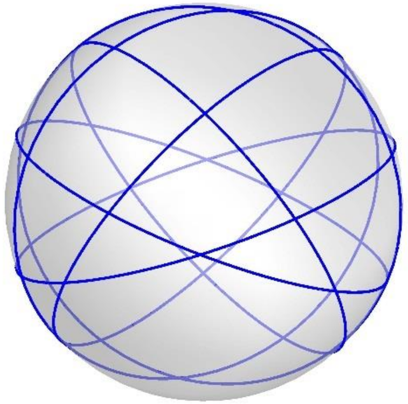


Great Circles Problem

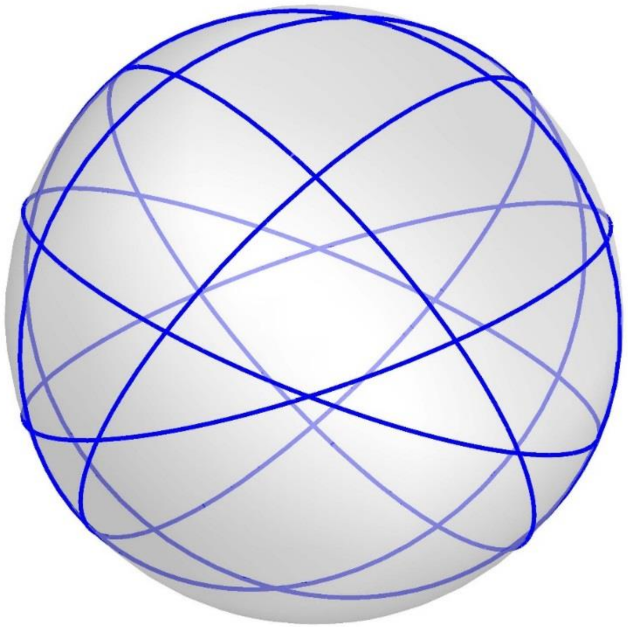
Kha Man

05/24/2015

Outline

- Problem Statement
- Lemma 1
- Lemma 2
- Theorem 1 – $\chi\left(G_{n\ special}\right) = 3$
- Some non-isomorphic graphs of 6 and 7 great circles

Problem Statement



(Great Circle Problem) A great circle is any circle on a sphere whose **radius is the same** as the radius of the sphere (so it is largest possible). A circle that goes through both the North and South poles is an example of a great circle on the Earth. Given n ($n \geq 3$) great circles on a sphere, **no three of which meet at a single point**, form a great circle graph G_n by making points of intersection into vertices, and connect two vertices by an edge if and only if there is an arc between them.

Problem: What is the largest chromatic number of any great circle graph?

You can find Stan Wagon's conjecture at this link

Is $\chi(G_n) = 3$???

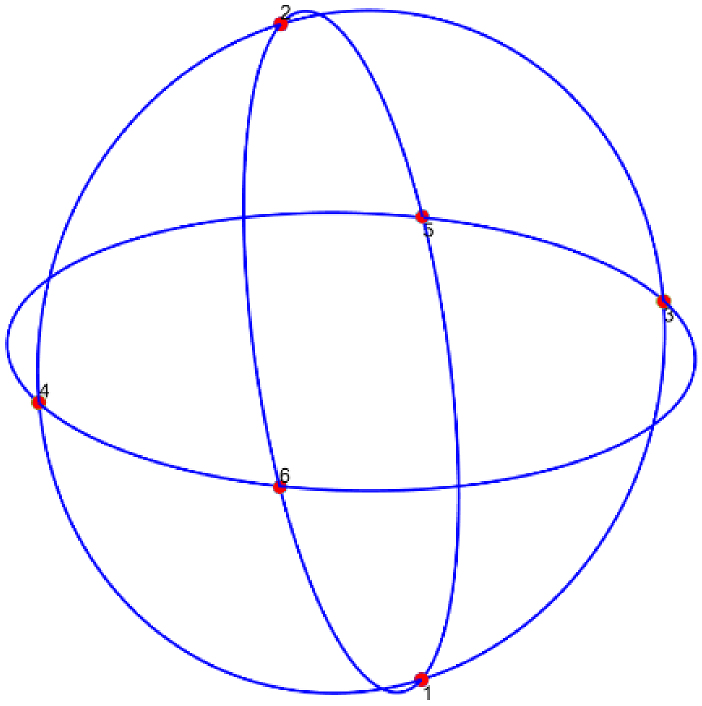
Model the problem

- G_n is the graph generated by n great circles
- Vertices are created by every pair of great circles
- V is the vertices set
- E is the edges set
- $\chi(G_n)$ implies the chromatic number (the least colors used for properly coloring vertices that a vertex V_i will have different colors from its neighbors which have a edge to V_i

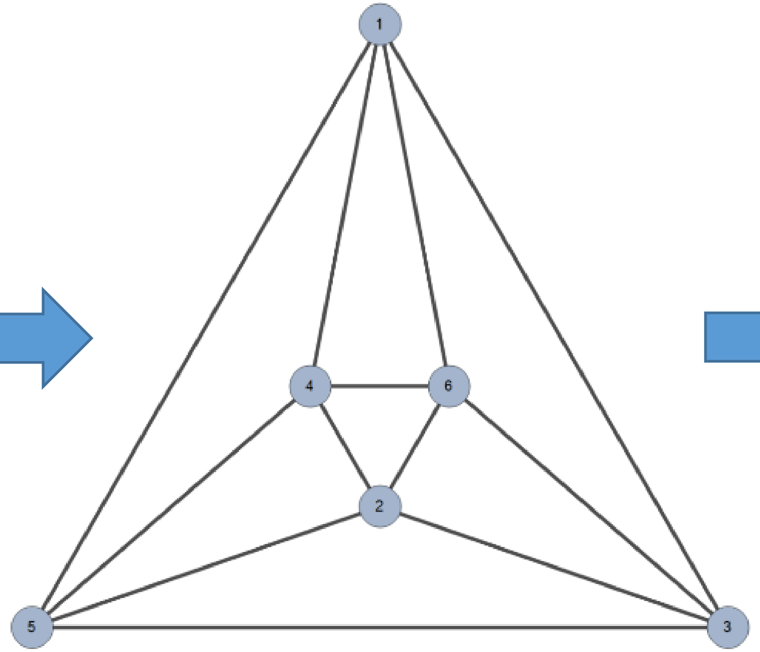
Some properties

- $\deg(V_i) = 4$
  - Since a vertex only allows 2 circles to pass through it, so every vertex will have 4 neighbors which means $\deg(V_i) = 4$
- The graph G_n is planar
  - All the vertices are formed by the intersections of the circles. So, there is no sudden arc may cut through the connection between vertices since by contradiction, it will keep forming the vertices continuously and it makes no sense
- $3 \leq \chi(G_n) \leq 4$
  - According to four-color theorem, $\chi(G) = 4$
  - A triangle can be formed by 3 random circles which means at that triangle, its vertices needs 3 different colors. Or the graph G contains a sub graph K_3

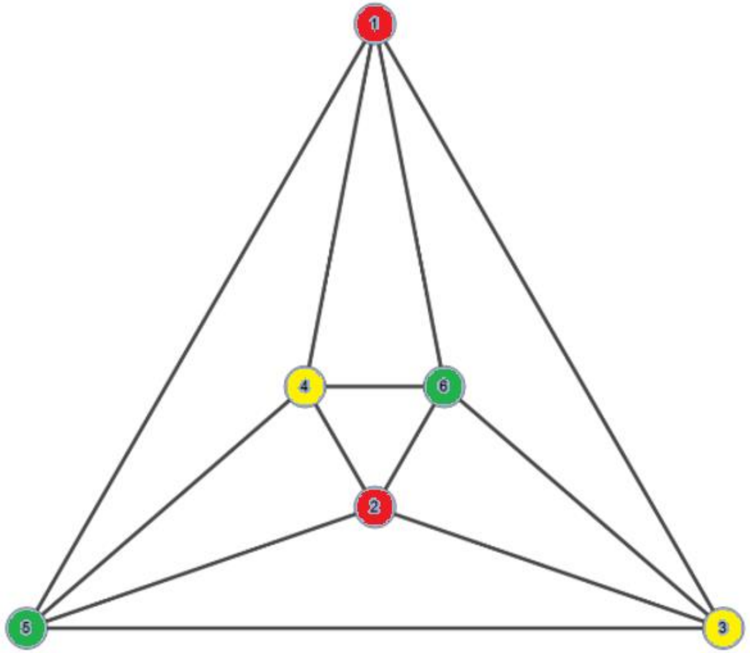
A warm start with some base cases



Generate 3 great circles
and get the intersections



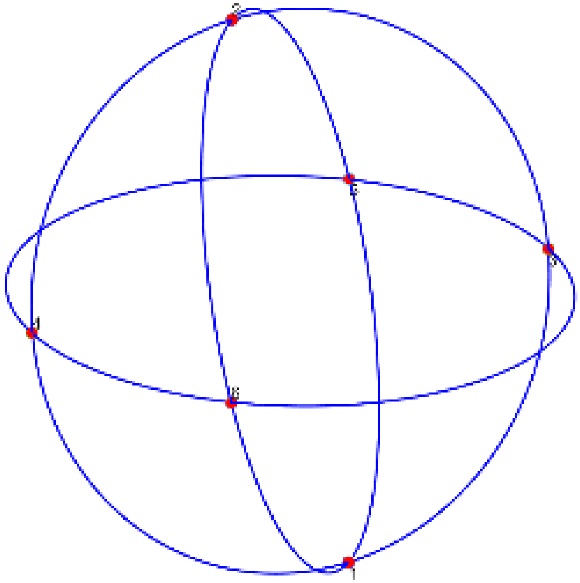
Create a 2D intuitive look
with the same vertices



Use least colors for vertices
(3)

There is only 1 non-isomorphic graph for G_3 (This can be verified manually)

Get G_4 from G_3



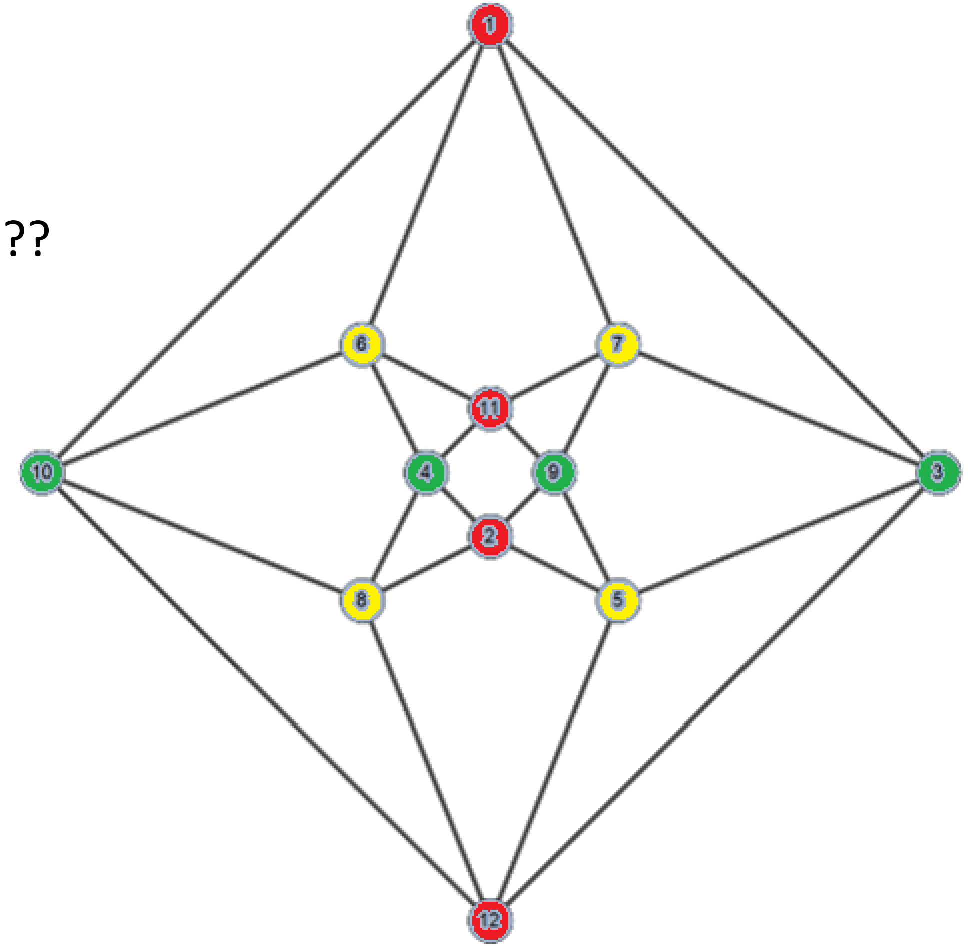
+ 1 great circle = ???

There are 3 cases happen when the 4th great circle is added:

- It cuts arc(1,5), arc(1,3), arc(2,6), arc(2,4), arc(3,6) and arc(4,5)
- It cuts arc(1,3), arc(3,5), arc(2,4), arc(4,6), arc(1,6) and arc(2,5)
- It cuts arc(1,5), arc(3,5), arc(2,6), arc(4,6), arc(2,6) and arc(1,5)

The outcome of 3 cases is the same as the figure on the right

➔ 1 non-isomorphic graph for G_4



G_4

$\chi(G_n) = 3$???

- Probably
- G_n is basically 4-regular planar graph without no intuitive and special properties
- We may create great circles more specifically to somehow reduce the randomness

The definition of a great circle

- The full definition.
- To simplify, I define a great circle with parameters:

  [Center] [Radius] [Roll] [Pitch] [Yaw]

  - Center O (x,y,z)
  - Radius
  - Roll: the rotation of the circle around x-axis
  - Pitch: the rotation of the circle around y-axis
  - Yaw: the rotation of the circle around z-axis (This parameter isn't necessary because we don't need the direction for the great circle)

- For example: C_0 = [O (1,2,3)] [1] [$\frac{\pi}{2}$] [$\frac{\pi}{2}$] [0]

An arrangement

- What happens when generating circles have roll same as pitch?
- Because $roll = pitch$, so that every great circle rotates around the plane $x = y$

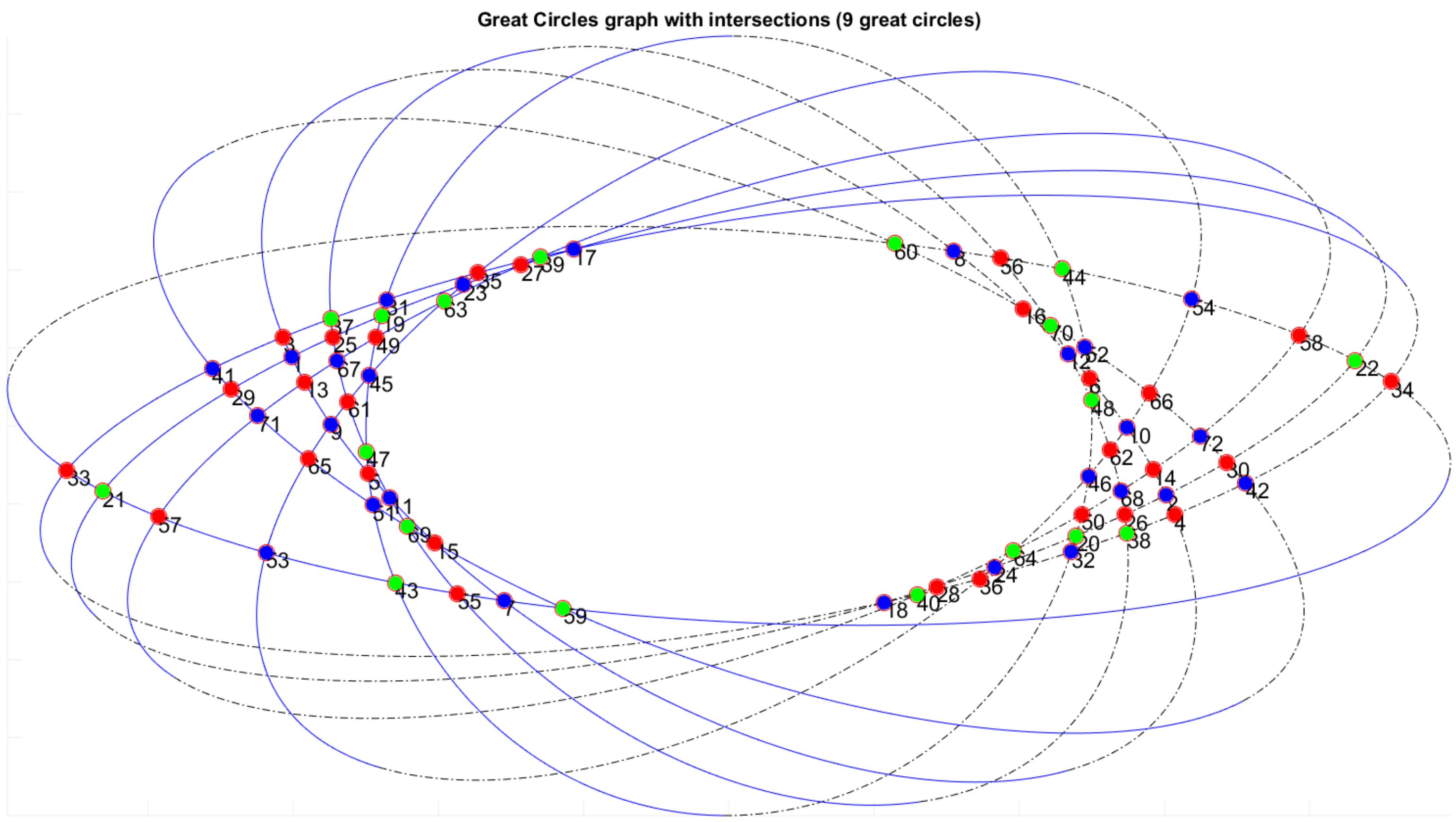
➔We always have an order of the rotations of great circles such that $roll_1 \leq roll_2 \leq roll_3 \leq \cdots \leq roll_n \,; roll_i \in (0\,, \pi]$

Redraw the graph in 2D

We can redraw the new arrangement by making some ellipses which have a half for “top” and another half for “bottom”

Every pair of “ellipses” with the same “top” or “bottom” will intersect

Since roll = pitch, “ellipses” rotate around an imaginary lines so that we always have the order



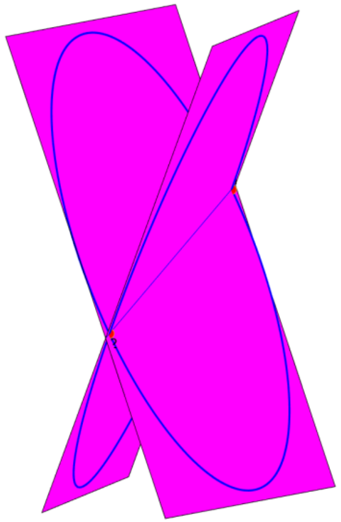
The special graph

- So I come with the definition for the new arrangement
- **Definition**:
  - G_n is special if it has the same shape as one consisting of roll=pitch great circles
- Annotation: $G_{n\ special}$

Some lemmas and theorems

A basic properties of G_n

- The center of all circles O is the **point symmetry** of vertices on both sides of the line connected 2 middle points
  - Proof:
    - 2 circles in 3D can be considered as 2 planes.
    - The intersection of 2 planes is a line
    - Therefore, all the intersections created by pairs of circles and O are on the line.



Lemma 1

Call n is the number of circles in the graph

1. There are $2(n-1)$ vertices and $2(n-1)$ edges on a circle
2. A pair of circles create 2 intersections. The distance between 2 intersections on a circle is $n-1$ edges on the circle

Lemma 1 - Proof

1. A circle will intersect $(n-1)$ other circles. A pair of circles will meet at 2 points. So the number of points on a circle is $2(n-1)$

   $\left|E\left(C_{2(n-1)}\right)\right| = 2(n-1)$ ➔ There are $2(n-1)$ edges on the circle
2. Assume the statement is correct with k great circles graphs which have $(2k-2)$ vertices on a circle.

Define $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \dots \rightarrow v_{k-1} \rightarrow \Psi(v_1) \rightarrow \Psi(v_2) \rightarrow \Psi(v_3) \rightarrow \dots \rightarrow \Psi(v_{k-1}) \rightarrow v_1$ is the circular path that has

$$d(v_i, \Psi(v_i)) = k - 1 \; ; \; i = 1,2,3, \dots, (k-1)$$

Lemma 1 - Proof

Now we add a new circle C_{k+1} into the graph. So on every circle C_1 to C_k, we have 2 new intersections made by C_{k+1}. Call it v_a and $\Psi(v_a)$

Without loss of generality, I consider v_a as the first vertex in my new circular path v_a -> v_1 -> v_2 -> v_3 -> … -> v_{k-1} - > $\Psi(v_a)$ -> $\Psi(v_1)$ -> $\Psi(v_2)$ -> $\Psi(v_3)$ -> …. -> $\Psi(v_{k-1})$ -> v_a

Because every vertex has O as the point symmetry, so if v_a is the first vertex that is close to $\Psi(v_{k-1})$ and v_1, $\Psi(v_a)$ must be close to v_{k-1} and $\Psi(v_1)$.

Lemma 1 - Proof

$d\big(v_a, \Psi(v_a)\big) = d(v_a, v_1) + d\big(v_1, \Psi(v_a)\big) = 1 + \big(d(v_1, \Psi(v_1) - d(\Psi(v_1), \Psi(v_a)\big)$
$= 1 + (k - 1) = k$

Call v_i is the vertex in the set $\{v_1, v_2, \ldots v_{k-1}\}$

$\Rightarrow d\big(v_i, \Psi(v_i)\big) = d(v_i, v_{k-1}) + d\big(v_{k-1}, \Psi(v_a)\big) + d(\Psi(v_a), \Psi(v_i))$

$= t + 1 + (k - 1 - t) = k$

Similarly, because we have (2k-2+2) = 2k edges on the new path, the other path of $d\big(v_i, \Psi(v_i)\big)$ that contains $\Psi(v_{k-1})$ is also equal to k

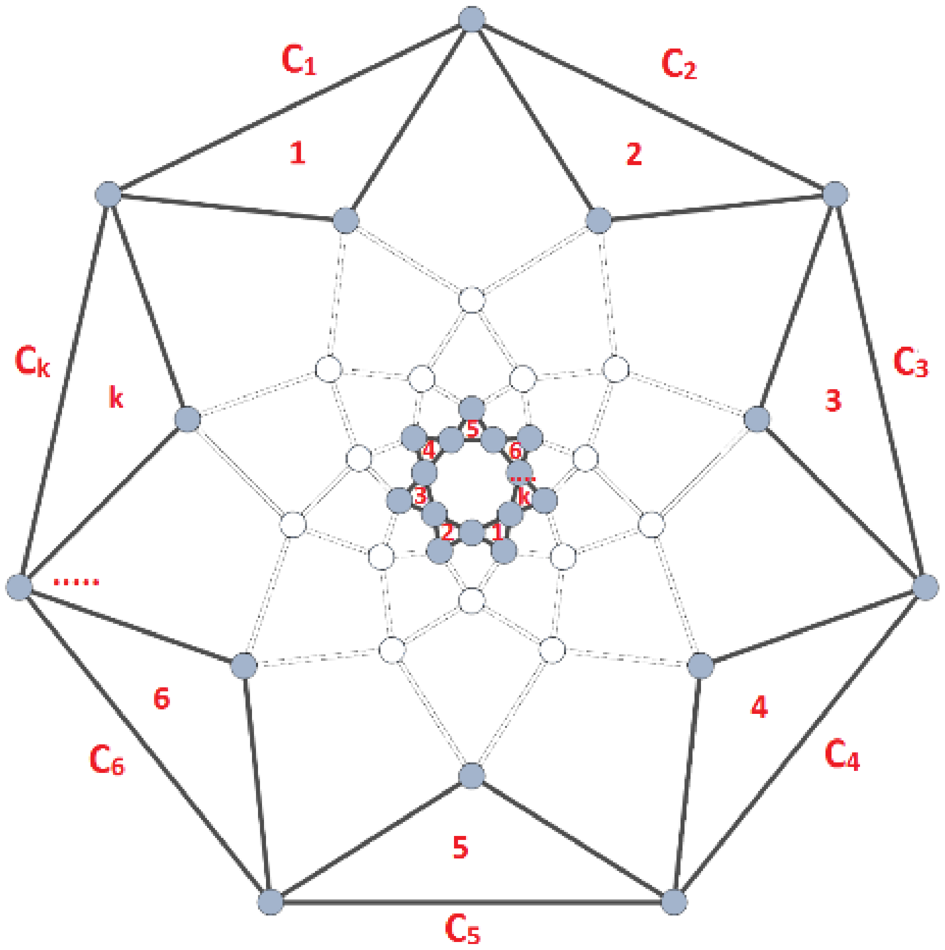
➔The induction hypothesis is correct with (k+1) circles

➔Q.E.D

Lemma 2

$G_{n\ special}$ can be seen in the form that has:

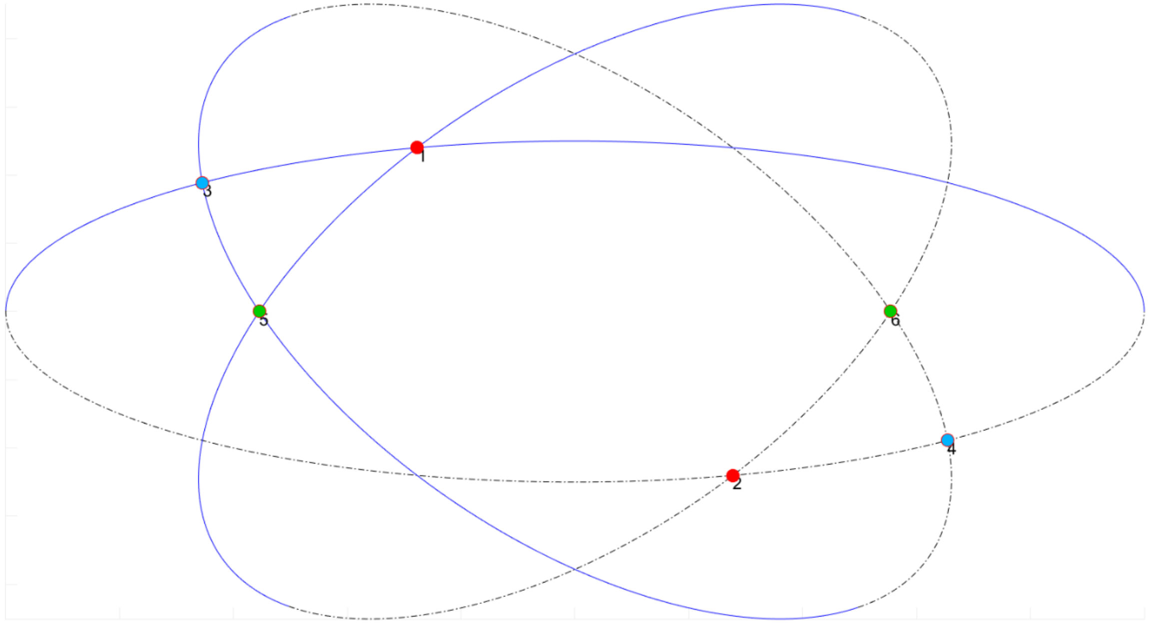
- k triangles at the "outer cycle"
- Another k triangles are made by the reflection and 1 polygon has k segments in the "middle"
- The other polygons are quadrilaterals



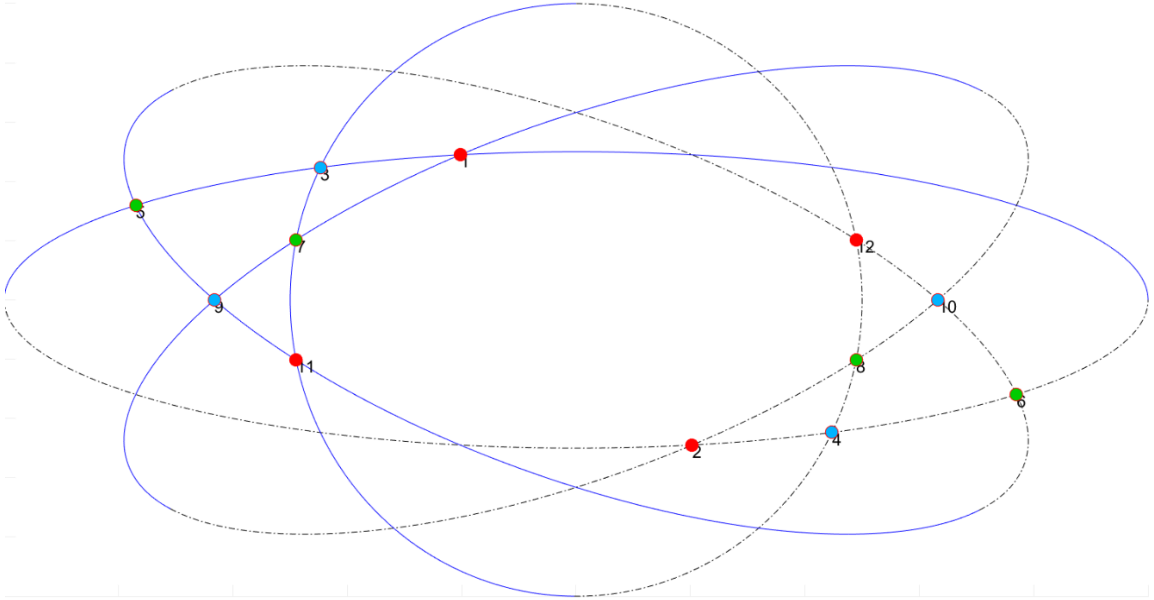
Lemma 2 - Proof

Before going into proof, let's see some special graphs $G_{3\ special}$, $G_{4\ special}$, $G_{5\ special}$, $G_{6\ special}$ in the next slides

$G_{3\text{ special}}$, $G_{4\text{ special}}$

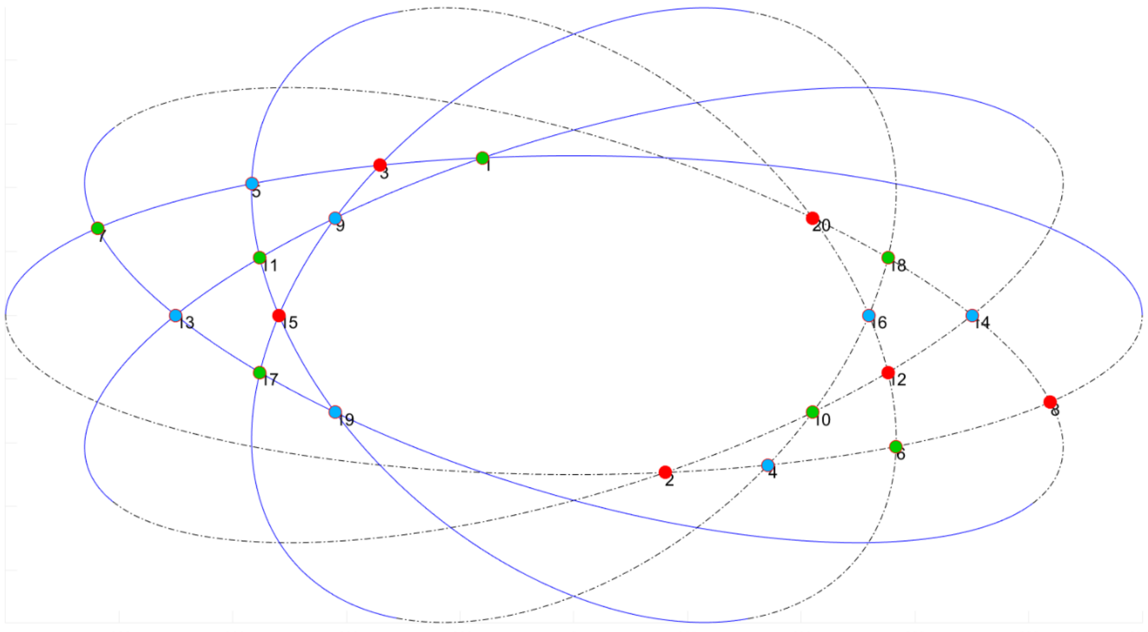


$G_{3\text{ special}}$

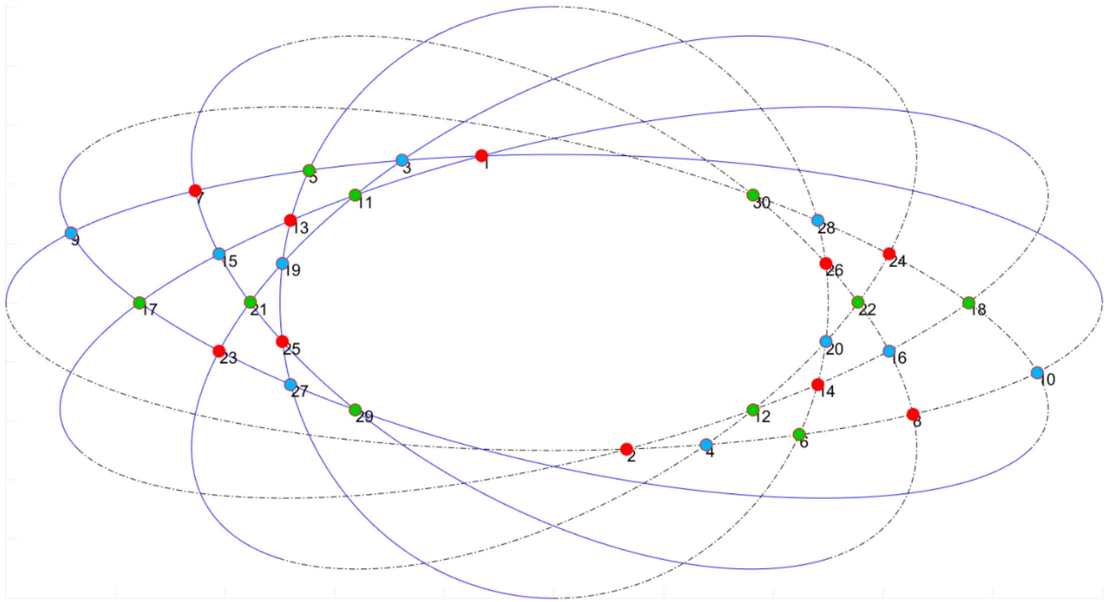


$G_{4\text{ special}}$

$G_{5\ special}$, $G_{6\ special}$



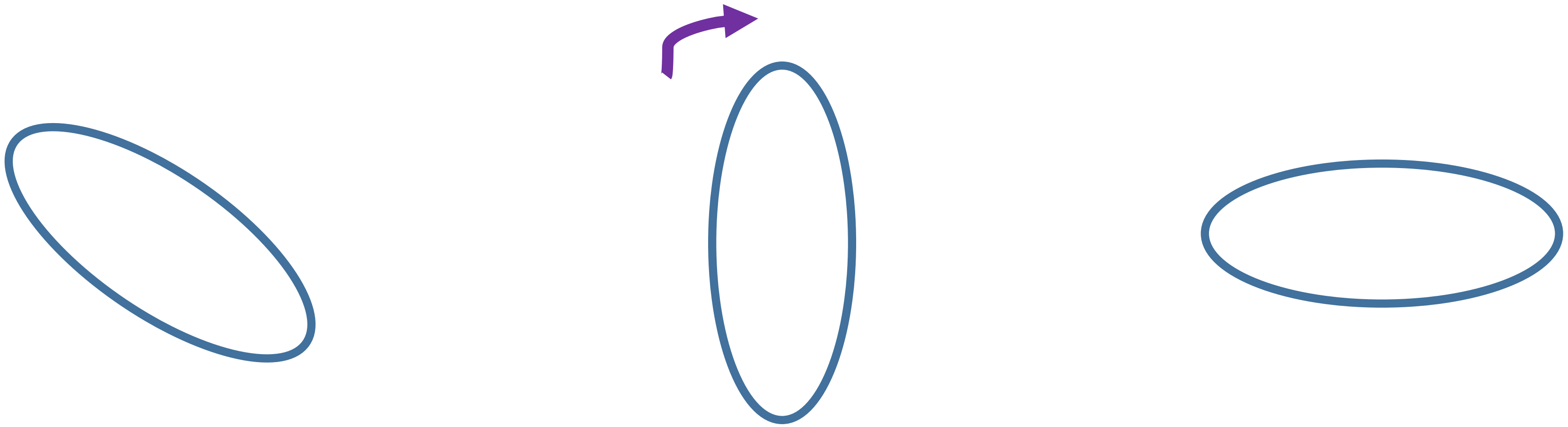
$G_{5\ special}$



$G_{6\ special}$

Lemma 2 - Proof

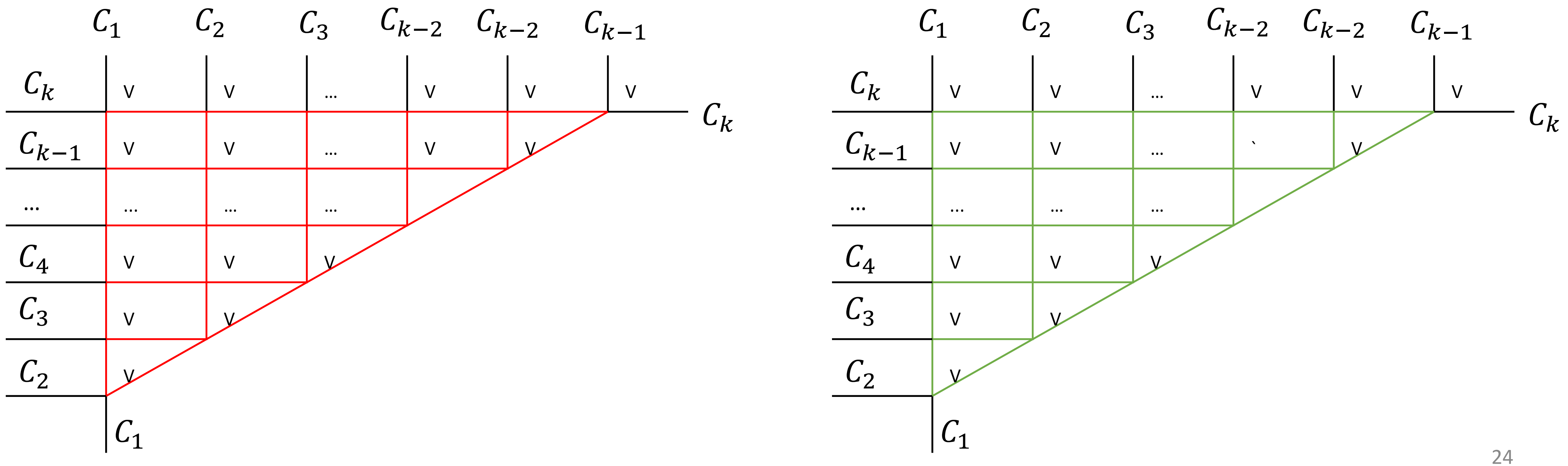
- Because the special graph has the "order", so WLOG, to build G_{k+1}, we only need to add a new great circle C_{k+1} that has $roll_{k+1} > roll_i$; $i \in [0, k]; roll_i \in (0, \pi]$



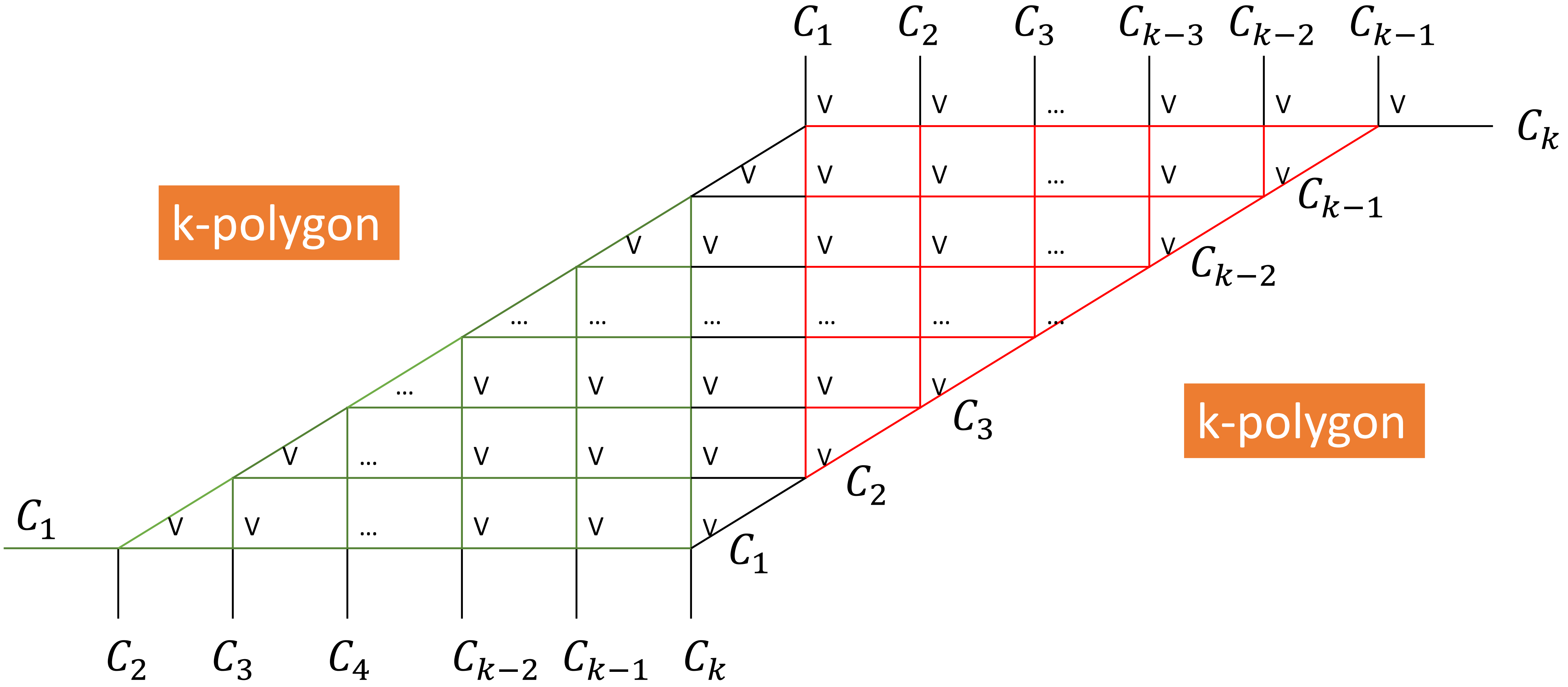
$Roll = \frac{\pi}{4}$ $Roll = \frac{\pi}{2}$ $Roll = \pi$

Lemma 2 - Proof

- Suppose we have $S_{k\ special}$ that has $C_1 \leq C_2 \leq \cdots \leq C_k$
- We split the graph into 2 parts by the plane made by C_1 and C_2



Lemma 2 - Proof

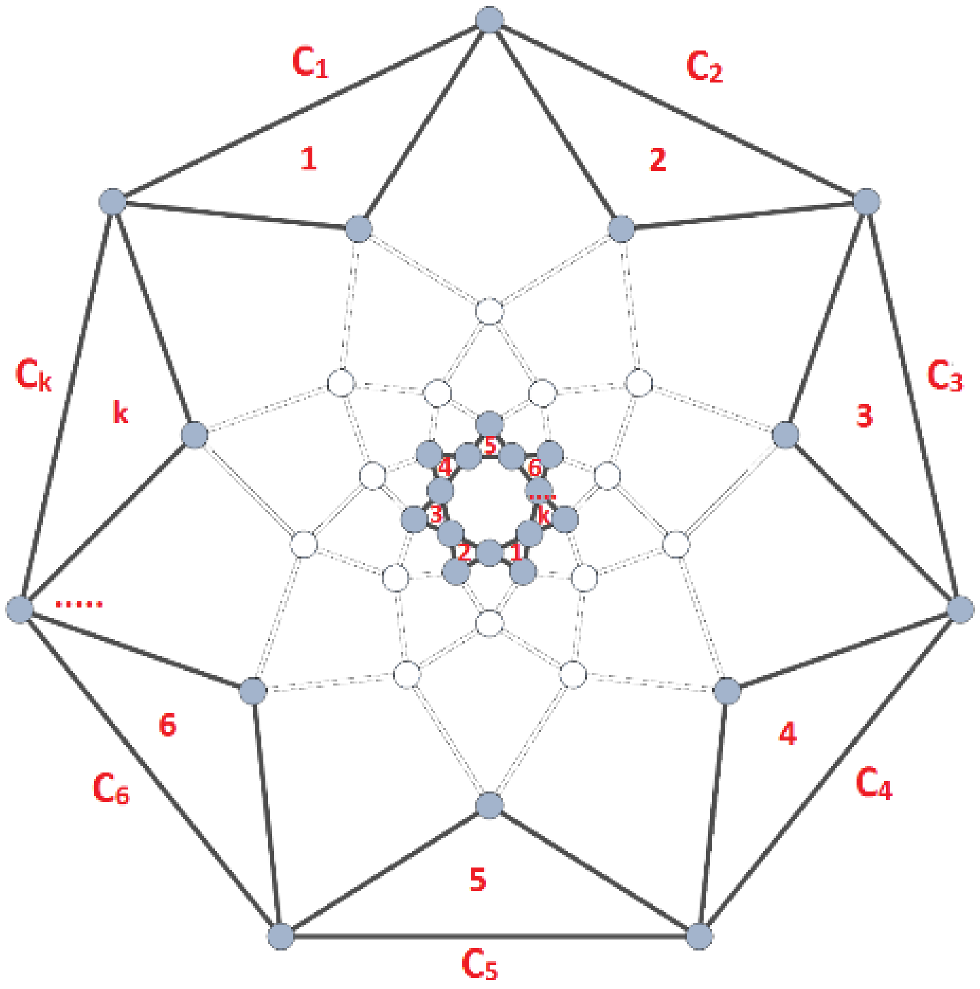


Lemma 2 - Proof

- So pick 1 k-polygon and consider it as the bound of the new form
- We have a place inside of graph for another k-polygon
- The quadrilaterals will be filled

➔ The new form is isomorphic to the original 3D form of $S_{n \text{ special}}$

(Lemma 2.2 will prove that the only way to get the new form of $G_{k+1 \text{ special}}$ is to add 1 great circle to the new form of $G_{k \text{ special}}$)



Theorem 1.

$$\chi\left(G_{n\ special}\right) = 3$$

Theorem 1 - Proof

- I split the problem into 4 sub-problems that have:
  - 3k great circles (3, 6, 9, 12, 15,….) (including (6k+3))
  - 2k great circles (4, 6, 8, 10, 12,…) (including 6k, (6k+2), (6k+4))
  - (6k+1) great circles (7, 13, 19, 25,…)
  - (6k+5) great circles (5, 11, 17, 23,…)

Chromatic number

- According to the techniques to color the graphs including 3k, 2k, 6k+1 and 6k+5 great circles, there are only 3 colors used

➔ $\chi\left(G_{n\ special}\right) = 3$

Chromatic number

- Some rules before coloring
  - Diagonal rule: The vertices on the same diagonal **should** have the same color (not a must because there are some places might have K_3 rule)
    - Proof: The vertices on the same diagonal are not connected together.
  - K_3 rule: 3 vertices that form a triangle **must** have 3 different colors

An annotation

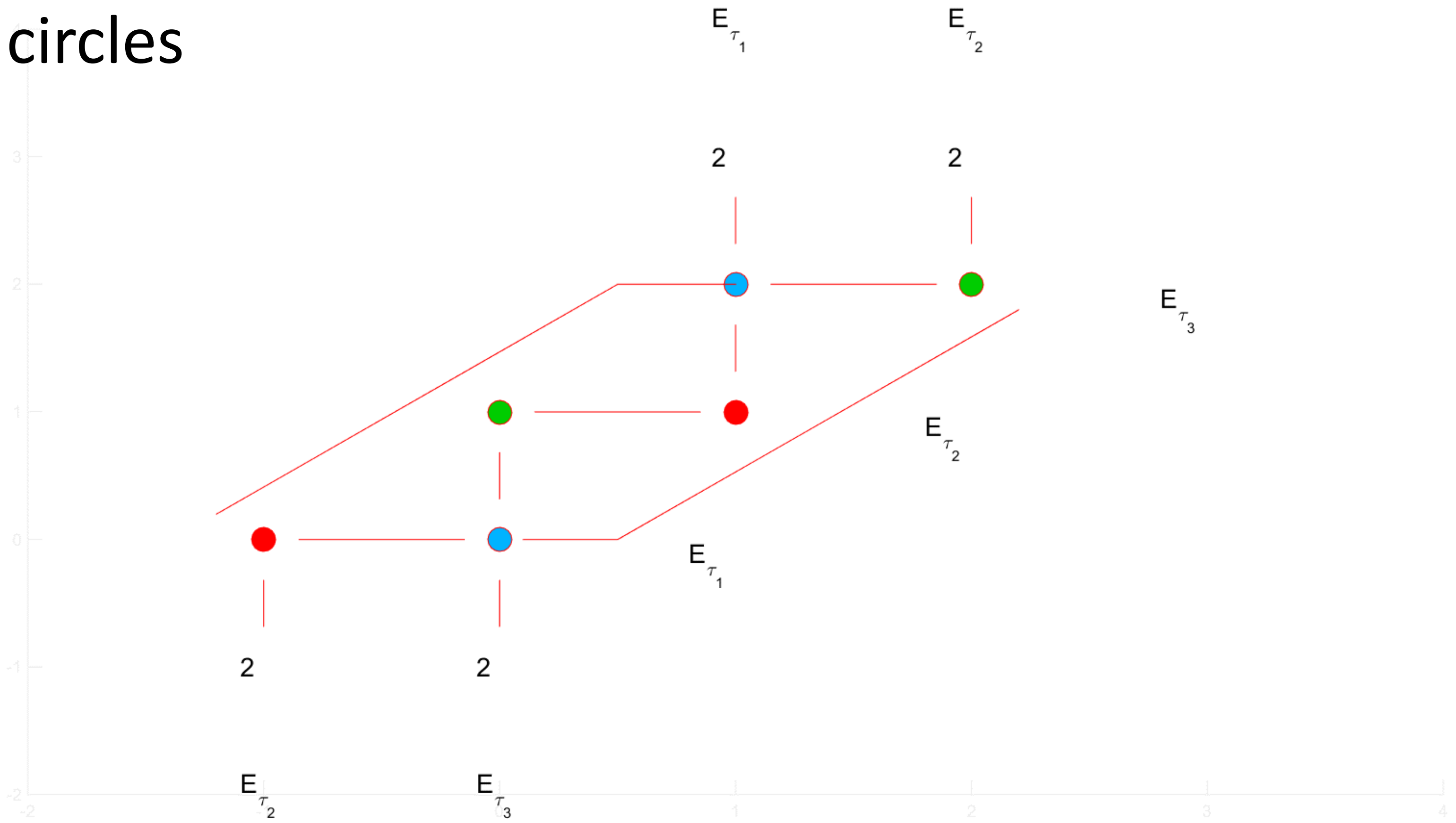
Because I copied my old slides, there are some old symbols not updated in my images

- That is the symbol E_{τ_i} equals to C_i

Chromatic number of 3k circles

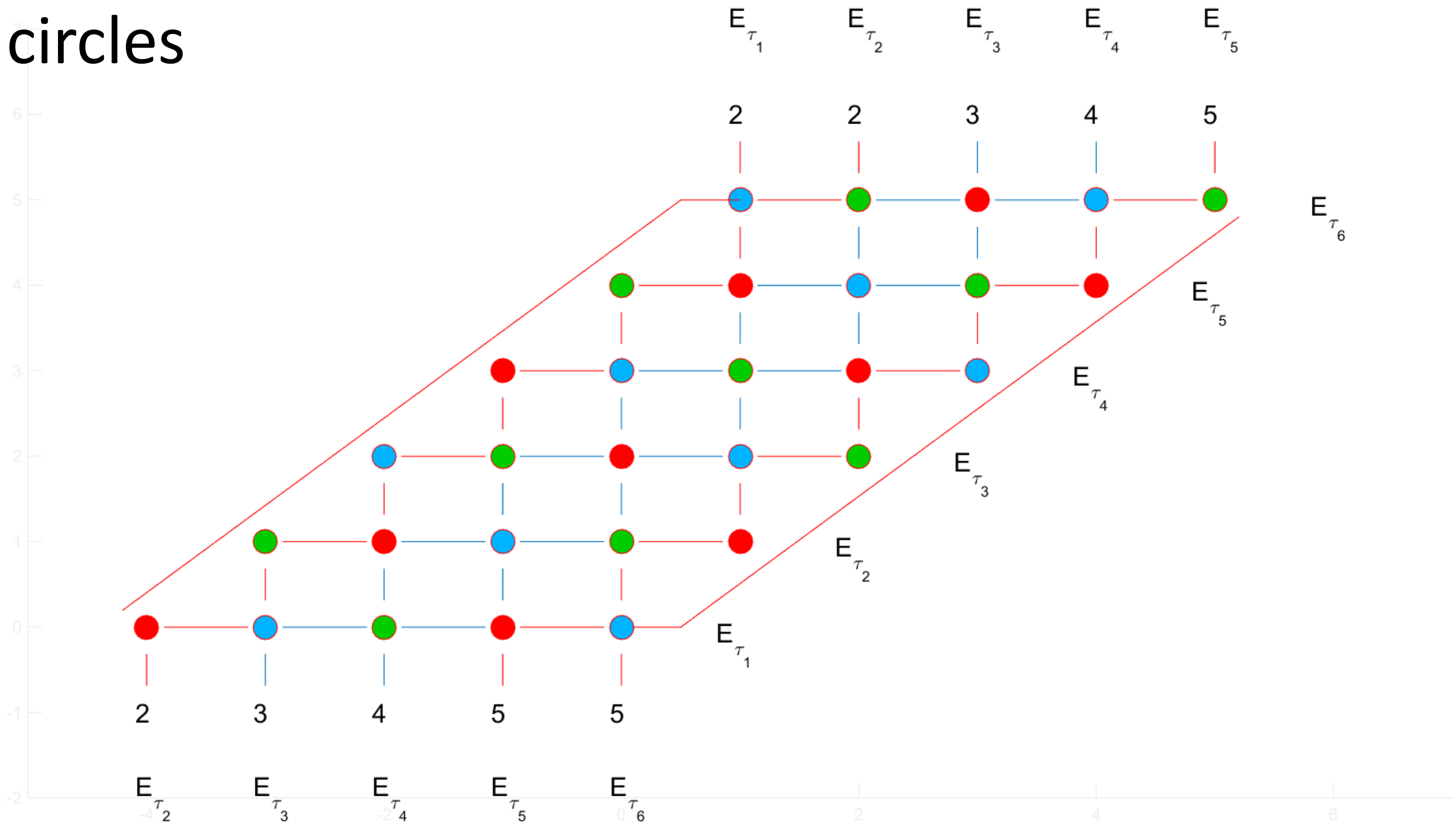
Chromatic number of 3k circles

- Base cases: 3 great circles



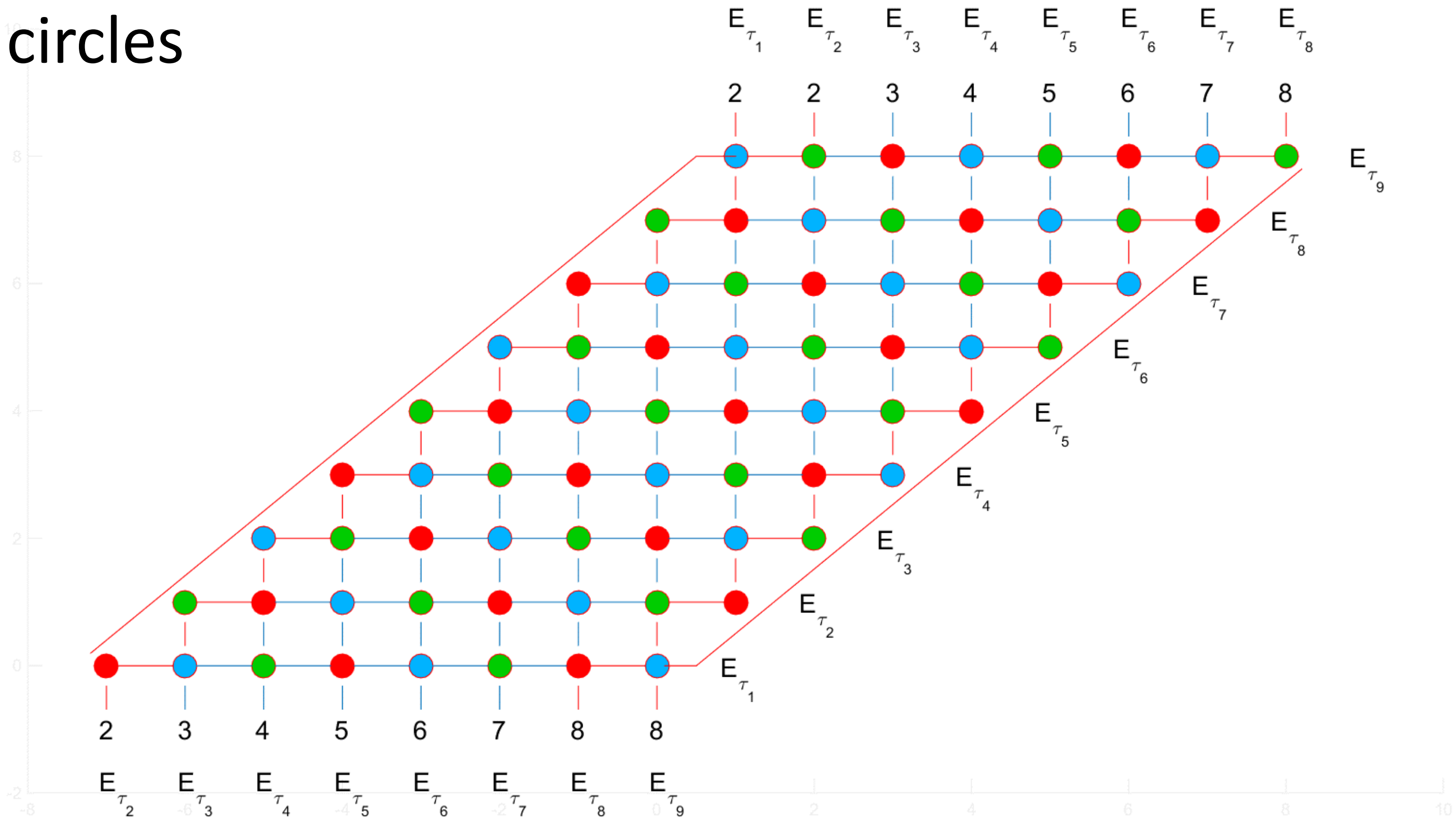
Chromatic number of 3k circles

- Base cases: 6 great circles



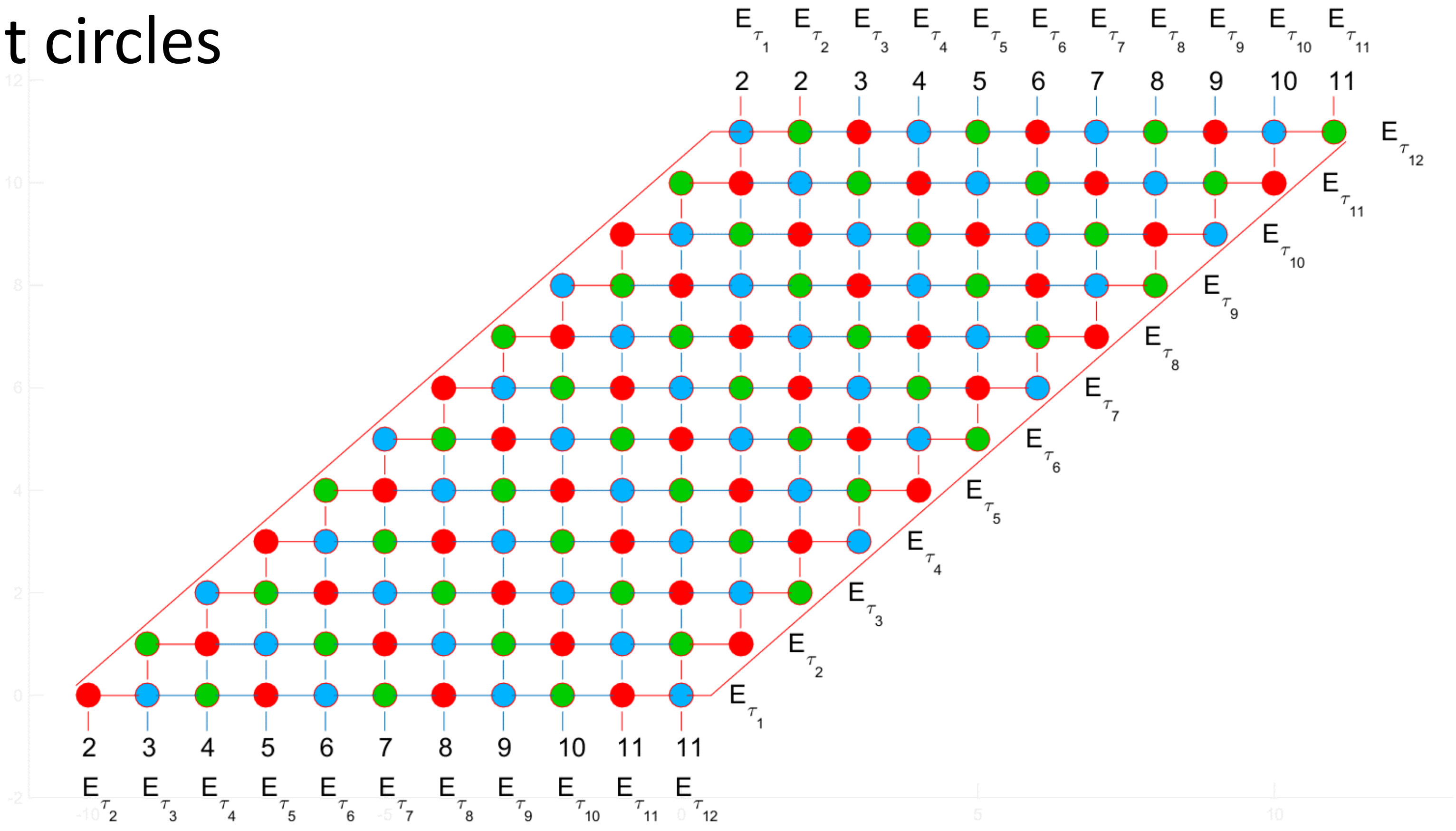
Chromatic number of 3k circles

- Base cases: 9 great circles



Chromatic number of 3k circles

- Base cases: 12 great circles



Chromatic number of 3k circles

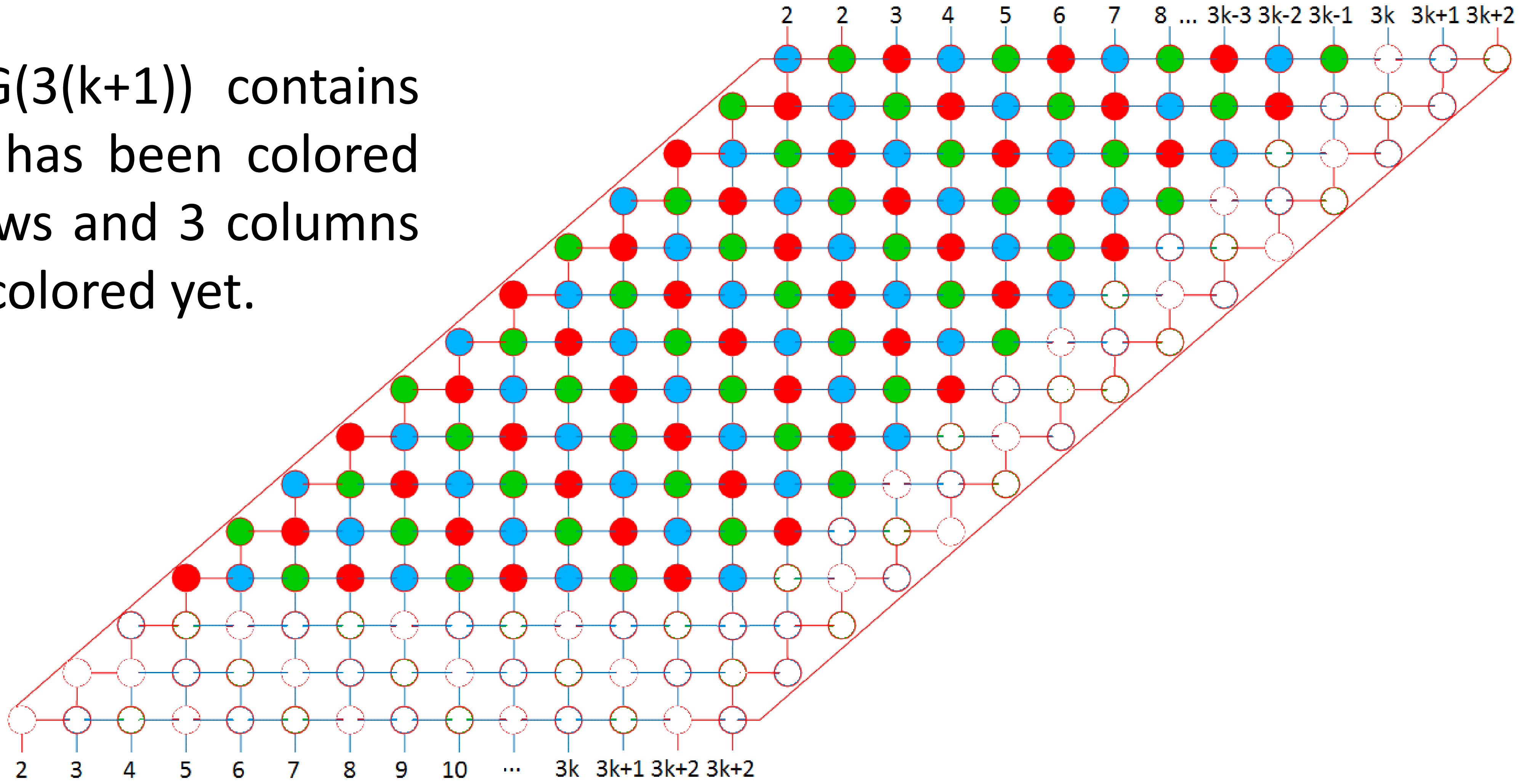
- According to the base cases with 3,6,9,12 great circles, the chromatic number is 3

➔Induction hypothesis: $\chi(G(3K)) = 3; (k > 0, k \in N)$ that has been correct with k=1,2,3,4 by spreading 3 colors from the bottom left to the top right of the equivalent graph.

➔Induction step: Prove that $\chi(G(3(K+1))) = 3$

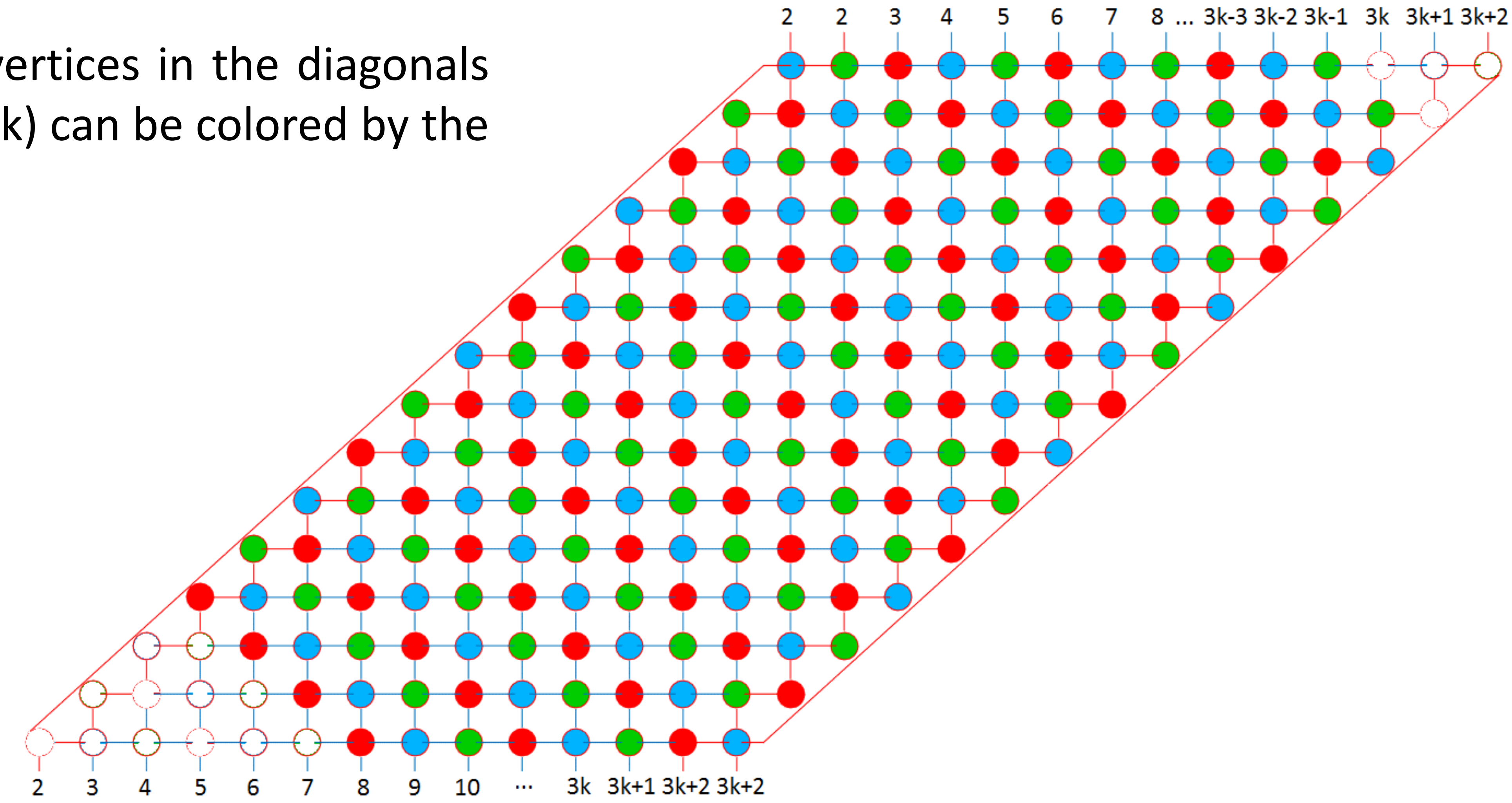
Chromatic number of 3k circles

The graph G(3(k+1)) contains G(3k) which has been colored already, 3 rows and 3 columns that are not colored yet.



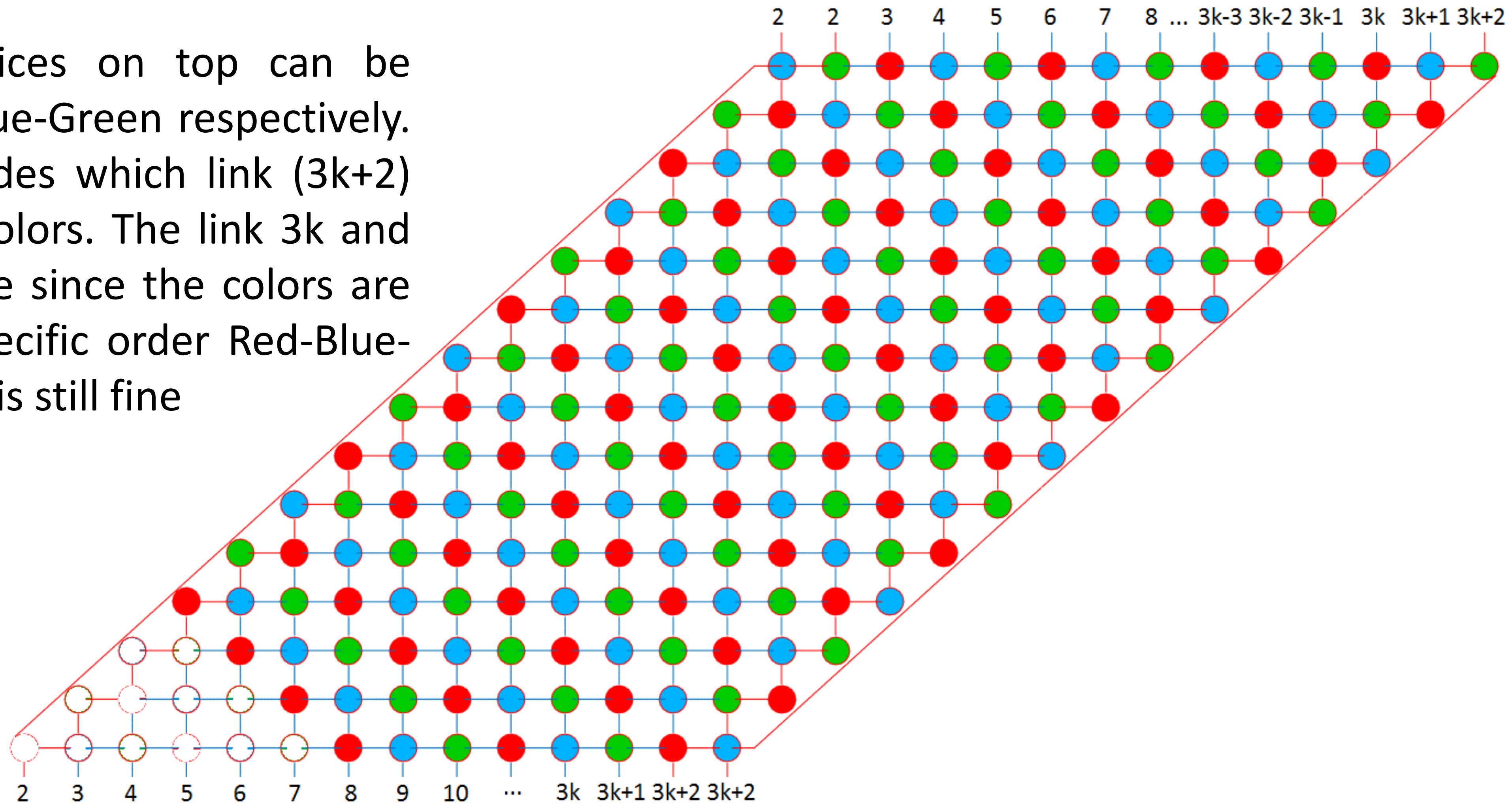
Chromatic number of 3k circles

The uncolored vertices in the diagonals of the graph G(3k) can be colored by the diagonal rule



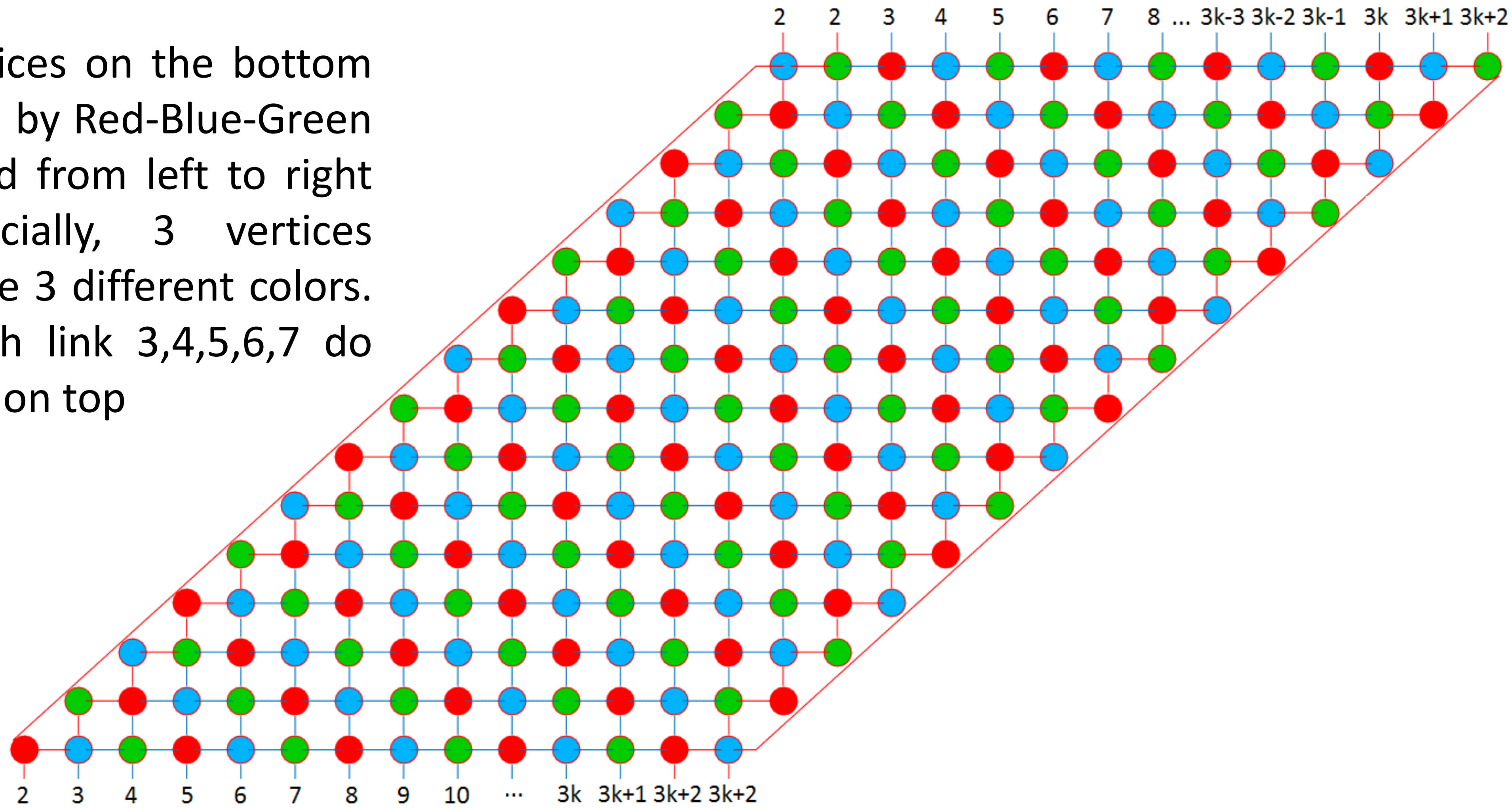
Chromatic number of 3k circles

4 uncolored vertices on top can be colored by Red-Blue-Green respectively. We can see 3 nodes which link (3k+2) have 3 different colors. The link 3k and (3k+1) are still fine since the colors are generated in a specific order Red-Blue-Green. Everything is still fine



Chromatic number of 3k circles

12 uncolored vertices on the bottom left can be colored by Red-Blue-Green respectively spread from left to right respectively. Specially, 3 vertices whose link (2) have 3 different colors. The vertices which link 3,4,5,6,7 do not conflict others on top



Chromatic number of 3k circles

➔ $\chi(G(3(K+1))) = 3$

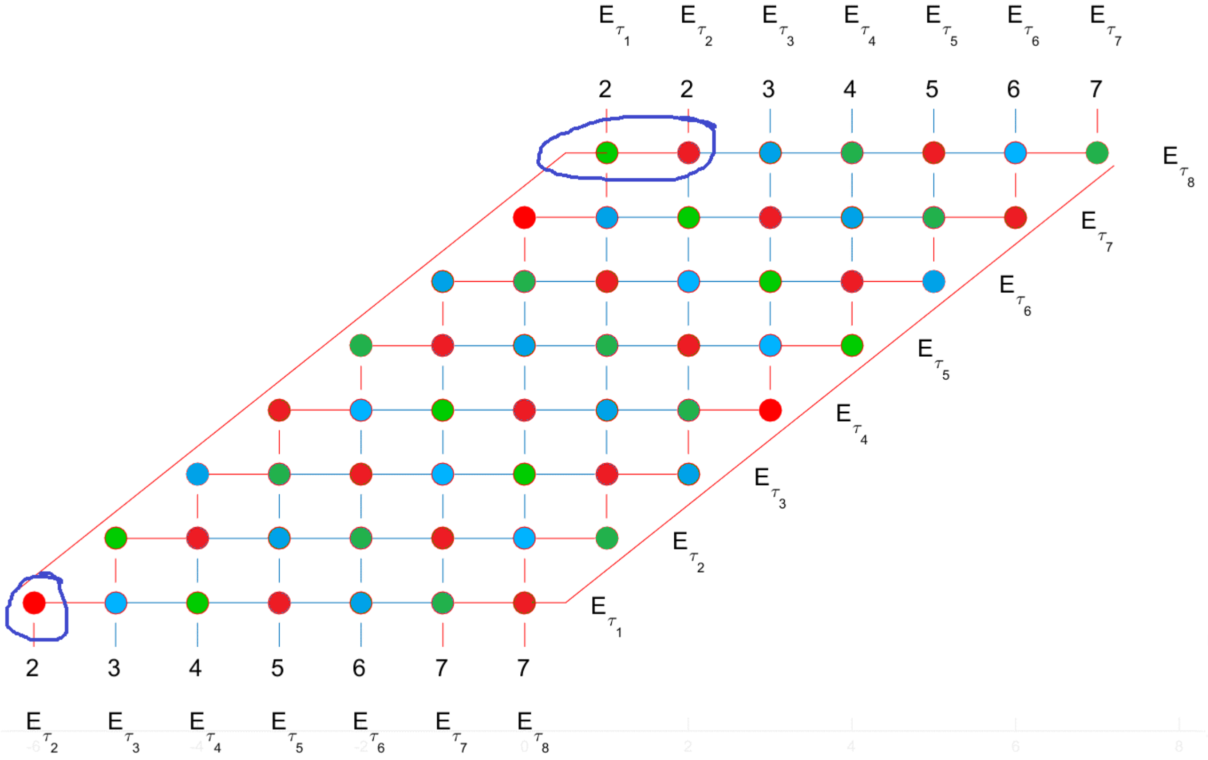
➔ The induction hypothesis is correct

➔ $\chi(G(3k)) = 3\,;\,(k > 0, k \in N)$

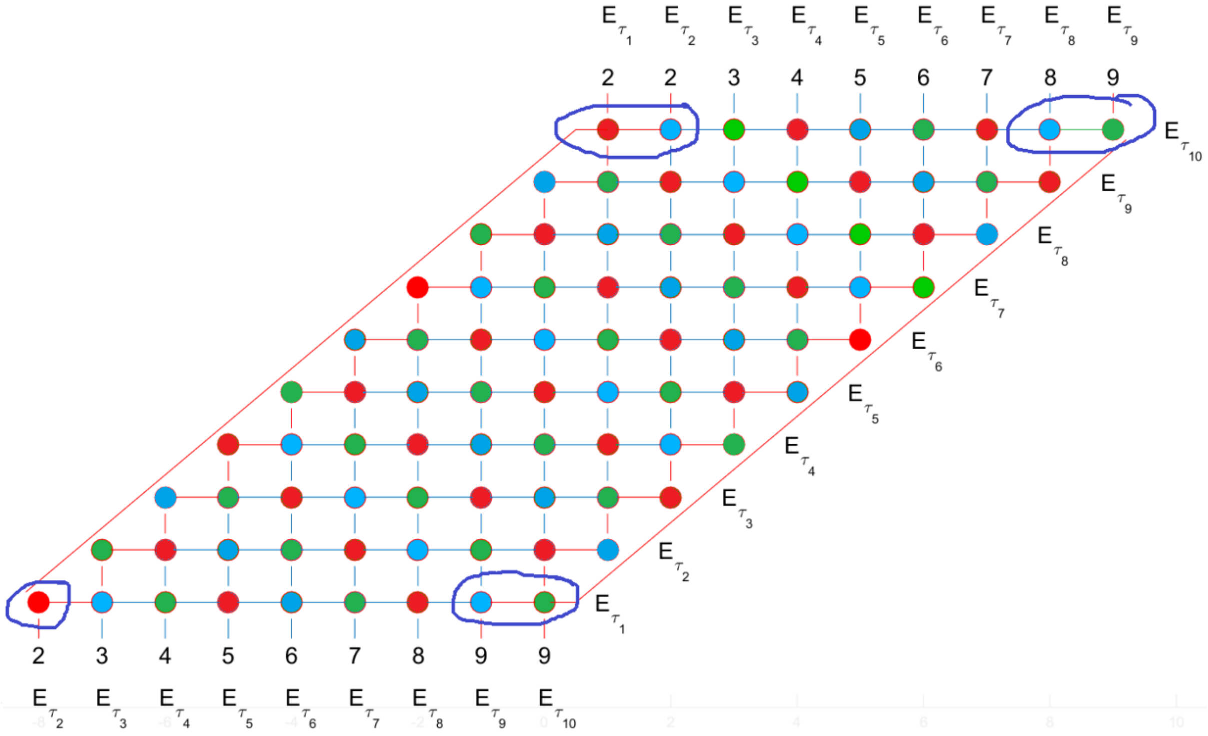
Chromatic number of 2k circles

Chromatic number of 2k circles

- With the graph 2k, I tried the same way to color the graph but it didn't work out



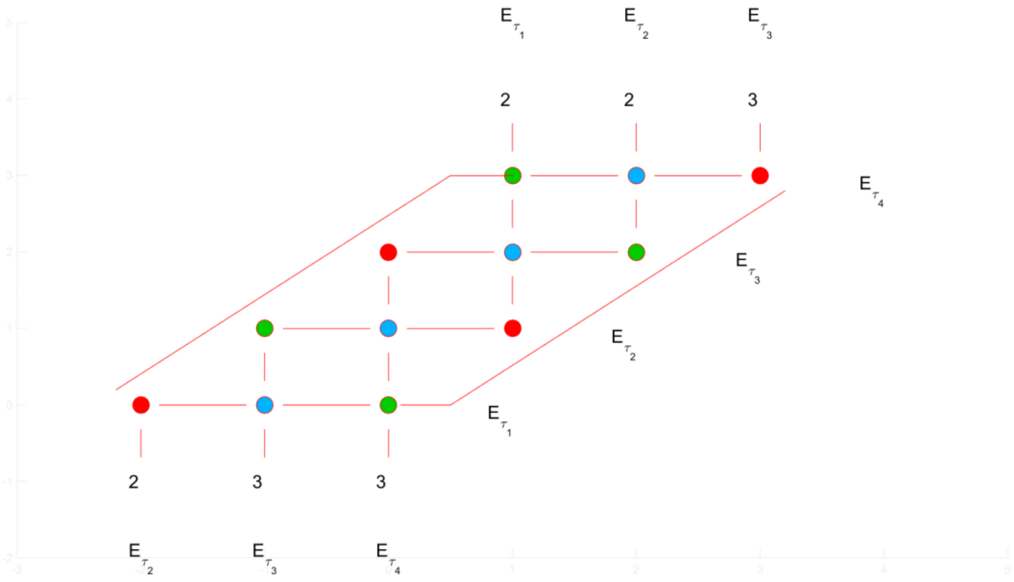
8 great circles



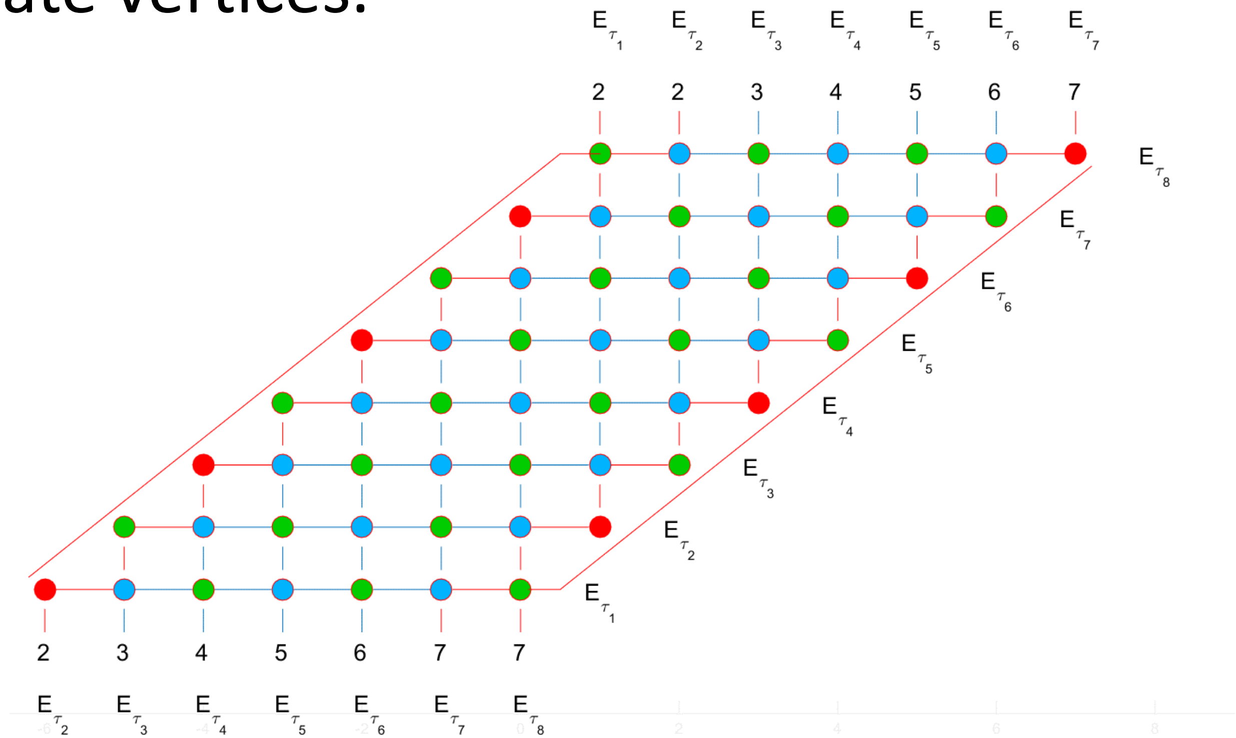
10 great circles

Chromatic number of 2k circles

- I tried to color it into a different way is to spread 2 colors Blue-Green from the bottom left to the top right of the equivalent graph and then put the color Red into the appropriate vertices.



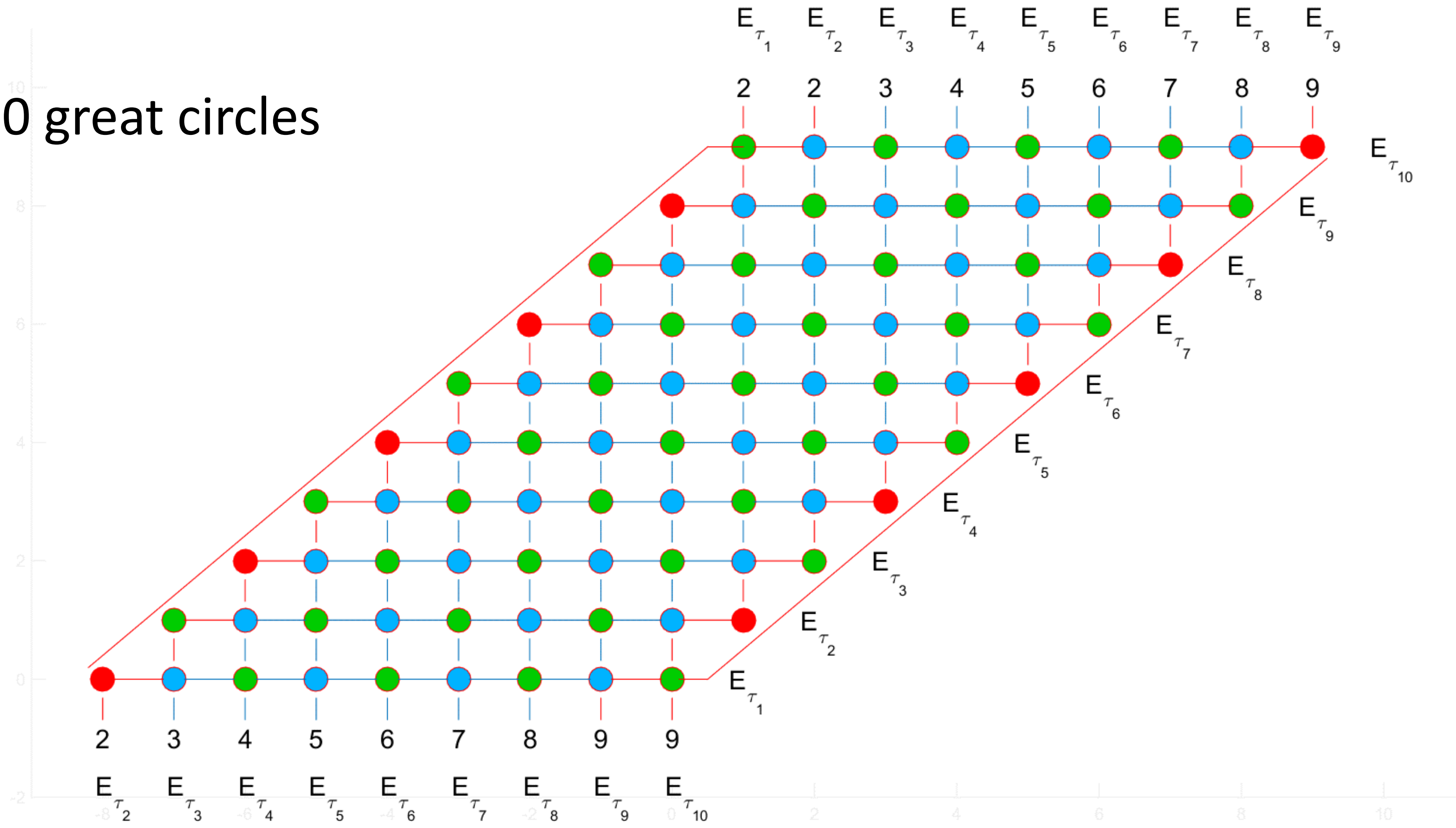
6 great circles



8 great circles

Chromatic number of 2k circles

- Base cases: 10 great circles



Chromatic number of 2k circles

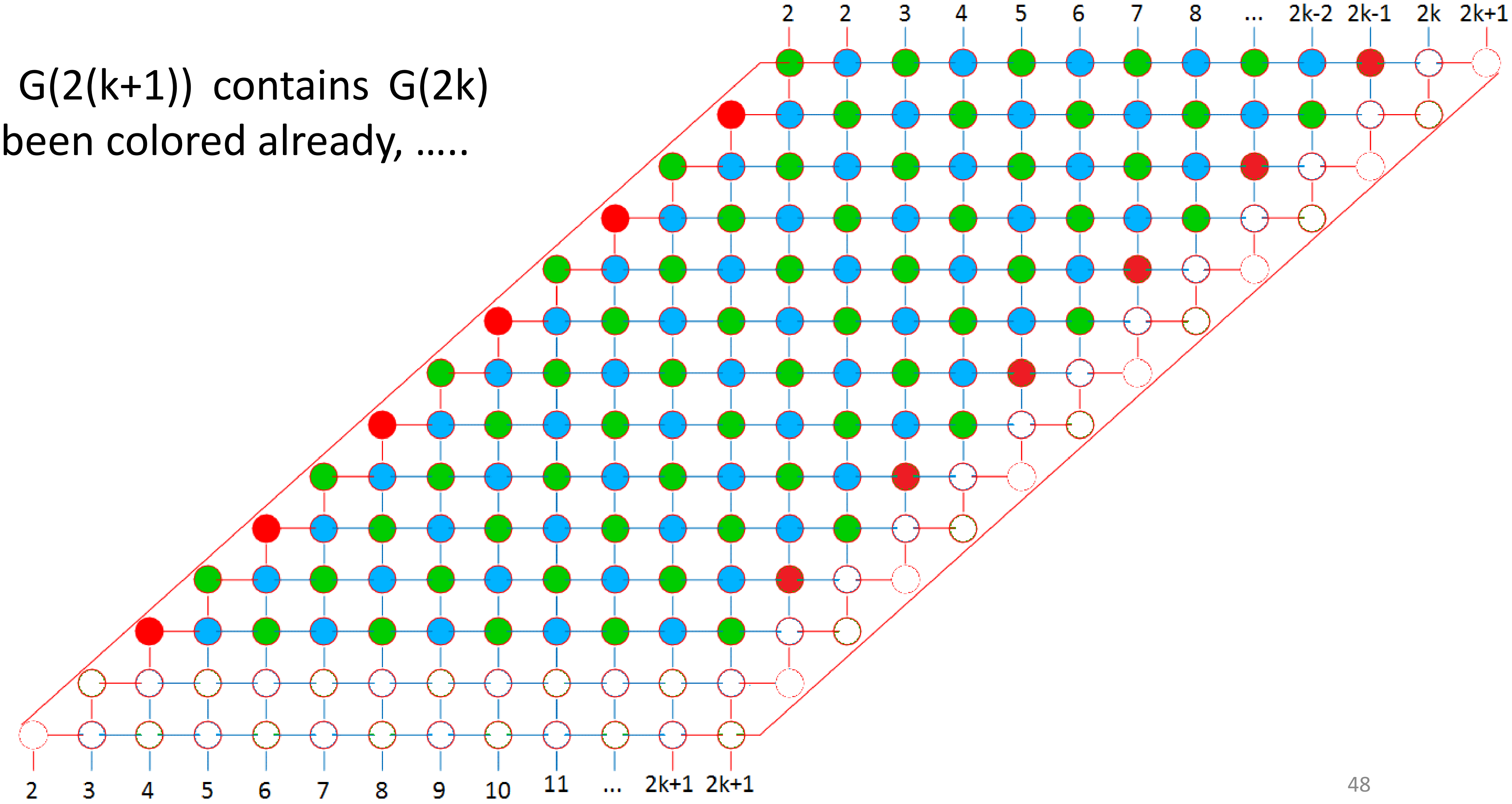
- According to the base cases with 4,8,10 great circles, the chromatic number is 3

➔Induction hypothesis: $\chi(G(2K)) = 3; (k > 1, k \in N)$ that has been correct with k=2,4,5 by spreading 2 colors Blue-Green from the bottom left to the top right of the equivalent graph and then putting the color Red into the appropriate vertices.

➔Induction step: Prove that $\chi(G(2(K+1))) = 3$

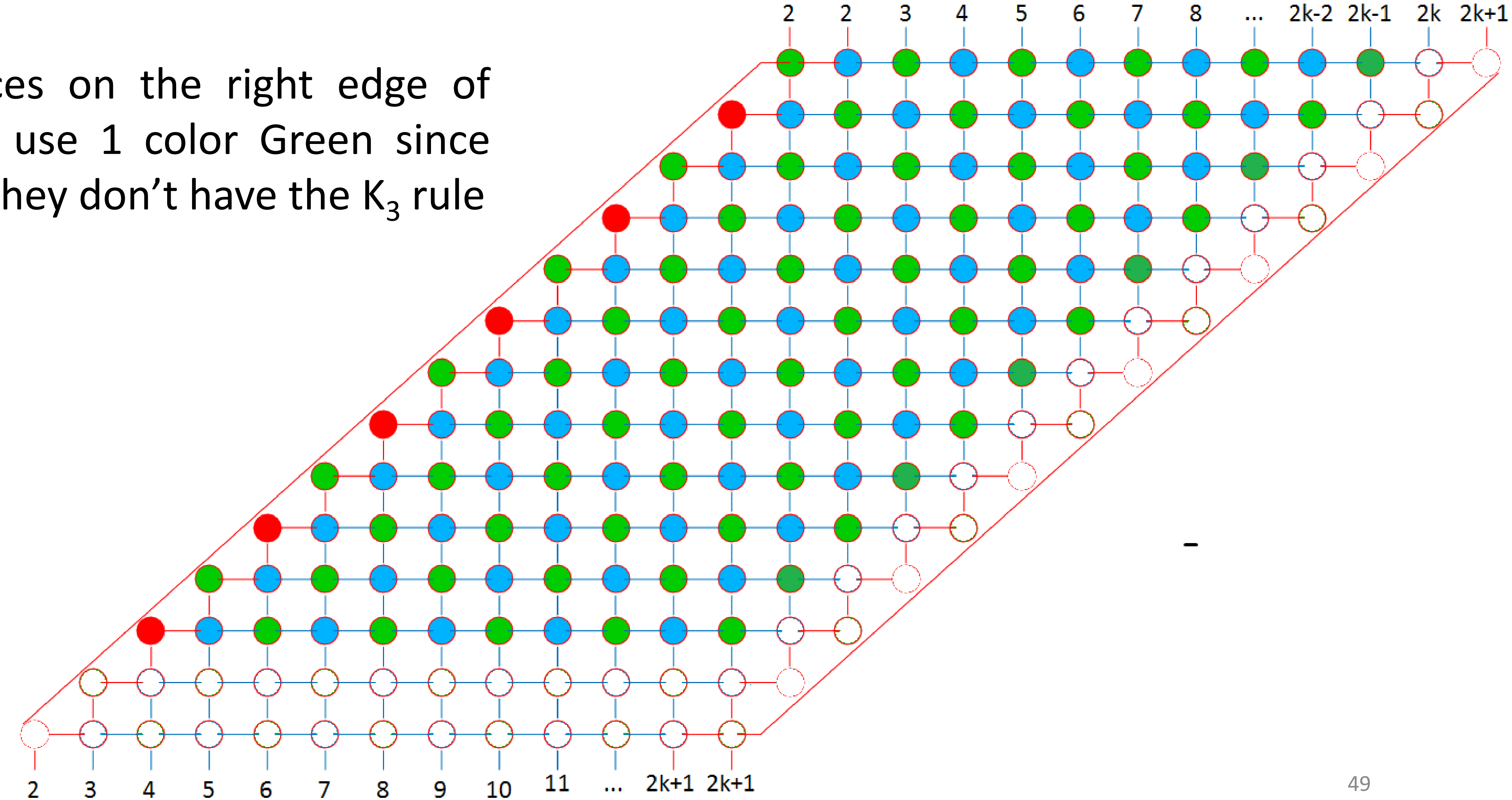
Chromatic number of 2k circles

The graph G(2(k+1)) contains G(2k) which has been colored already, …..



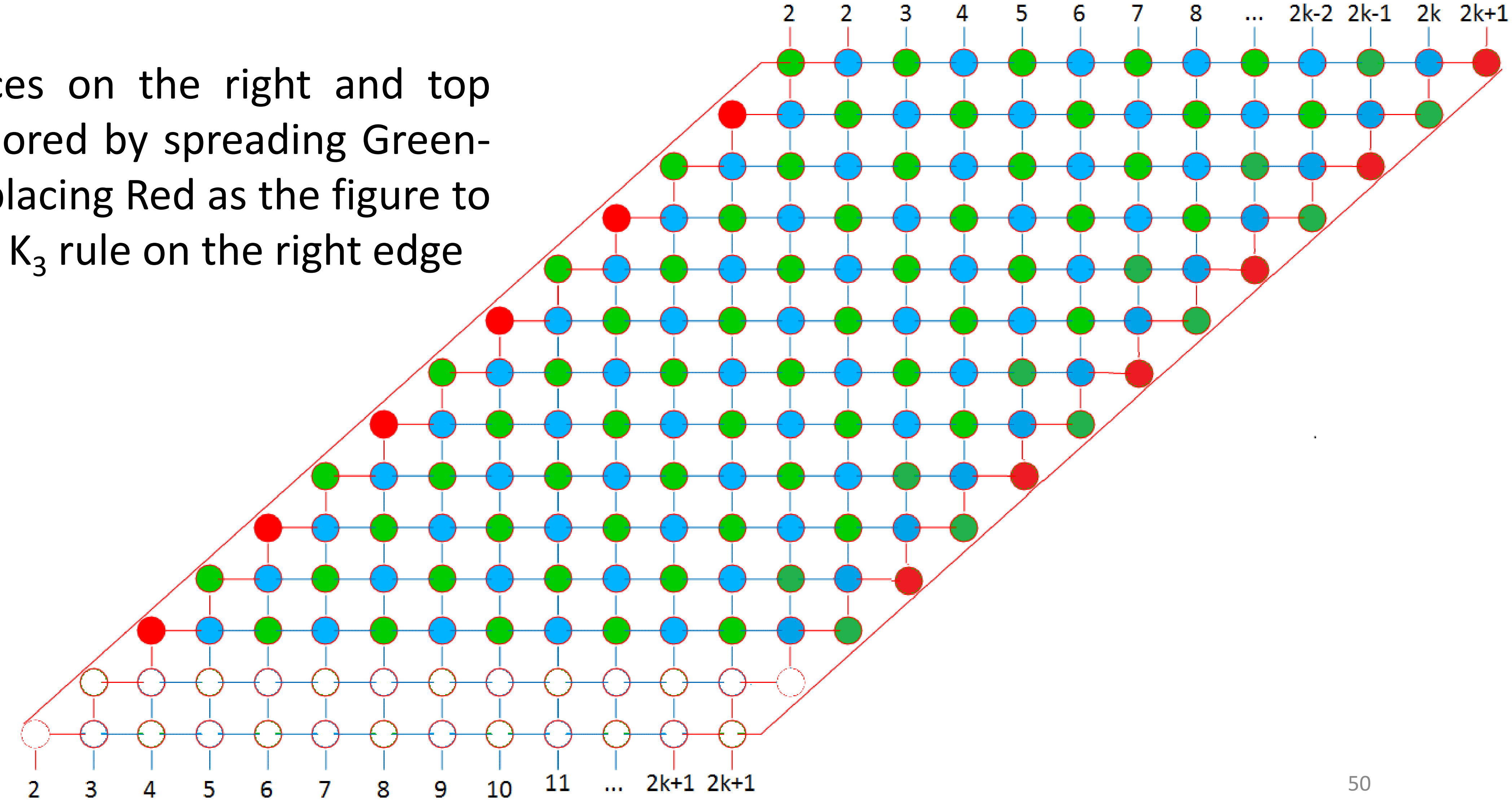
Chromatic number of 2k circles

The vertices on the right edge of G(2k) can use 1 color Green since currently they don't have the K_3 rule



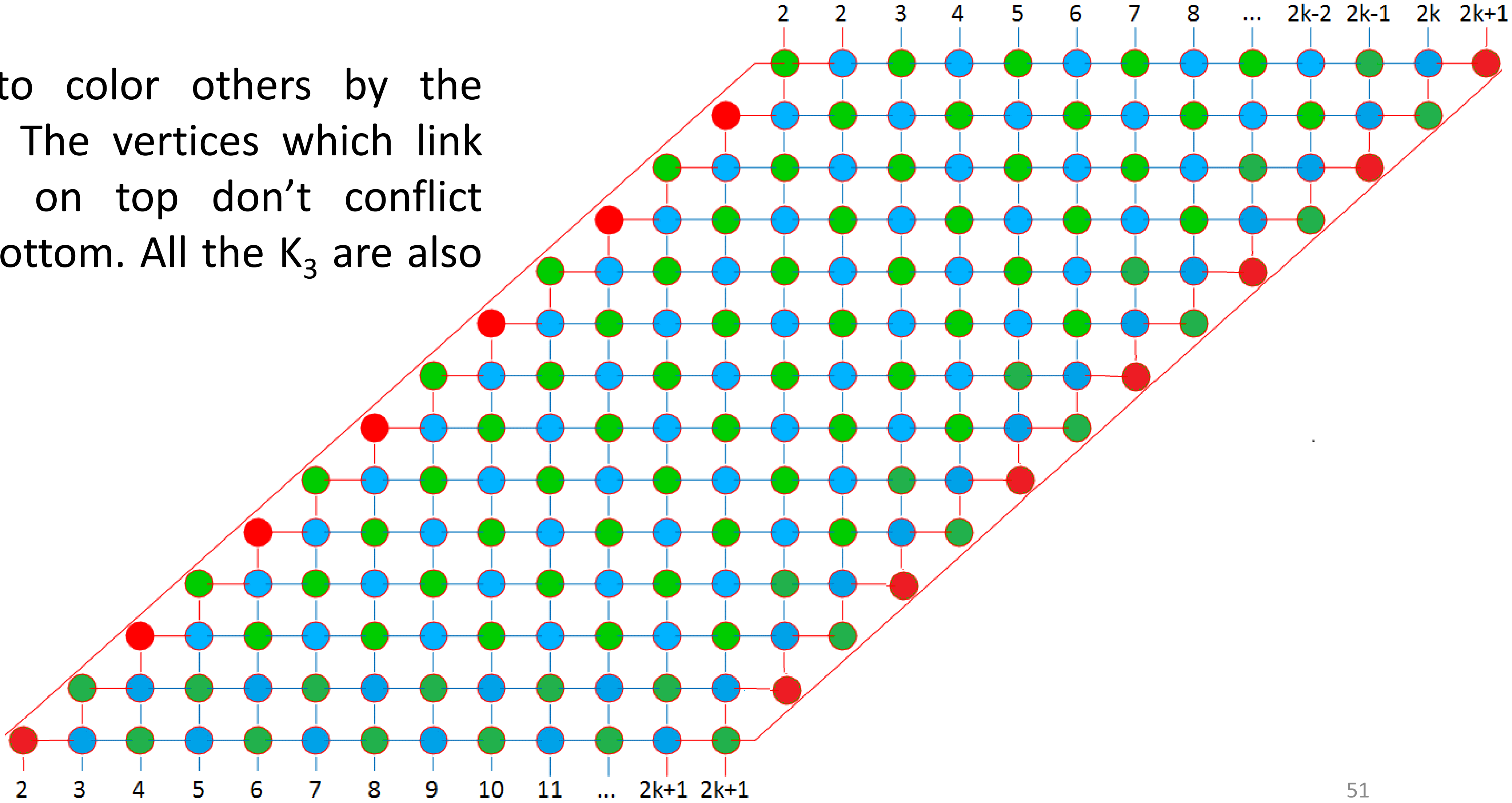
Chromatic number of 2k circles

The vertices on the right and top can be colored by spreading Green-Blue and placing Red as the figure to satisfy the K_3 rule on the right edge



Chromatic number of 2k circles

Continue to color others by the same way. The vertices which link (3,4,5,…2k) on top don't conflict those on bottom. All the K_3 are also satisfied



Chromatic number of 2k circles

➔ $\chi(G(2(K+1))) = 3$

➔ The induction hypothesis is correct

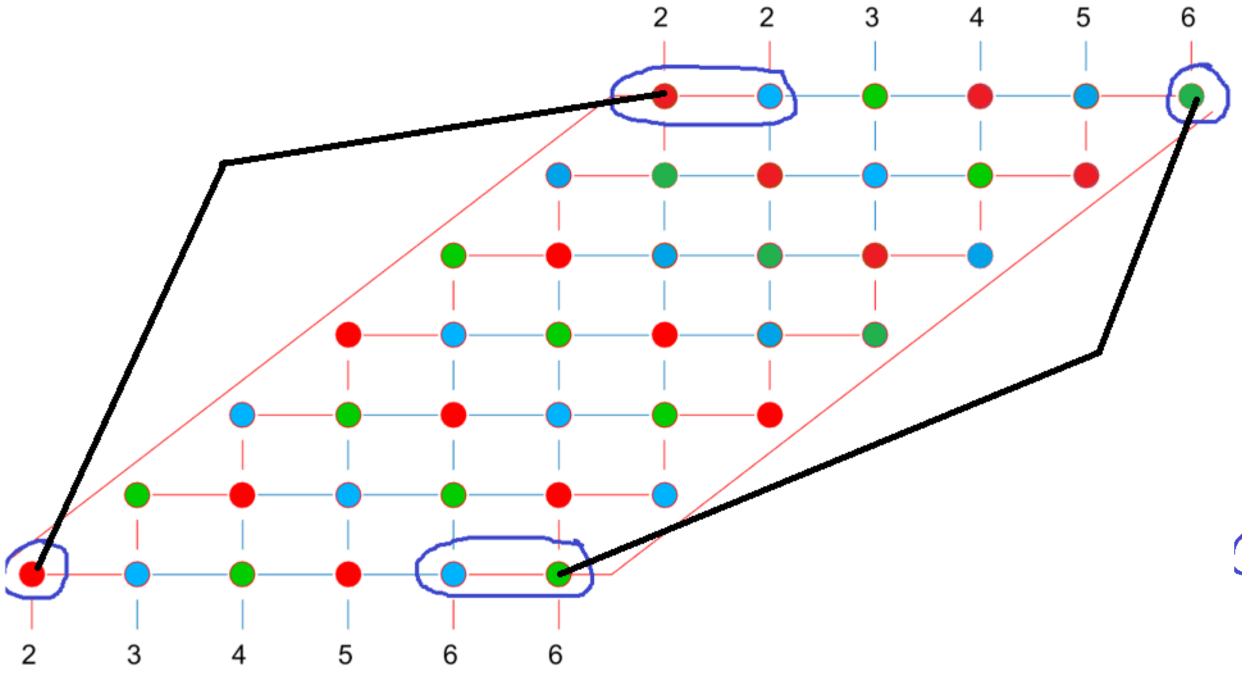
➔ $\chi(G(2k)) = 3\,; (k > 1, k \in N)$

Chromatic number of (6k+1) circles

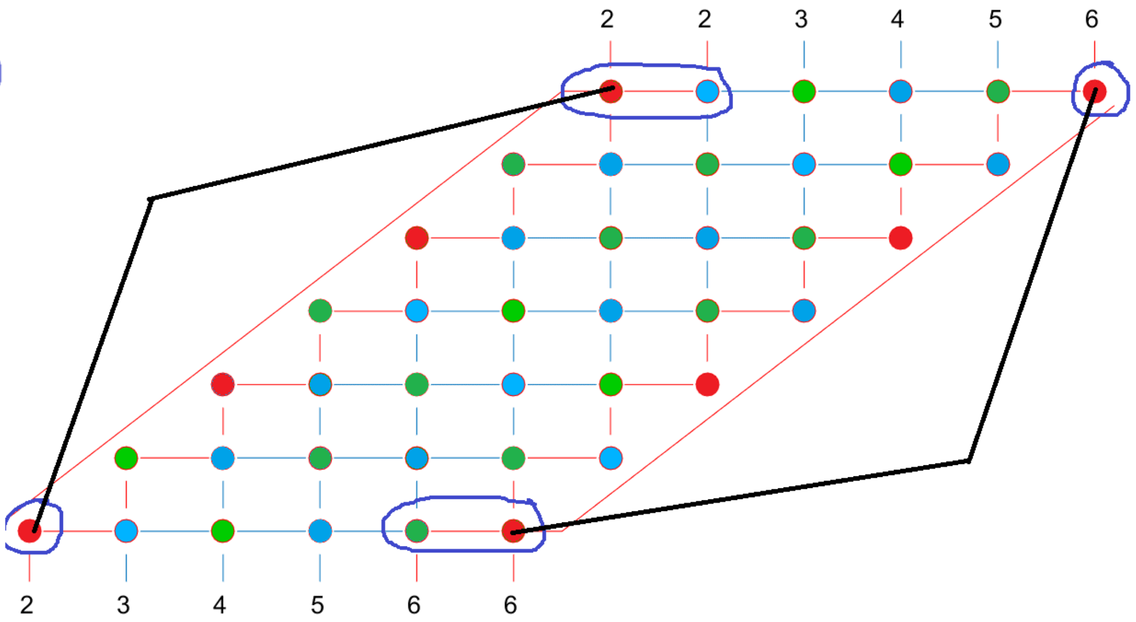
Chromatic number of (6k+1) circles

This type of graph doesn't suit 2 previous coloring ways because:

- 3k: There is always a K_3 (2) that contains 2 vertices have the same color because $(6k + 1) \equiv 1(mod\ 3)$
- 2k: $(6k + 1) \equiv 1(mod\ 2)$, so K_3 (contains link 2) always has 2 vertices have the same color



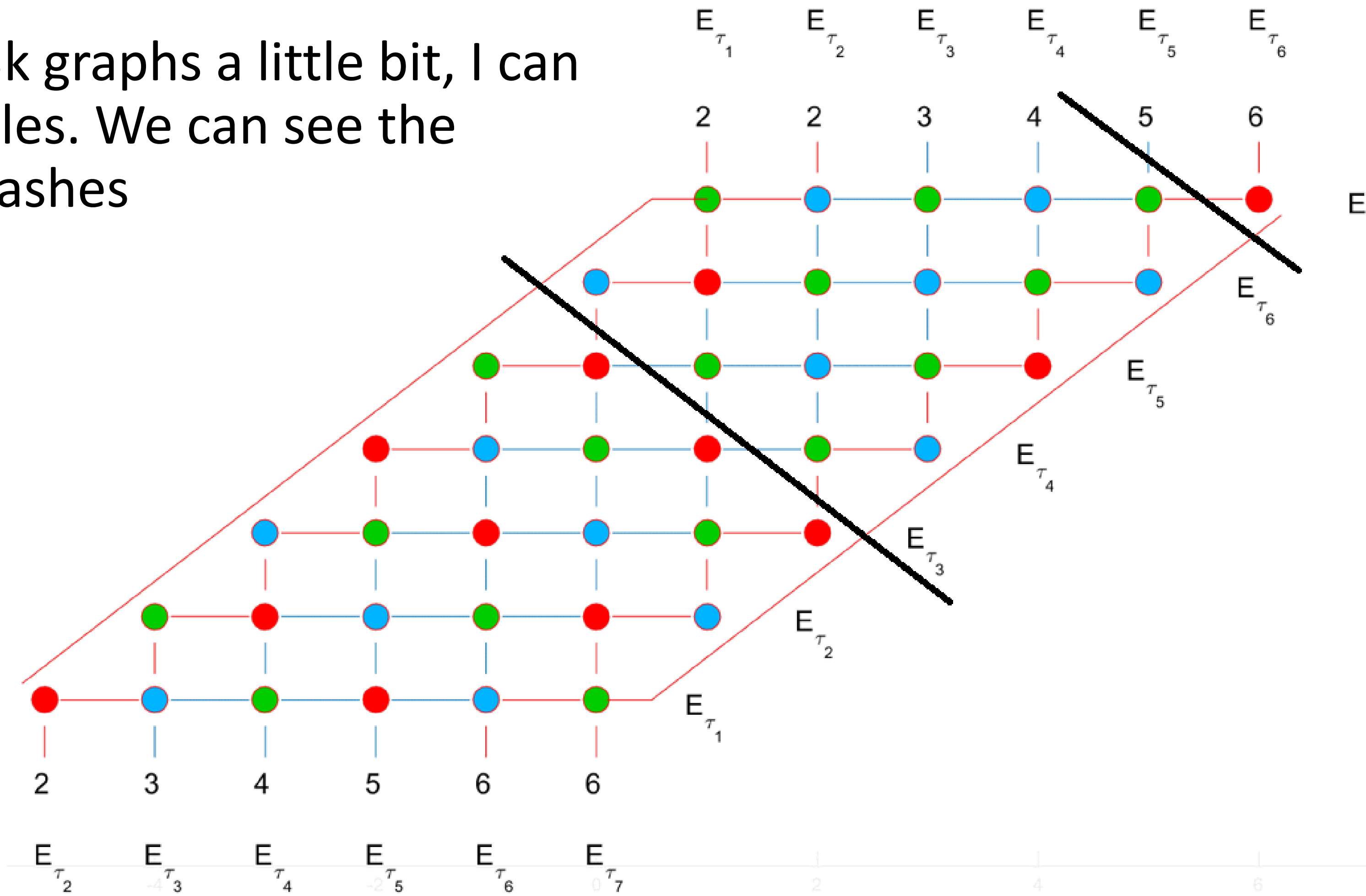
Color the graph by the way used on 3k graphs



Color the graph by the way used on 2k graphs

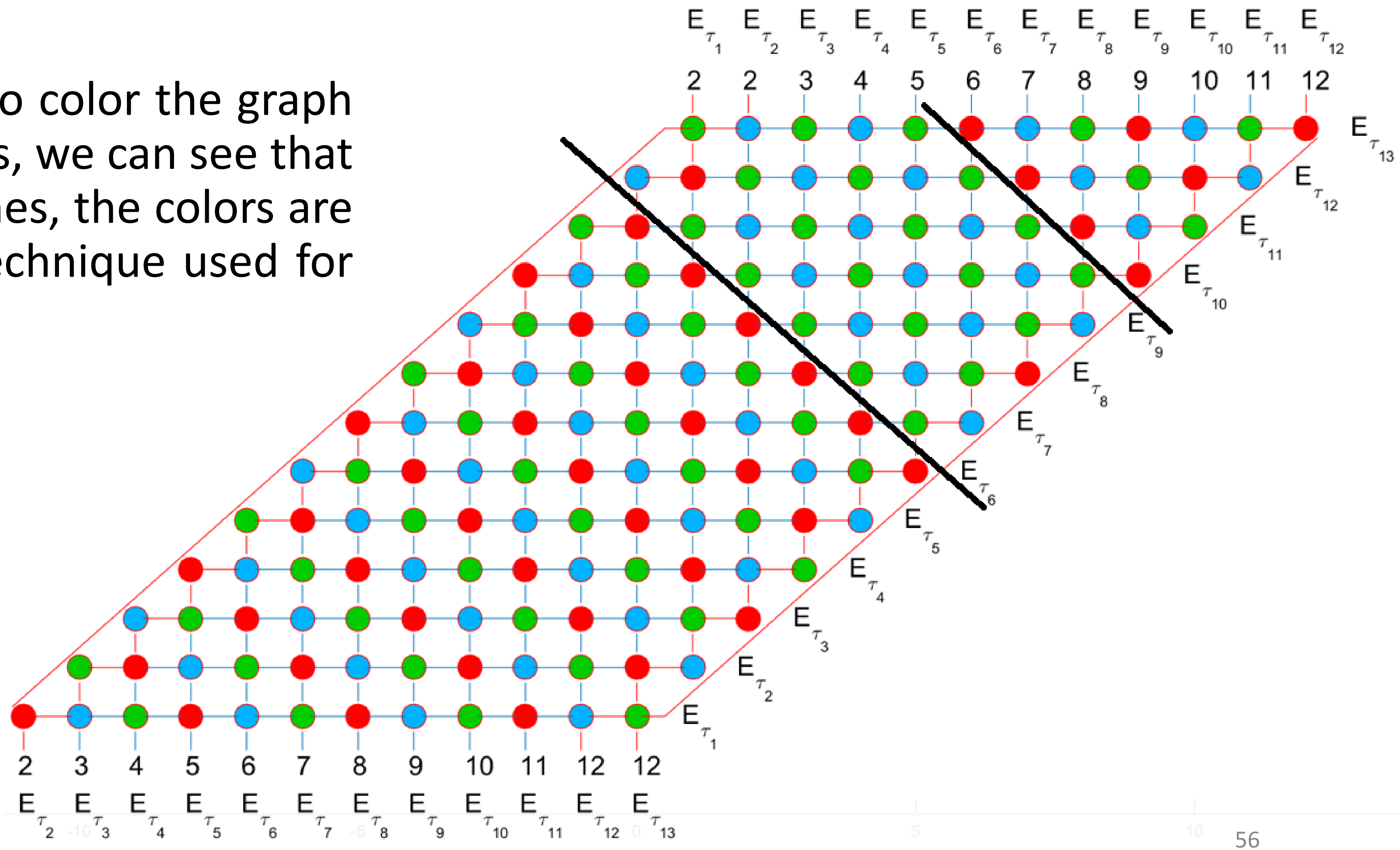
Chromatic number of (6k+1) circles

- By modifying the technique of 3k graphs a little bit, I can color the graph with 7 great circles. We can see the difference in between 2 black slashes



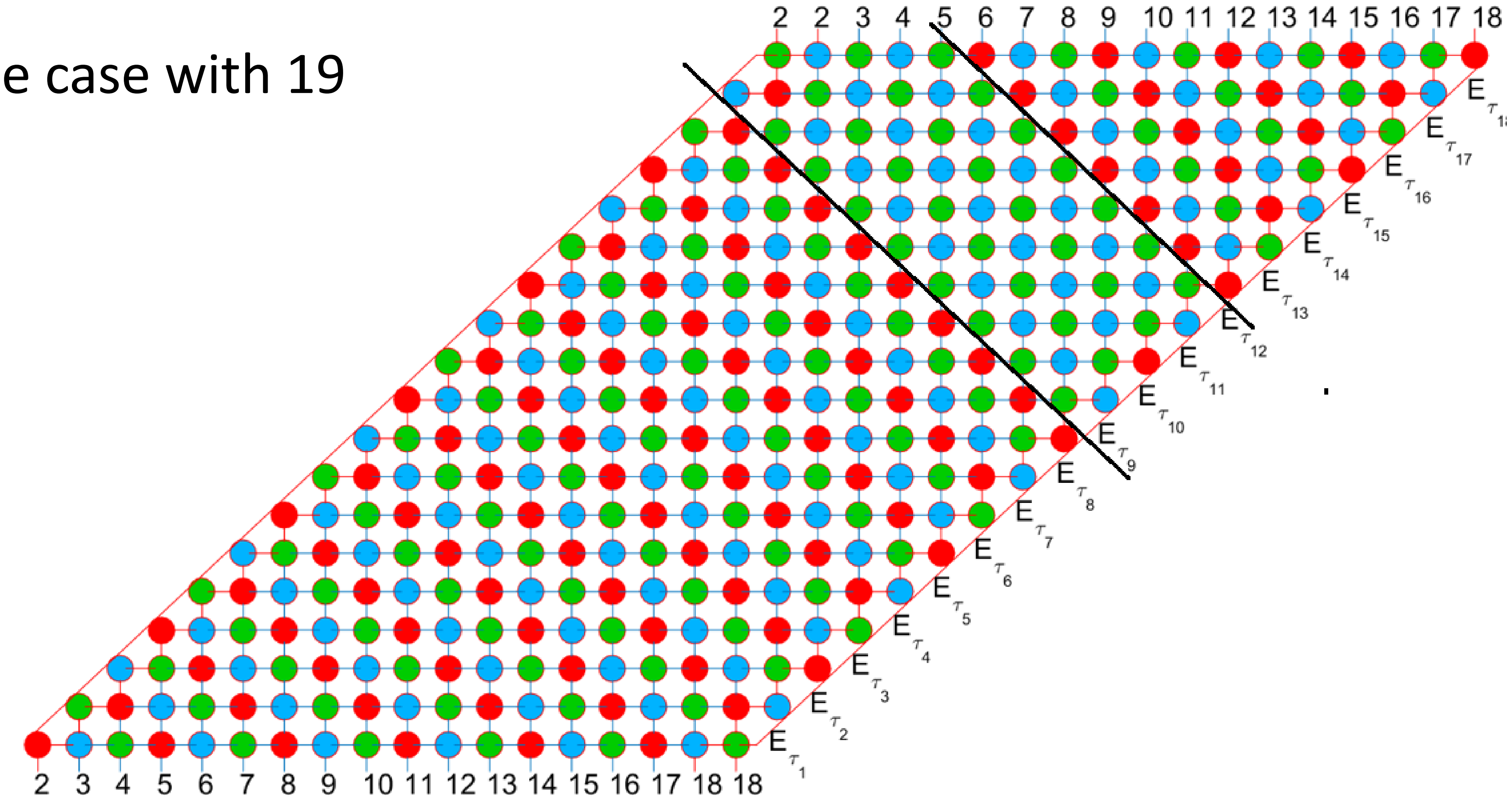
Chromatic number of (6k+1) circles

- Greedily continue to color the graph with 13 great circles, we can see that outside of the slashes, the colors are similar to the 1st technique used for 3k graphs



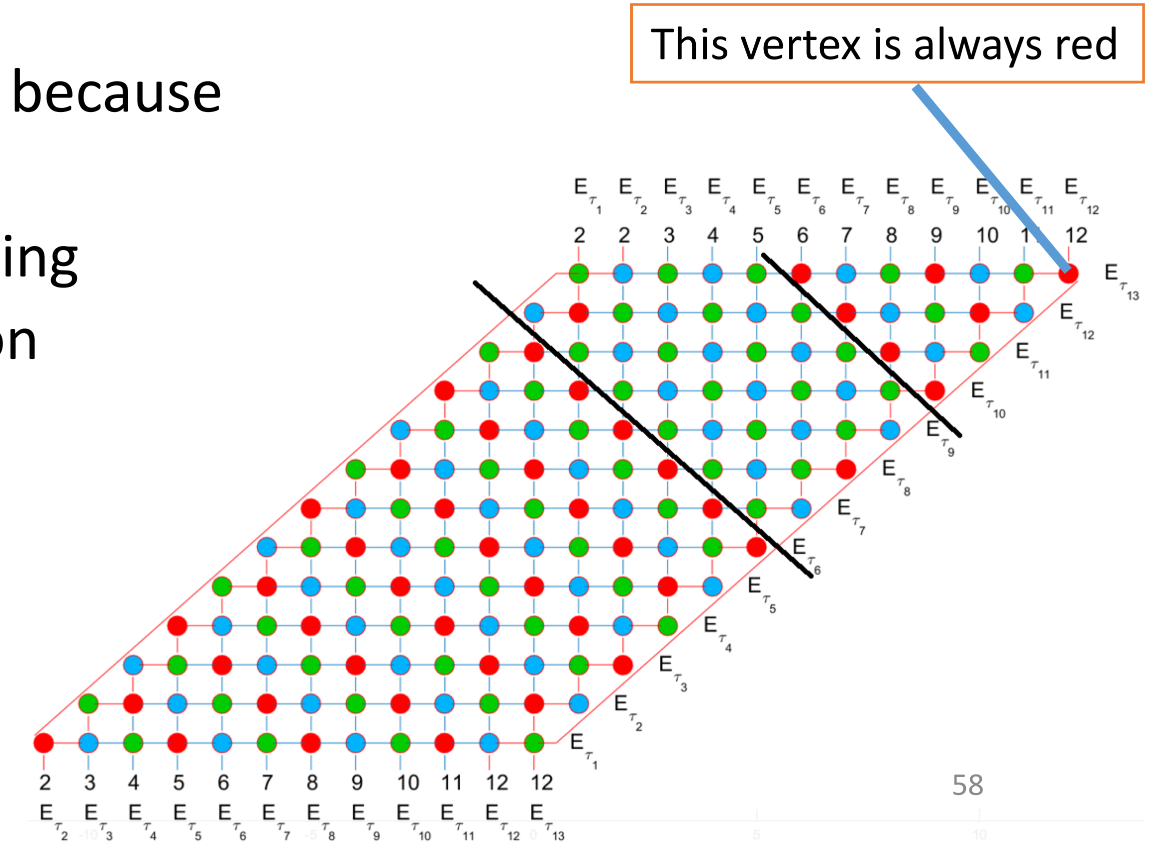
Chromatic number of (6k+1) circles

- Another base case with 19 great circles



Chromatic number of (6k+1) circles

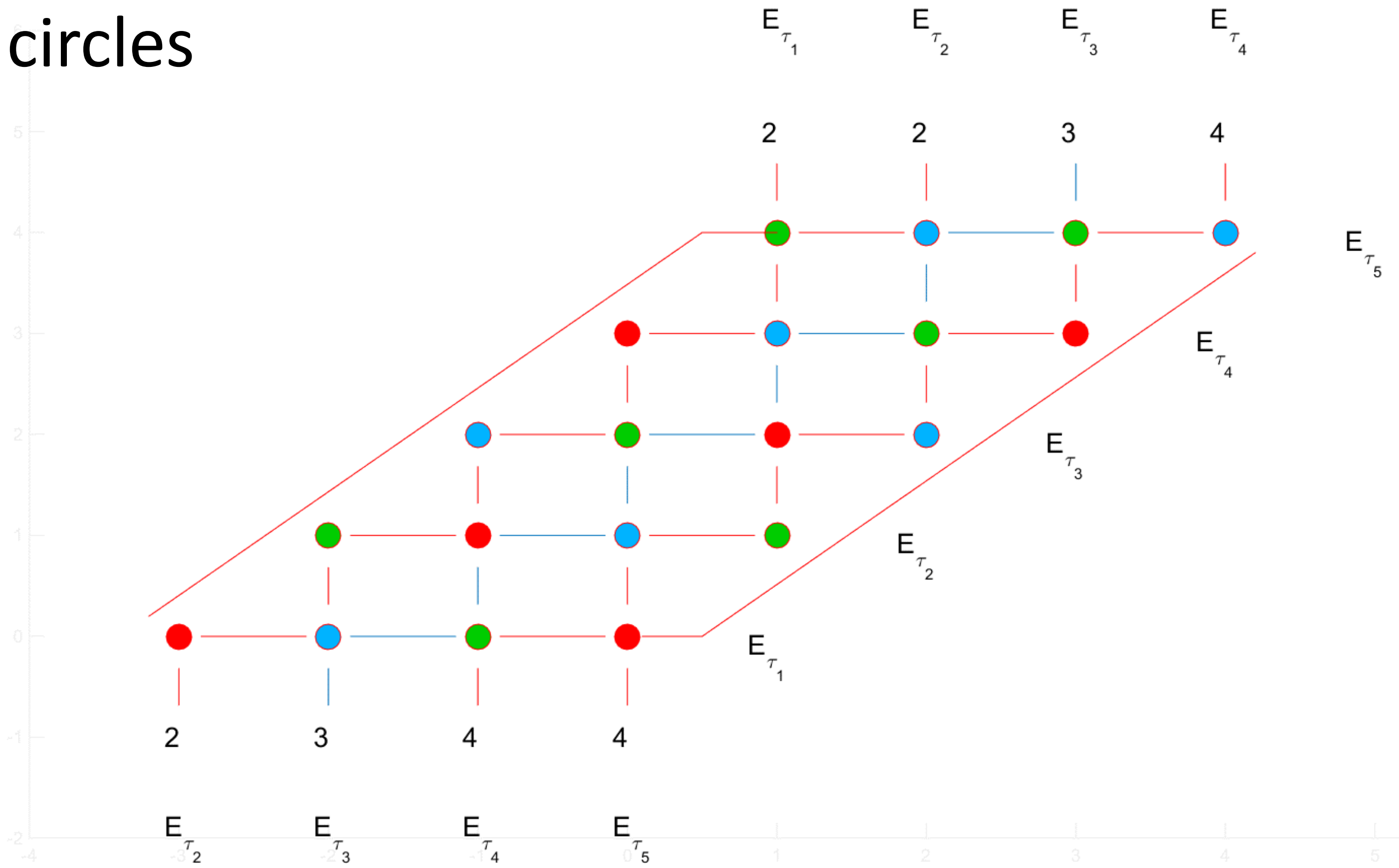
- With k>2, by spreading RED-BLUE-GREEN respectively on the additional diagonals, I can color all this type of graph based on the case k=2
  - The vertex on the top right is always left because step $6k \equiv 0(mod3)$
  - The other additional vertices use spreading technique which doesn't create conflicts on the links at edges
  - Similar to the coloring way of 3k graphs, all additional K_3 are also satisfied



Chromatic number of (6k+5) circles

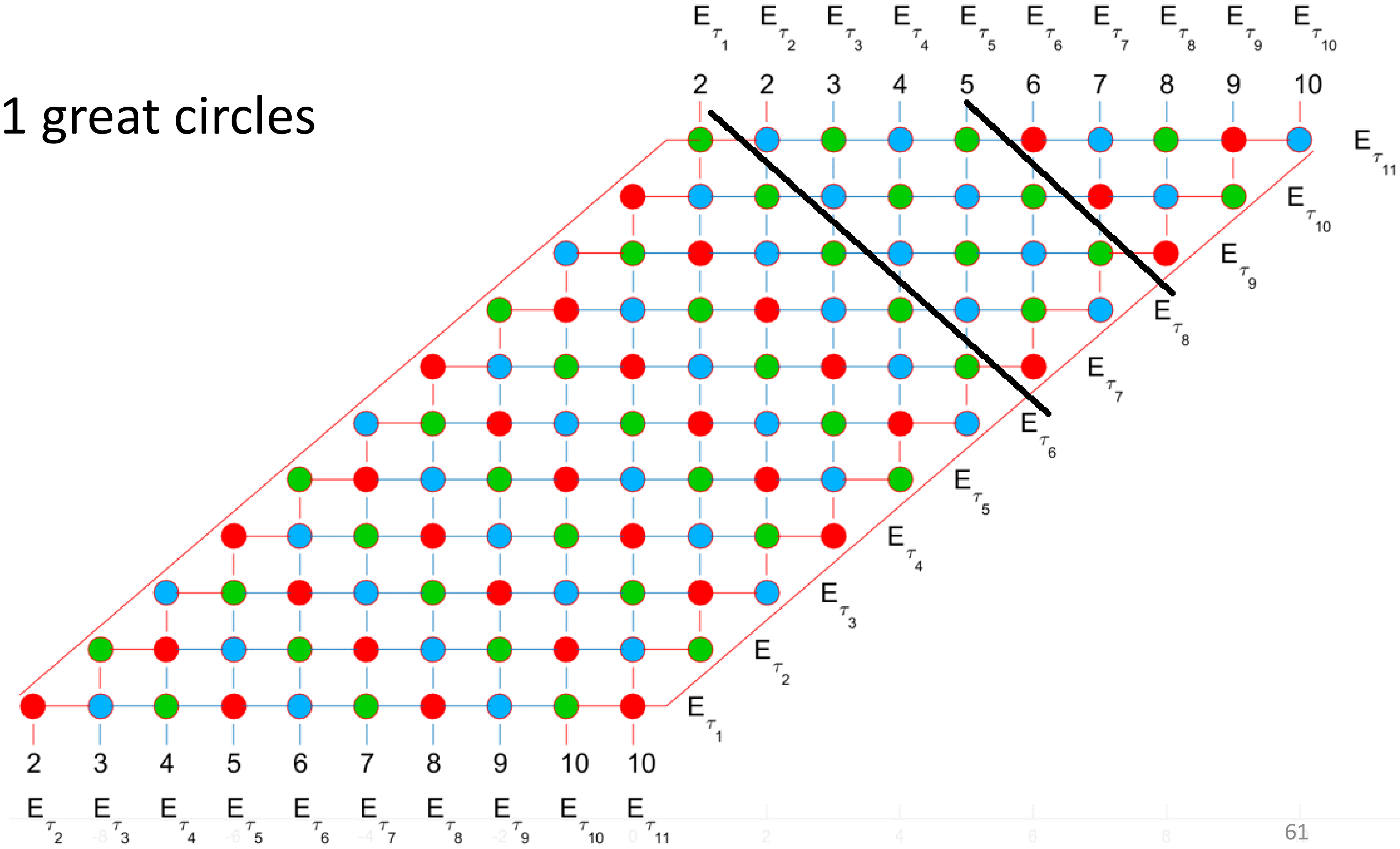
Chromatic number of (6k+5) circles

- Base cases: 5 great circles



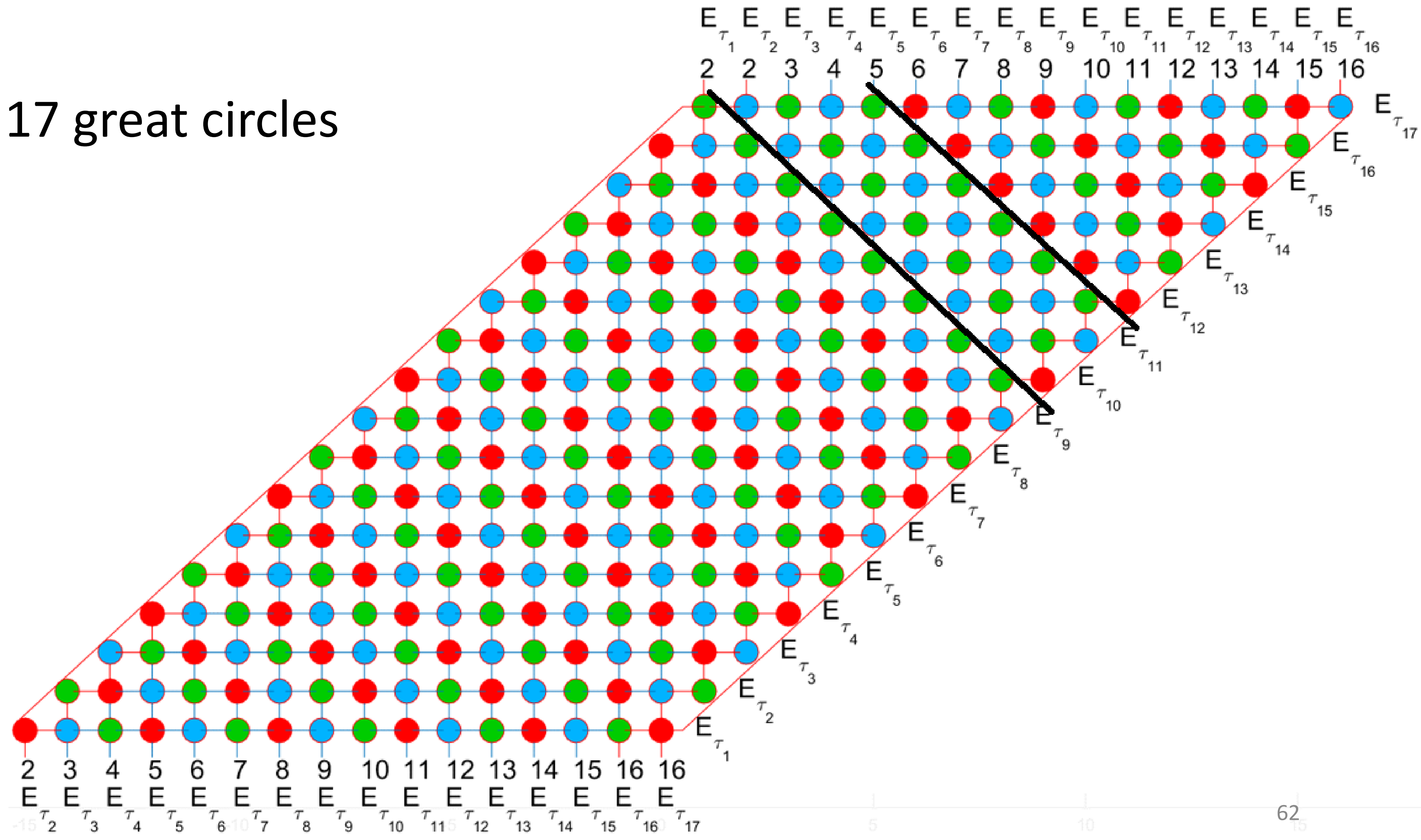
Chromatic number of (6k+5) circles

- Base cases: 11 great circles



Chromatic number of (6k+5) circles

- Base cases: 17 great circles



Chromatic number of (6k+5) circles

- This type of graph is similar to (6k+1) graph. With cases k>1, by spreading RED-BLUE-GREEN respectively on the diagonals, I can color all this type of graph based on the case k=1

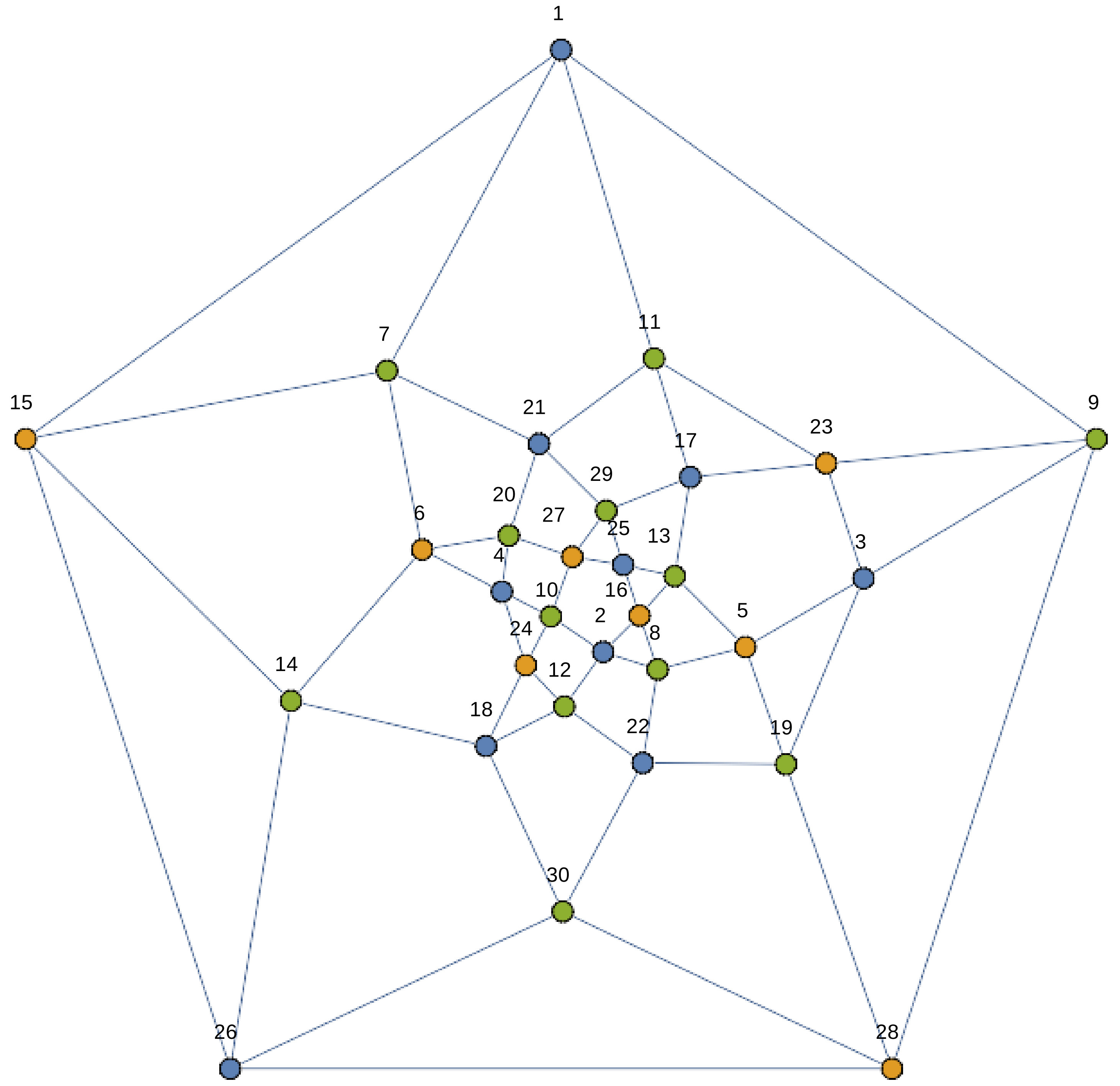
Generated 1000 great circles of 5, 6, 7, 8 great circles

Non-isomorphic graphs

- I generated **1000** graphs for each 6,7,8 great circles.
  - **6** great circles: have **4** non-isomorphic graphs.
  - **7** great circles: have **11** non-isomorphic graphs.
  - **8** great circles: have **114** non-isomorphic graphs.

- These graphs might be helpful for us to see how they have been changed.

Non-isomorphic graphs made by 6 great circles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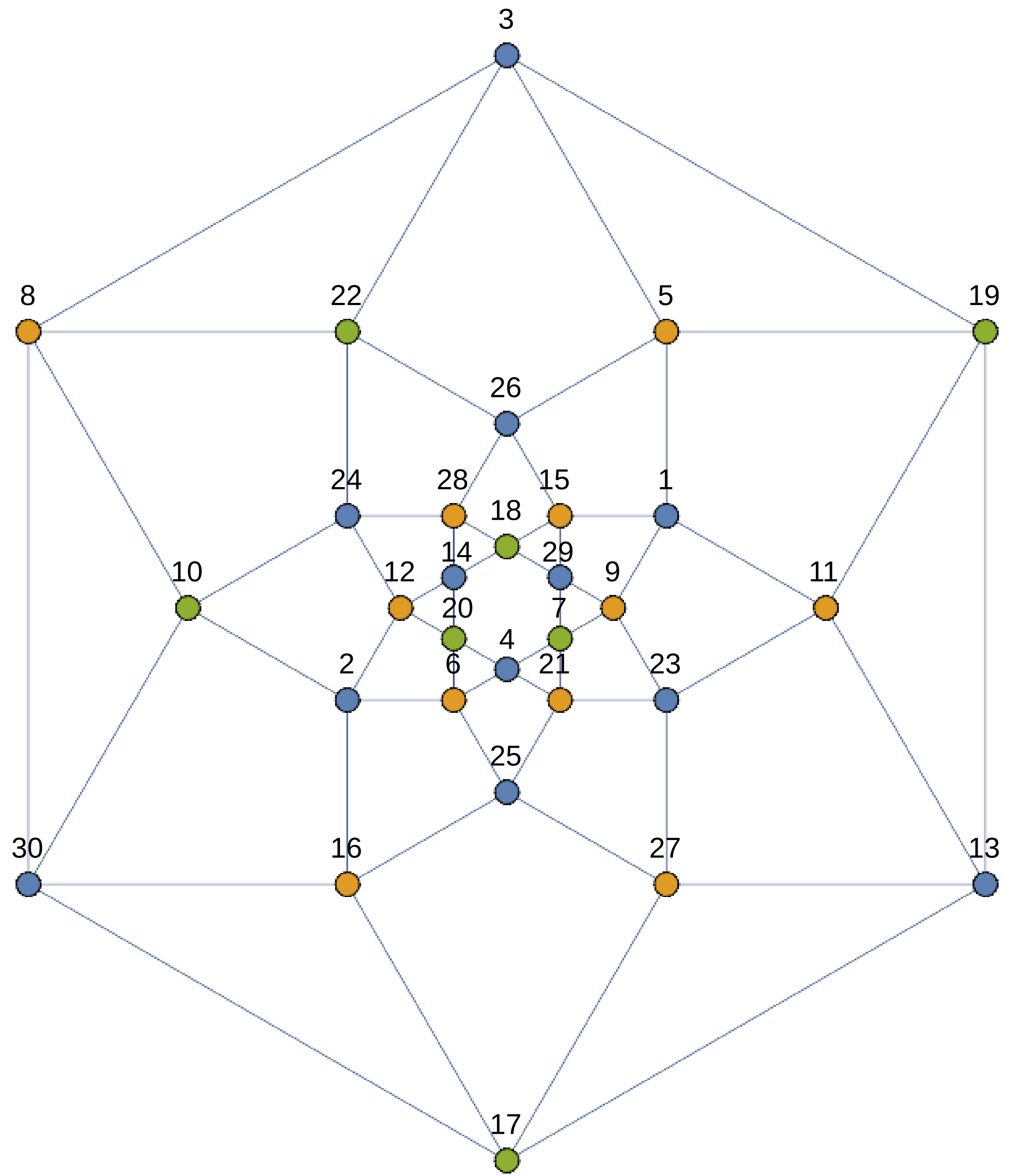
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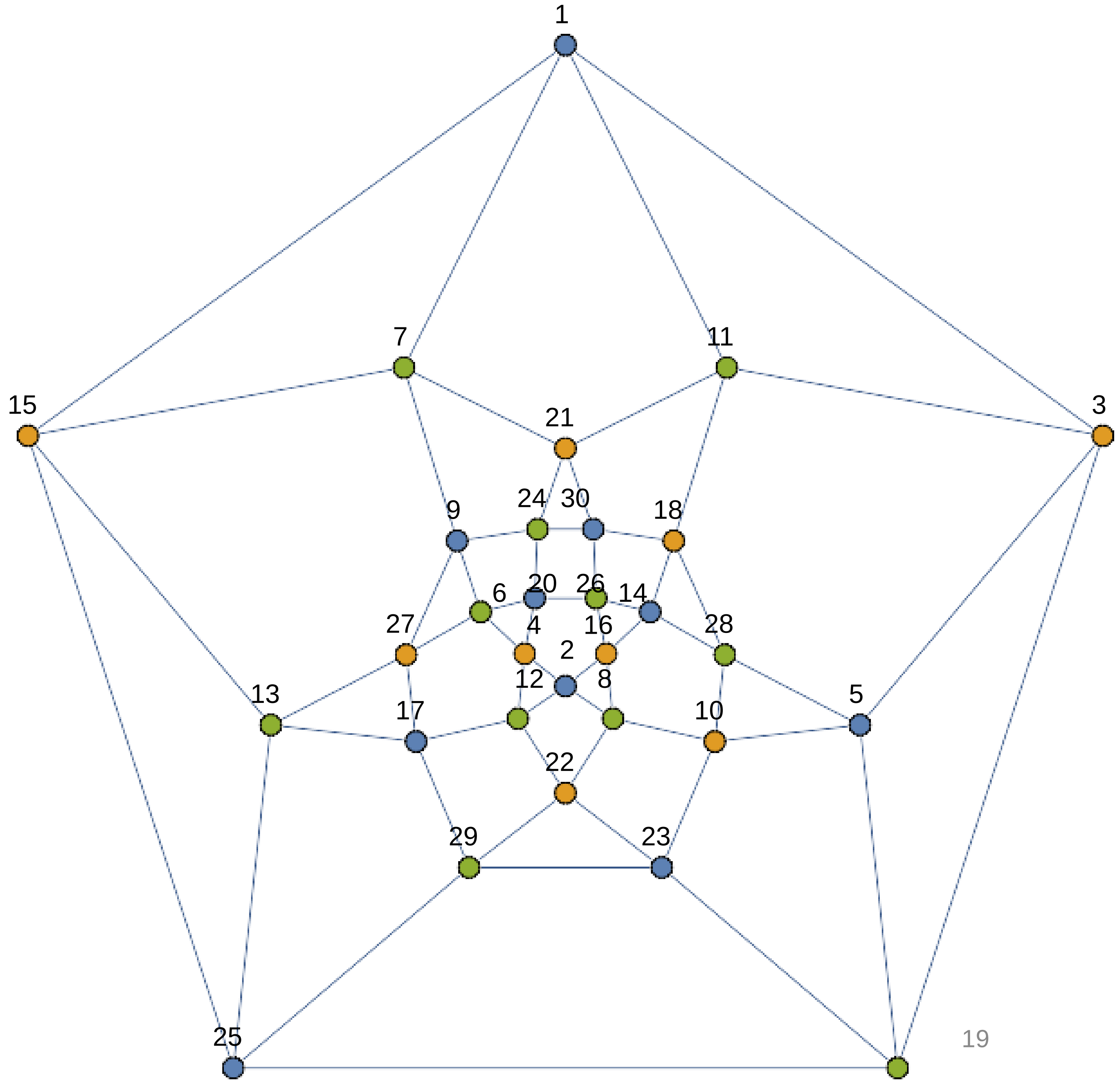
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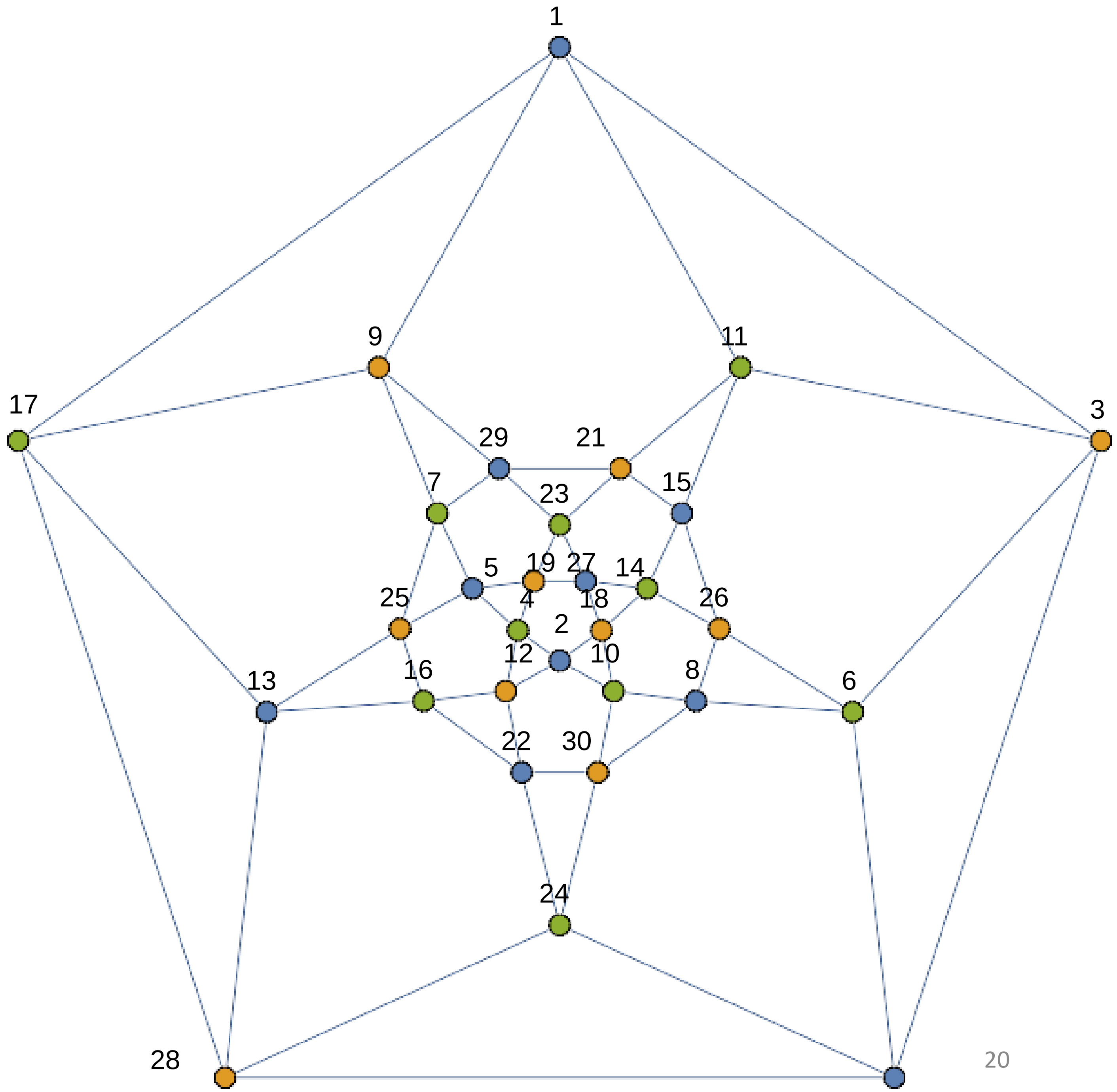
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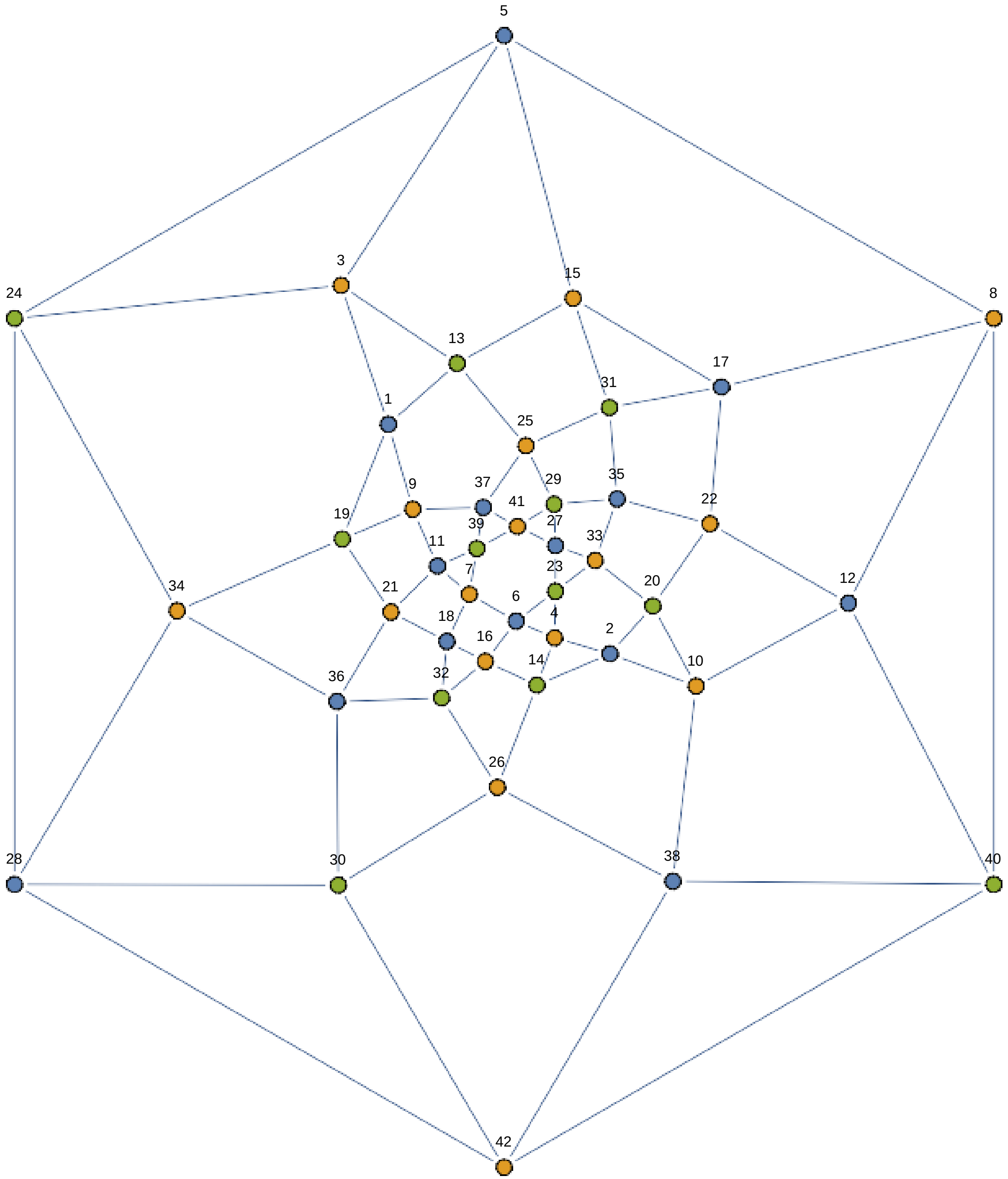
Non-isomorphic graphs made by 7 great circles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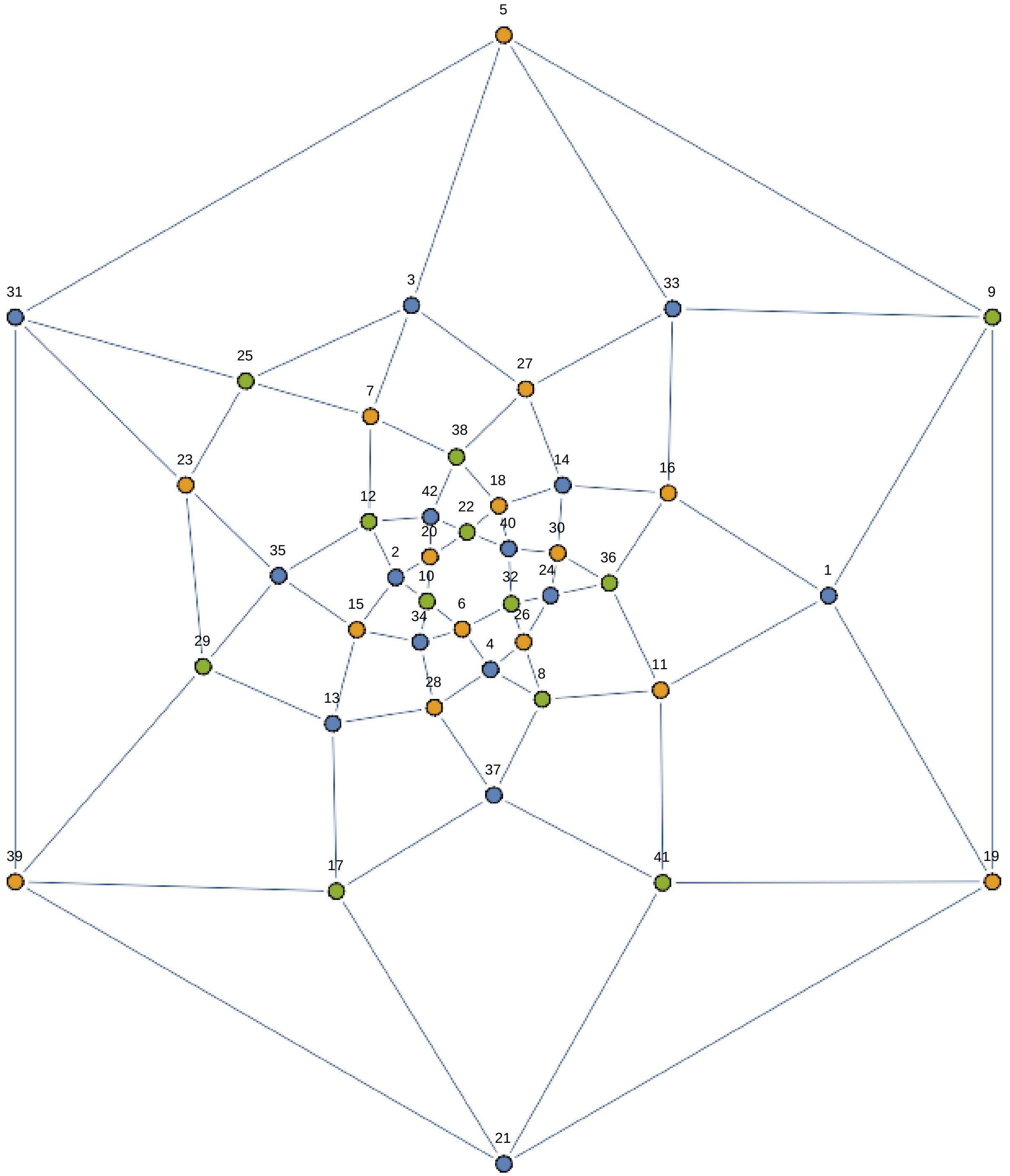
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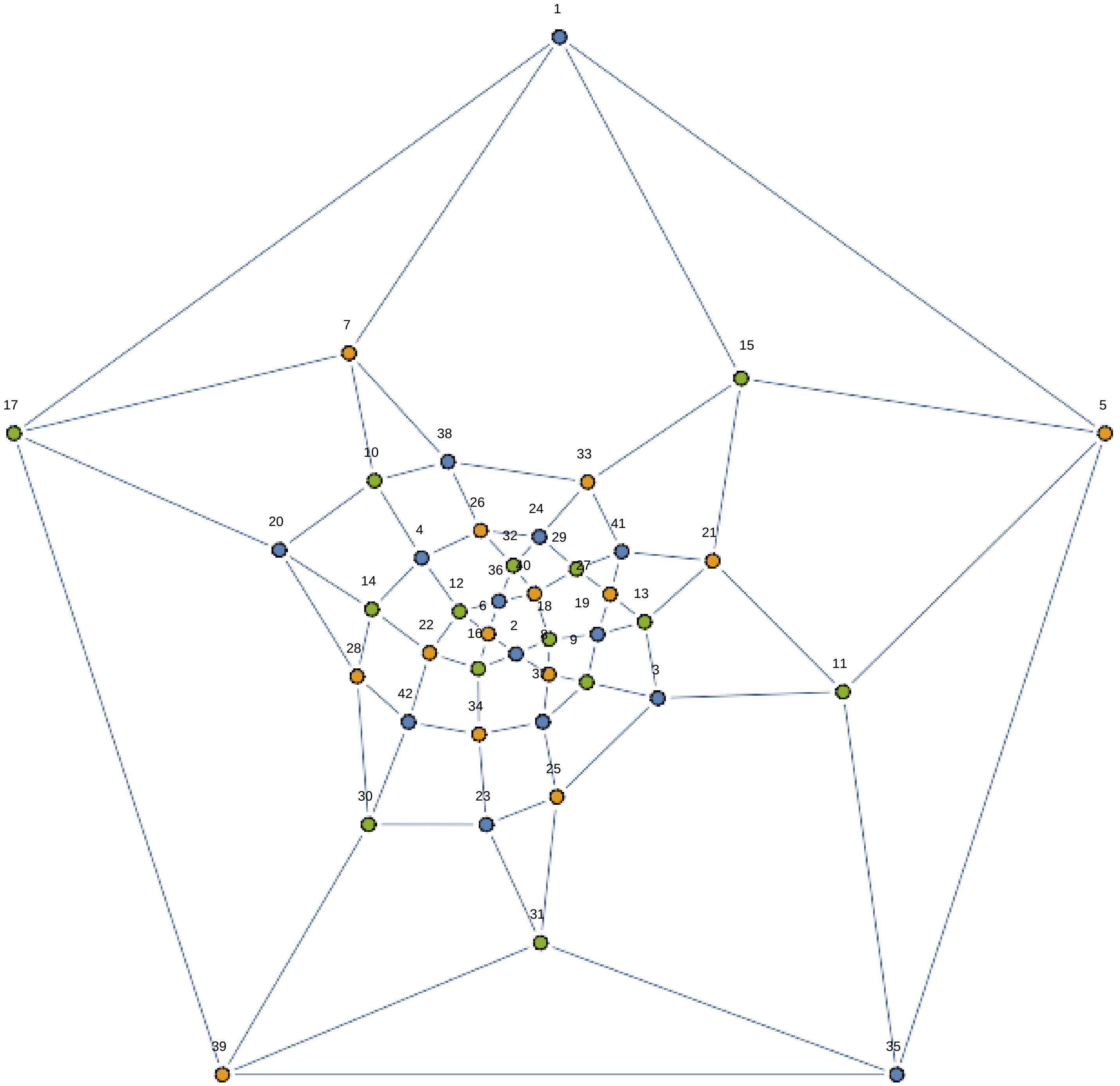

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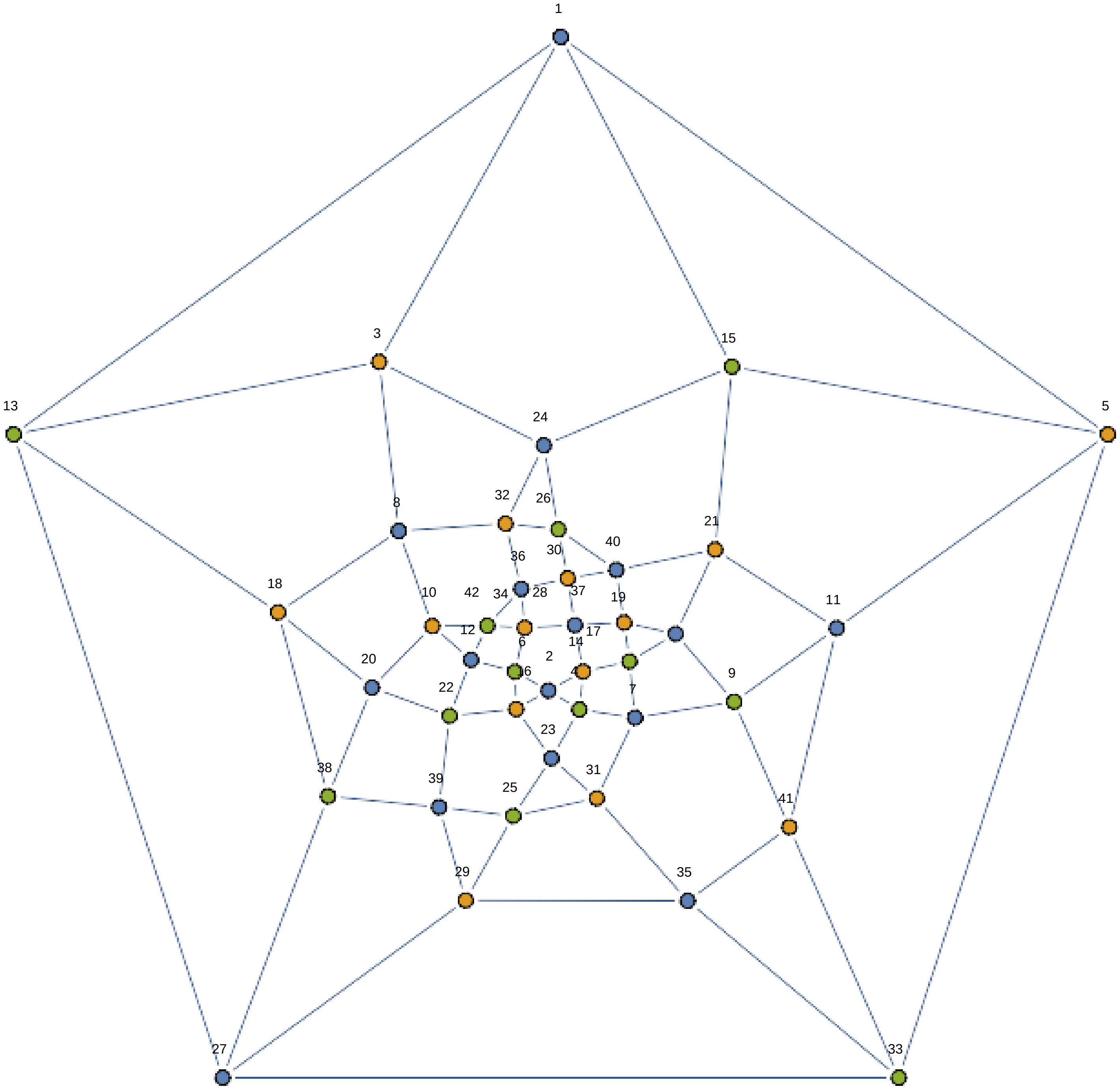
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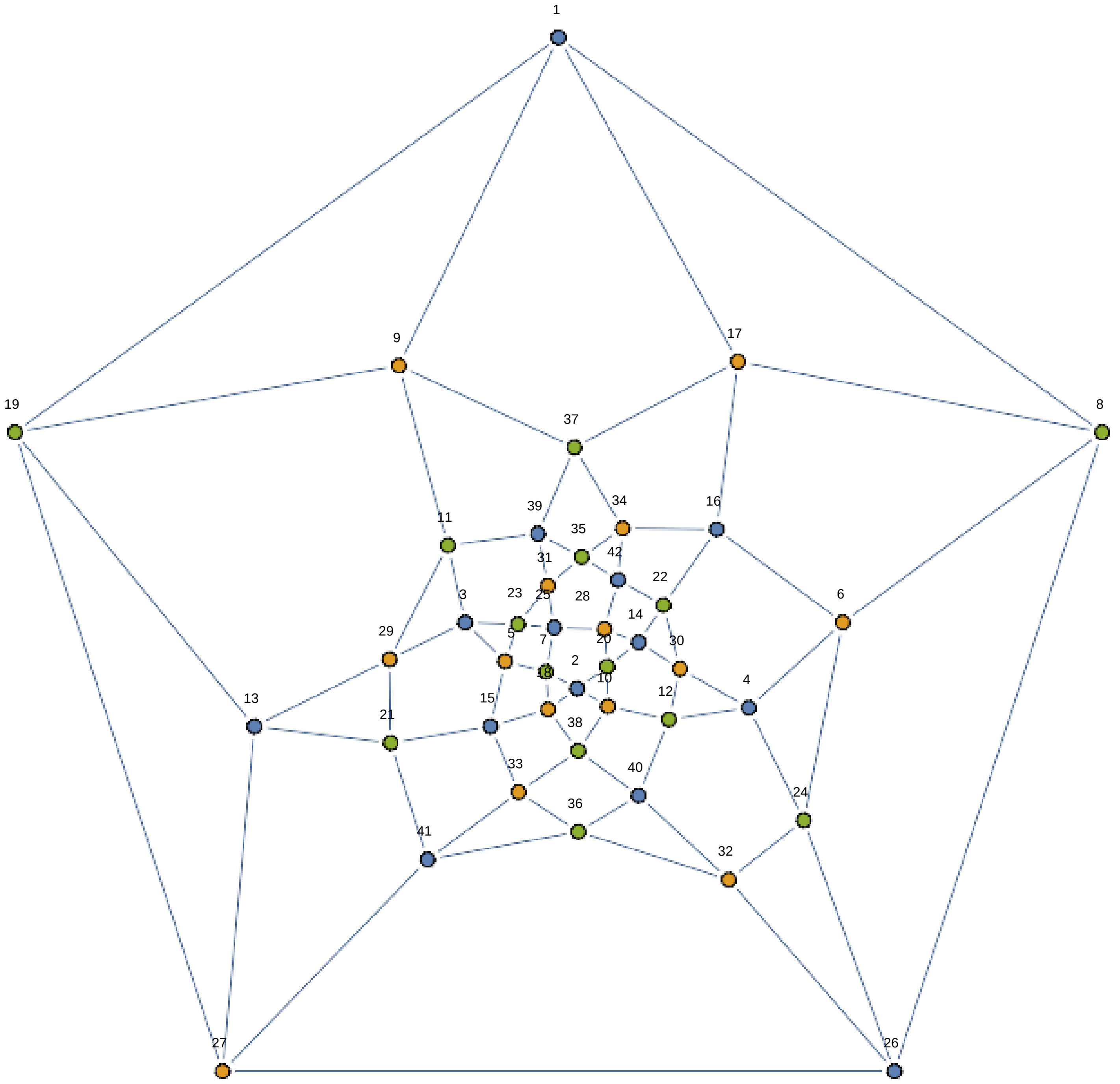
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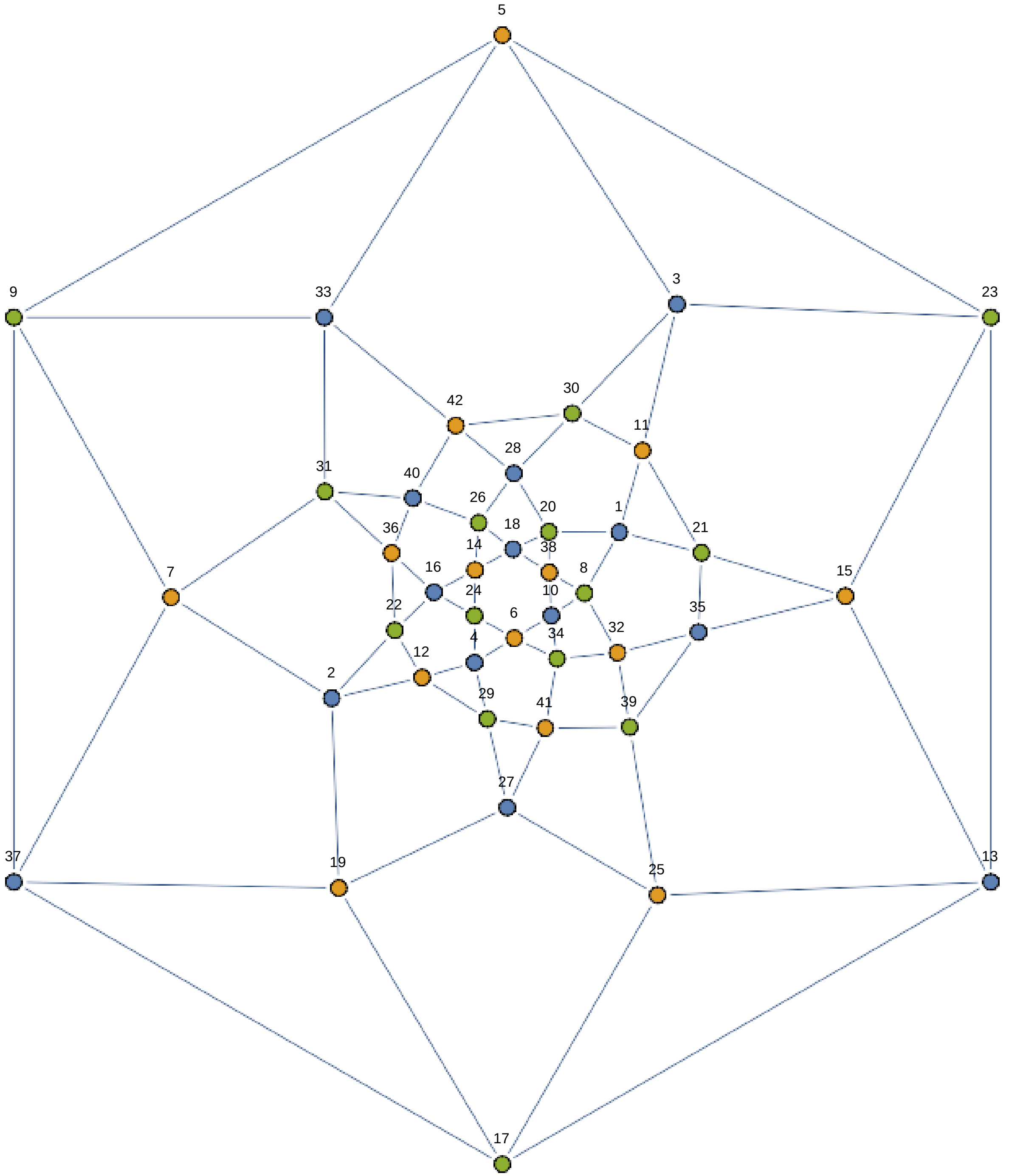
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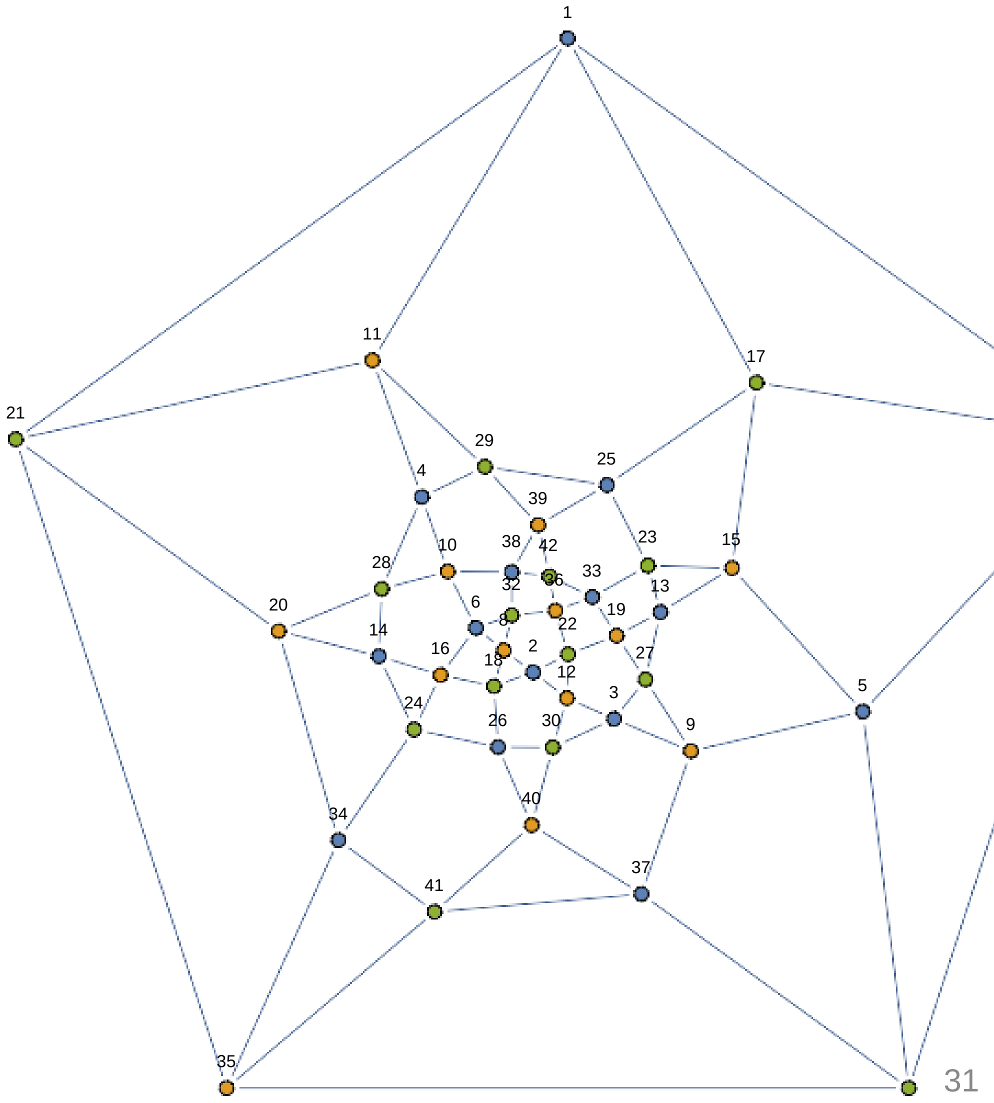
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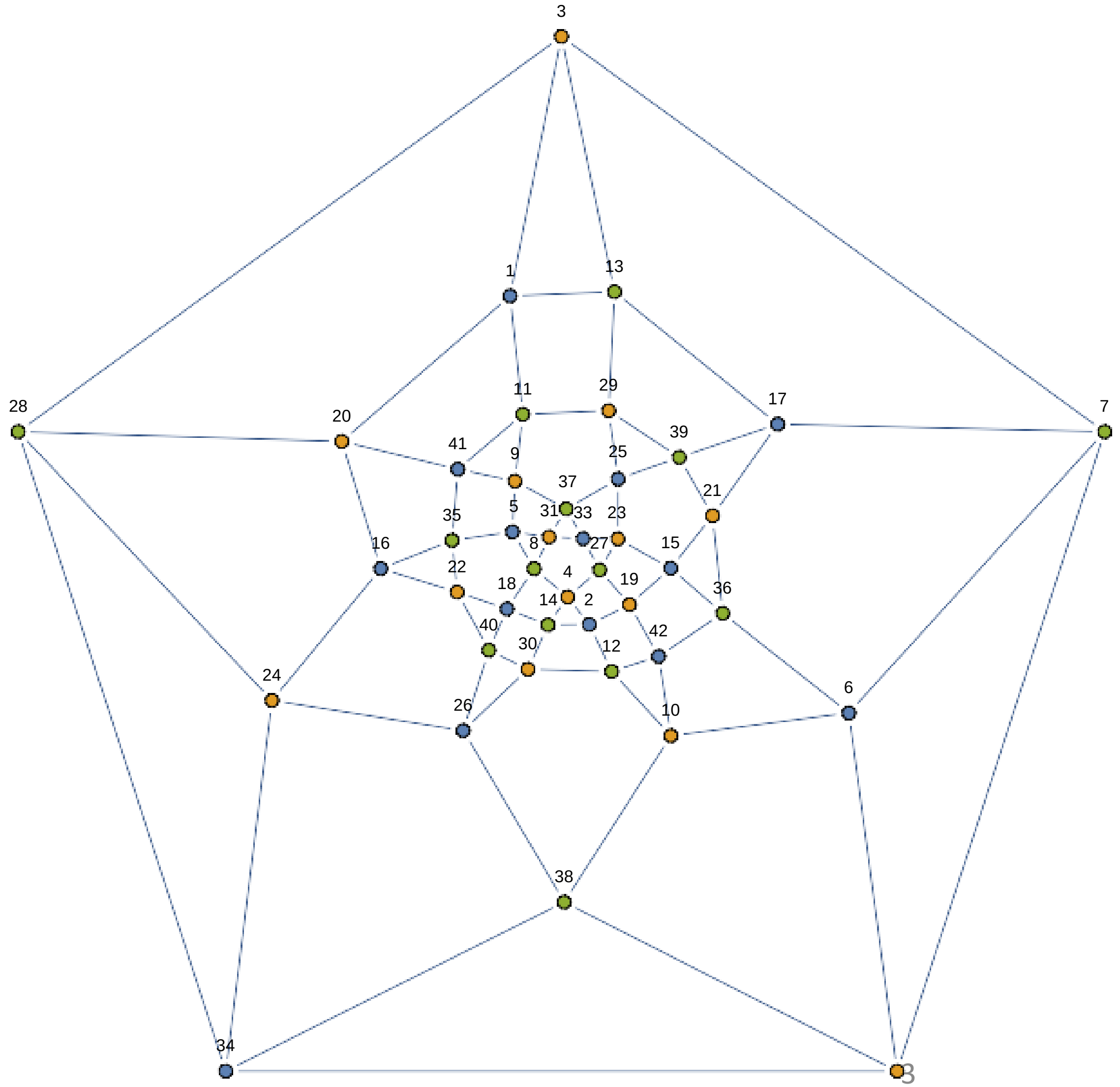
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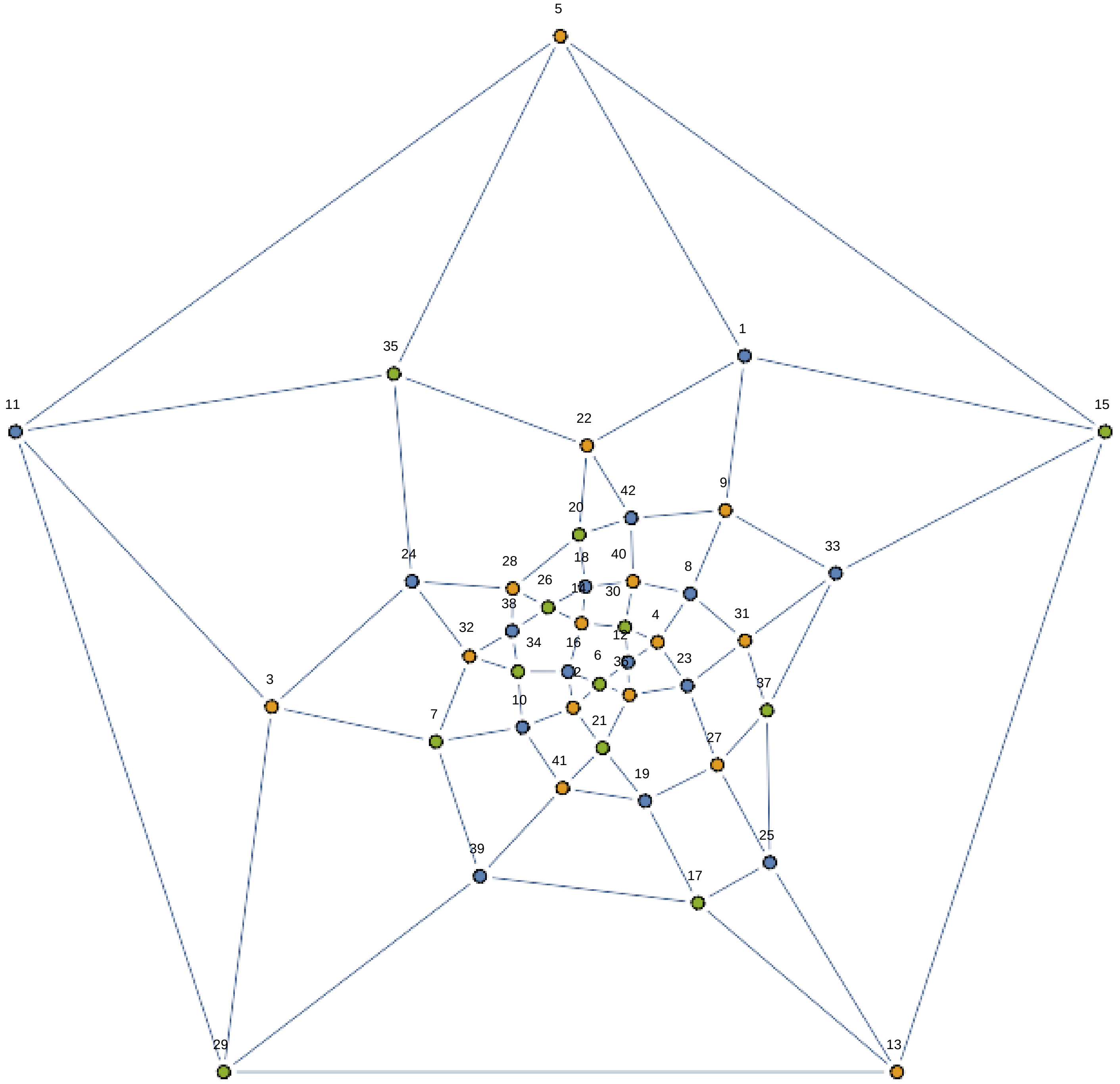

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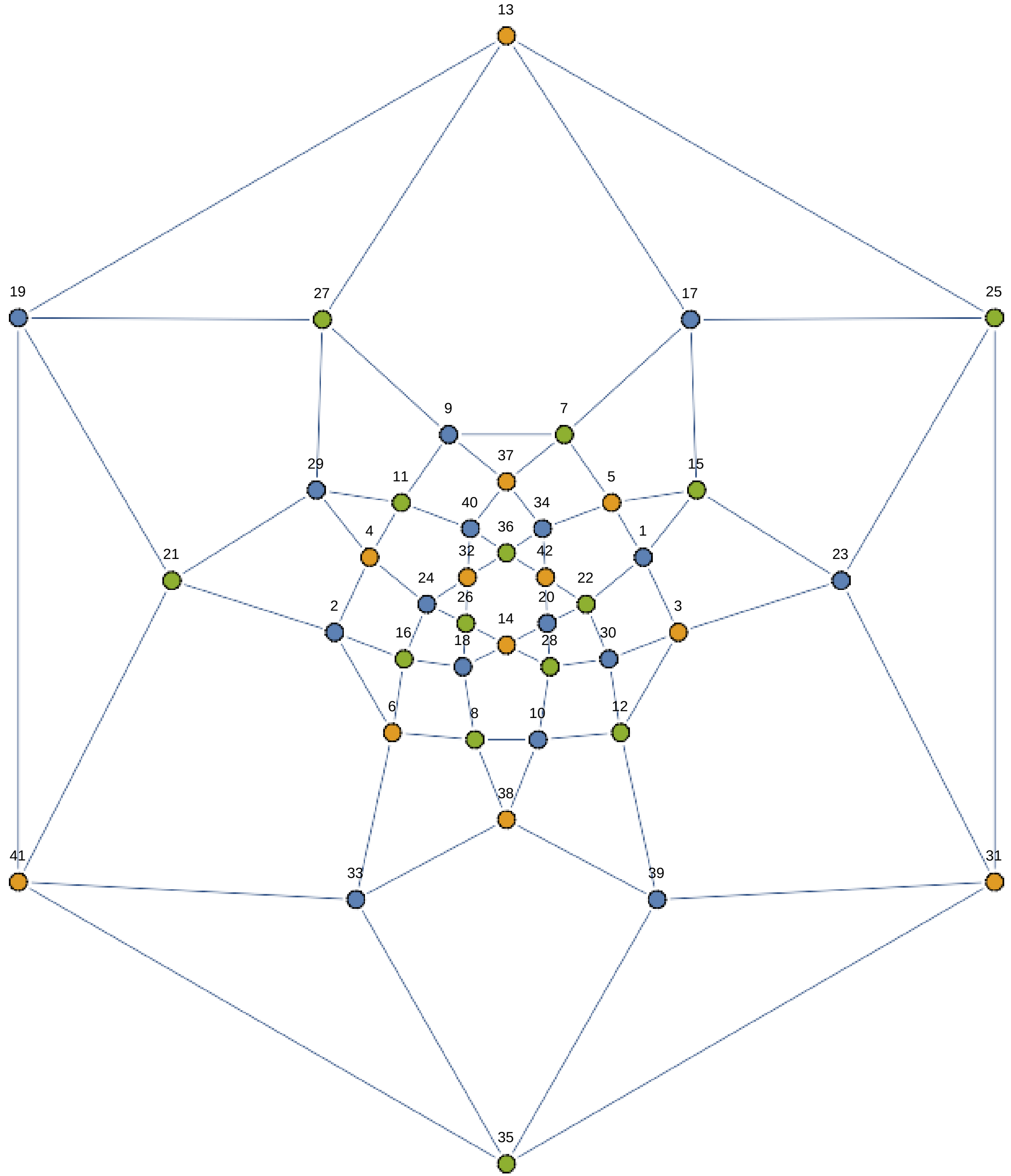
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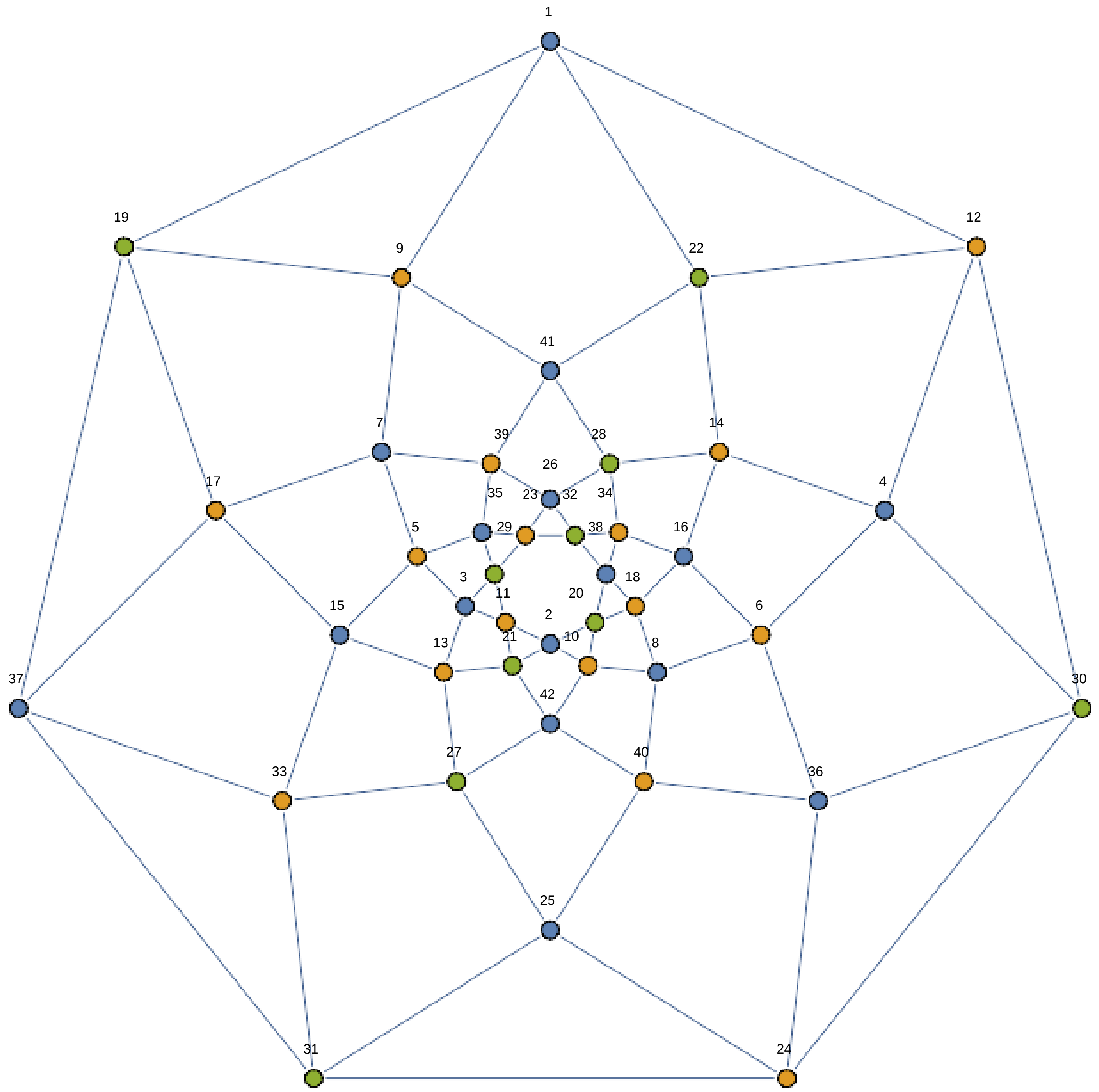
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