

Proof By Induction

① Base case: ($n=0$)

$$\sum_{i=0}^n i^2 = \frac{1}{6} n(n+1)(2n+1) \quad \text{for all } n \in \mathbb{N}$$

$$\begin{aligned} (0)^2 &= \frac{1}{6} (0)(0+1)(2(0)+1) \\ 0 &= \frac{1}{6} (0)(1)(1) \\ 0 &= 0 \quad \checkmark \end{aligned}$$

b. Inductive hypothesis: (Assume $n=k$ is true)

$$\text{Assume } \sum_{i=0}^k i^2 = \frac{1}{6} k(k+1)(2k+1) \quad \text{for some } k \in \mathbb{N}.$$

c. Inductive Step: (show that $n=k$ is true $\Rightarrow n=k+1$ is also true)

$$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^k i^2 + (k+1)^2$$

$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 \quad (\text{assumption } n=k)$$

$$= \frac{1}{6} k(k+1)(2k+1) + \frac{6}{6} (k+1)(k+1)$$

$$= \frac{1}{6} (k+1) [k(2k+1) + 6(k+1)]$$

$$= \frac{1}{6} (k+1) (2k^2 + k + 6k + 6)$$

$$= \frac{1}{6} (k+1) (2k^2 + 7k + 6) = \frac{1}{6} (k+1) (2k^2 + 4k + 3k + 6)$$

$$= \frac{1}{6} (k+1) [2k(k+2) + 3(k+2)] = \frac{1}{6} (k+1) (k+2) (2k+3)$$

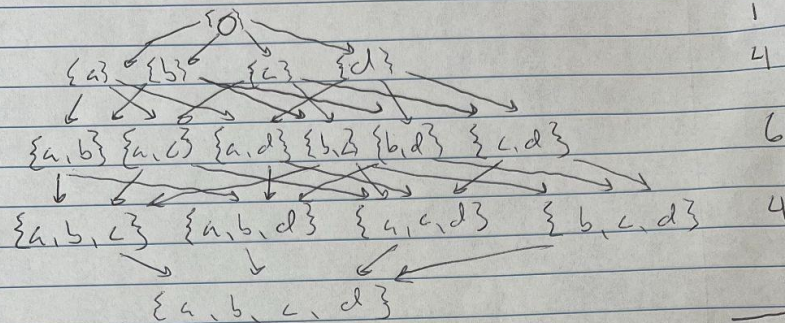
$$\text{target } \frac{1}{6} (k+1) ((k+1)+1) (2(k+1)+1) = \frac{1}{6} (k+1) (k+2) (2k+3)$$

Since the statement is true for $k=0$, and truth for $n=k$ implies that $n=k+1$ is also true, the statement is true by mathematical induction.

Combinatorics of Subsets

2. a. set $A = \{a, b, c, d\}$

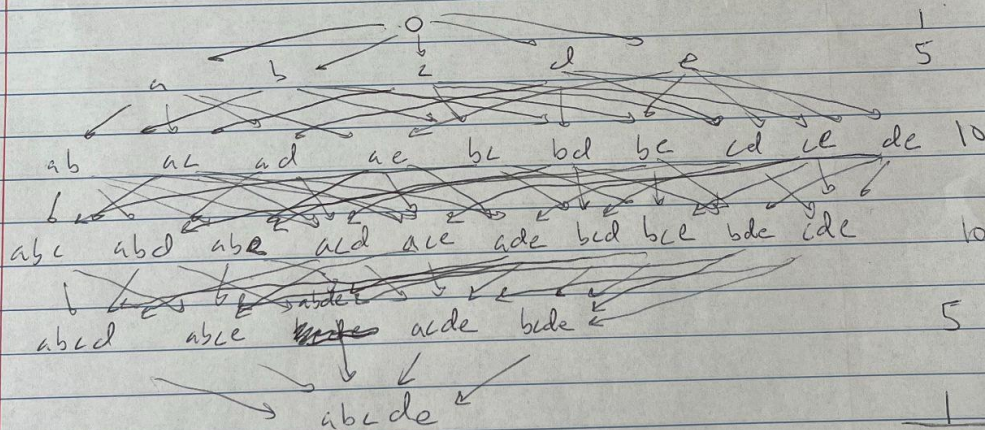
b. # of subsets



c. total = 16

d. set $B = \{a, b, c, d, e\}$

subsets



total = 32

g.

e. 1, 4, 6, 4, 1

f. 1, 5, 10, 10, 5, 1

h. Pascal's triangle

						1
					1	1
				1	2	1
			1	3	3	1
		1	4	6	4	1
	1	5	10	10	5	1

3. Proving Set Equalities (direct proof)

show $A^c \cup B = (A \cap B^c)^c$

$$\begin{aligned} x \in (A \cap B)^c &\Rightarrow x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B \\ &\Rightarrow x \in A^c \text{ or } x \in B^c \Rightarrow x \in A^c \cup B^c \end{aligned}$$

4. Euclid's algorithm for GCD

a. $\text{gcd}(81, 64) = 1$

$$81 = 64 \cdot q + r$$

$$81 = 64(1) + 17$$

$$64 = 17(3) + 13$$

$$17 = 13(1) + 4$$

$$13 = 4(3) + 1$$

$$4 = 1(4) + 0$$

~~1~~

b. $\text{gcd}(201, 100) = 1$

$$201 = 100(2) + 1$$

$$100 = 1(100) + 0$$

c. $\text{gcd}(1025, 65) = 5$

$$1025 = 65(15) + 50$$

$$65 = 50(1) + 15$$

$$50 = 15(3) + 5$$

$$15 = 5(3) + 0$$

d. $\text{gcd}(40320, 2101311) = 9$

$$\begin{aligned} 2101311 &= 40320(52) + 4671 \\ 40320 &= 4671(8) + 2952 \\ 4671 &= 2952(1) + 1719 \\ 2952 &= 1719(1) + 1233 \\ 1719 &= 1233(1) + 486 \\ 1233 &= 486(2) + 261 \\ 486 &= 261(1) + 225 \\ 261 &= 225(1) + 36 \\ 225 &= 36(6) + 9 \\ 36 &= 9(4) + 0 \end{aligned}$$