Proof By Induction

① a. Base case: $(n=0)$

$$\sum_{i=0}^{n} i^2 = \frac{1}{6}\, n(n+1)(2n+1) \quad \text{for all } n \in \mathbb{N}$$

$$(0)^2 = \frac{1}{6}(0)(0+1)(2(0)+1)$$
$$0 = (0)(1)(1)$$
$$0 = 0 \ \checkmark$$

b. Inductive hypothesis : (Assume $n=k$ is true)

Assume $$\sum_{i=0}^{k} i^2 = \frac{1}{6}\, k(k+1)(2k+1)$$ for some $k \in \mathbb{N}$.

c. Inductive Step: (Show that $n=k$ is true $\Rightarrow n=k+1$ is also true)

$$\sum_{i=0}^{k+1} i^2 = \underbrace{\sum_{i=0}^{k} i^2}\, + (k+1)^2$$
$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 \quad \text{(assumption in 2)}$$
$$= \frac{1}{6} k(k+1)(2k+1) + \frac{6}{6}(k+1)(k+1)$$
$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$
$$= \frac{1}{6}(k+1)(2k^2+k+6k+6)$$
$$= \frac{1}{6}(k+1)(2k^2+7k+6) = \frac{1}{6}(k+1)(2k^2+4k+3k+6)$$
$$= \frac{1}{6}(k+1)\left[2k(k+2)+3(k+2)\right] = \frac{1}{6}(k+1)(k+2)(2k+3)$$

target $$\frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$$

Since the statement is true for $k=0$, and truth for $n=k$, implied that $n=k+1$ is also true, the statement is true by mathematical induction

Combinatorics of Subsets

2. a. set A = {a, b, c, d}

b. # of subsets

{ }

{a} {b} {c} {d}

{a, b} {a, c} {a, d} {b, c} {b, d} {c, d}

{a, b, c} {a, b, d} {a, c, d} {b, c, d}

{a, b, c, d}

of subsets
1
4
6
4
1

c. total = 16

d. set B = {a, b, c, d, e}

subsets

o

a b c d e

ab ac ad ae bc bd be cd ce de

abc abd abe acd ace ade bcd bce bde cde

abcd abce ~~abde~~ acde bcde

abcde

subsets
1
5
10
10
5
1

total = 32

g.

e. 1, 4, 6, 4, 1

f. 1, 5, 10, 10, 5, 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

h. pascal's triangle

3. Proving Set Equalities (direct proof)

show $A^c \cup B = (A \cap B^c)^c$

$x \in (A \cap B)^c \Rightarrow x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A^c$ or $x \in B^c \Rightarrow x \in A^c \cup B^c$

4. Euclid's algorithm for GCD

a. gcd $(81, 64) = 1$

$81 = 64 \cdot q + r$

$81 = 64(1) + 17$

$64 = 17 \cdot (3) + 13$

$17 = 13 \cdot (1) + 4$

$13 = 4(3) + 1$

$4 = 1(4) + 0$

b. gcd $(201, 100) = 1$

$201 = 100(2) + 1$

$100 = 1(100) + 0$

c. gcd $(1025, 65) = 5$

$1025 = 65(15) + 50$

$65 = 50(1) + 15$

$50 = 15(3) + 5$

$15 = 5(3) + 0$

d. gcd $(40320, 2101311)$

$= 9$

$2101311 = 40320 \cdot (52) + 4671$

$40320 = 4671(8) + 2952$

$4671 = 2952(1) + 1719$

$2952 = 1719(1) + 1233$

$1719 = 1233(1) + 486$

$1233 = 486(2) + 261$

$486 = 261(1) + 225$

$261 = 225(1) + 36$

$225 = 36(6) + 9$

$36 = 9(4) + 0$