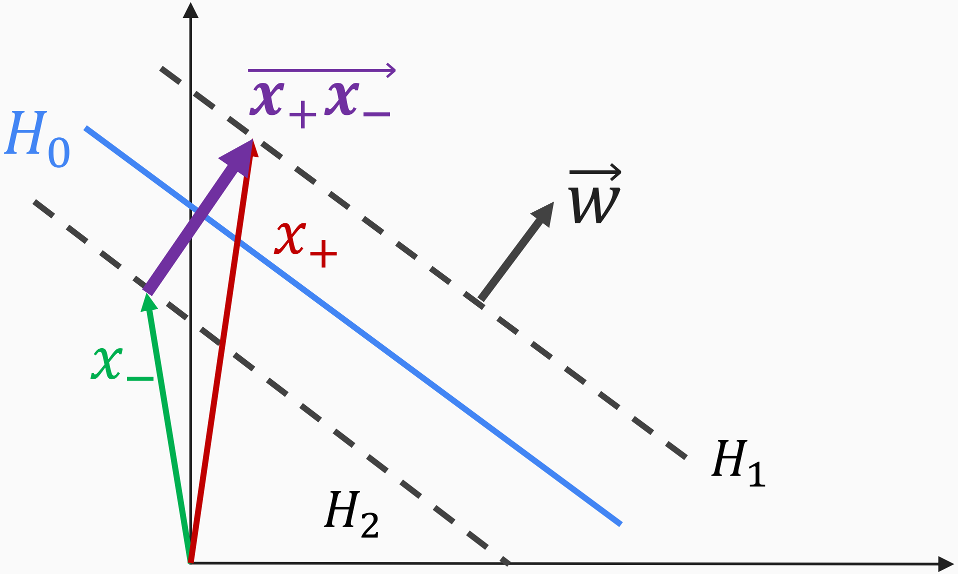

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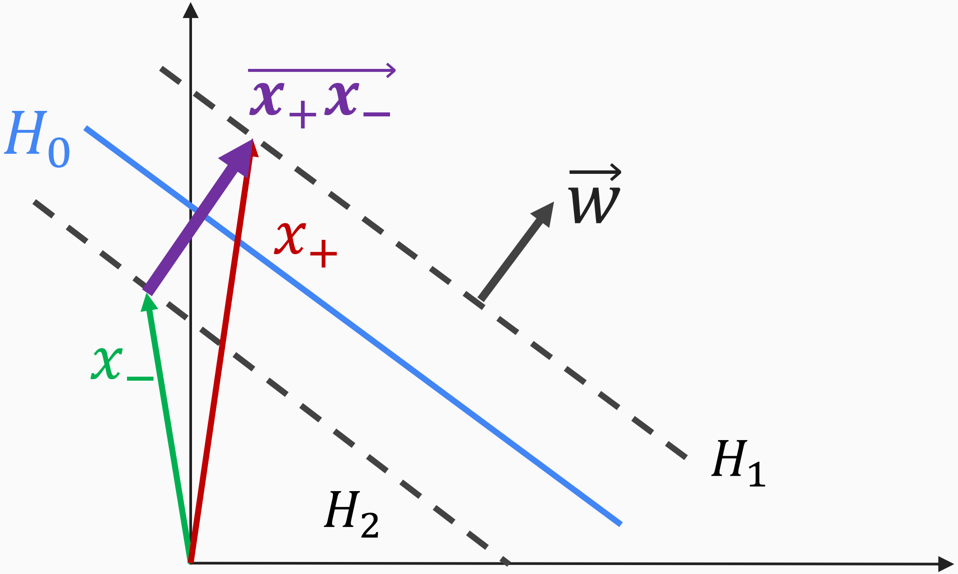

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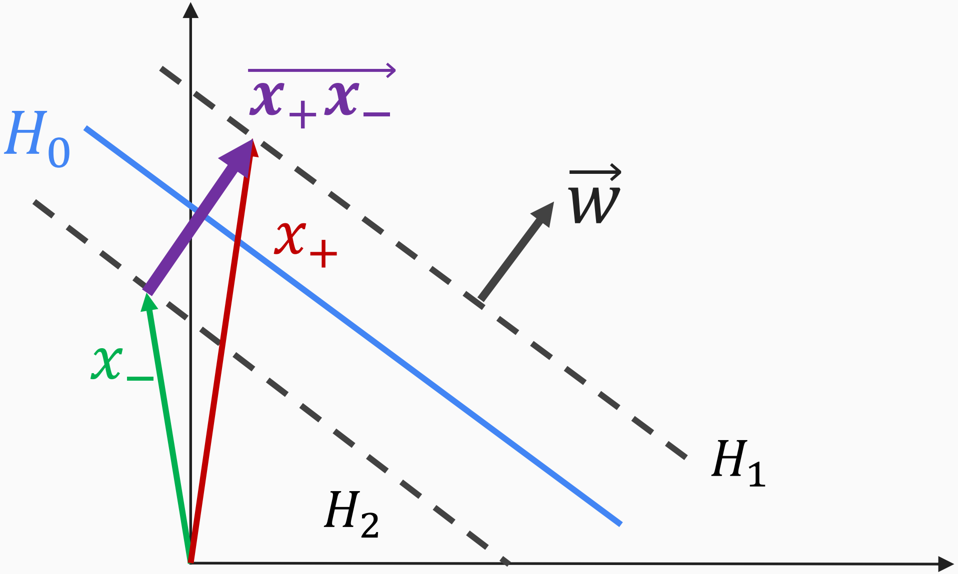

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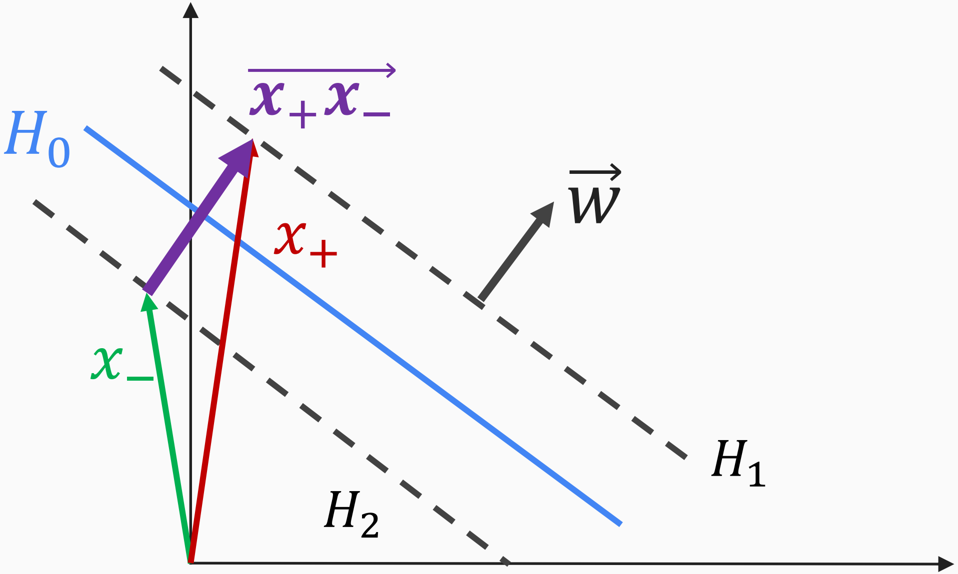

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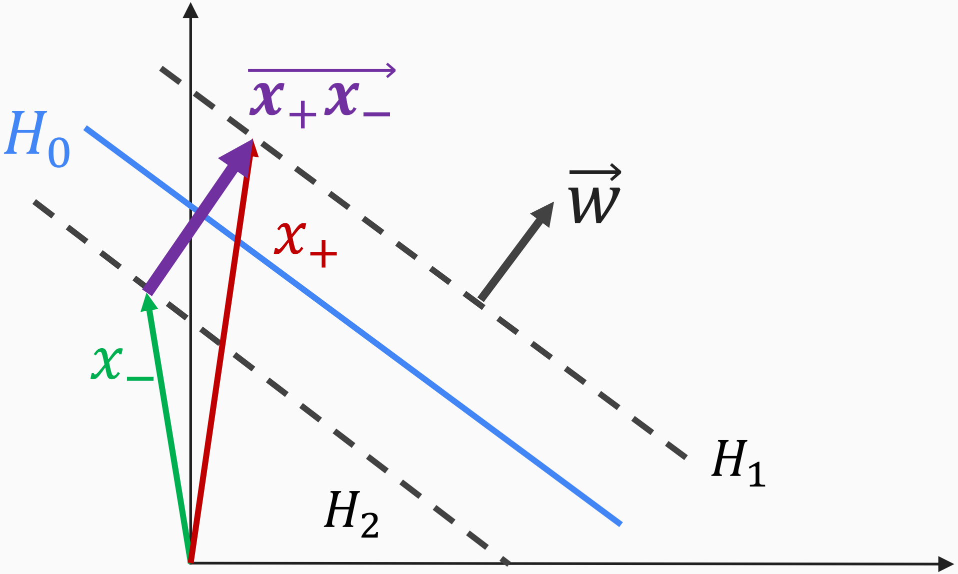

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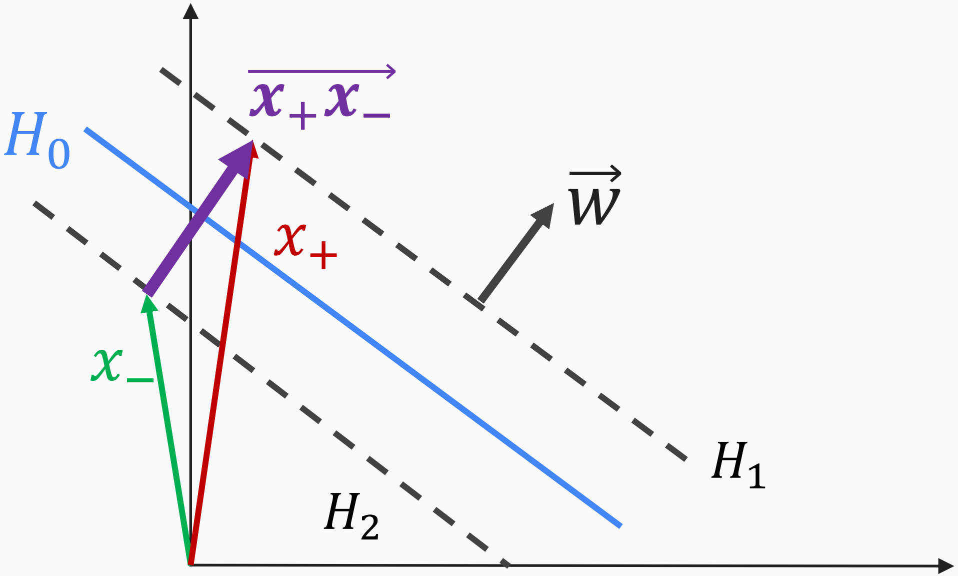

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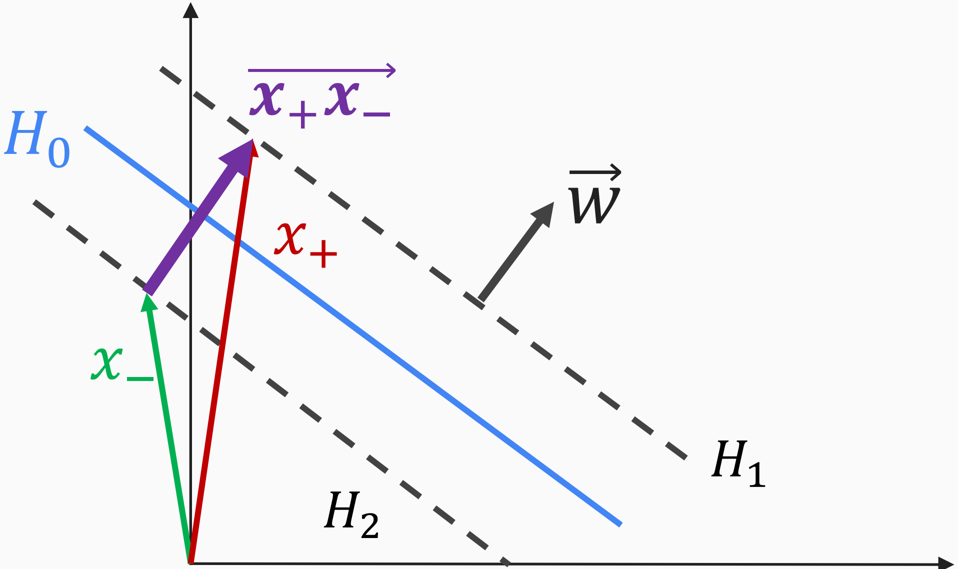

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Suppose we have

$$y_i = 1 = \blacktriangle$$

$$y_i = -1 = \bullet$$

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Problem solved ☺ : we found the equation of the optimal hyperplane

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Optimization algorithms work much better on differentiable functions

So our program becomes a quadratic program

$$minimize \quad \frac{1}{2}\|w\|^2$$

$$subject\ to: \quad y_i(wx_i + b) \geq 1 \qquad i\ goes\ from\ 1\ to\ N$$

Problem

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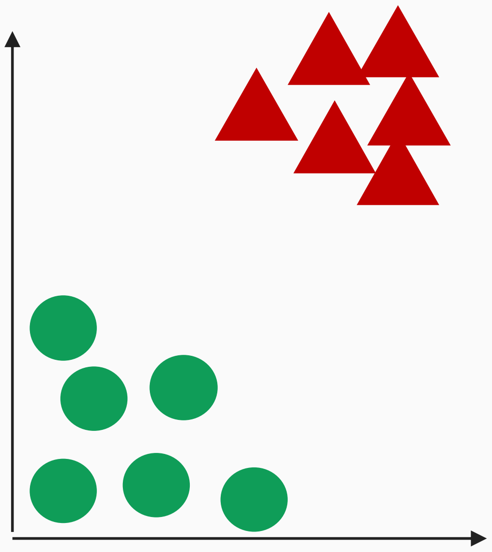

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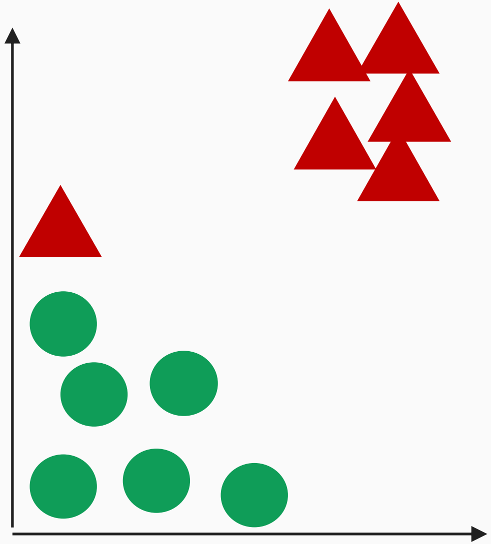

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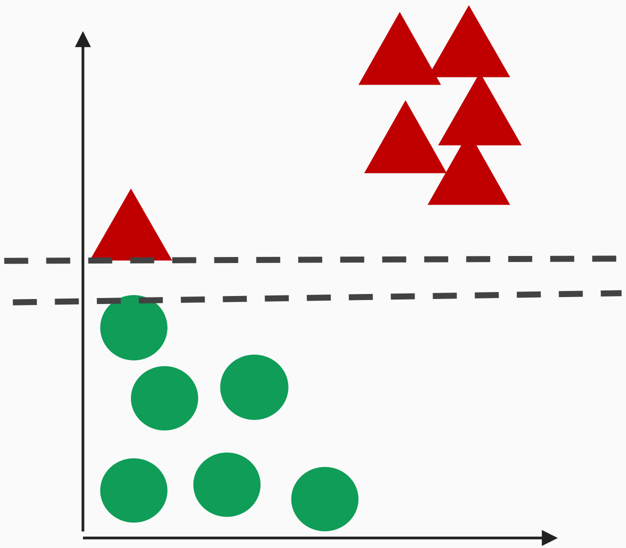

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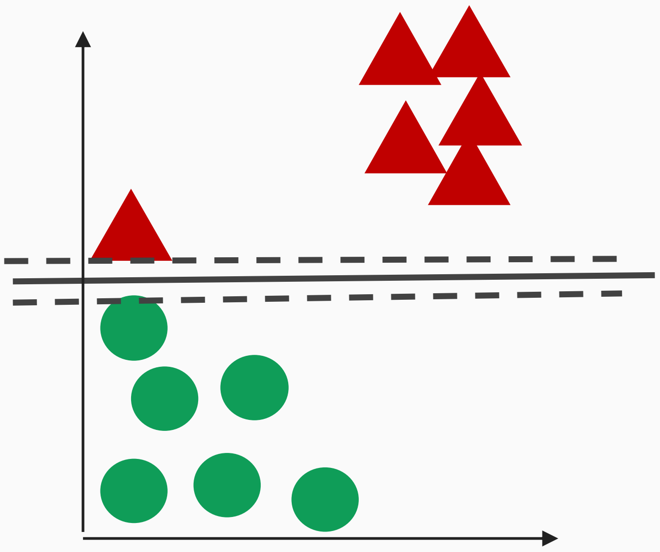
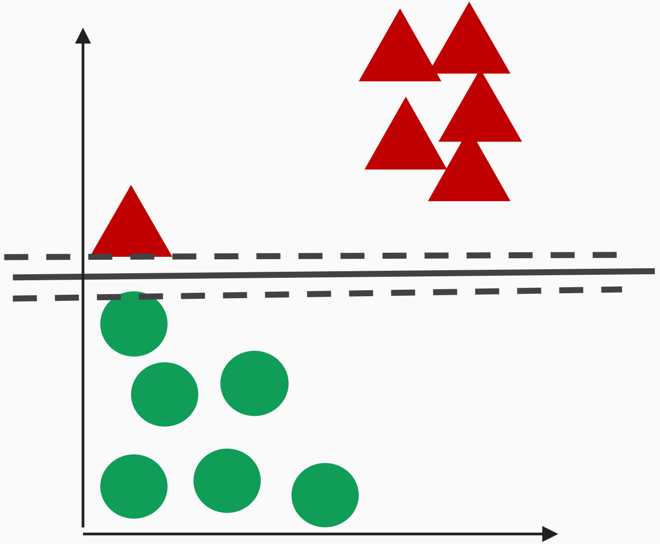


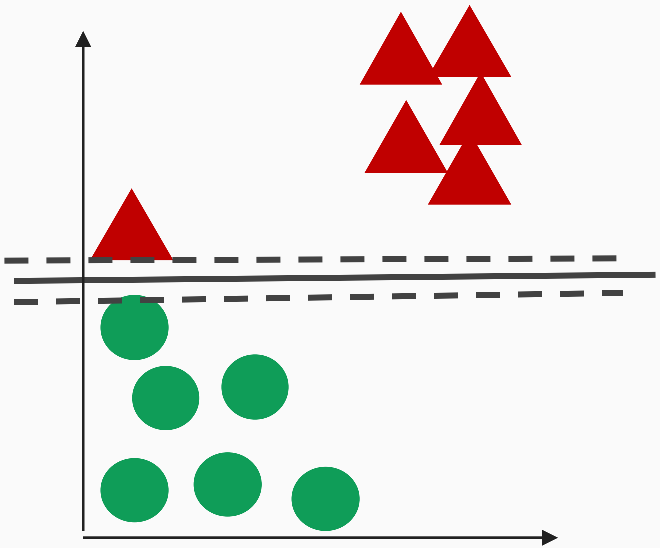
Problem

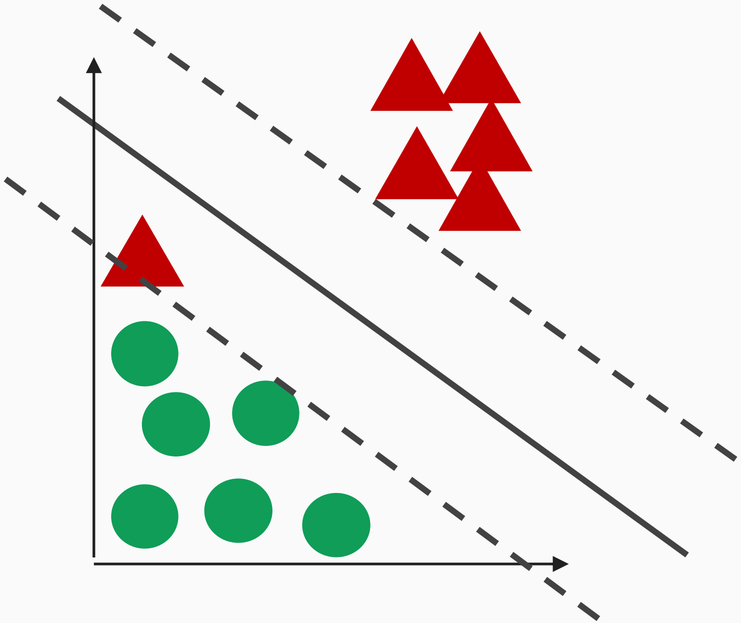


Hard SVM is sensitive to outliers !

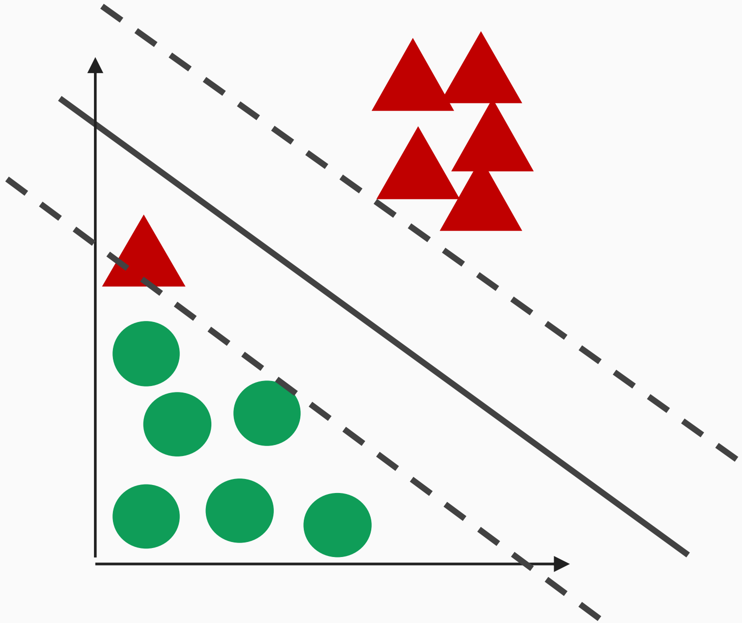
Solution is : ....

(linear) Soft margin SVMs





This is Soft margin SVMs



mathematically ☹ :

We introduce a slack variable $\zeta^{(\mathrm{i})} \geq 0$ that tells us how much the i^{th} data point is allowed to violate the margin .

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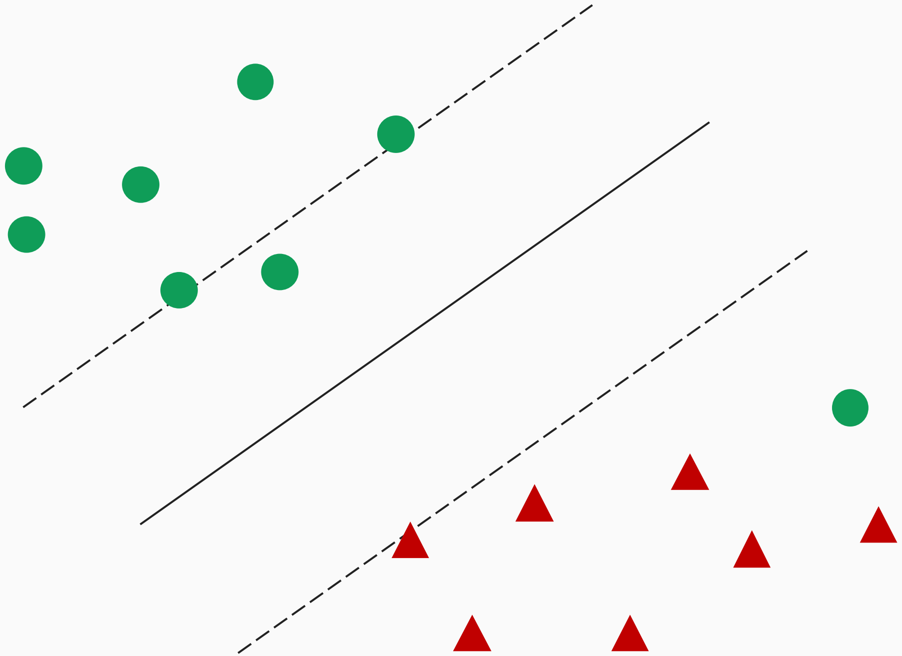

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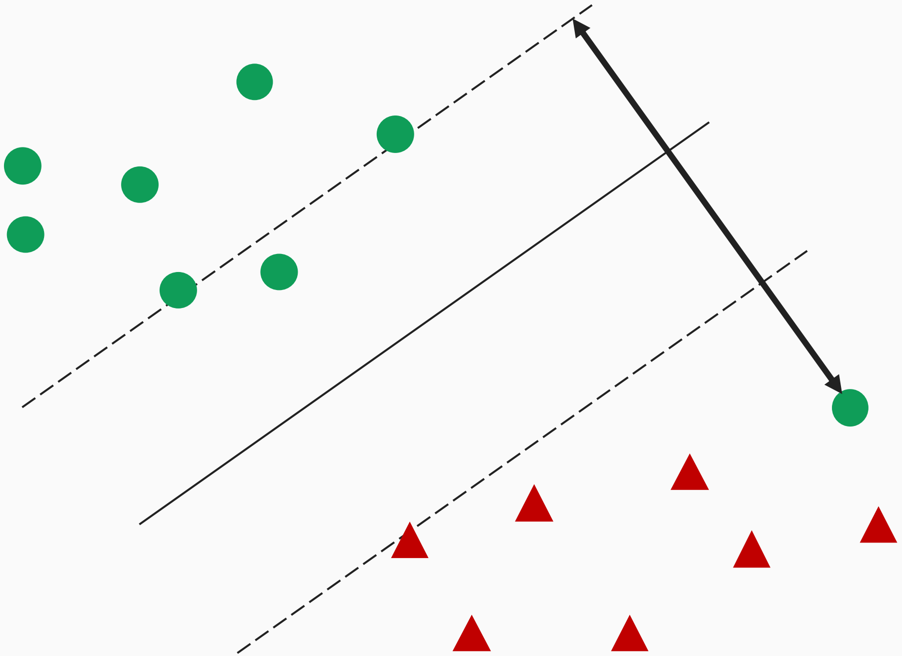

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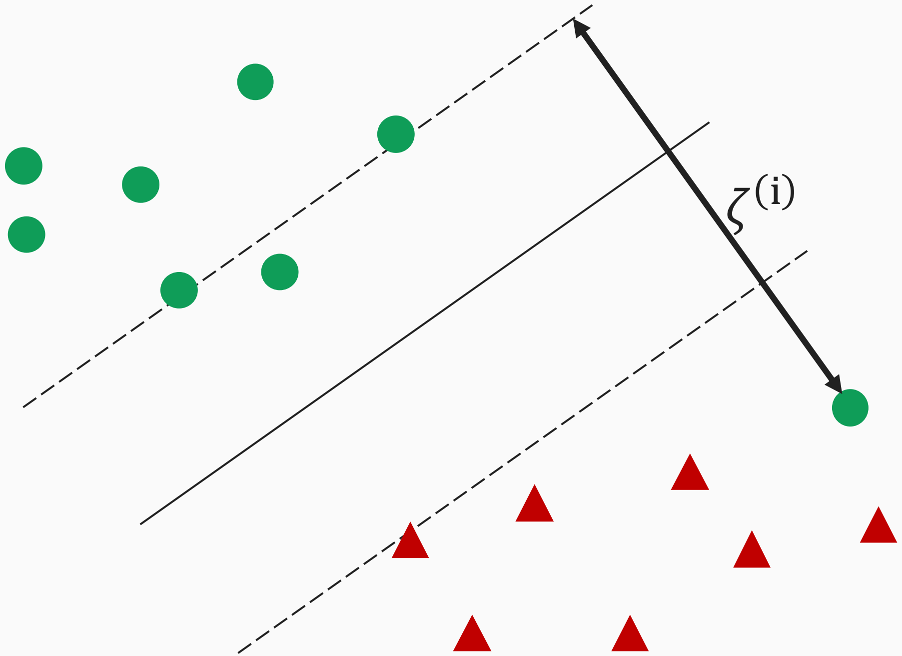

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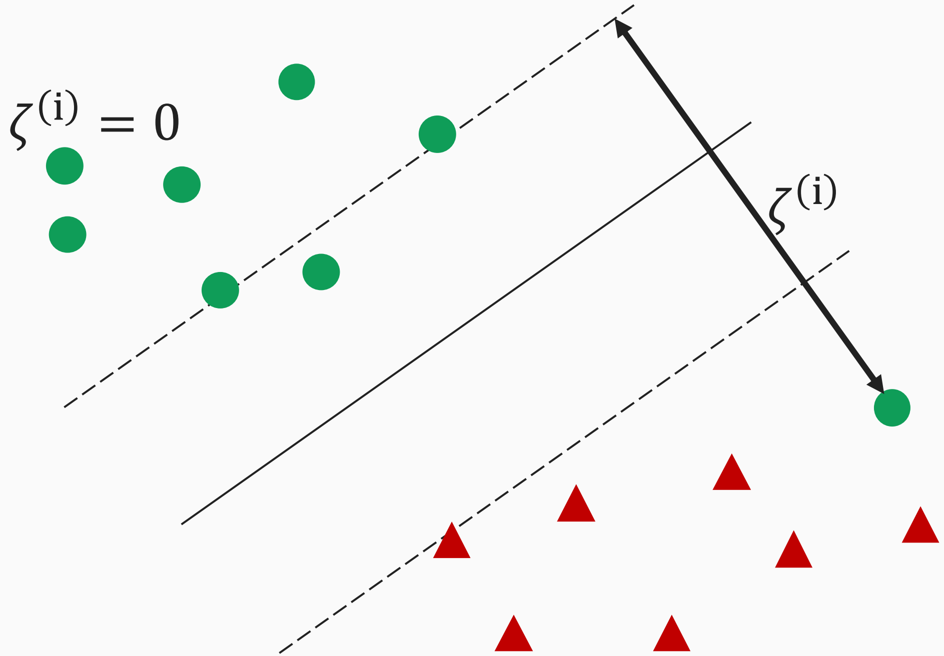

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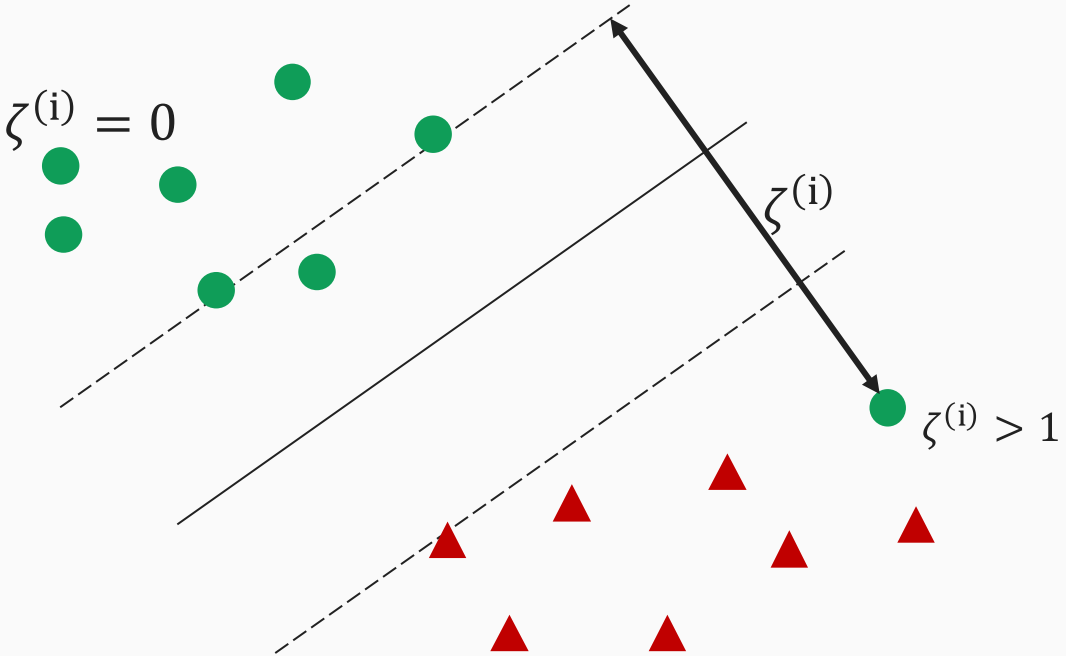

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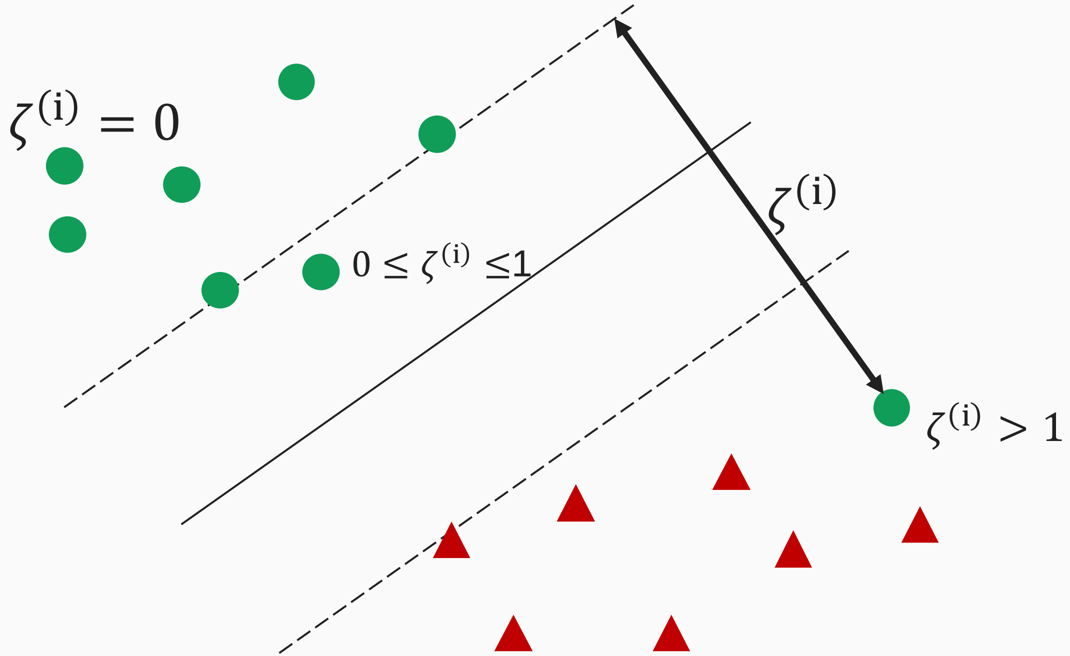

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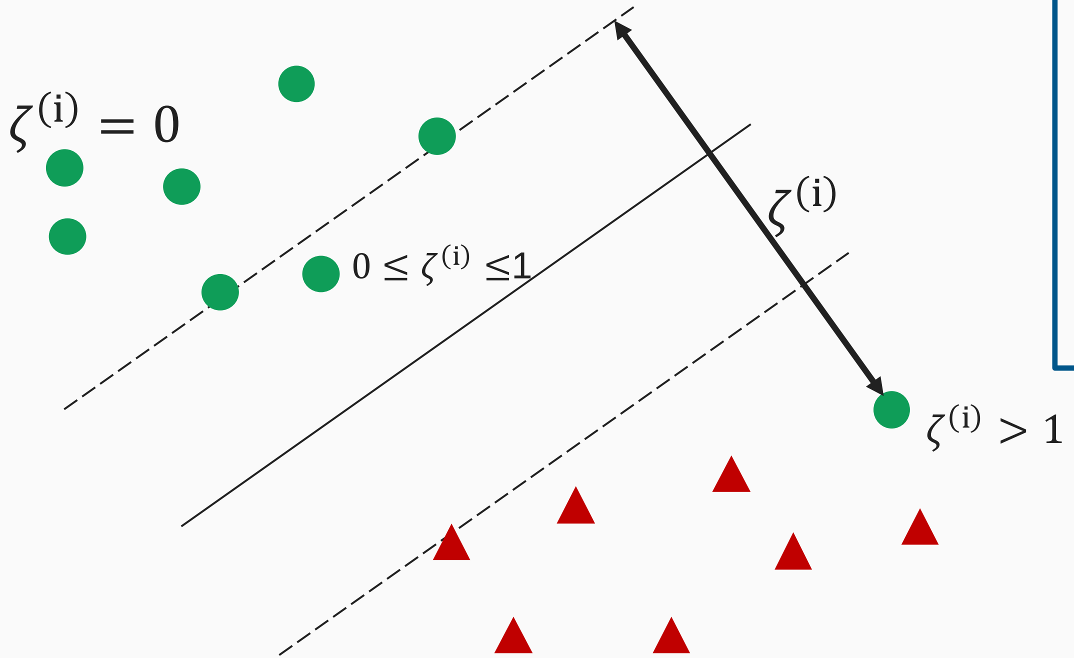

$\zeta^{(\mathrm{i})} > 1$ incorrectly classified

$0 \leq \zeta^{(\mathrm{i})} \leq 1$ correctly classified but inside the margin

$\zeta^{(\mathrm{i})} = 0$ correctly classified

Trade off :

Maximize the margin

Minimize the violations

$$\begin{aligned} &minimize \quad \frac{1}{2}\|w\|^2 \\ &subject\ to: \quad y_i(wx_i + b) \geq 1 \qquad i \in \{1, \cdots, N\} \end{aligned}$$

$$\textit{minimize} \qquad \frac{1}{2}\|w\|^2 + \sum_{i=1}^{N} \zeta^{(i)}$$

$$\textit{subject to :} \qquad y_i(wx_i + b) \geq 1 \qquad i \in \{1, \cdots, N\}$$

$$\text{minimize} \quad \frac{1}{2}\|w\|^2 + \sum_{i=1}^{N} \zeta^{(i)}$$

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$$\zeta^{(i)} \geq 0$$

Soft margin

$$minimize \quad \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{N}\zeta^{(i)}$$

$$subject\ to: \quad y_i(wx_i + b) + \zeta^{(i)} \geq 1 \quad i \in \{1, \cdots, N\}$$

$$\zeta^{(i)} \geq 0$$

What if our data is not linearly separable?

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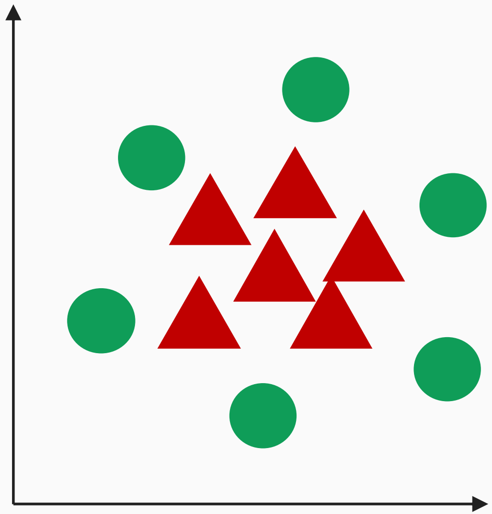


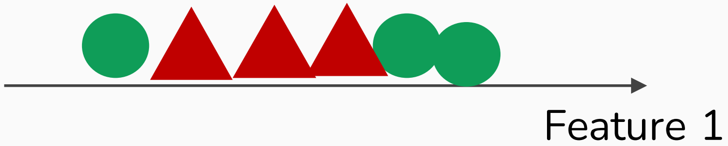
Solution is : ....

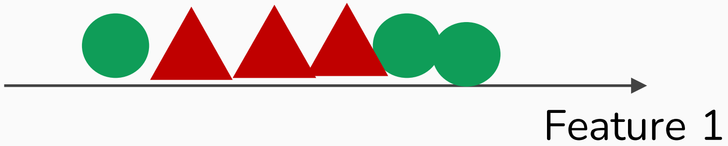
Or Solutions are : ....

We can find a non linear decision boundary to separate data using multiple ideas :

1. Adding features
2. Adding features using similarity function and deleting the old ones
3. Kernel trick

Adding features


Feature 1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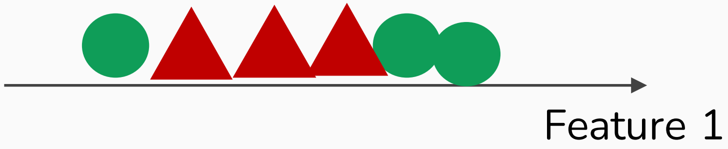
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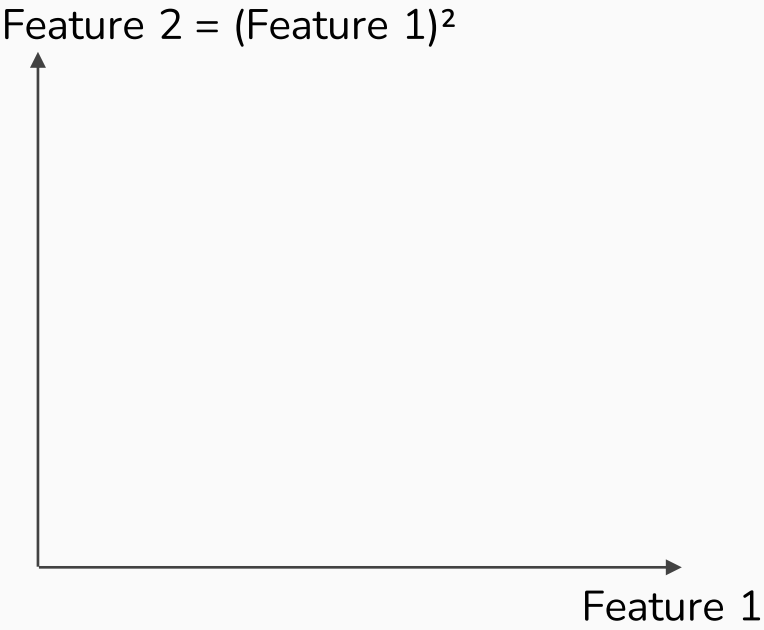
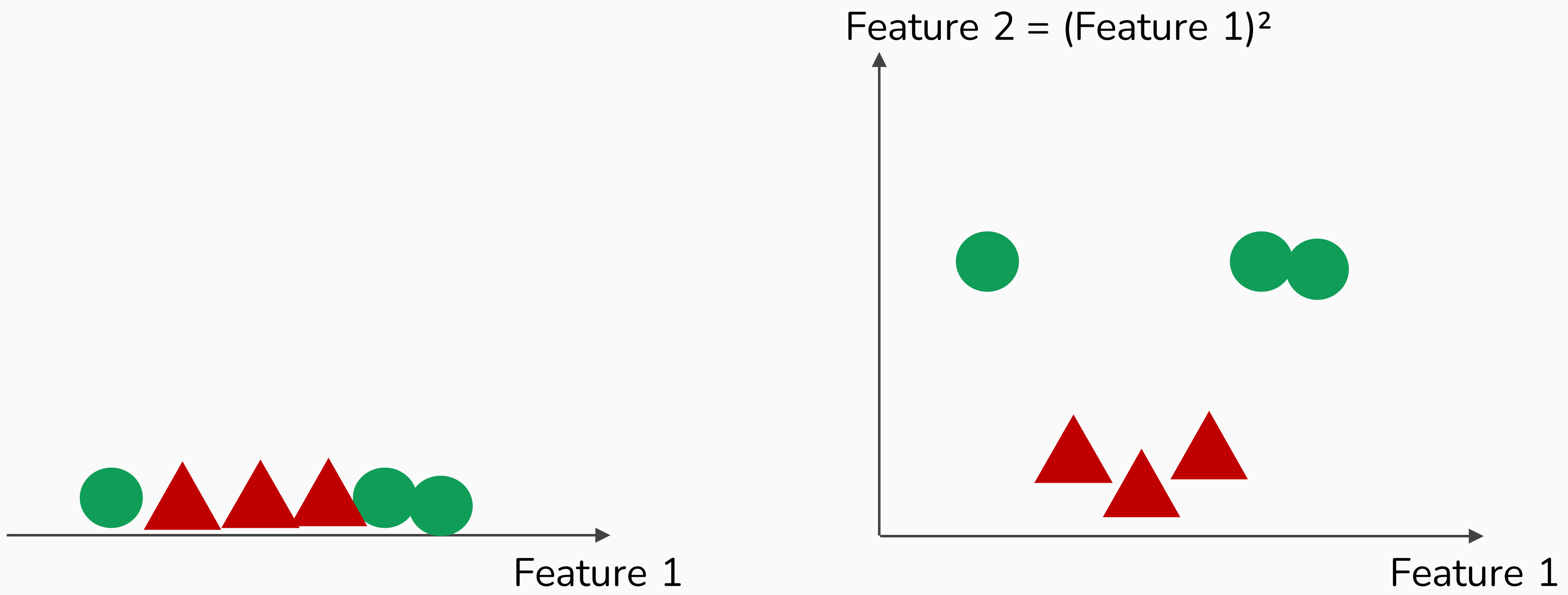
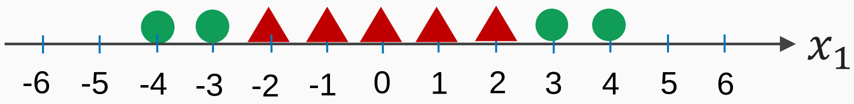

Feature 2 = (Feature 1)²
Feature 1

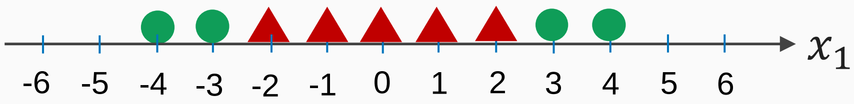


Feature 1
Feature 2 = (Feature 1)²
Feature 1

Adding features using similarity functions


-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
x_1

Choose landmarks



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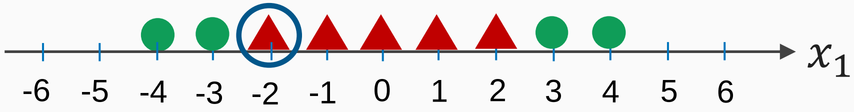

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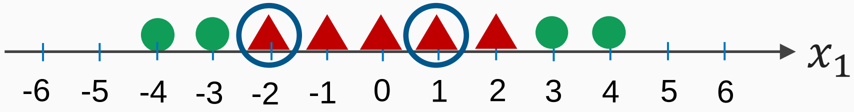

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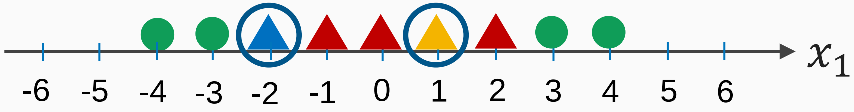
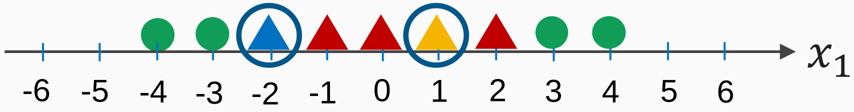


Choose landmarks

Our landmarks are :

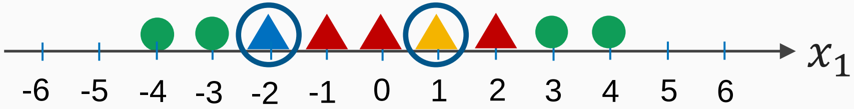


Choose landmarks

Our landmarks are :

 $= -2$

 $= 1$

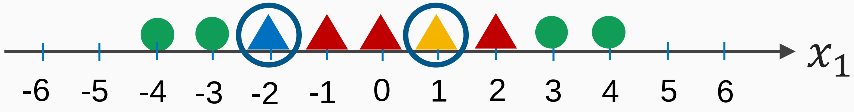


Next we measure the similarity between each point and the landmarks

Our landmarks are :

$= -2$

$= 1$

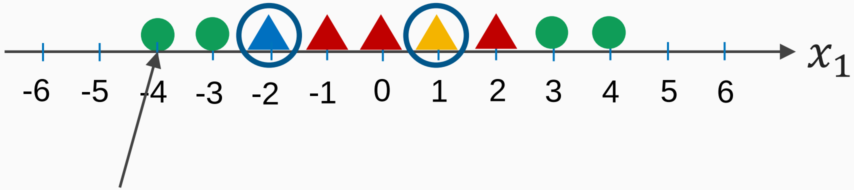


Next we measure the similarity between each point and the landmarks

Our landmarks are :

Blue triangle $= -2$

Yellow triangle $= 1$



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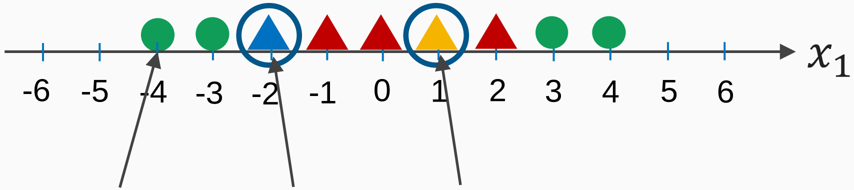

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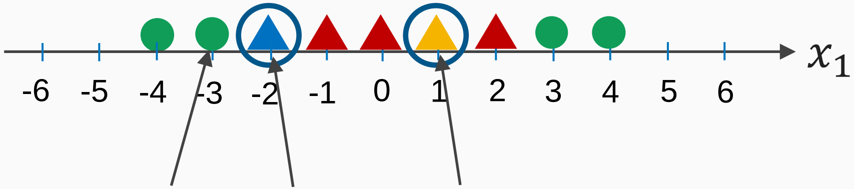
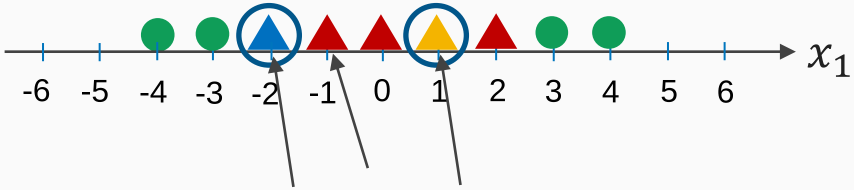


Next we measure the similarity between each point and the landmarks

Our landmarks are :

$\blacktriangle = -2$

$\blacktriangle = 1$



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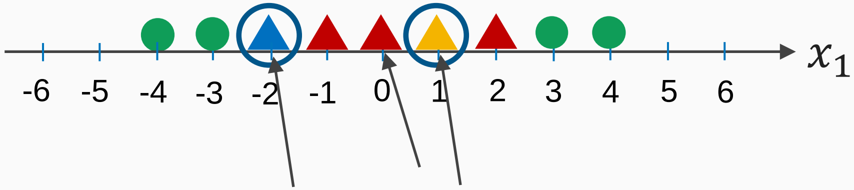

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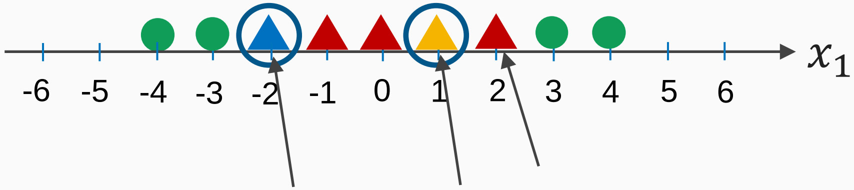

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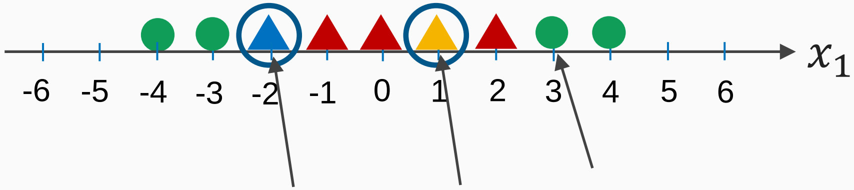

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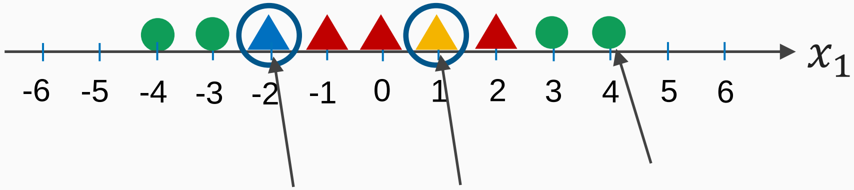

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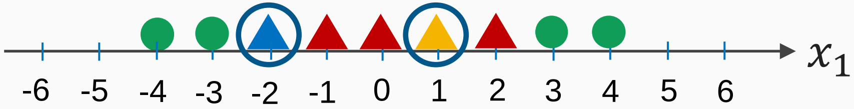

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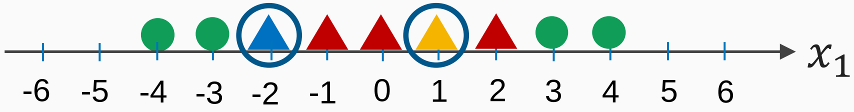
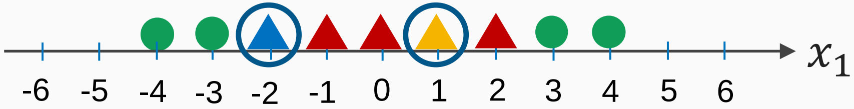


Next we measure the similarity between each point and the landmarks

Our landmarks are :

 $= -2 \longrightarrow x_2$

 $= 1$



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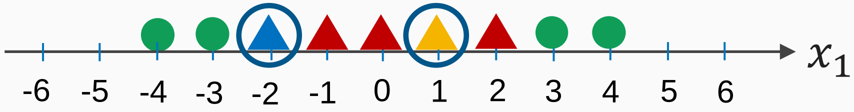

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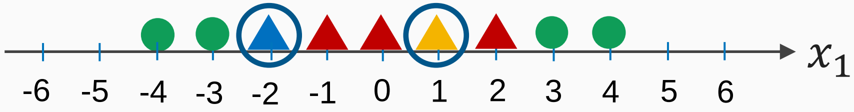
Our landmarks are :

$= -2 \longrightarrow x_2$

$= 1 \longrightarrow x_3$



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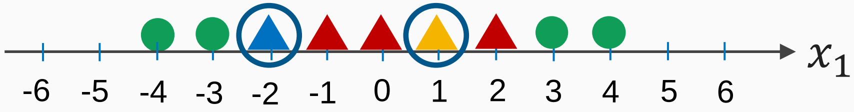
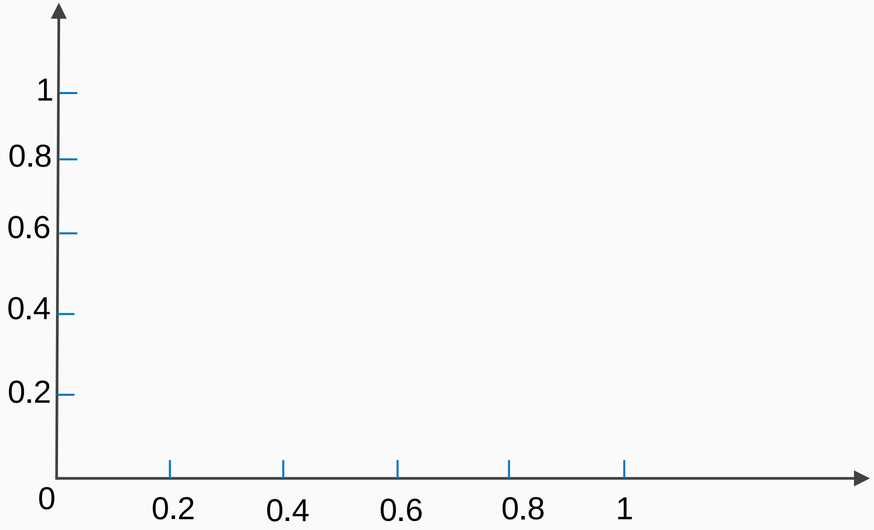
Our landmarks are :

$\blacktriangle = -2 \longrightarrow x_2$

$\blacktriangle = 1 \longrightarrow x_3$





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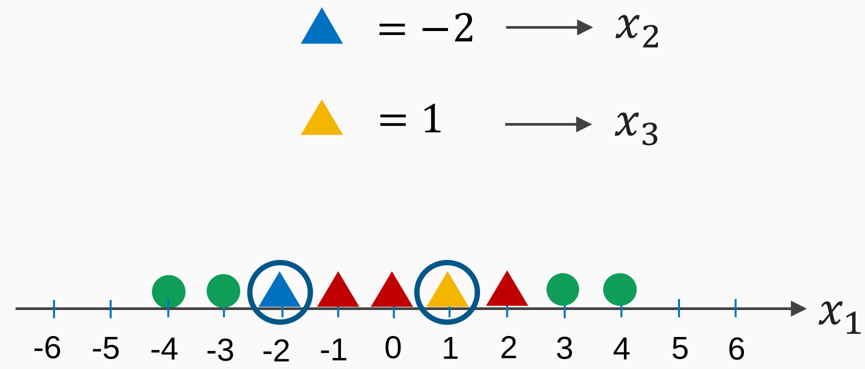

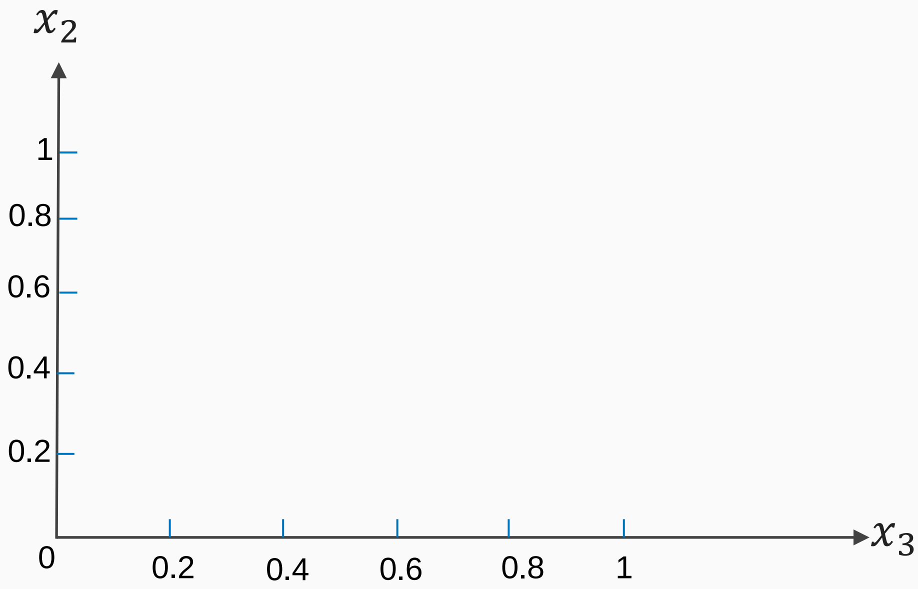

The question is : How do we measure the similarity ?

The answer is :
using a **kernel function**

The kernel function measures the similarity between each data point and a specific landmark

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We will use as a kernel function the (Gaussian) Radial Basis function (RBF)

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$$K\left(x_i, l_j\right) = e^{\left(-\gamma \left\| x_i - l_j \right\|^2\right)}$$

Our landmarks are :

$\blacktriangle = -2 \longrightarrow x_2$

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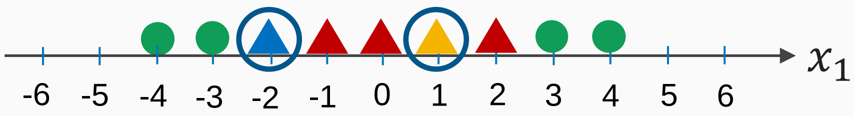

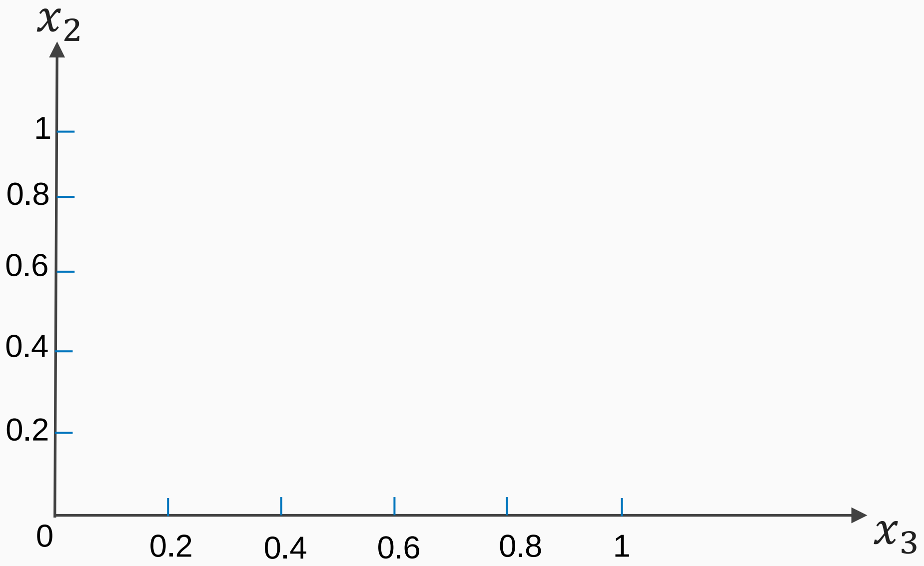

$$K(x_i, l_j) = e^{\left(-\gamma \|x_i - l_j\|^2\right)}$$

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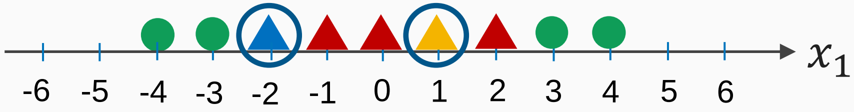
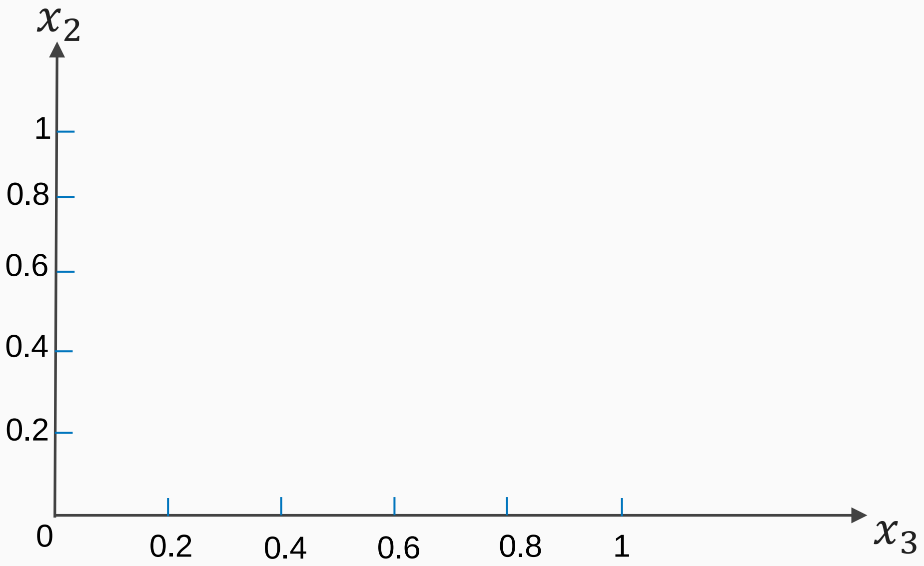
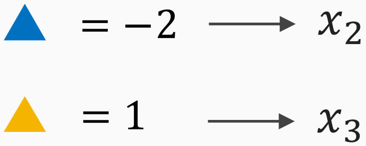


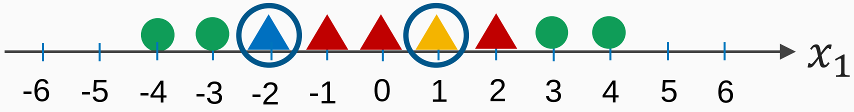


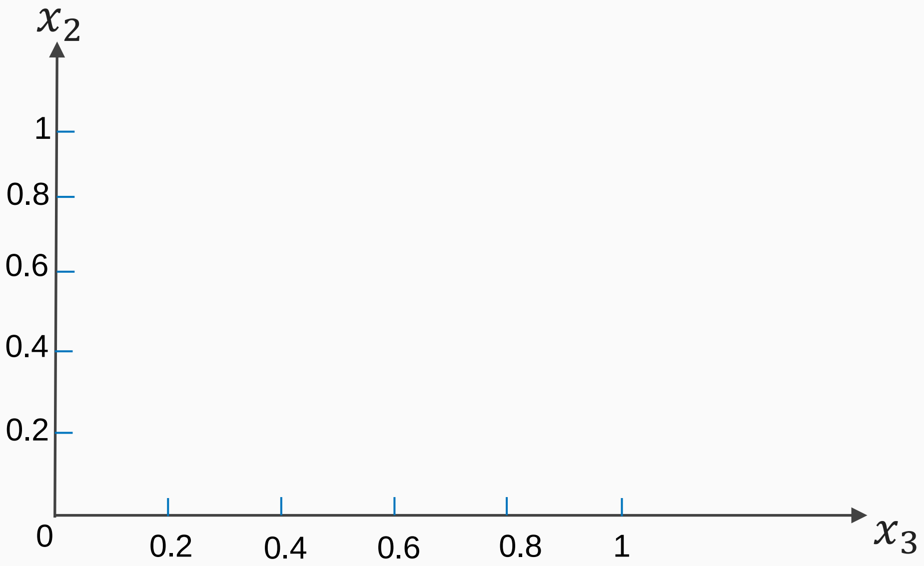
$$K(x_i, l_j) = e^{\left(-\gamma \|x_i - l_j\|^2\right)}$$

Let $\gamma = 0.3$

Our landmarks are :







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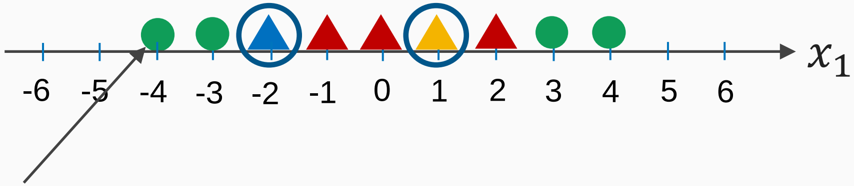

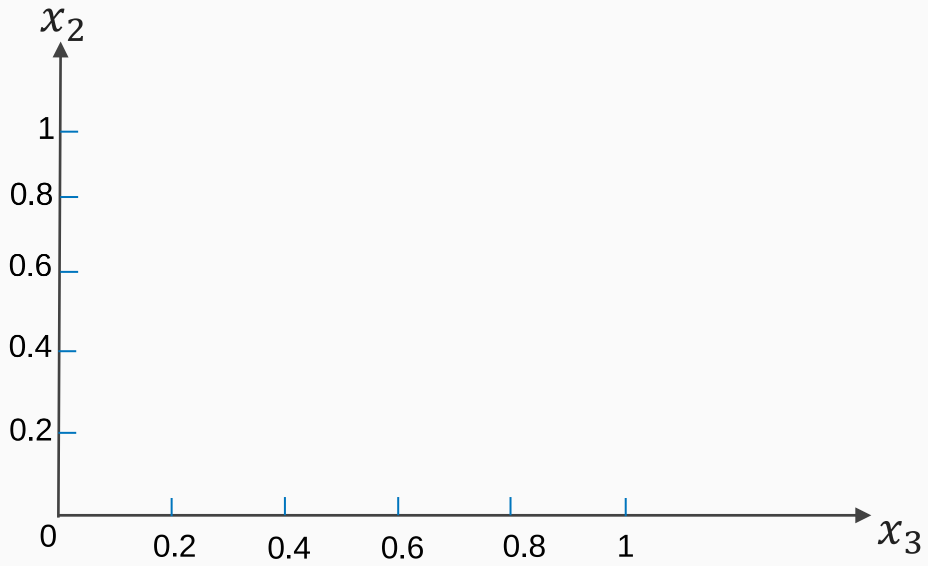

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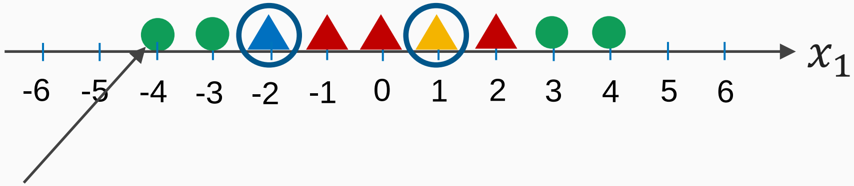

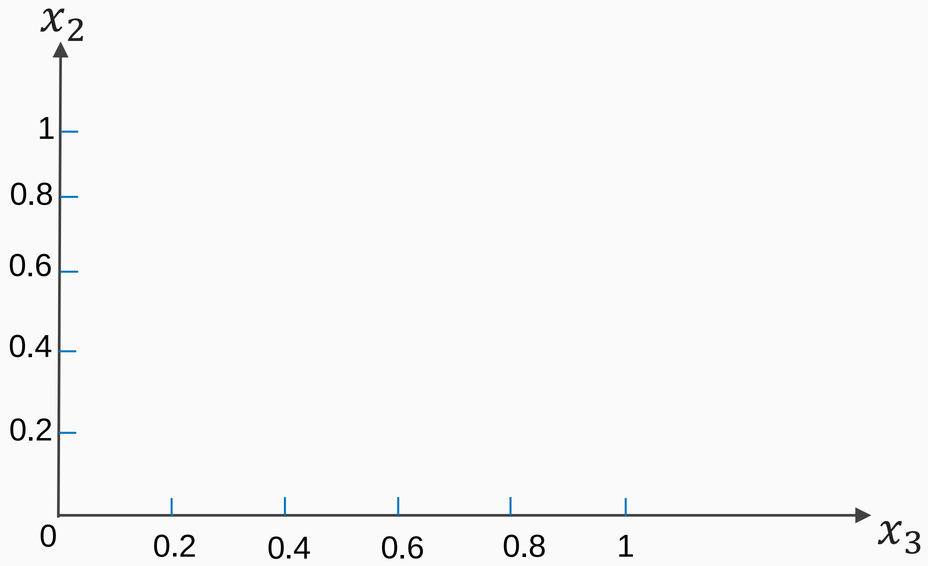

$$x_2 = K(\bullet, \blacktriangle) = e^{\left(-0.3\|-4-(-2)\|^2\right)}$$

$$K(x_i, l_j) = e^{\left(-\gamma \|x_i - l_j\|^2\right)}$$

Let $\gamma = 0.3$

Our landmarks are :

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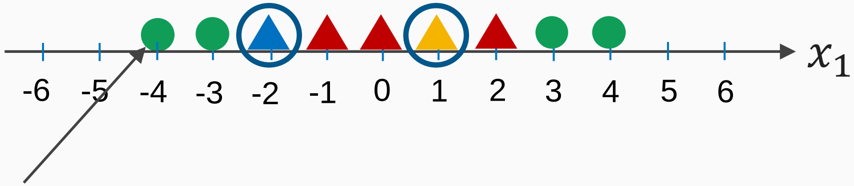

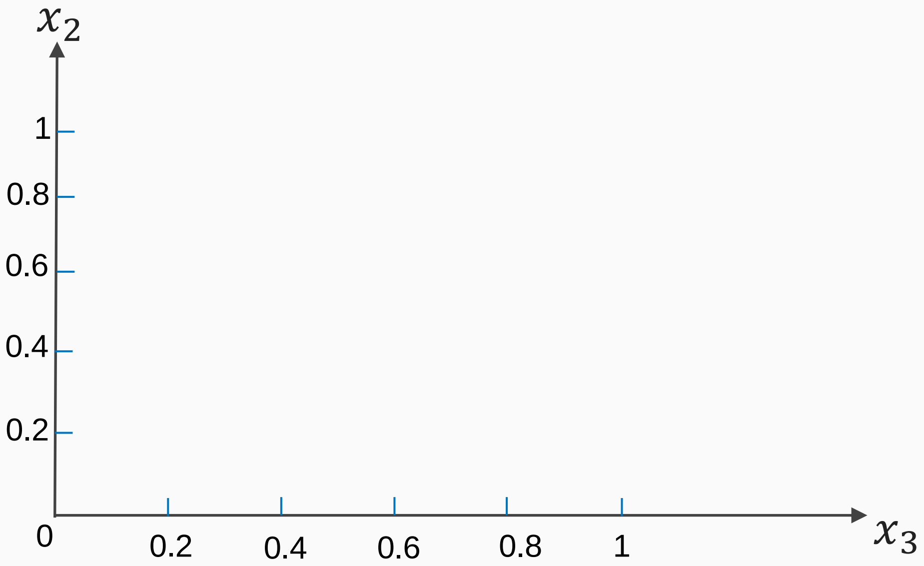

$$x_2 = K(\bullet, \blacktriangle) = e^{\left(-0.3\|-4-(-2)\|^2\right)} = 0.30$$

$$K(x_i, l_j) = e^{\left(-\gamma\|x_i - l_j\|^2\right)}$$

Let $\gamma = 0.3$

Our landmarks are :

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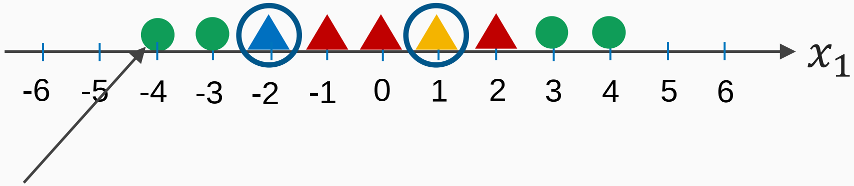

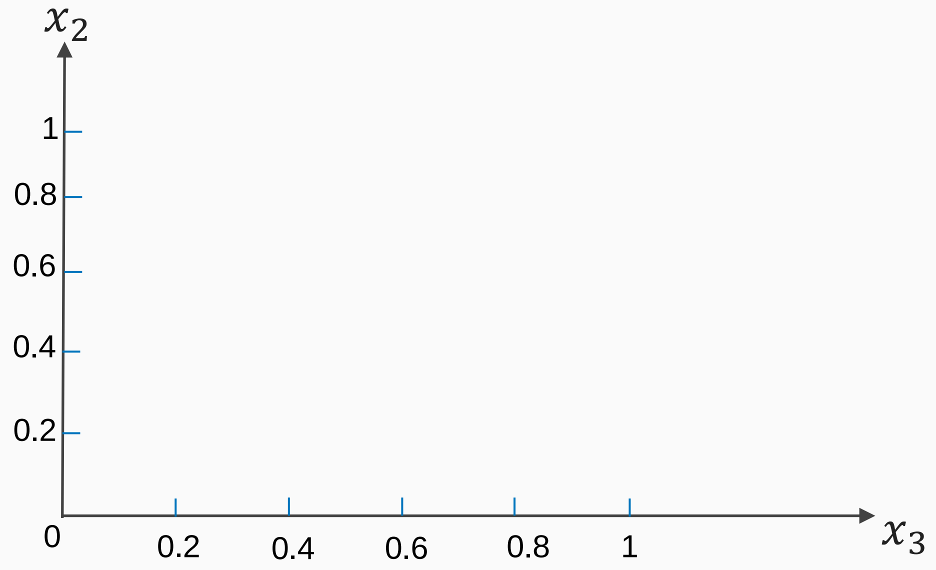

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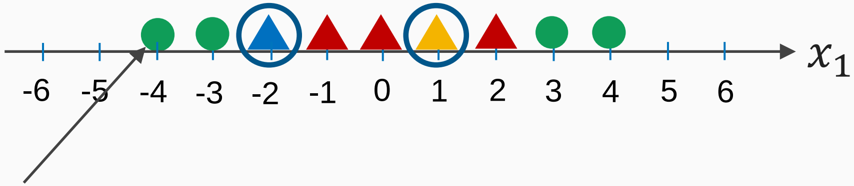

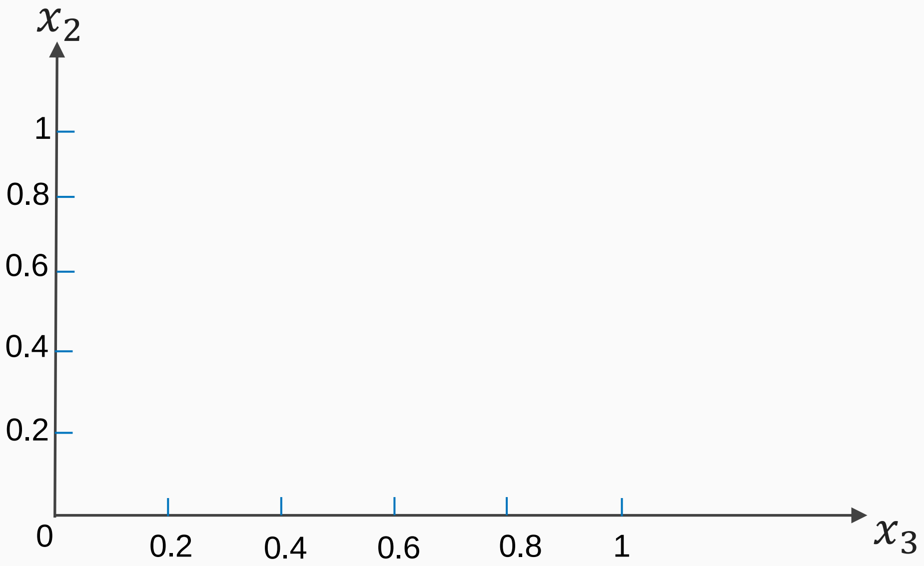

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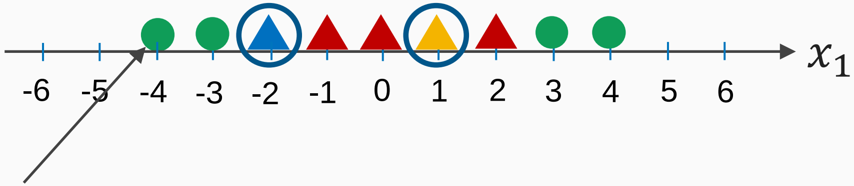

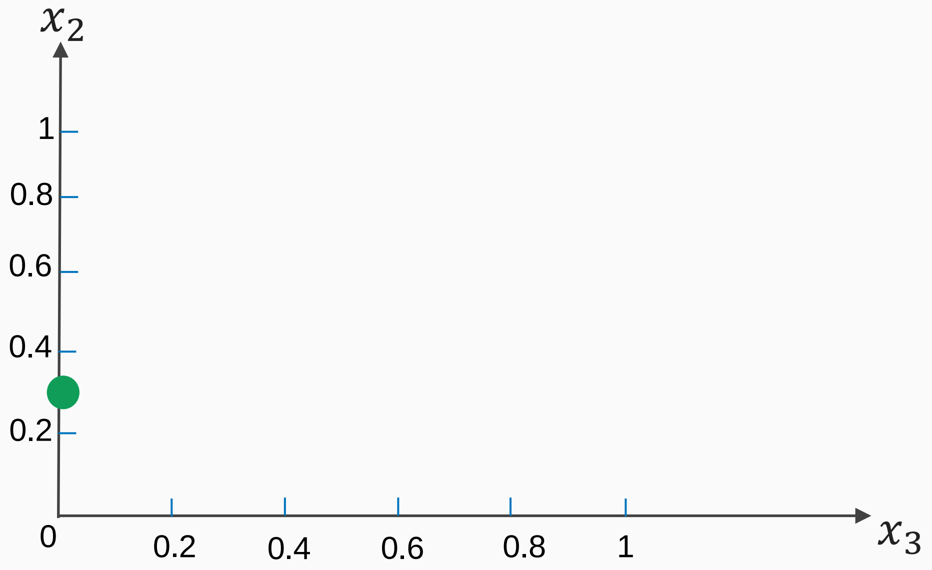

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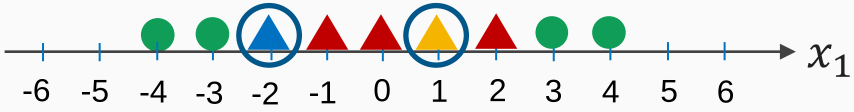

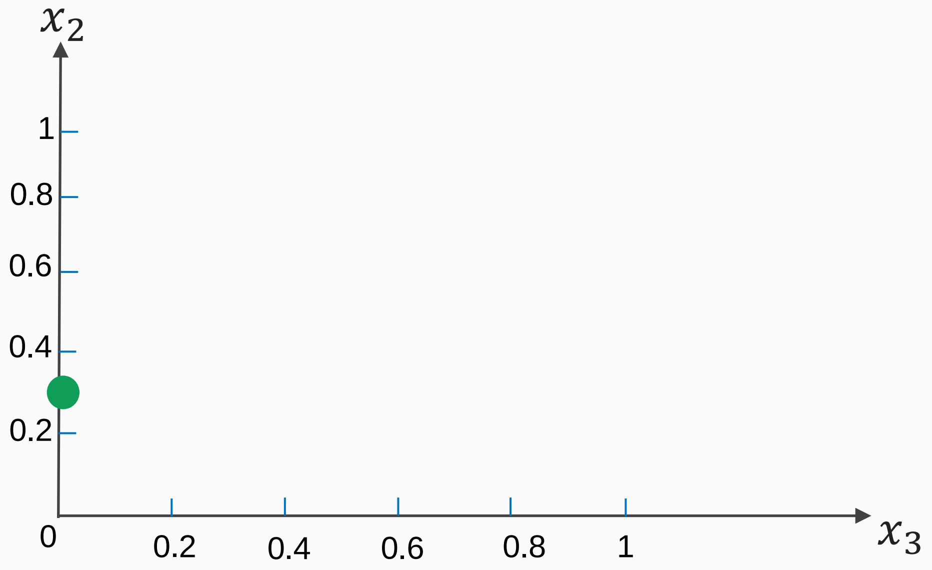

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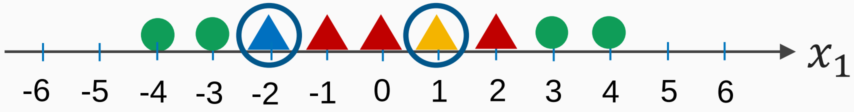

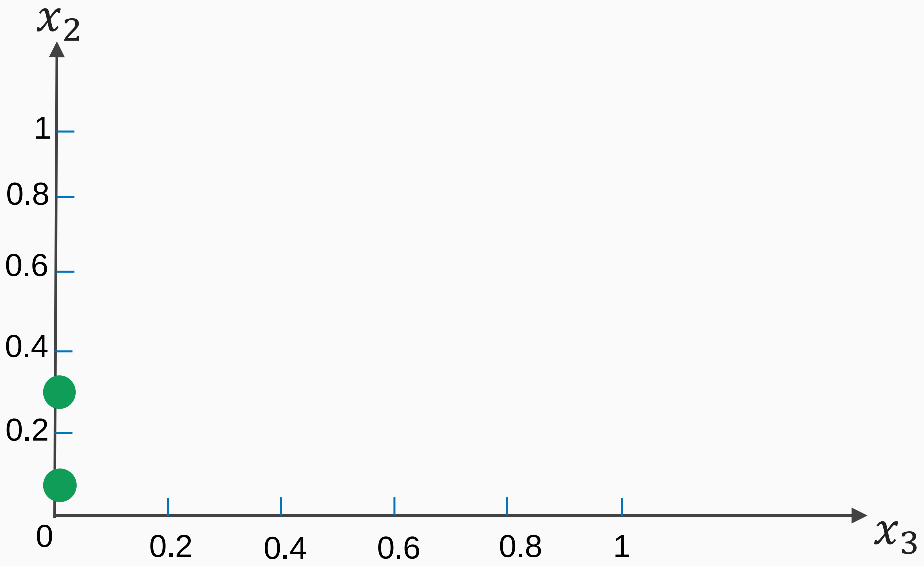

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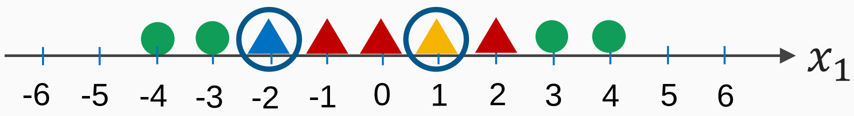

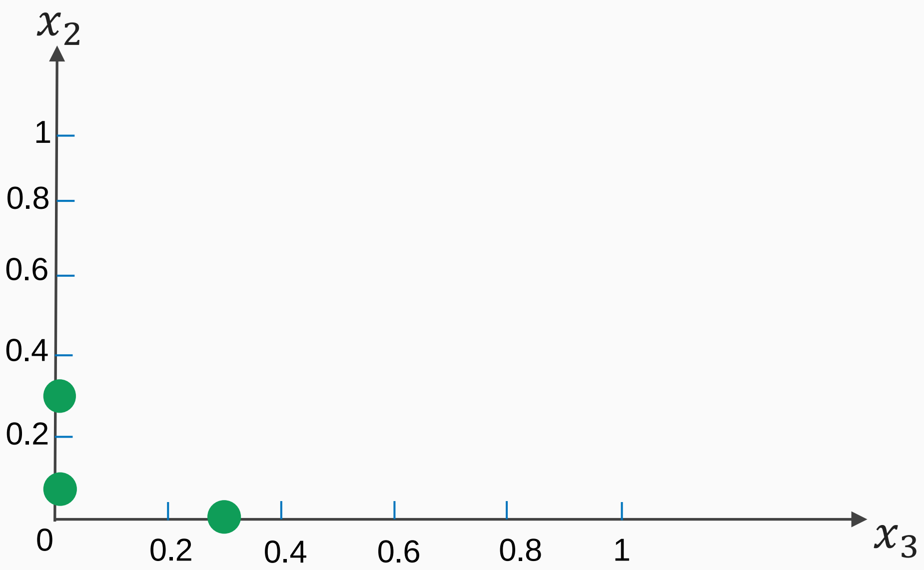

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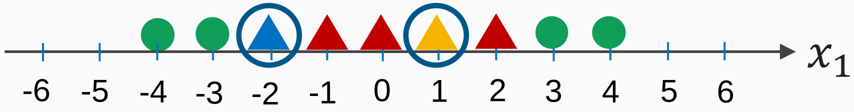

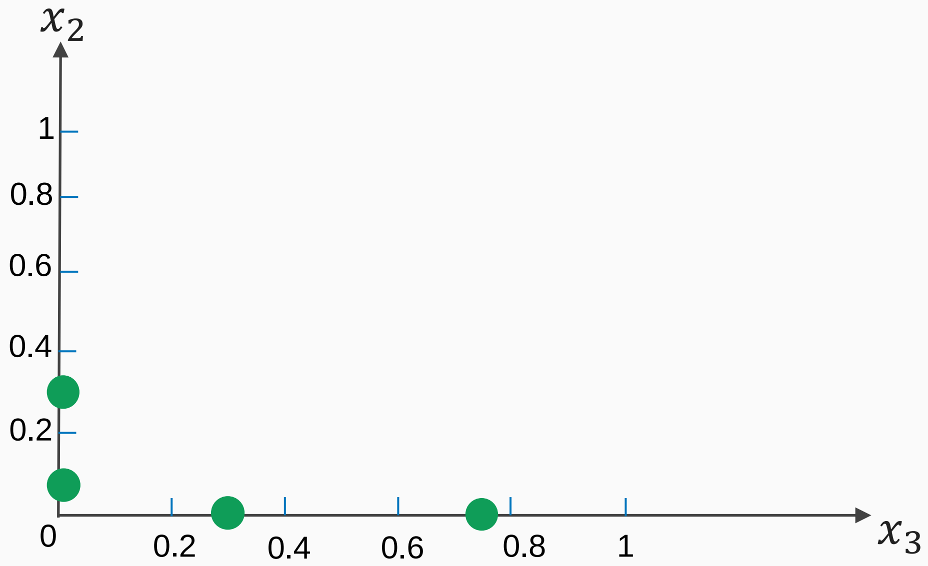

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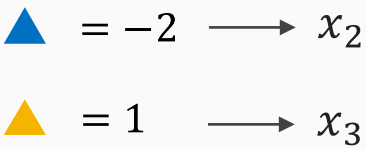

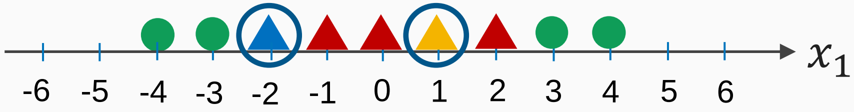

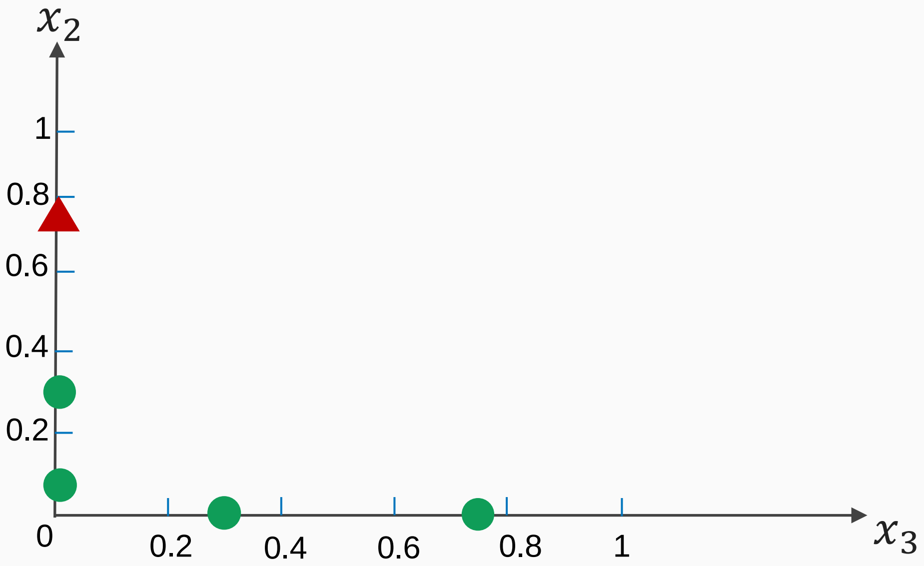

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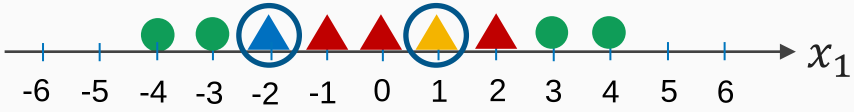

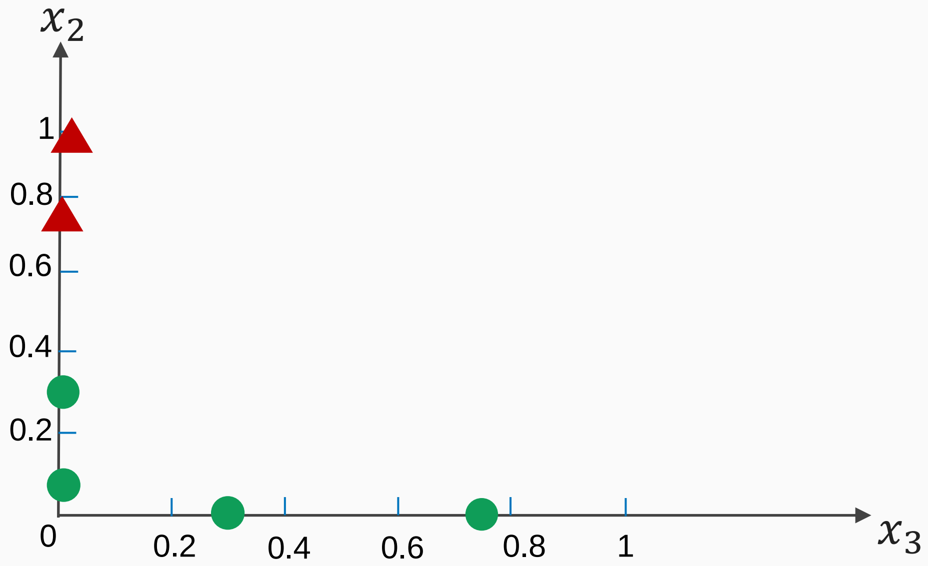

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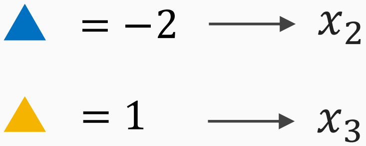

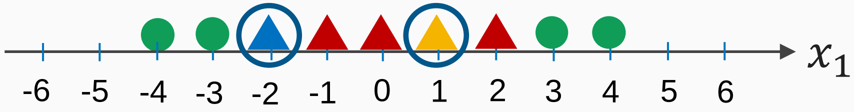

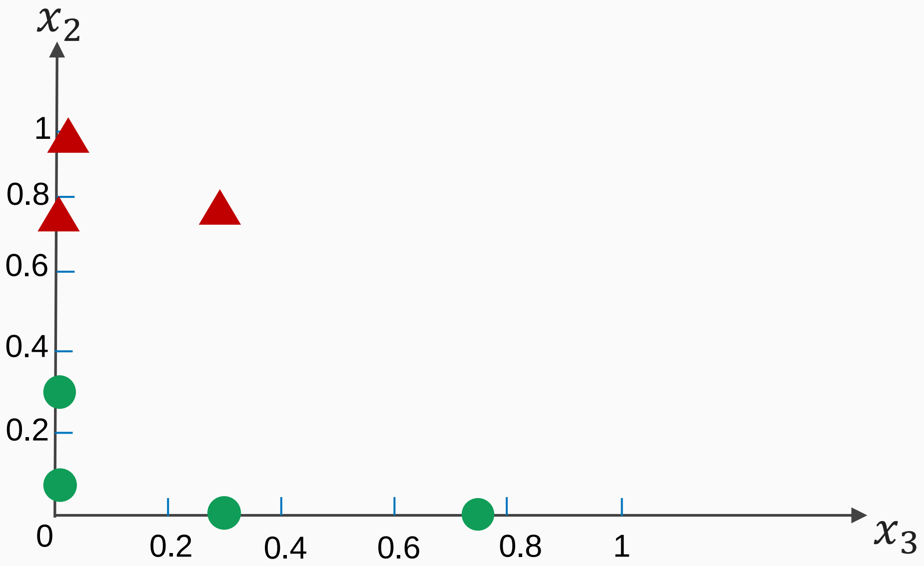

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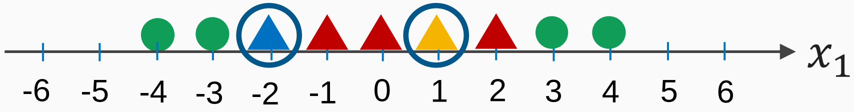

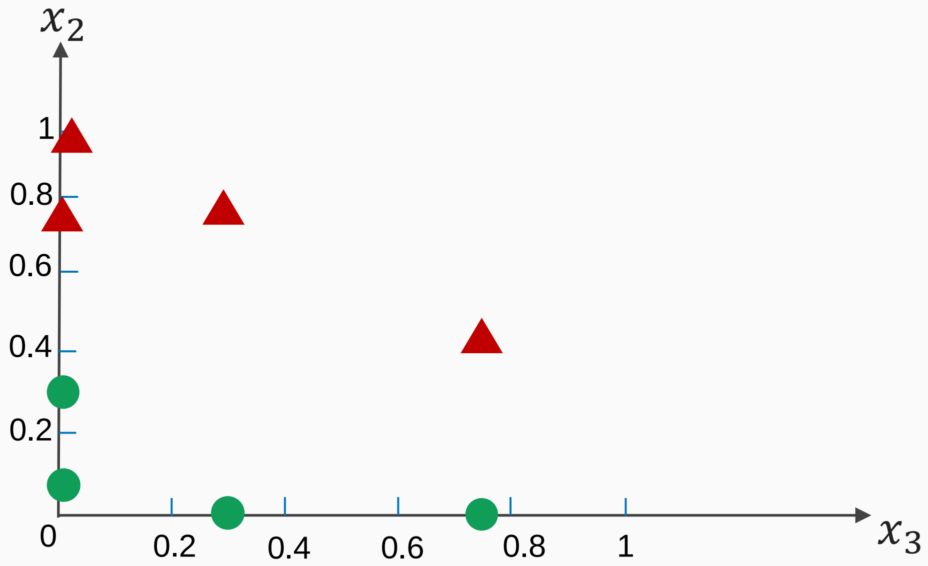

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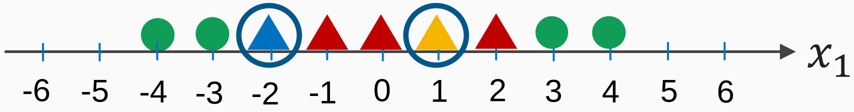

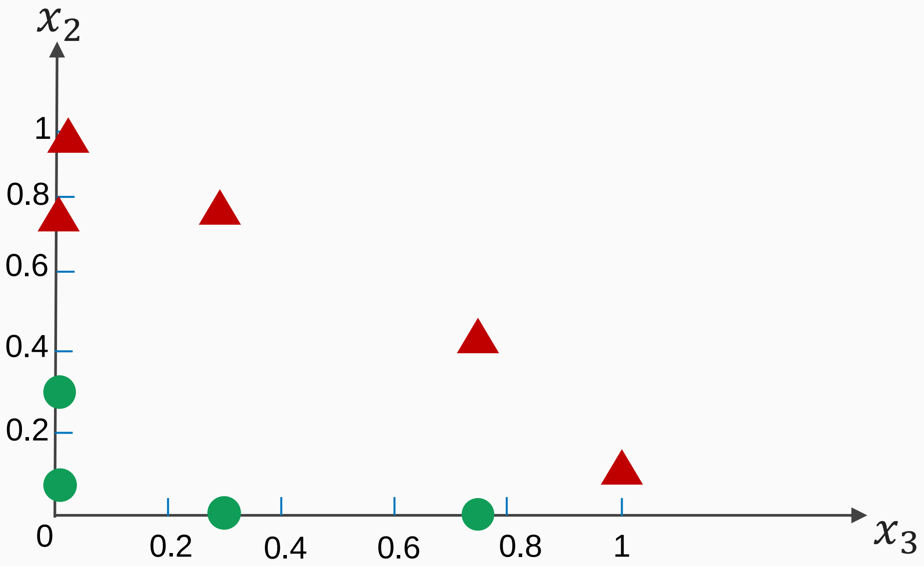

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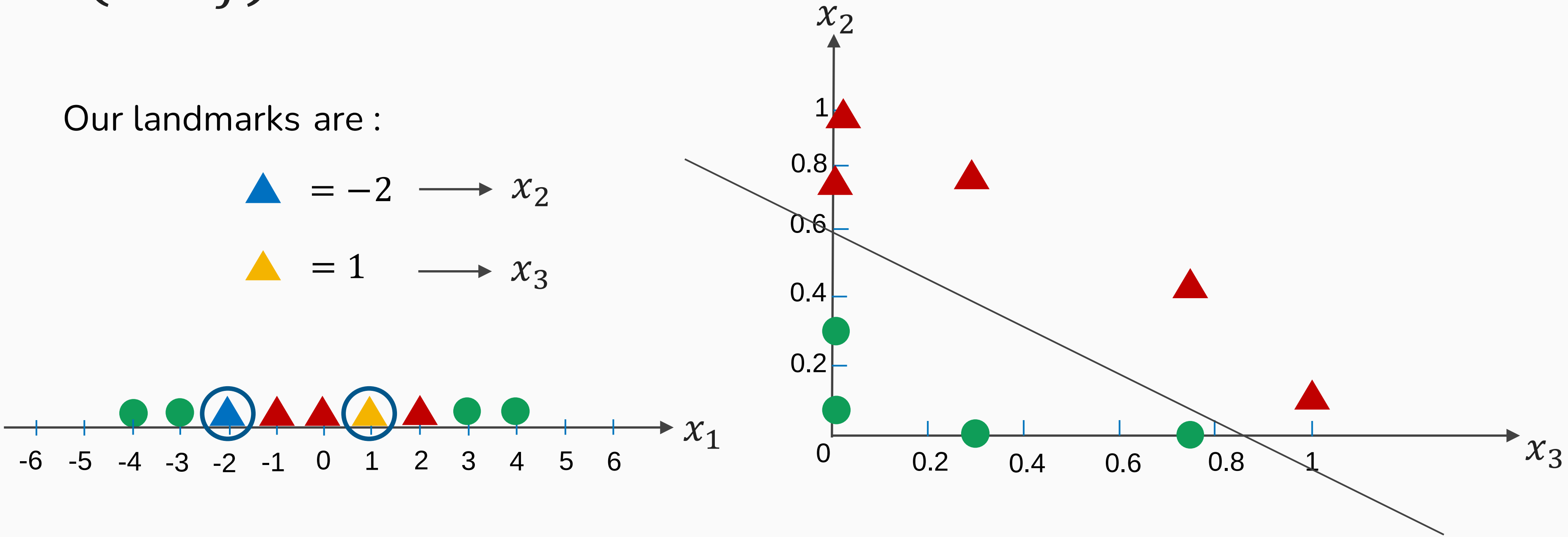

Linearly separable

How do we select the landmarks ?

We choose every data point as a landmark

Downsides ?

Computationally expensive

Kernel trick

Instead of the primal problem :

$$\textit{minimize} \quad \frac{1}{2}\|w\|^2 + \mathrm{C}\sum_{i=1}^{N}\zeta^{(i)}$$

$$\textit{subject to}: \quad y_i(wx_i + b) + \zeta^{(i)} \geq 1 \quad i \in \{1, \cdots, N\}$$

$$\zeta^{(i)} \geq 0$$

Using KKT conditions we obtain the dual problem

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$$\text{maximize} \sum_{i=1}^{N} \alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \cdot \alpha_j \cdot y_i \cdot y_j \cdot \langle {x_i}^T, x_j \rangle$$

$$\text{subject to :} \qquad 0 \le \alpha_i \le \mathrm{C} \qquad i = 1, \dots, \mathrm{N}$$

$$\textit{and} \sum_{i=1}^{N} \alpha_i \cdot y_i = 0$$

Now we can transform our features x_i using some transformation to overcome the non linear data and SVM can learn using some new features

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Or we could use a kernel function to calculate the inner product without even visiting the high dimensional space

For example

Suppose we have a 2nd transformation ϕ :

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$$\phi(x_i) = \begin{pmatrix} {x_{i1}}^2 \\ x_{i1}x_{i2} \\ x_{i2}x_{i1} \\ {x_{i2}}^2 \end{pmatrix} \quad \phi(x_j) = \begin{pmatrix} {x_{j1}}^2 \\ x_{j1}x_{j2} \\ x_{j2}x_{j1} \\ {x_{j2}}^2 \end{pmatrix}$$

$$x_i = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad , \quad x_j = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

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$$\langle \phi({x_i}^T), \phi(x_j) \rangle = 9 + 30 + 30 + 100 = 169$$

$$\langle \phi({x_i}^T), \phi(x_j) \rangle = K\left(x_j, x_i\right) = \left(\begin{pmatrix}1\\2\end{pmatrix}^T \cdot \begin{pmatrix}3\\5\end{pmatrix}\right)^2$$

$$= (3 + 10)^2 = 169$$

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This kernel is called : linear kernel

And finally our program becomes :

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$$\text{maximize} \sum_{i=1}^{N} \alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \cdot \alpha_j \cdot y_i \cdot y_j \cdot K(x_j, x_i)$$

$$\text{subject to :} \qquad 0 \leq \alpha_i \leq C \qquad i = 1, \ldots, N$$

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Once we find the corresponding α_i using a QP solver we use the following equation to find w and b that minimizes the primal problem

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$$w = \sum_{i=1}^{N} \alpha_i \cdot y_i \cdot x_i$$

$$b = \frac{1}{N} \sum_{i=1}^{N} (y_i - w^T x_i)$$

One special kernel function is the RBF kernel :

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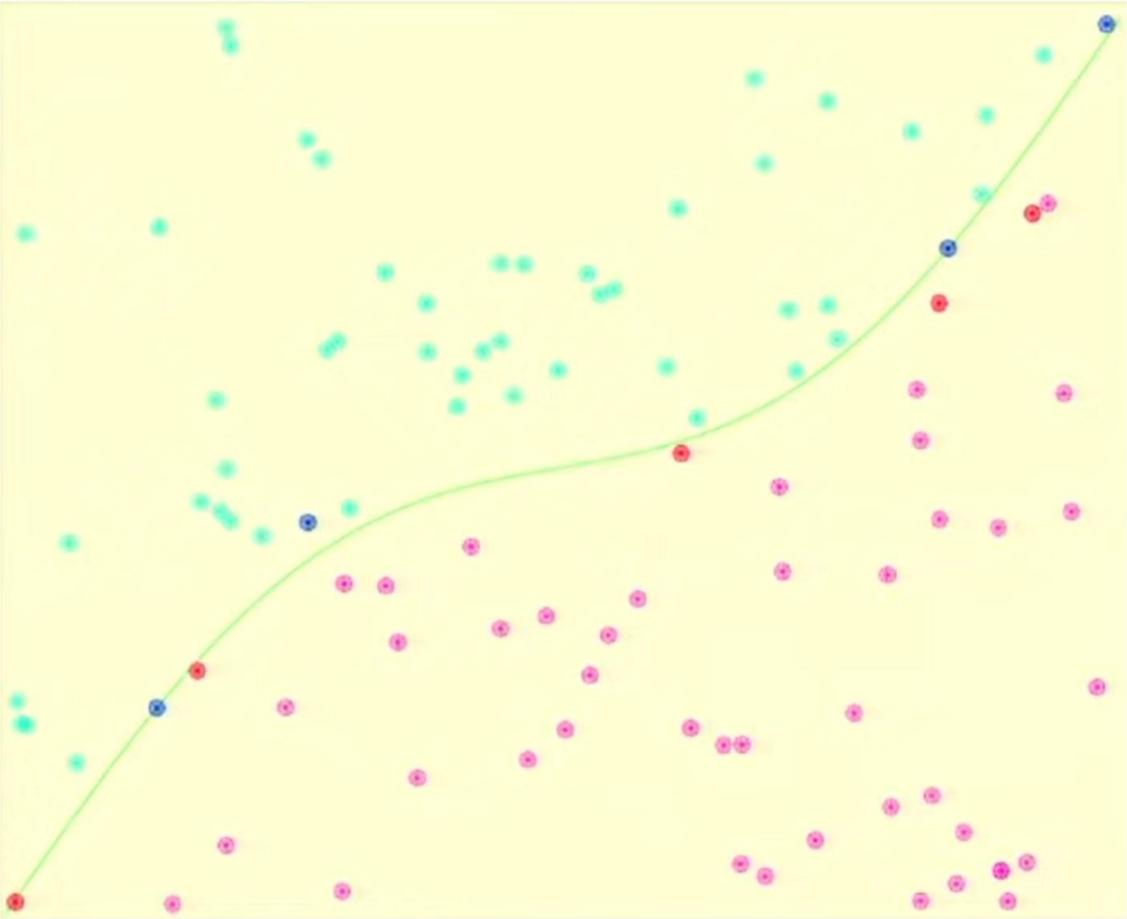
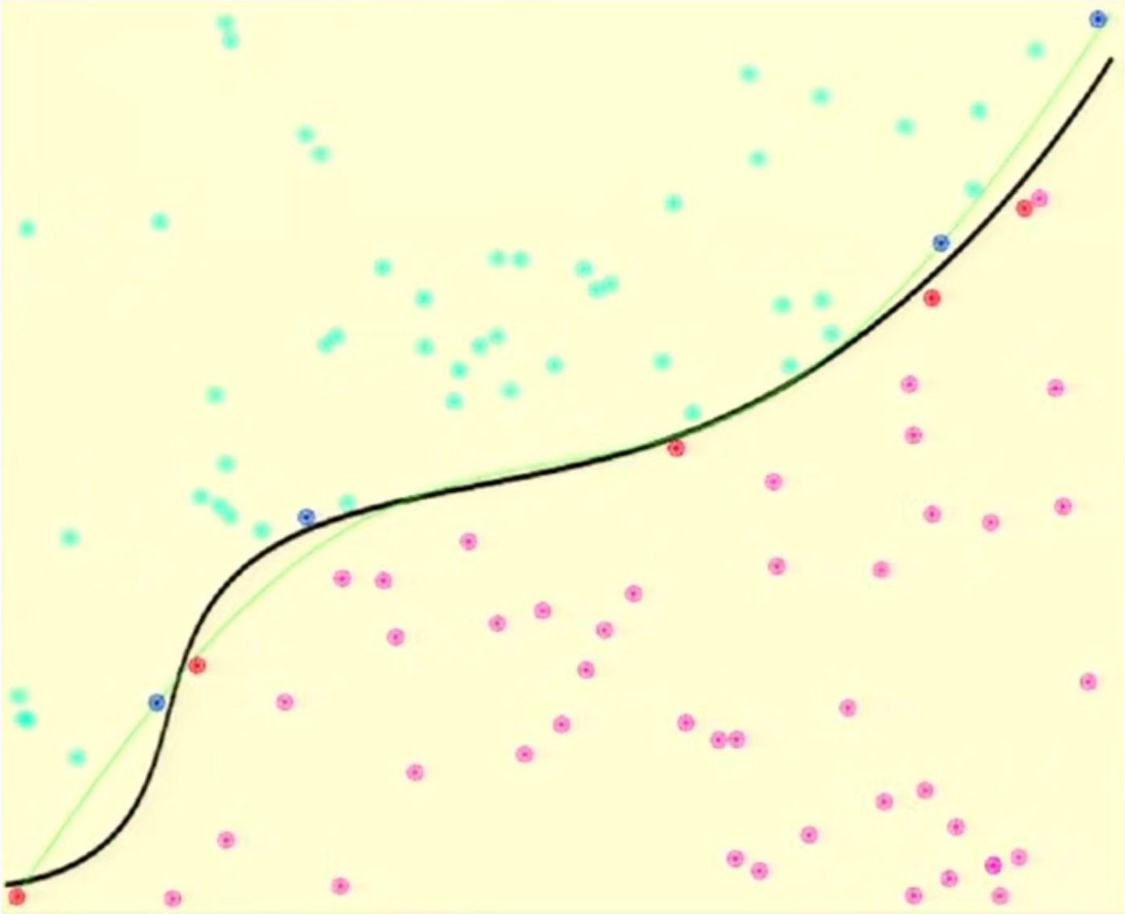
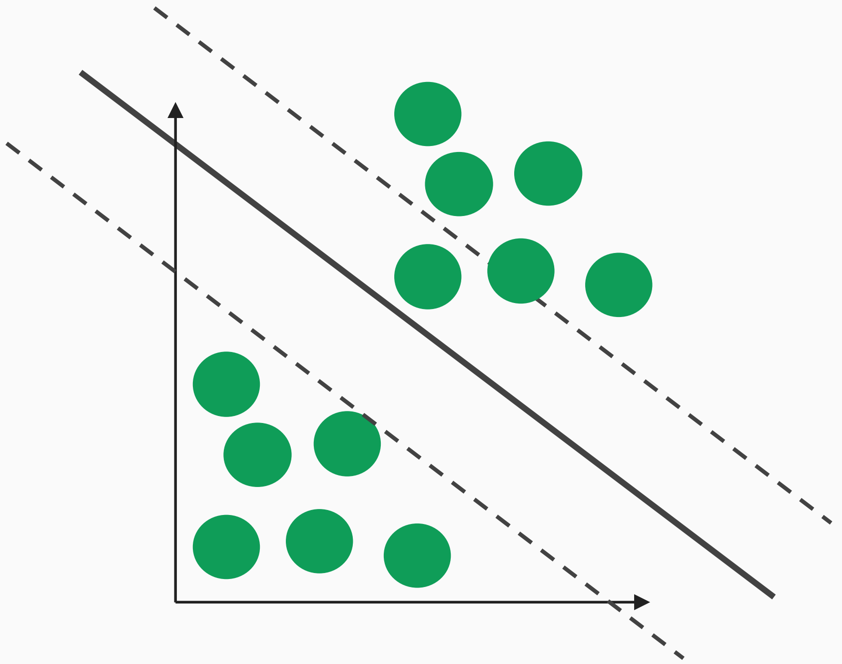
$$K\left(x_i, x_j\right) = e^{\left(-\gamma\|x_i - x_j\|^2\right)}$$



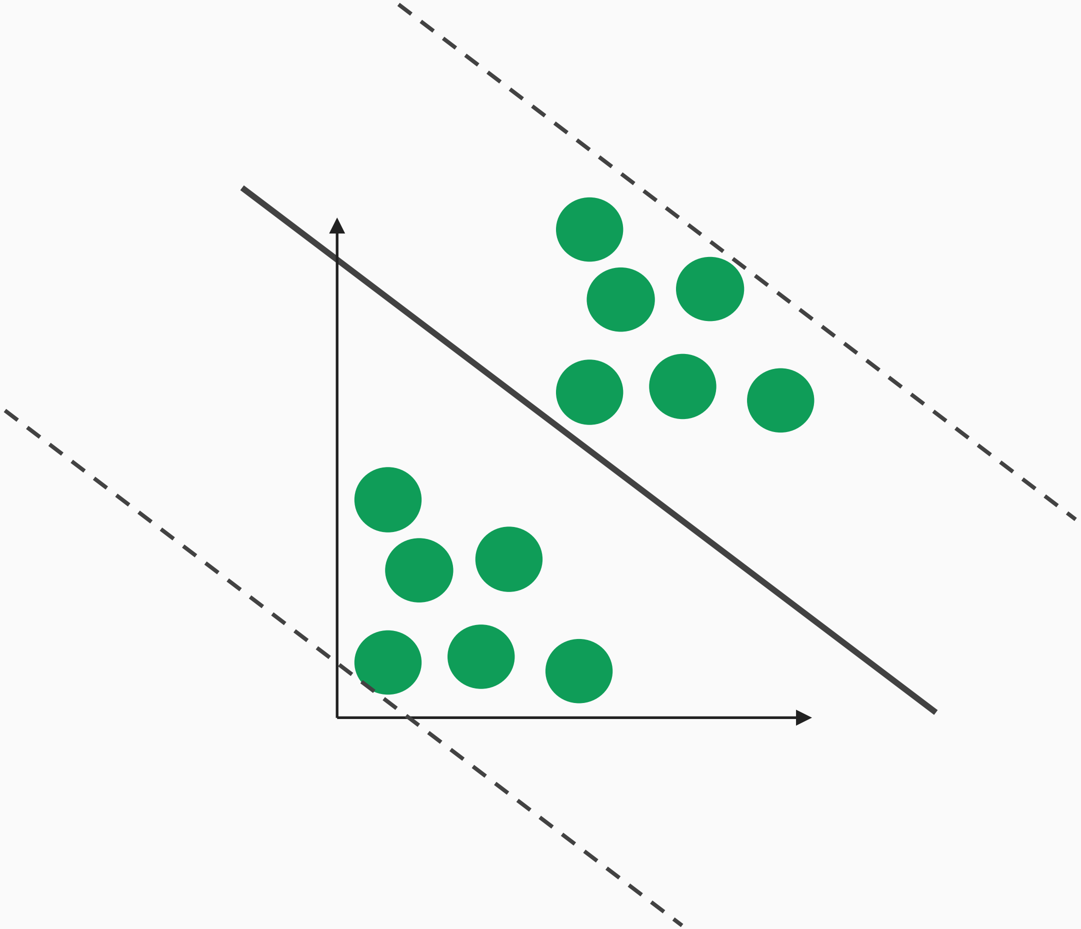


SVMs for regression

Instead of :



we maximize the instances on the street :



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