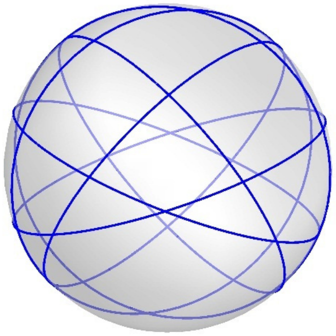


Great Circles Problem

Kha Man

04/21/2015

Outline

- Lemma 1
- Lemma 2 – The uniqueness of the special graph S_k
- Lemma 3 – The equivalent graph of S_k
- Lemma 4 – The properties of S_k
- Theorem 1 – The chromatic number of S_k
- The next steps

Lemma 1

Call n is the number of circles in the graph

1. There are $2(n-1)$ vertices and $2(n-1)$ edges on a circle
2. A pair of circles create 2 intersections. The distance between 2 intersections on a circle is $n-1$ edges on the circle

Lemma 1 - Proof

1. A circle will intersect $(n-1)$ other circles. A pair of circles will meet at 2 points. So the number of points on a circle is $2(n-1)$

   $|E(C_{2(n-1)})| = 2(n-1)$ ➔ There are $2(n-1)$ edges on the circle
2. Assume the statement is correct with k great circles graphs which have $(2k-2)$ vertices on a circle.

Define $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \ldots \rightarrow v_{k-1} \rightarrow \Psi(v_1) \rightarrow \Psi(v_2) \rightarrow \Psi(v_3) \rightarrow \ldots. \rightarrow \Psi(v_{k-1})$

$\rightarrow v_1$ is the circular path that has

$$d(v_i, \Psi(v_i)) = k - 1 \; ; \; i = 1,2,3, \ldots, (k-1)$$

Lemma 1 - Proof

Now we add a new circle C_{k+1} into the graph. So on every circle C_1 to C_k, we have 2 new intersections made by C_{k+1}. Call it v_a and $\Psi(v_a)$

Without loss of generality, I consider v_a as the first vertex in my new circular path v_a -> v_1 -> v_2 -> v_3 -> … -> v_{k-1} - > $\Psi(v_a)$ -> $\Psi(v_1)$ -> $\Psi(v_2)$ -> $\Psi(v_3)$ -> …. -> $\Psi(v_{k-1})$ -> v_a

Because every vertex has O as the point symmetry, so if v_a is the first vertex that is close to $\Psi(v_{k-1})$ and v_1, $\Psi(v_a)$ must be close to v_{k-1} and $\Psi(v_1)$.

Lemma 1 - Proof

$$d\big(v_a, \Psi(v_a)\big) = d(v_a, v_1) + d\big(v_1, \Psi(v_a)\big) = 1 + \big(d(v_1, \Psi(v_1) - d(\Psi(v_1), \Psi(v_a)\big)$$
$$= 1 + (k-1) = k$$

Call v_i is the vertex in the set $\{v_1, v_2, \ldots v_{k-1}\}$

$$\Rightarrow d\big(v_i, \Psi(v_i)\big) = d(v_i, v_{k-1}) + d\big(v_{k-1}, \Psi(v_a)\big) + d(\Psi(v_a), \Psi(v_i))$$
$$= t + 1 + (k - 1 - t) = k$$

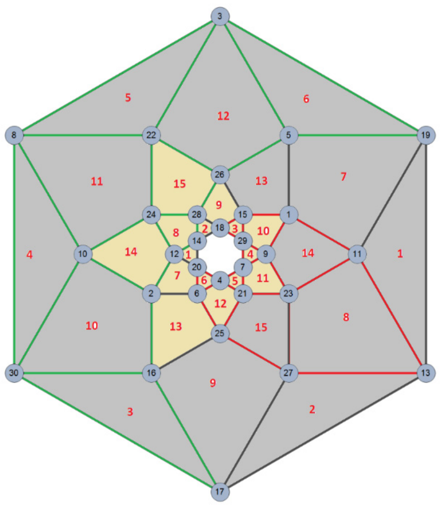
Similarly, because we have (2k-2+2) = 2k edges on the new path, the other path of $d\big(v_i, \Psi(v_i)\big)$ that contains $\Psi(v_{k-1})$ is also equal to k

➔ The induction hypothesis is correct with (k+1) circles

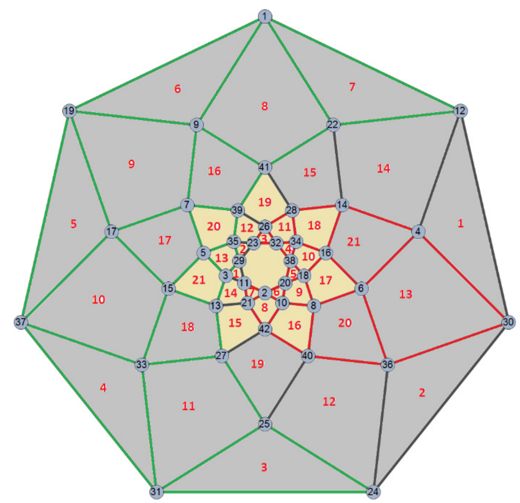
The special graph

Definition: A graph of k great circles $\mathcal{S}_k$ is **special** if it contains even number of triangles, quadrilaterals and 2 polygon that has k segments.

Lemma 2 will prove the special graph only has 1 unique structure



6 great circles

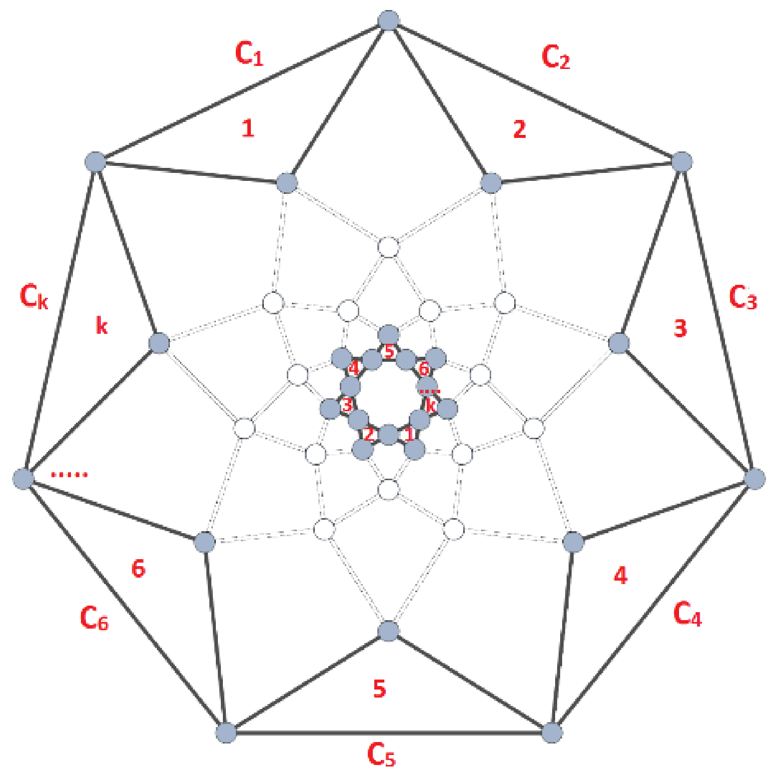


7 great circles

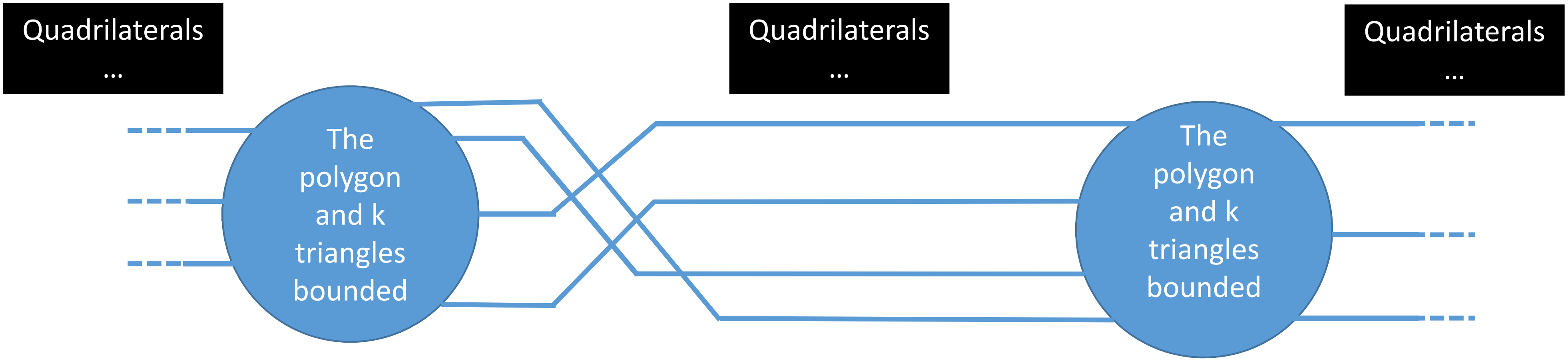
Lemma 2. (The uniqueness of the special graph)

S_k is unique and it has the following form:

- K triangles at the "outer cycle"
- Another k triangles are made by the reflection and 1 polygon has k segments in the "middle"
- The other polygons are quadrilaterals



Lemma 2 – Another drawing

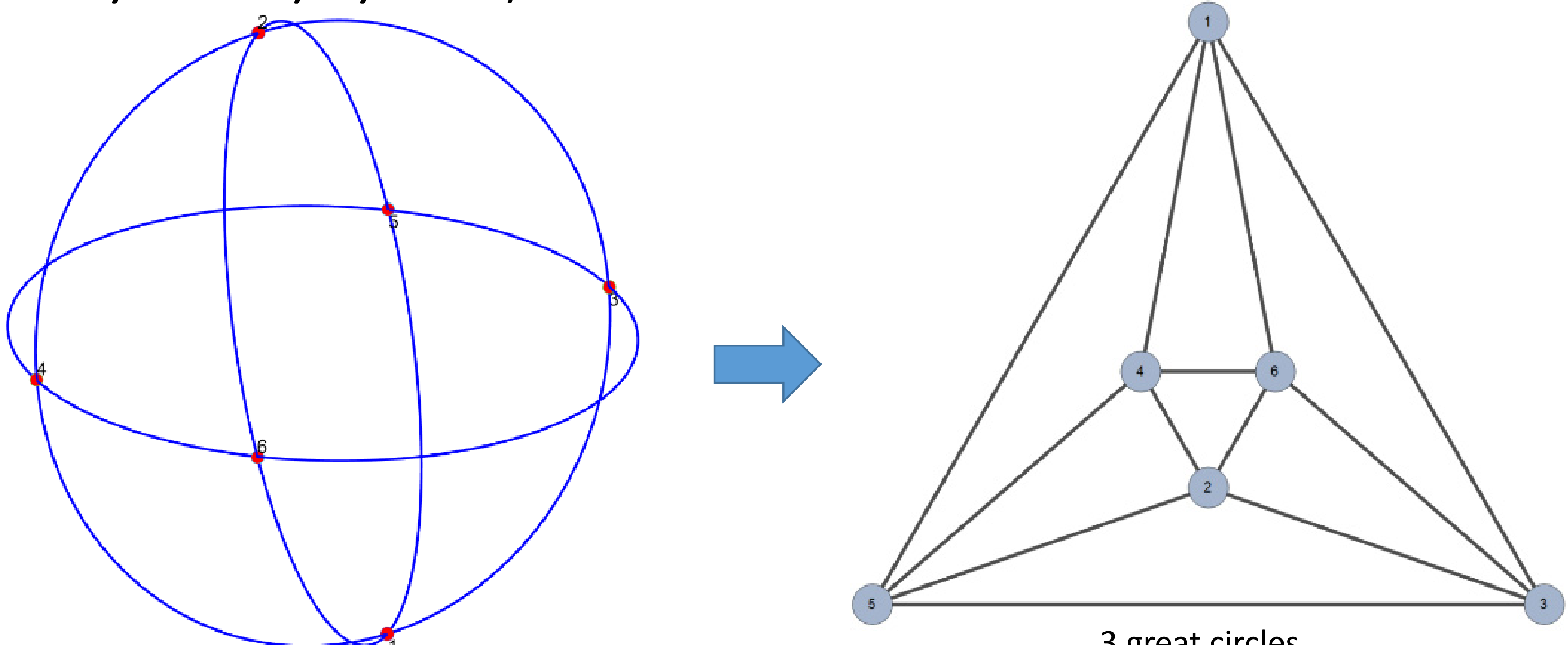


Lemma 2 – Proof

By definition, $\mathcal{S}_\mathrm{k}$ contains triangles, quadrilaterals and 2 polygon that has k segments.

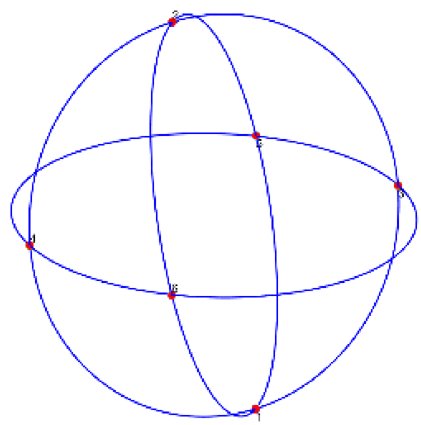
Lemma 2 – Proof

- 3 great circles has 1 non-isomorphic graph and it’s special (We can verify it easily by hand)

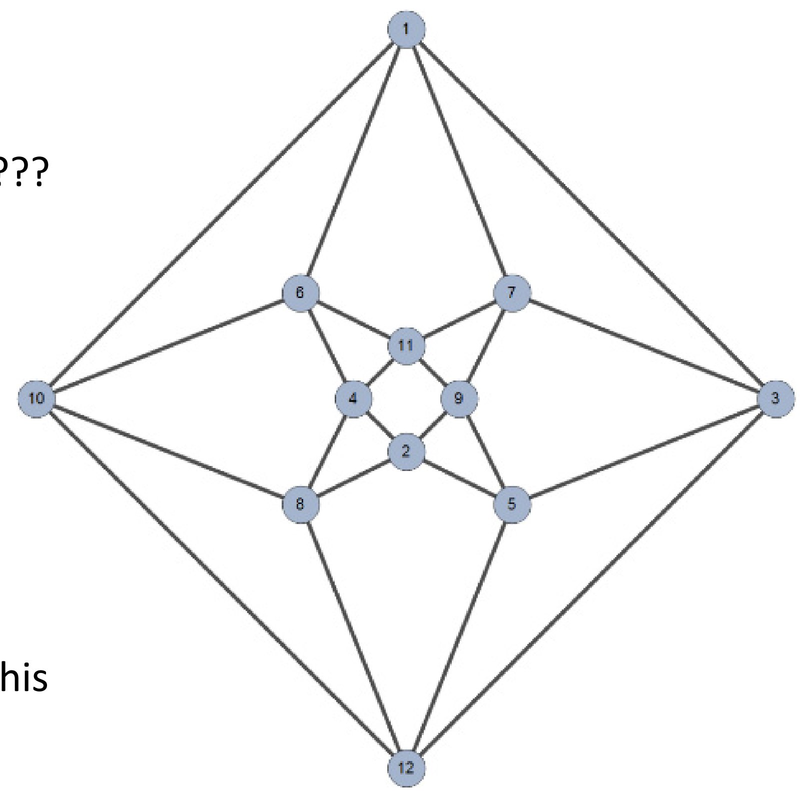


3 great circles

Lemma 2 – Proof



+ 1 great circle = ???



4 great circles

There are 3 cases happen when the 4th great circle is added:

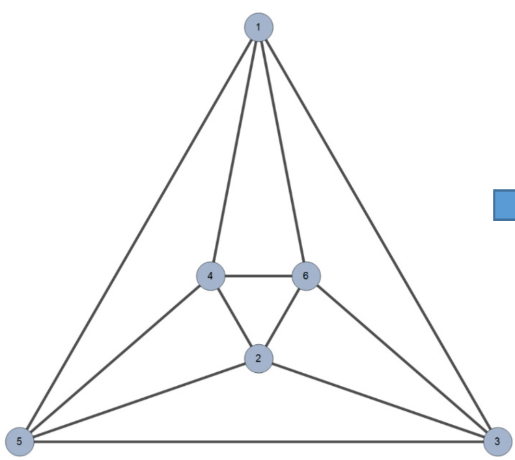
- It cuts arc(1,5), arc(1,3), arc(2,6), arc(2,4), arc(3,6) and arc(4,5)
- It cuts arc(1,3), arc(3,5), arc(2,4), arc(4,6), arc(1,6) and arc(2,5)
- It cuts arc(1,5), arc(3,5), arc(2,6), arc(4,6), arc(2,6) and arc(1,5)

The outcome of 3 cases is the same as the figure on the right. This graph is special and unique

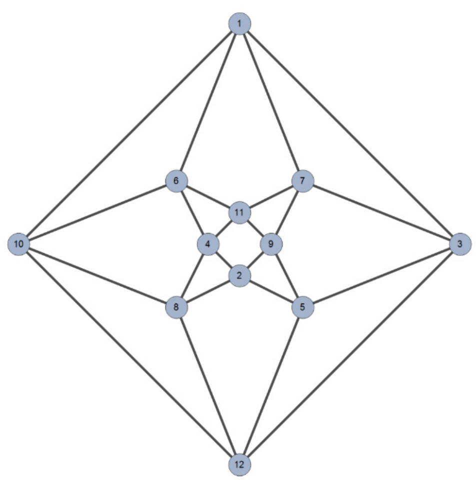
Lemma 2 – Proof

- 3 and 4 great circles only have 1 non-isomorphic graph
- They are both S_3 and S_4 respectively

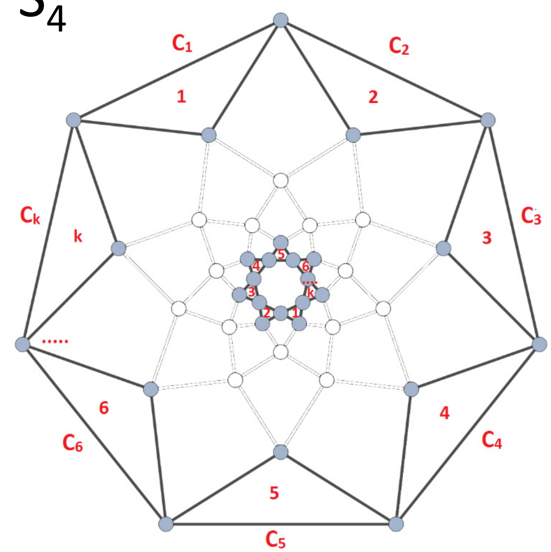
➔ We can assume S_k is somehow similar to S_3 and S_4



3 great circles



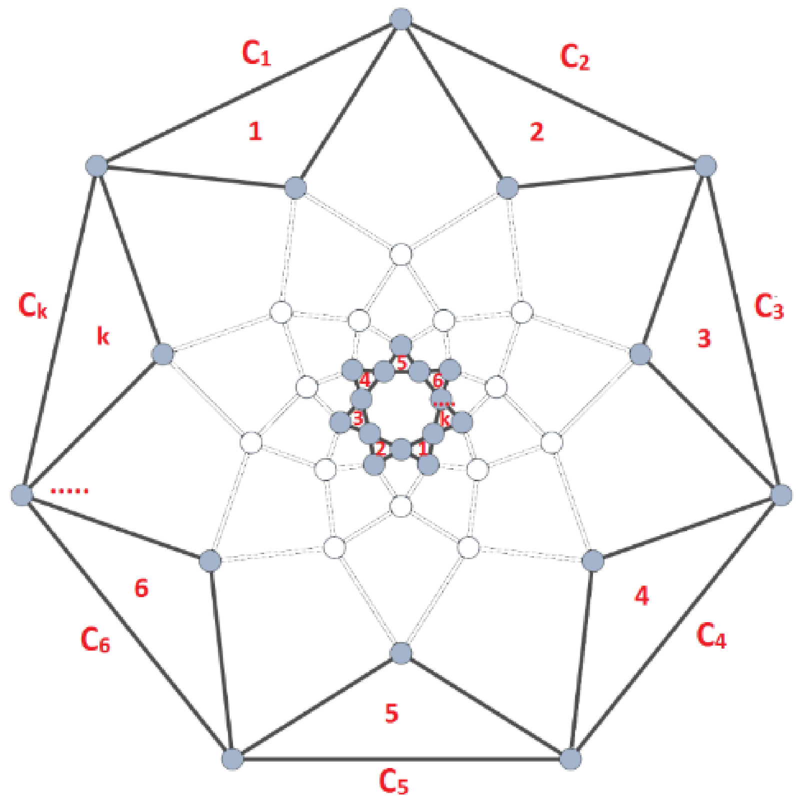
4 great circles



k great circles

Lemma 2 – Proof

- We may have a form of S_k
- If we may have different ways to create non-isomorphic graphs for S_k, it's also true in case of S_{k+1}
- In case of S_4 was created by S_3 , we can do believe that S_{k+1} may be derived from S_k
- This form of S_k is capable of generating all non-isomorphic graphs of S_{k+1}



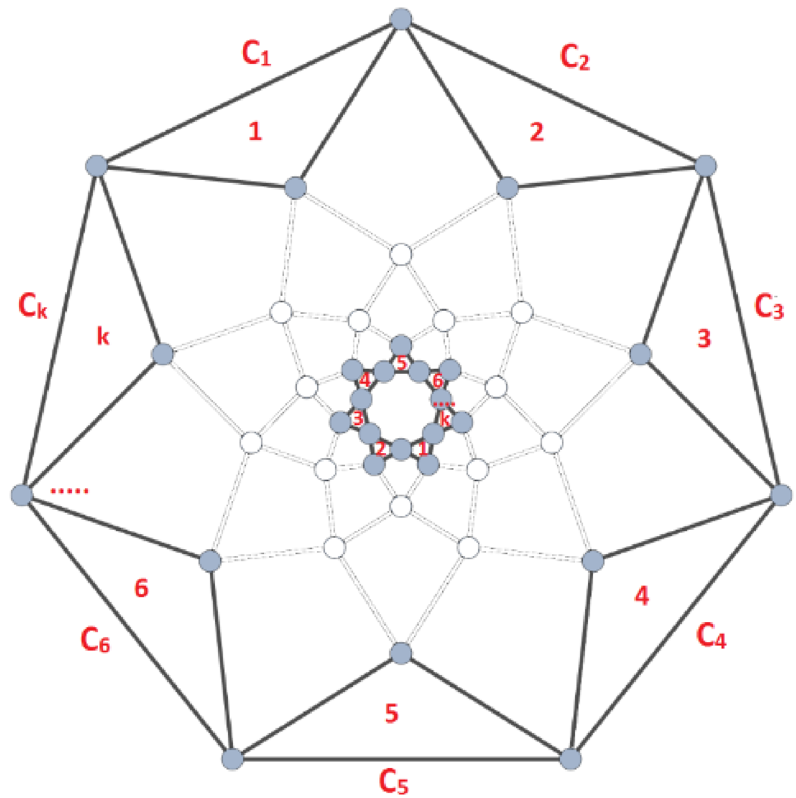
k great circles

Lemma 2 – Proof

Now, we will "draw" the (k+1)th great circle to this form.

Call V is a point we start to draw (k+1)th great circle

Call $\mathfrak{C}(V)$ is the set of 2 great circles that (k+1)th intersect first

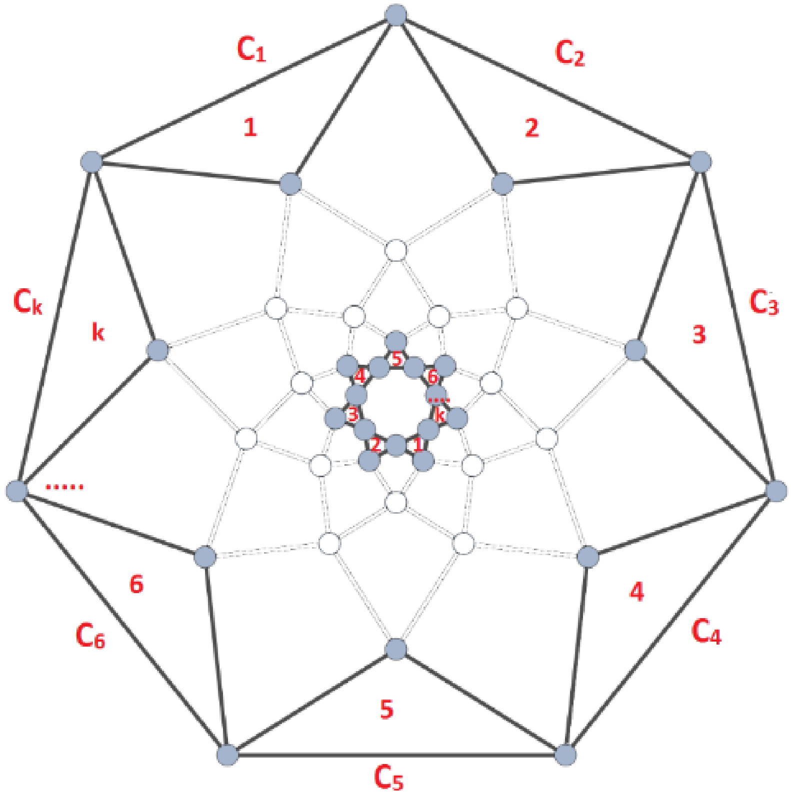


k great circles

Lemma 2 – Proof

The rule to make S_{k+1}:

- Don't make any quadrilaterals into pentagonals
- A circle will intersect C_{k+1} at 2 intersections
- 2*k intersections are added into the current graph after C_{k+1} is implemented

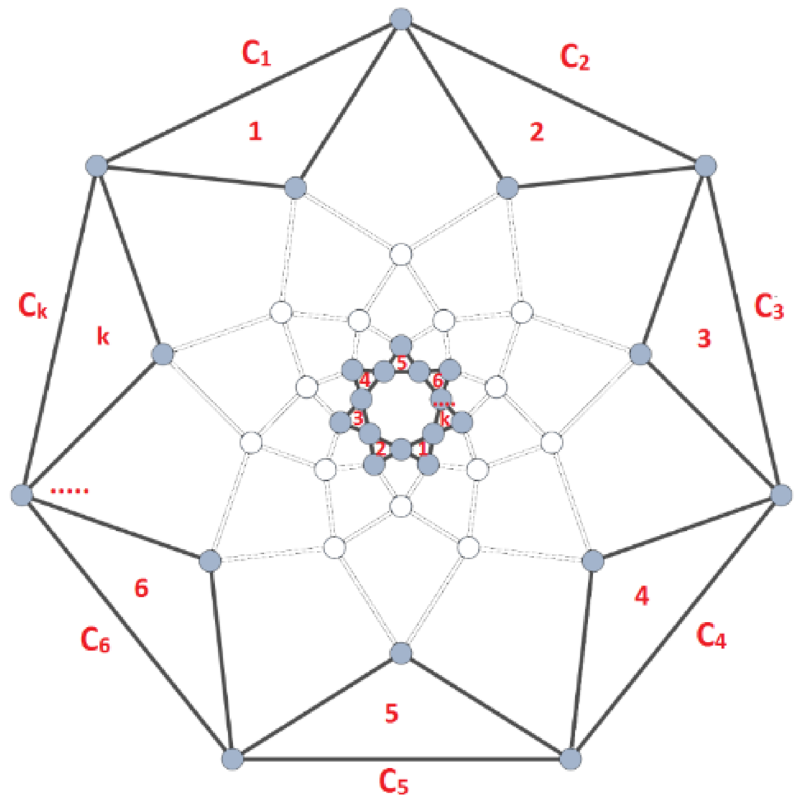


k great circles

Lemma 2 – Proof

We have 2 cases for the drawing:

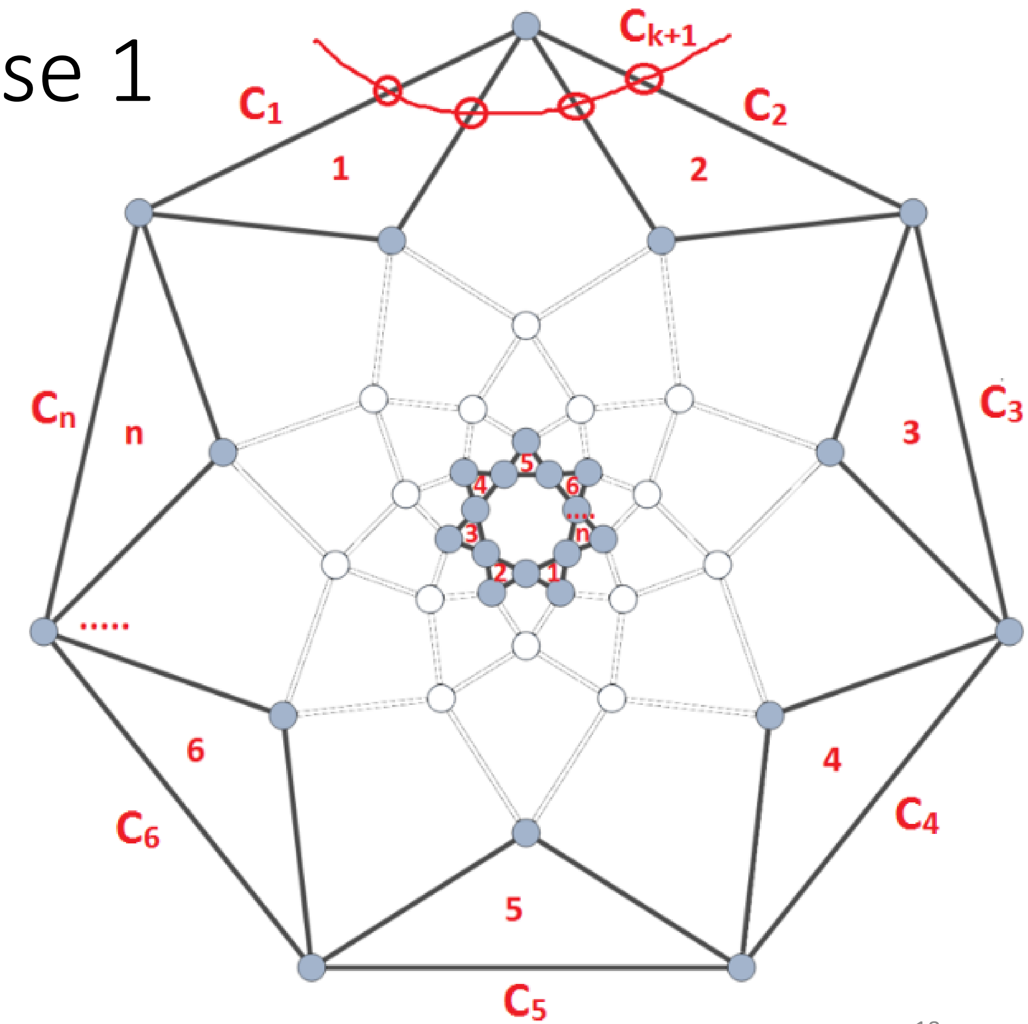
1. V is located at the outer face. It means (k+1)th will change the outer cycle that's supposed to be bounded by triangles
2. V is located at a face inside of the graph and (k+1)th will not include any part of outer face



k great circles

Lemma 2 – Proof – Case 1

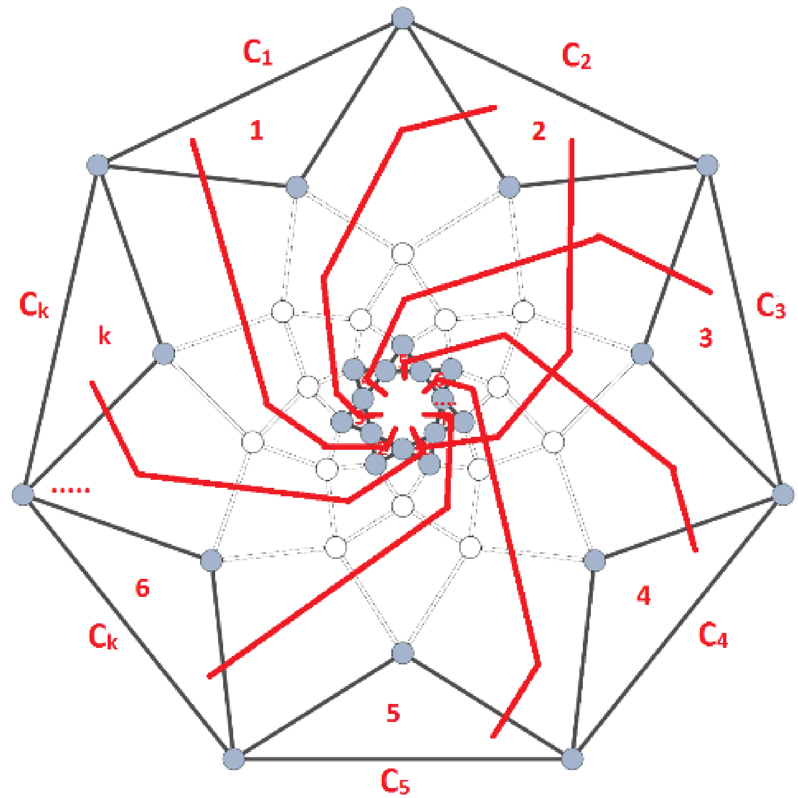
An example of incorrect drawing



Lemma 2 – Proof

Some definitions before going into details of the proof

- A special path is the path from a triangle at the outer cycle to the middle polygon such that it doesn't make any
- There are 2 special paths starting at a triangle at the outer cycle
- The special path will intersect at (k-1) points to be in the middle polygon (By lemma 1.1)

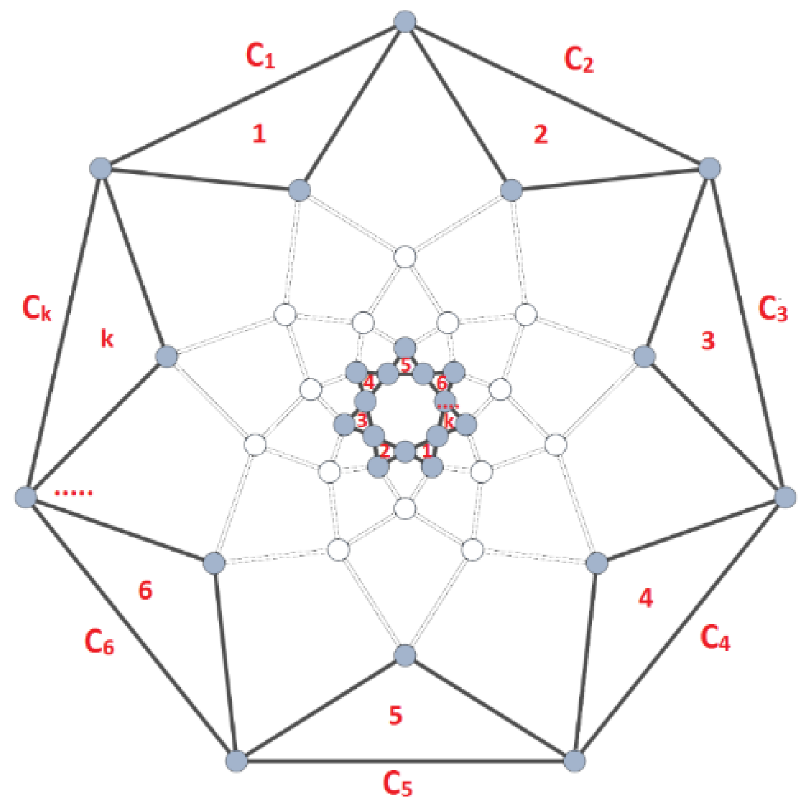


k great circles

Lemma 2 – Proof – Case 1

- We have 2 subcases for case 1:

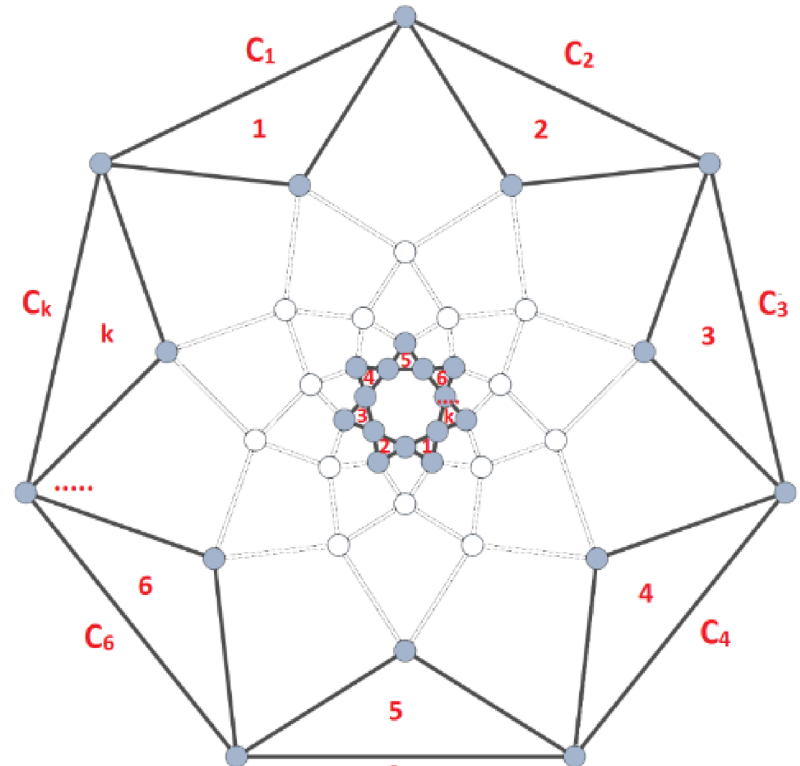
1. $\mathfrak{C}(V) = \{C_i, C_{i+1}\}$
2. $\mathfrak{C}(V) \neq \{C_i, C_{i+1}\}$



k great circles

Lemma 2 – Proof – Case 1.1

- Due to the special geometry, we can only work with the case when $\mathfrak{C}(V) = \{C_1, C_2\}$

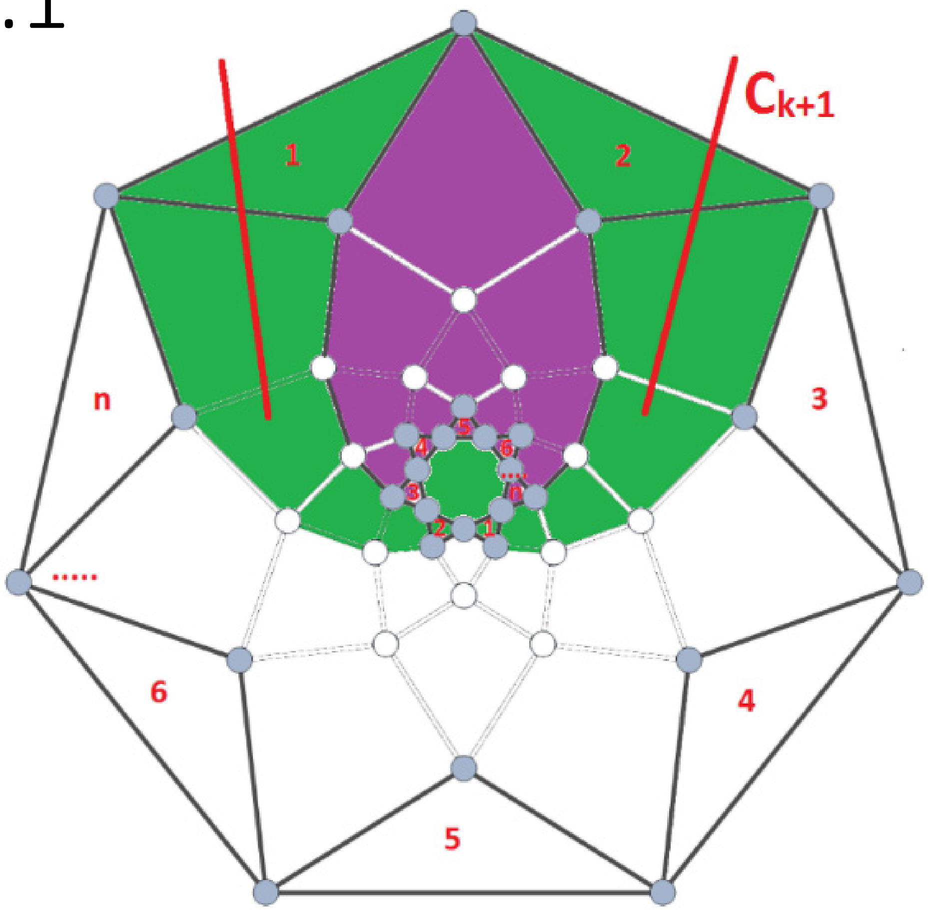


k great circles

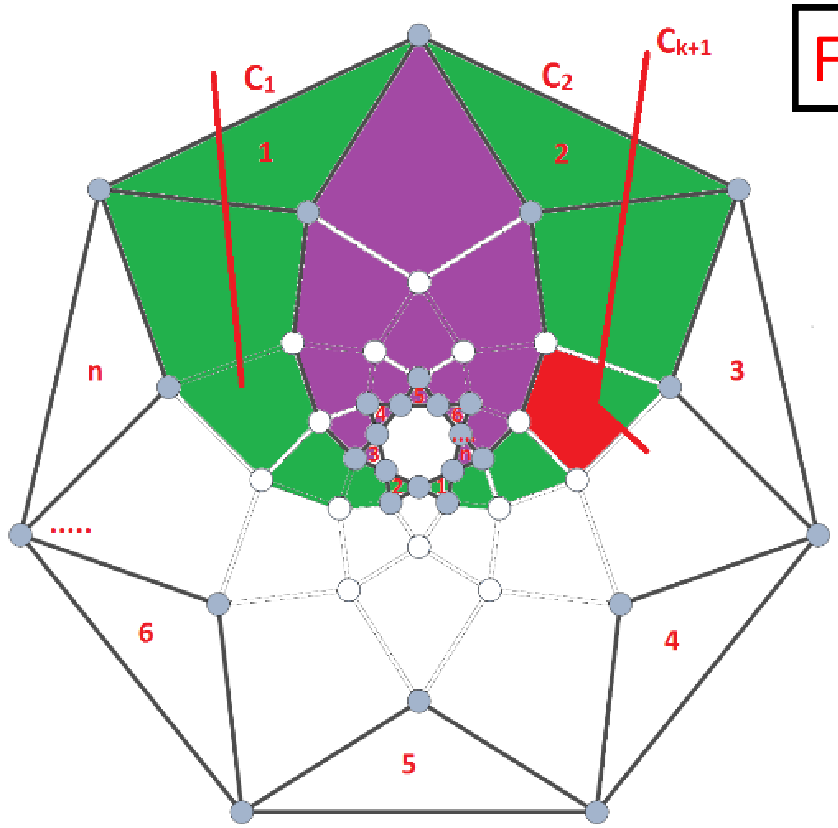
Lemma 2 – Proof – Case 1.1

We realize that C_{k+1} needs to follow the green region to make S_{k+1} (???)

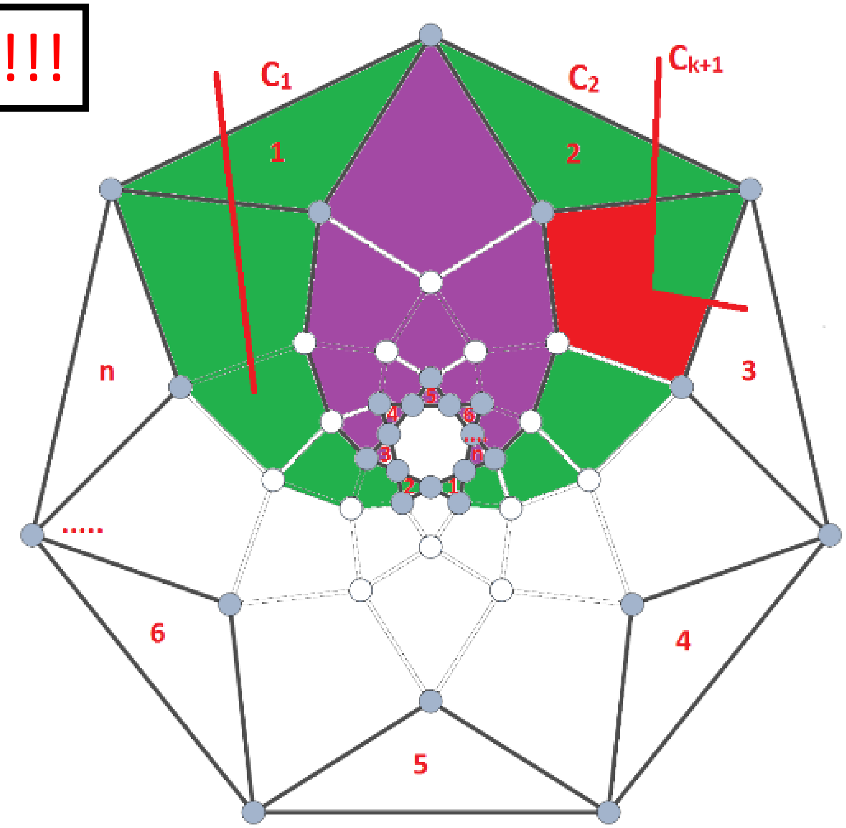
- $\mathcal{R}_{purple} = \{C_1 \cap C_2\} \setminus (C_1, C_2, \ldots, C_n)$
- $\mathcal{R}_{green} = \{C_1 \cup C_2\} \setminus \mathcal{R}_{purple}$



Lemma 2 – Proof – Case 1.1



FAILED!!!



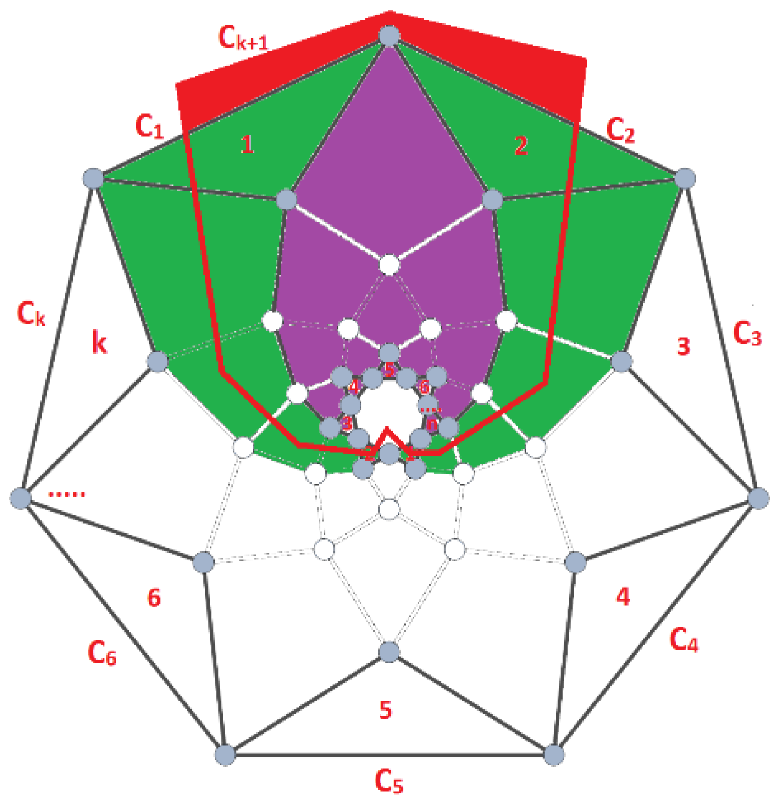
Lemma 2 – Proof – Case 1.1

Finally, this form is the only one for S_{k+1} we can get in case 1.1

- C_{k+1} created 2k intersections
  - Every special path created (k-1)

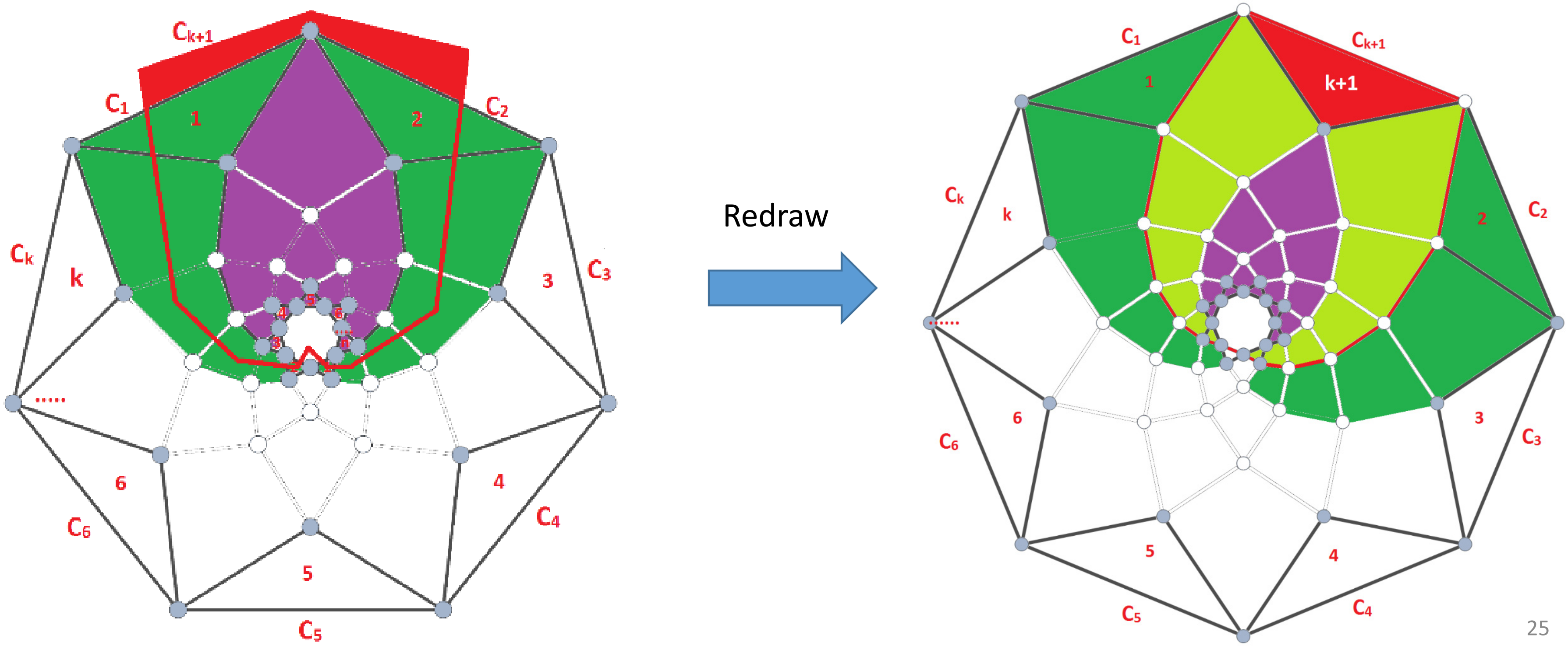
  ➔C_{k+1} created: 2*(k-1) + 2 at the outer cycle = 2k

- Lemma 1.2 is still true in the new S_{k+1}



Lemma 2 – Proof – Case 1.1

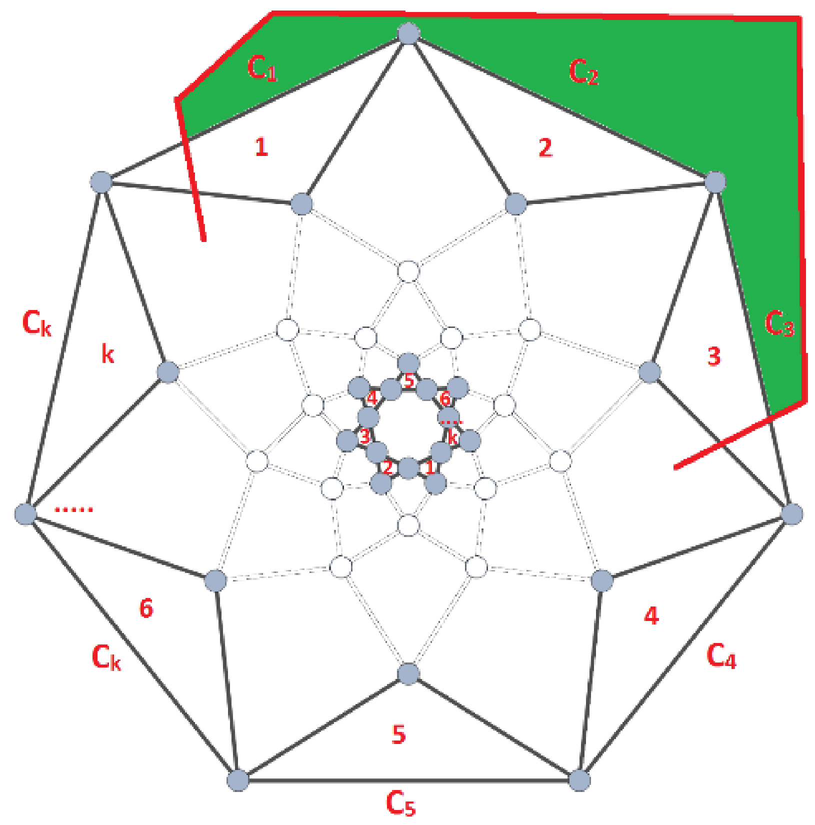
- The outcome has exactly the same form for (k+1) circles we assume



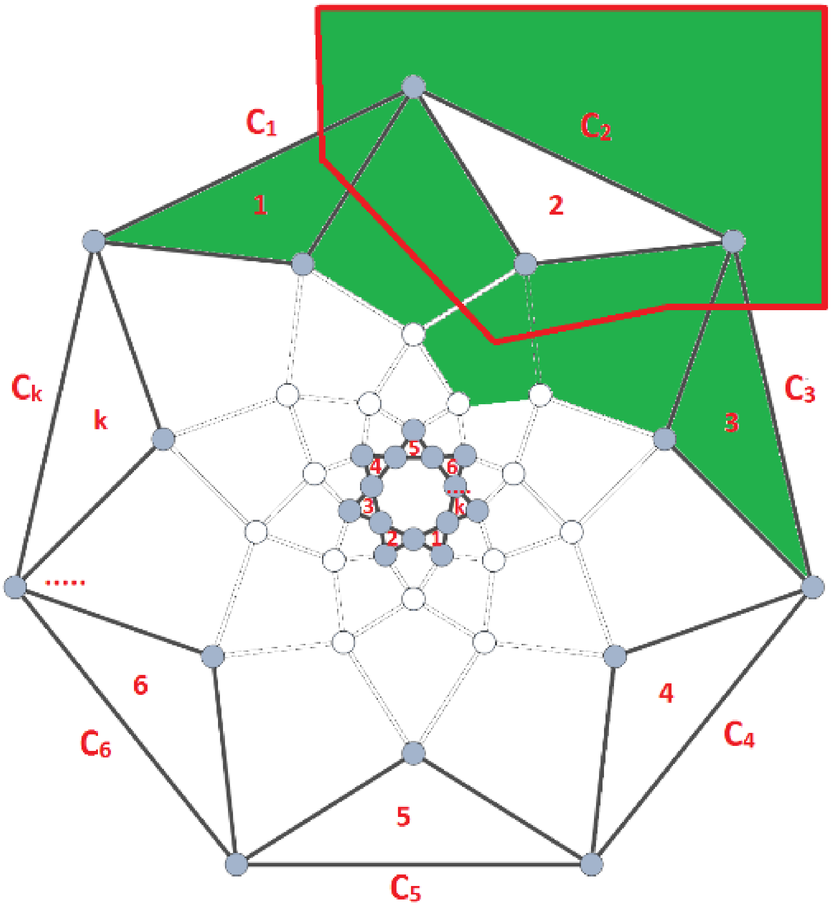
Lemma 2 – Proof – Case 1.2

In case 1.2, we have $\mathfrak{C}(V) \neq \{C_i, C_{i+1}\}$

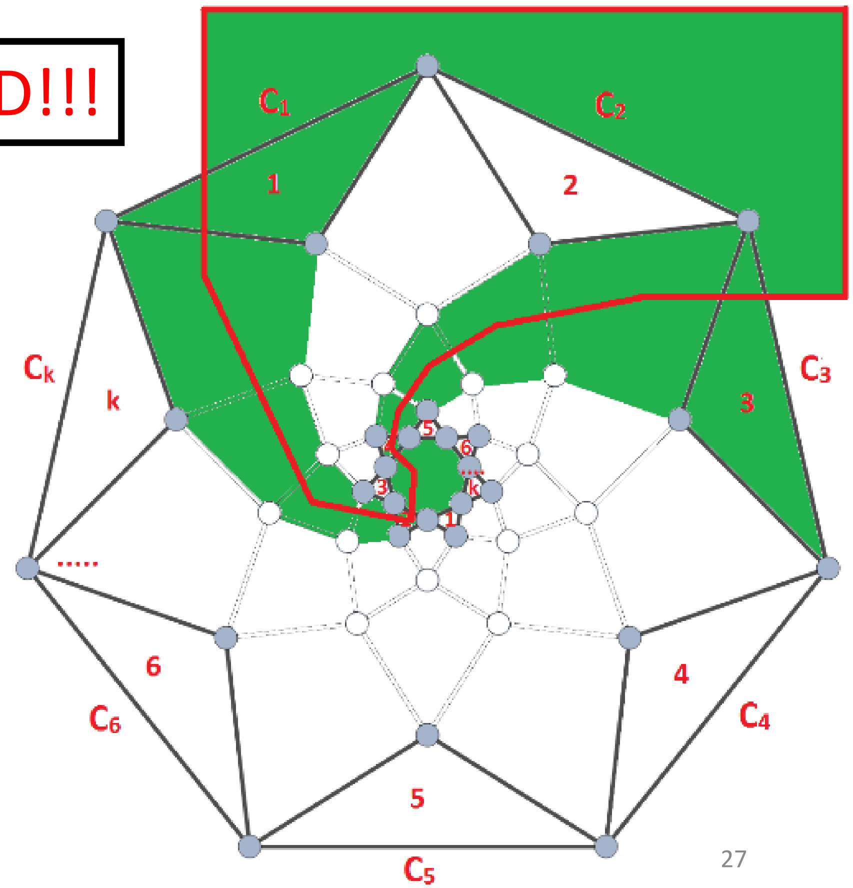
- Assume $\mathfrak{C}(V) = \{C_1, C_3\}$
- C_{k+1} then intersects $\{C_1, C_2\}$ and creates a quadrilateral at the outer cycle (somehow it's creating another form of S_k)



Lemma 2 – Proof – Case 1.2



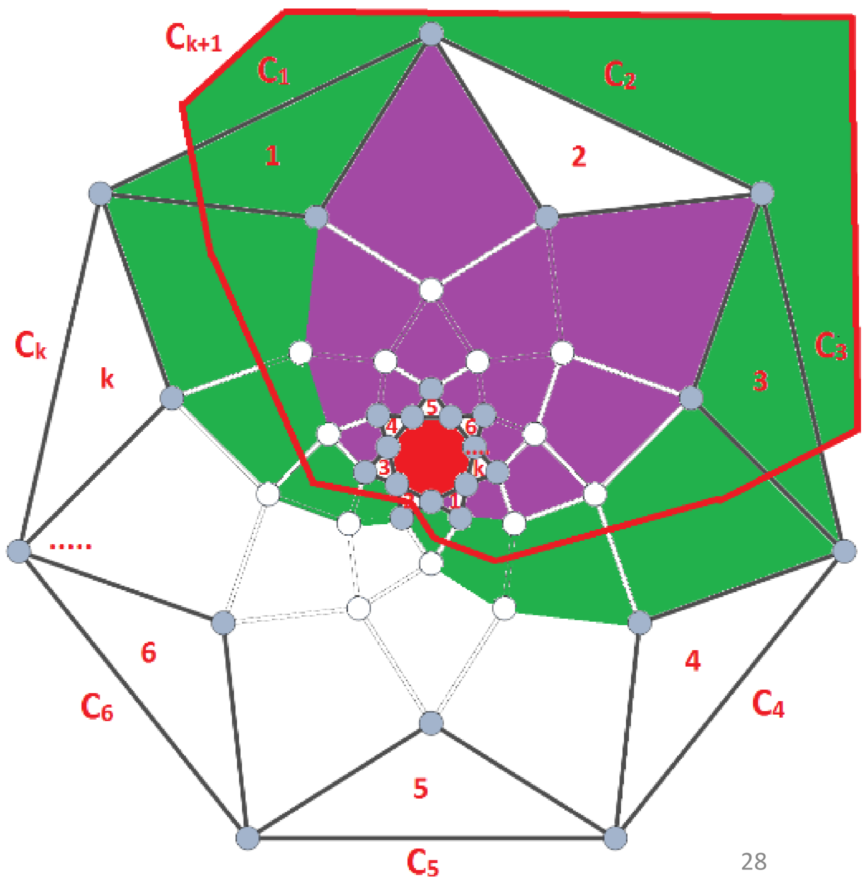
FAILED!!!



Lemma 2 – Proof – Case 1.2

FAILED!!!

- They both meet at the triangle (C_1, C_2, C_3) in the middle and only have (2k-2) points



Lemma 2 – Proof – Case 1.2

➔ There is no S_k in case when $\mathfrak{C}(\mathrm{C}_{\mathrm{k+1}})$ = {C_1, C_3}

Similarly, when $\mathfrak{C}(V) =$ {C_i, C_j}, $j \neq \{i-1, i+1\}$

Call P_i is the special path containg the triangle i

So, we have {P_{i1} , P_{i2}} , {P_{j1}, P_{j2}}

➔ $C_2^1 . C_2^1 = 4$ possible cases

Lemma 2 – Proof – Case 1.2

1. C_{k+1} will be closed at a quadrilateral before going to the middle polygon ➔ It doesn't have enough intersections

2. P_j will intersect with C_{j+1} at the vertex V^* that has $d\left(\left(C_j \cap C_{j+1}\right) \in \textit{the outer cycle}, V^*\right) = k - 1$

   Since P_i doesn't start at the triangle (j+1), so $d\left(V^*, \Psi(V^*)\right) \neq (k - 1 + 1) = k$

➔Contradict the Lemma 2.1

3. Due to the geometry, there is another case that's similar to 2nd case

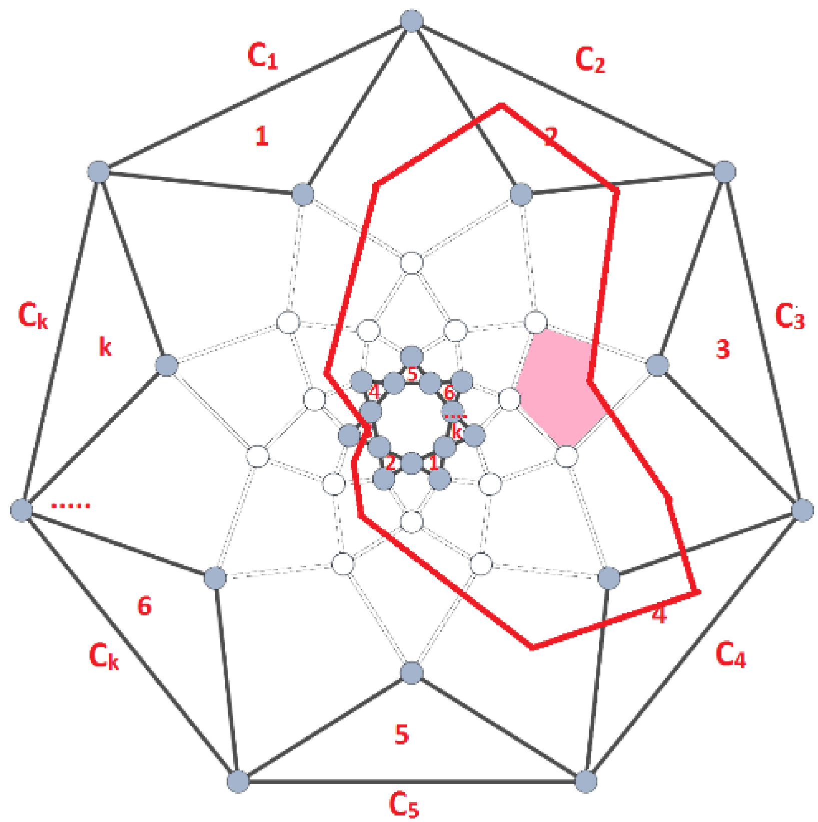
4. The last case is when 2 special paths don't meet at the polygon ➔ Then C_{k+1} won't have enough 2k intersections

➔ Finally, there is no S_{k+1} in case 1.2

Lemma 2 – Proof – Case 2

In case 2, the $(k+1)^{th}$ circle will only travel inside of the current form of S_k .

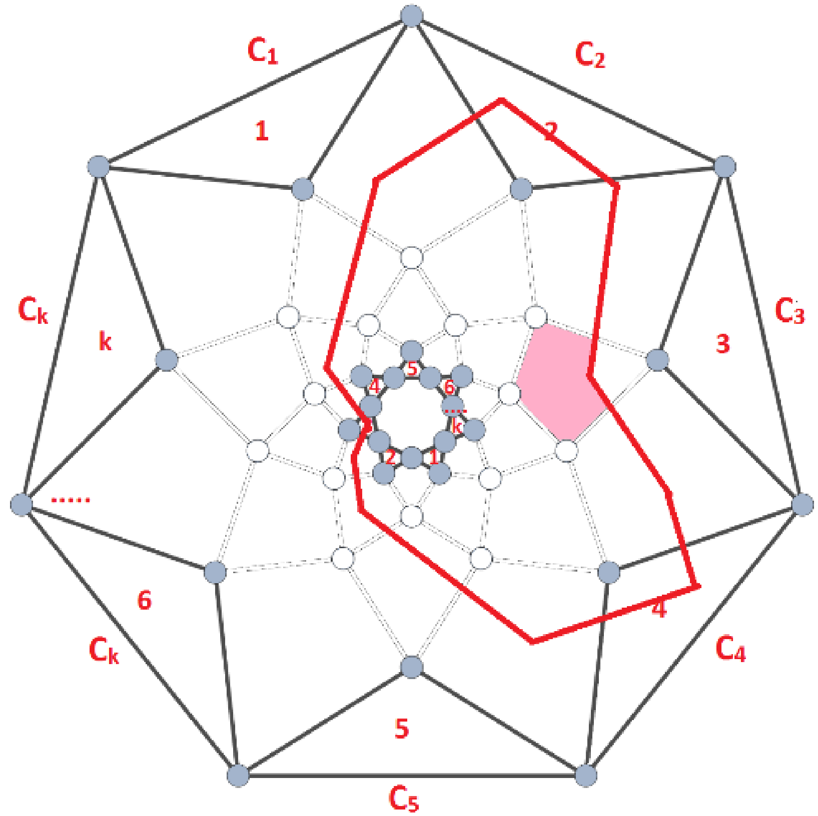
We will prove there is no such S_{k+1} made in this case.



An example of drawing in this case but it isn't satisfied

Lemma 2 – Proof – Case 2

- Starting at the center of the graph, we have paths tend to the triangles at the outer cycle. So WLOG, set V is in any triangle at the outer cycle.
- There are 2 cases happen
  1. The starting point and ending point are in the different triangle.
  2. The starting point and ending point are in the same triangle.



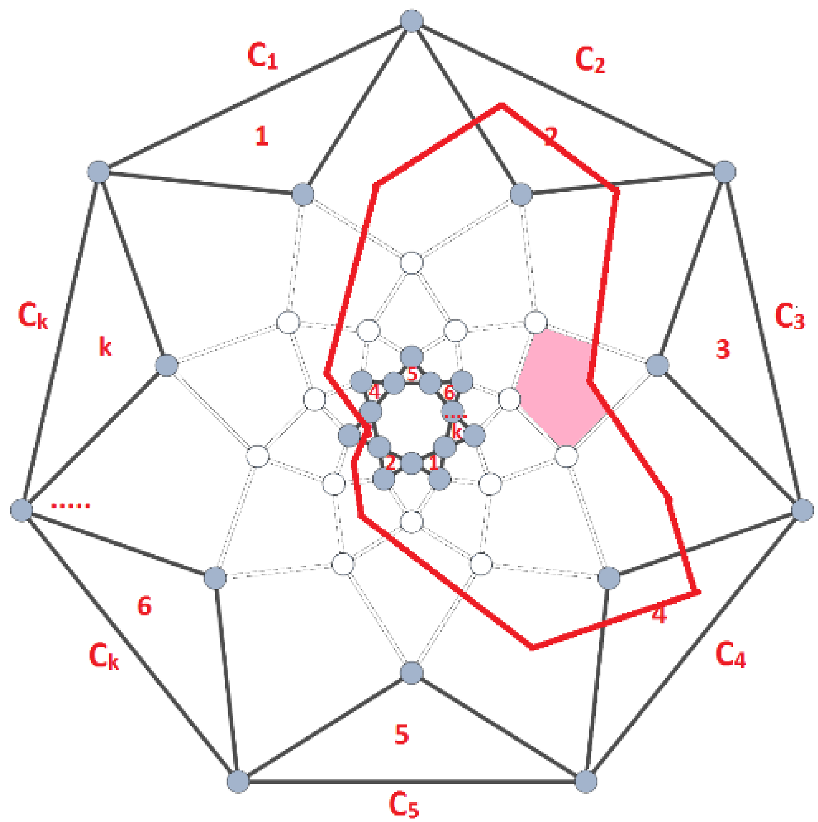
An example of drawing in Case 2.1

Lemma 2 – Proof – Case 2

Case 2.1: The starting point and ending point are in the different triangle.

- C_{k+1} must start at a triangle at the outer cycle, go to center then return back to another triangle at the outer cycle
- There is only path as defined
- The path connecting 2 triangles must create a pentagonal !!!

➔ There is no S_{k+1} in Case 2.1



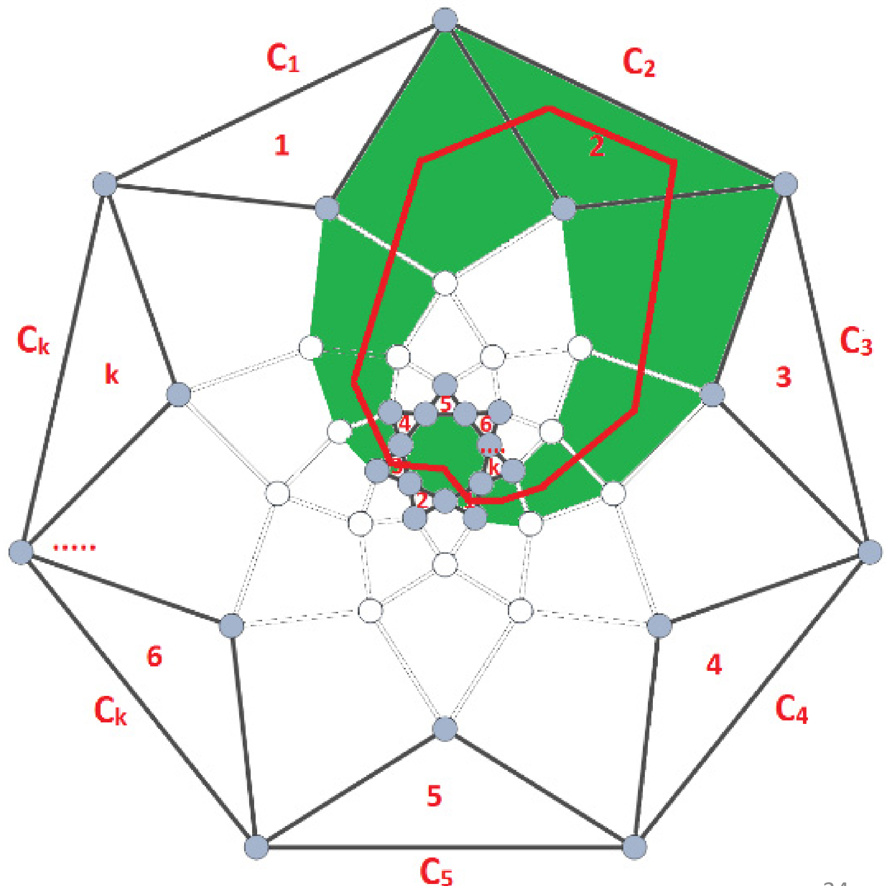
An example of drawing in Case 2.1

Lemma 2 – Proof – Case 2

Case 2.2: The starting point and ending point are in the same triangle.

- C_{k+1} doesn't intersect C_2 !!!

➔ There is no S_{k+1} in Case 2.2



Lemma 2 – Proof

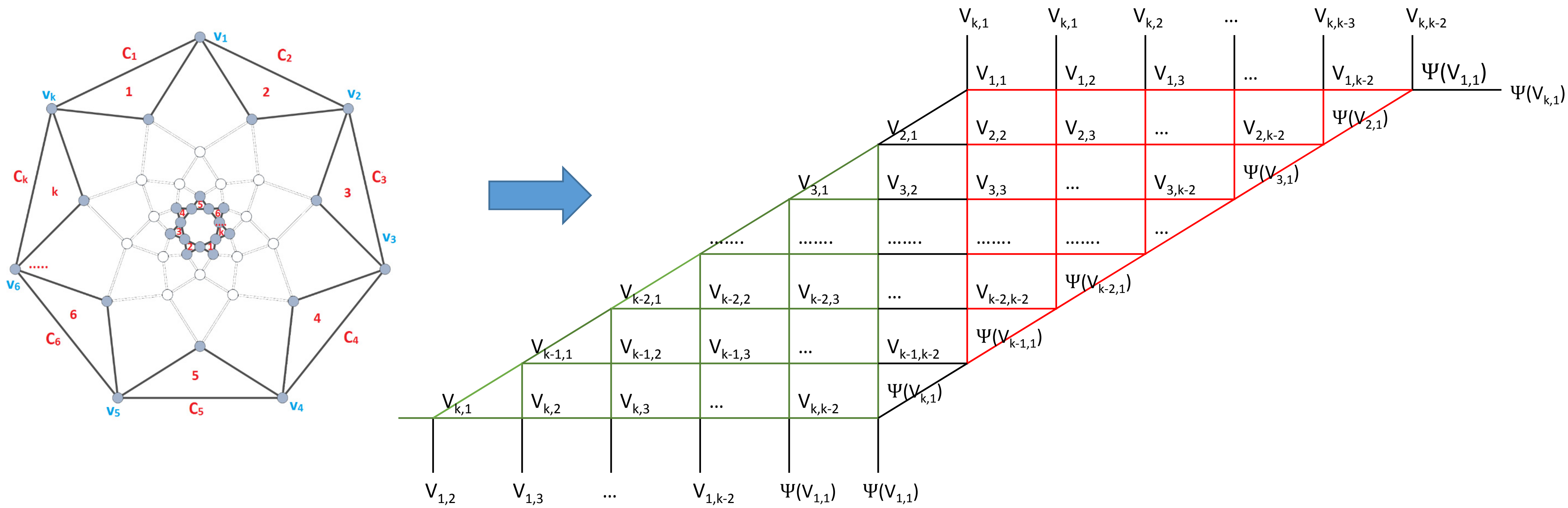
So there is only the case 1.1 creates a S_{k+1}

➔We only found a graph for S_{k+1} that is exactly depicted by the induction hypothesis for S_{k+1}

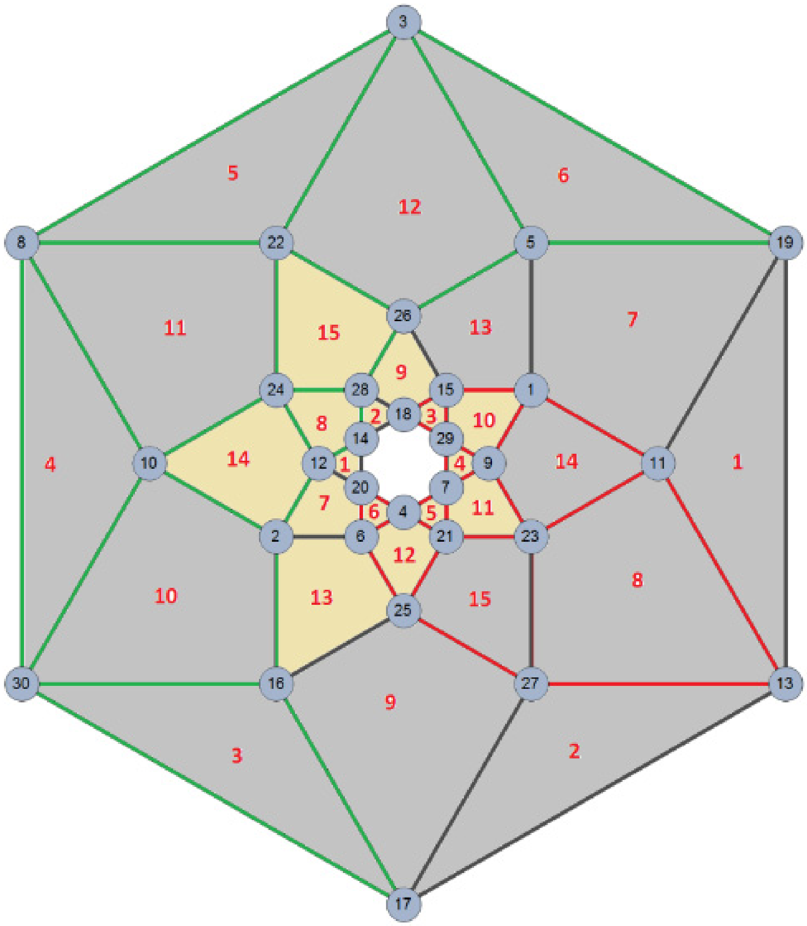
➔S_{k+1} is unique

Lemma 3.

S_k can be transform into the following equivalent graph:



Lemma 3 – Proof – A base case with 6 circles

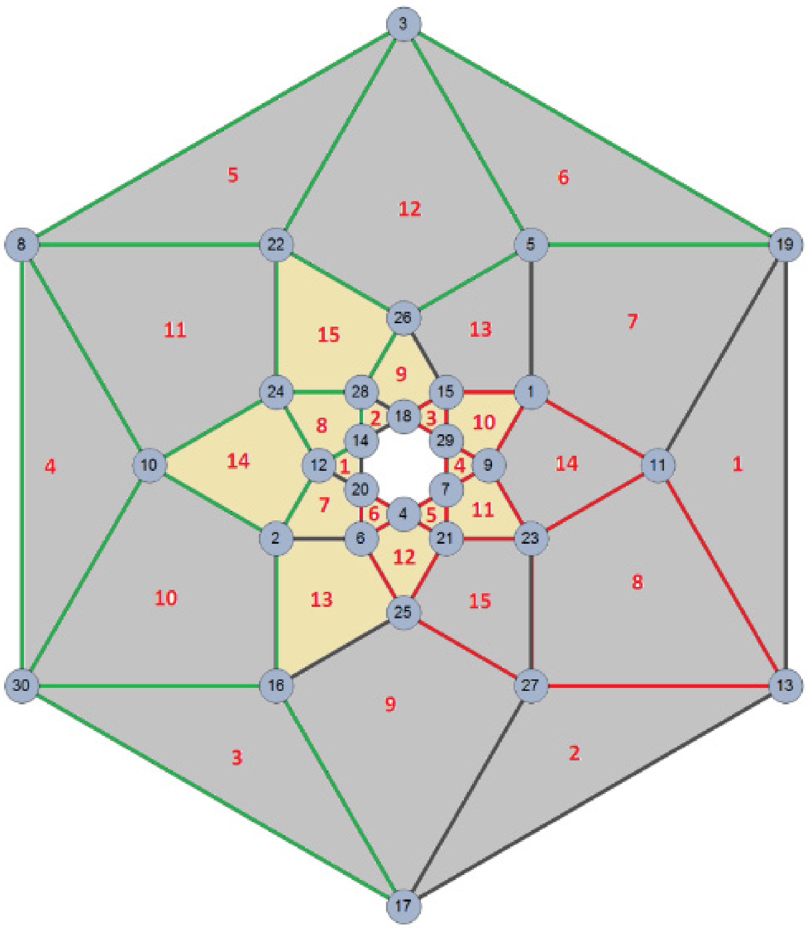


Here is the special graph of **6** great circles.

Annotation:

- Red edges ∈ 1^{st} side
- Green edges ∈ 2^{nd} side
- **Black** edges are the external links of the 1^{st} side and the 2^{nd} side
- Every region has a number in the middle
- There are 2 regions have the same number. One has 1 set of vertices while the other has the reflection of that set via O
- Grey/Yellow regions contain numbers distinctly

Lemma 3 – Proof – A base case with 6 circles

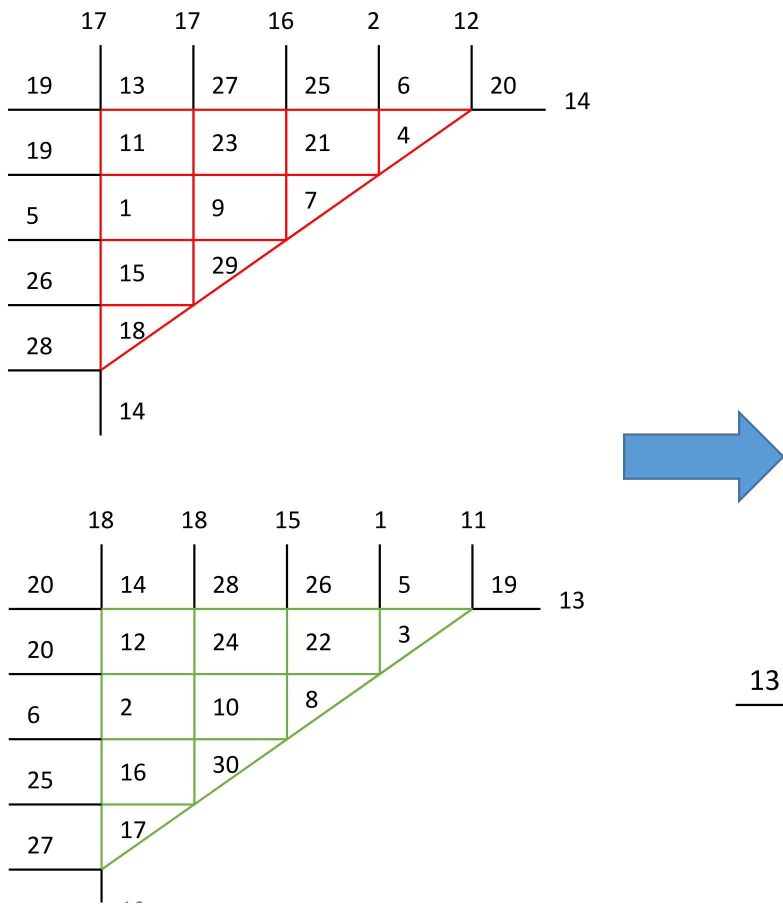


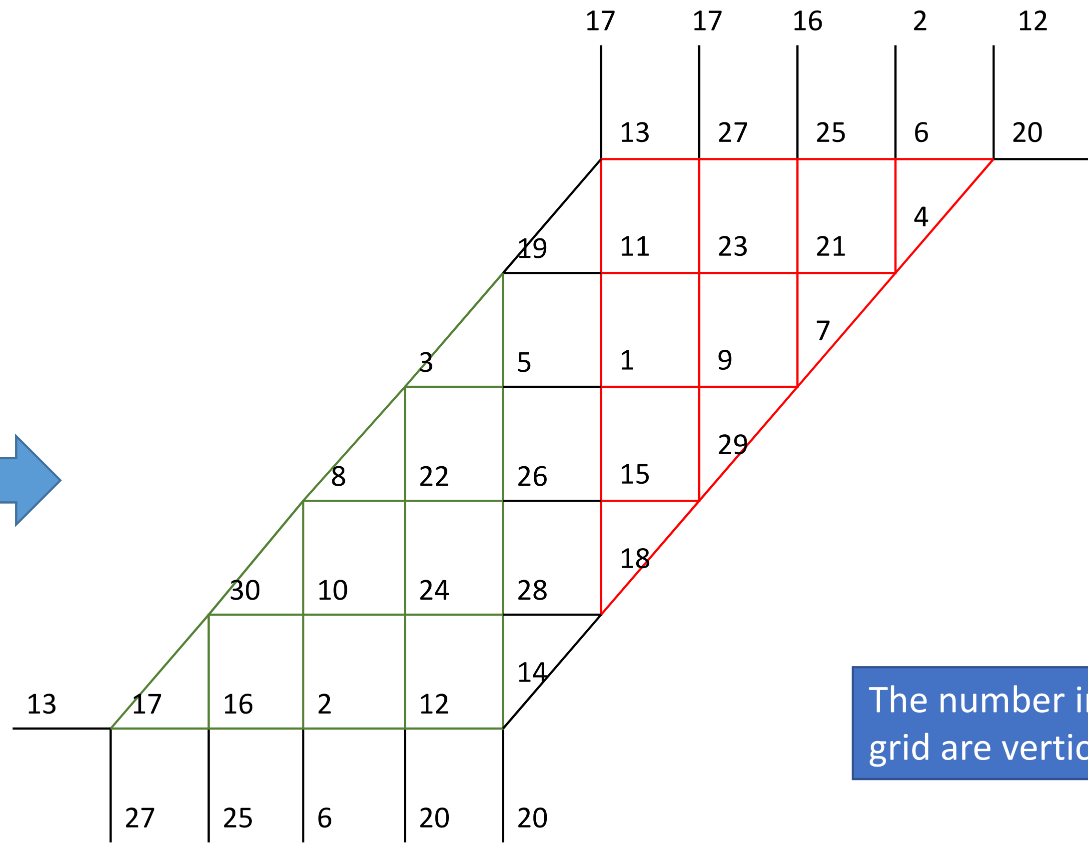
	17	17	16	2	12	
19	13	27	25	6	20	14
19	11	23	21	4		
5	1	9	7			
26	15	29				
28	18					
	14					

	18	18	15	1	11	
20	14	28	26	5	19	13
20	12	24	22	3		
6	2	10	8			
25	16	30				
27	17					
	13					

The number in this grid are vertices

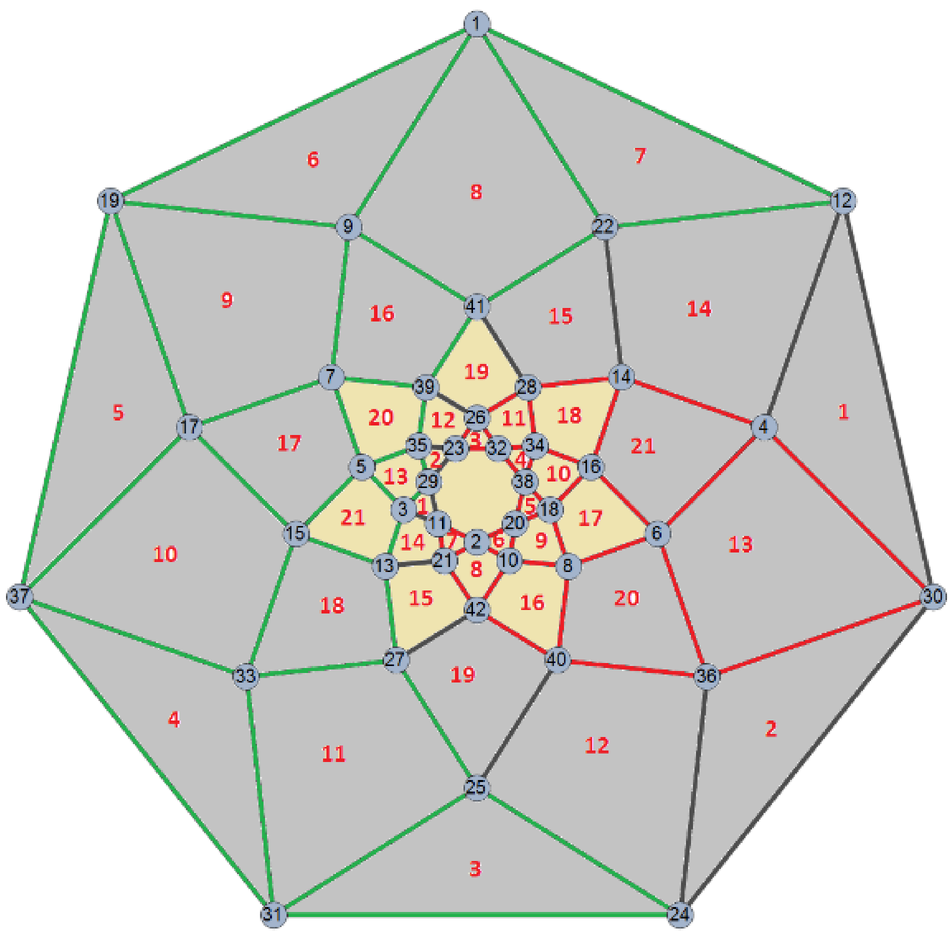
Lemma 3 – Proof – A base case with 6 circles





The number in this grid are vertices

Lemma 3 – Proof – A base case with 7 circles



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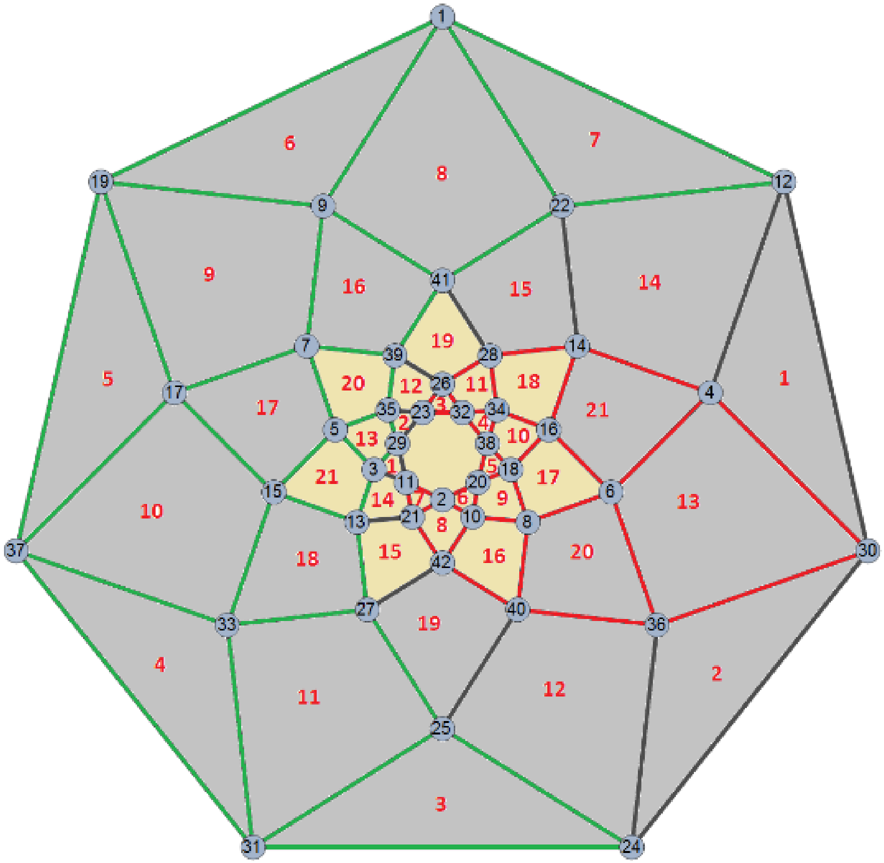
- Grey/Yellow regions contain numbers distinctly

Lemma 3 – Proof – A base case with 7 circles

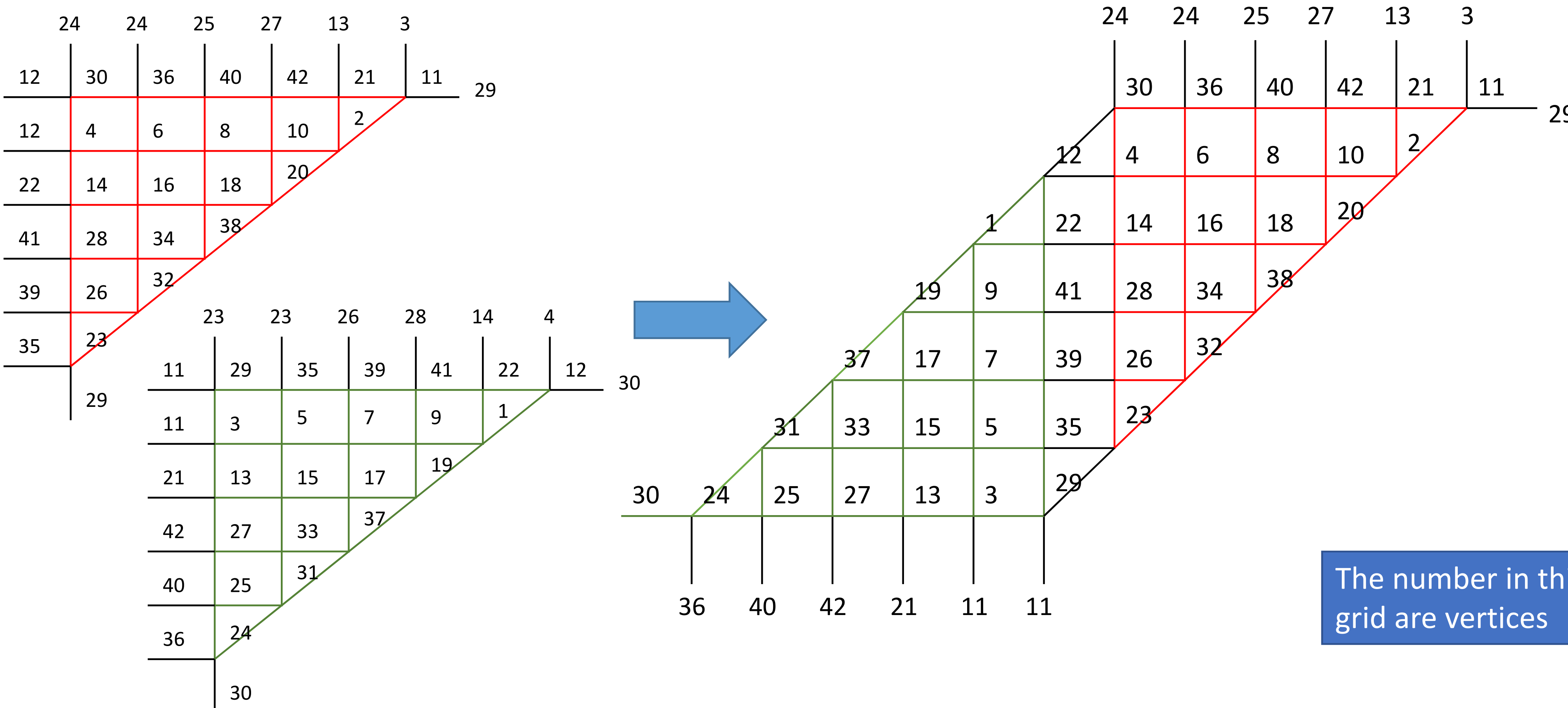


	24	24	25	27	13	3	
12	30	36	40	42	21	11	29
12	4	6	8	10	2		
22	14	16	18	20			
41	28	34	38				
39	26	32					
35	23						
	29						

	23	23	26	28	14	4	
11	29	35	39	41	22	12	30
11	3	5	7	9	1		
21	13	15	17	19			
42	27	33	37				
40	25	31					
36	24						
	30						

The number in this grid are vertices

Lemma 3 – Proof – A base case with 7 circles

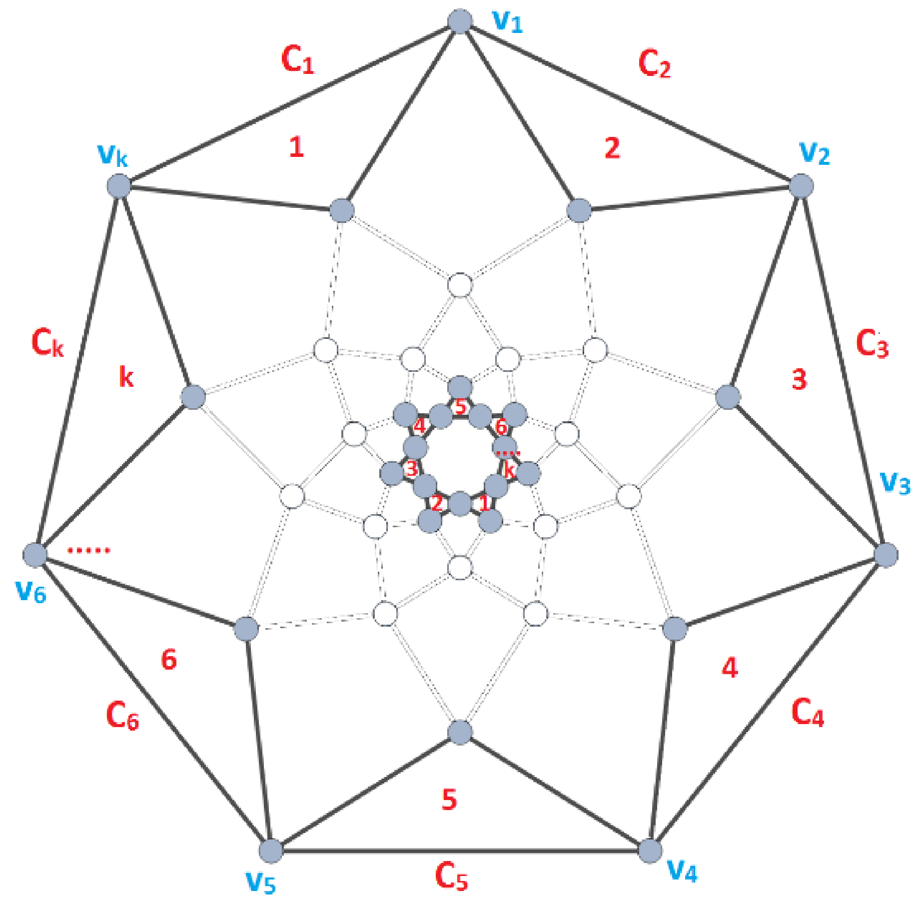


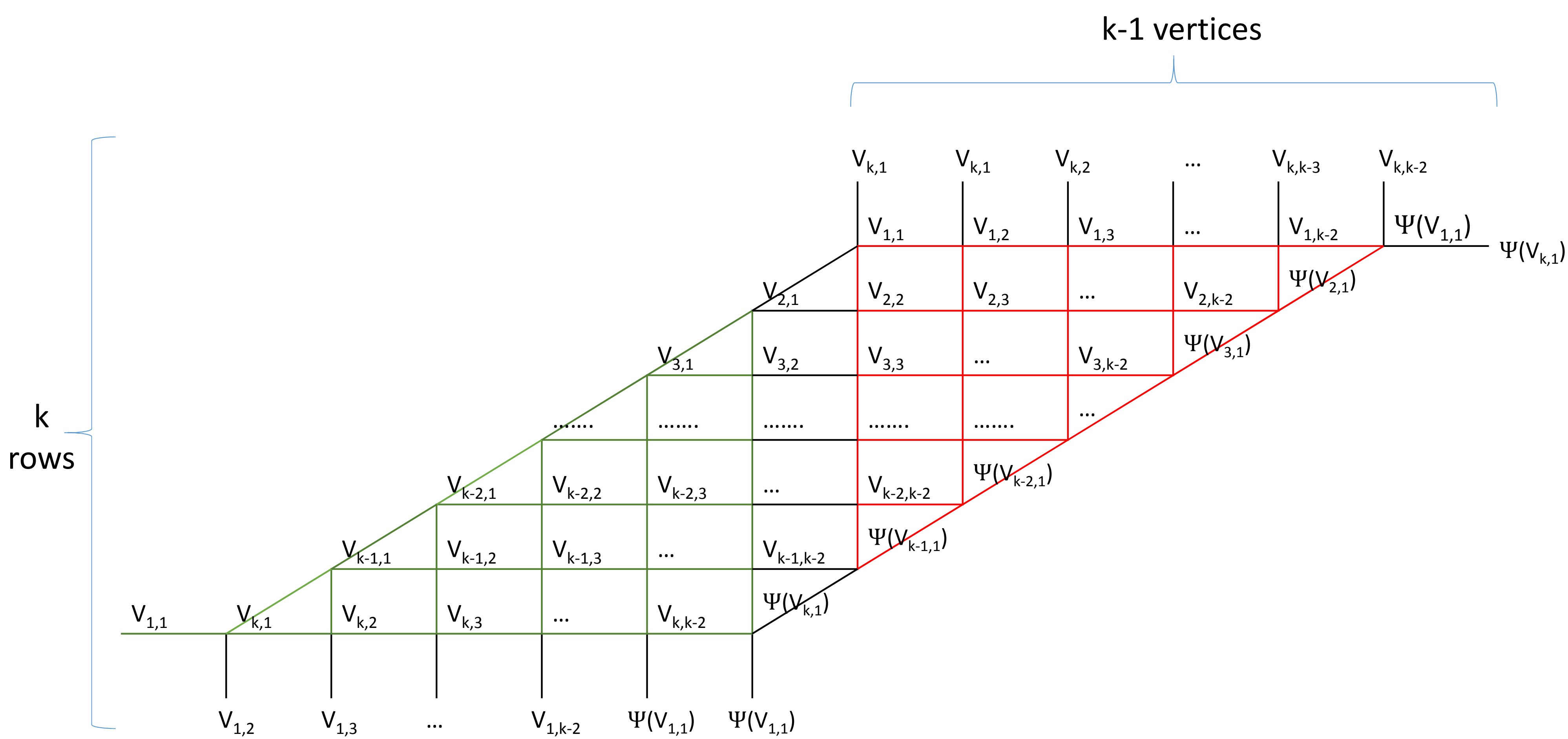
The number in this grid are vertices

Lemma 3

Call

- $V_{i,1}$ are the vertices made by C_i and C_{i+1} on the unbounded cycle.
- $V_{i,1}$, $V_{i,2}$, $V_{i,3}$, … $V_{i,2k-2}$ are the vertices on the circle C_i in the order that ($V_{i,1}$, $V_{i,2}$, $V_{i+1,1}$) is a triangle




k-1 vertices
k
rows
V k,1
V k,1
V k,2
...
V k,k-3
V k,k-2
V 1,1
V 1,2
V 1,3
...
V 1,k-2
Ψ(V 1,1)
Ψ(V k,1)
V 2,1
V 2,2
V 2,3
...
V 2,k-2
Ψ(V 2,1)
V 3,1
V 3,2
V 3,3
...
V 3,k-2
Ψ(V 3,1)
....
....
....
....
....
...
V k-2,1
V k-2,2
V k-2,3
...
V k-2,k-2
Ψ(V k-2,1)
V k-1,1
V k-1,2
V k-1,3
...
V k-1,k-2
Ψ(V k-1,1)
V 1,1
V k,1
V k,2
V k,3
...
V k,k-2
Ψ(V k,1)
V 1,2
V 1,3
...
V 1,k-2
Ψ(V 1,1)
Ψ(V 1,1)

Lemma 4. The properties of S_k

There are 2k triangles, (k+1)*(k-3) quadrilaterals and 2 polygon has k segments.

Proof: By Lemma 2 and Lemma 3, we can easily count this

Theorem 1

The chromatic number of $\mathcal{S}_{\mathrm{k}}$ is 3

Theorem 1 – Proof

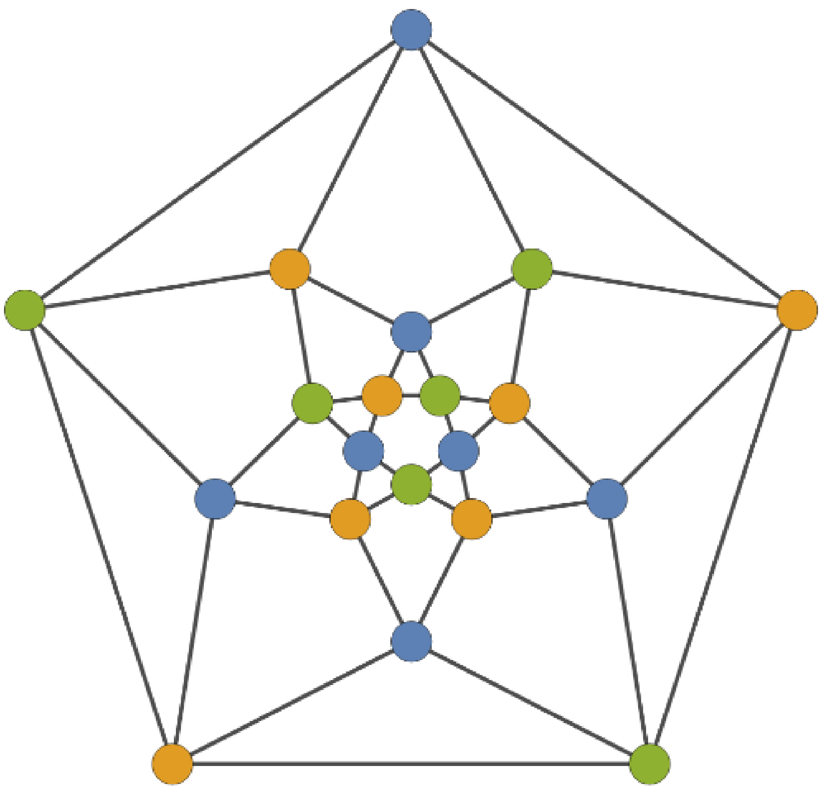
- By Lemma 2.2, S_k is unique
- By Lemma 3, S_k can be transformed into an equivalent parallelogram
- The proof to prove the equivalent parallelogram is 3-colorable is in the presentation in 01-25-2015

Adding a circle

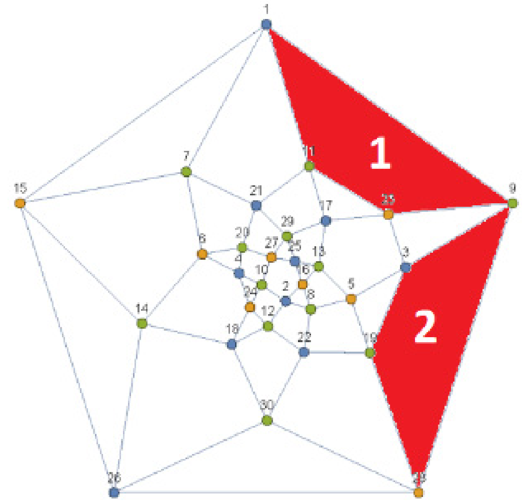
Adding a circle

- This part presents how a graph is changed after adding a random great circle into it
- I will show the transitions from graphs of 5, 6 great circles to 6, 7 great circles respectively

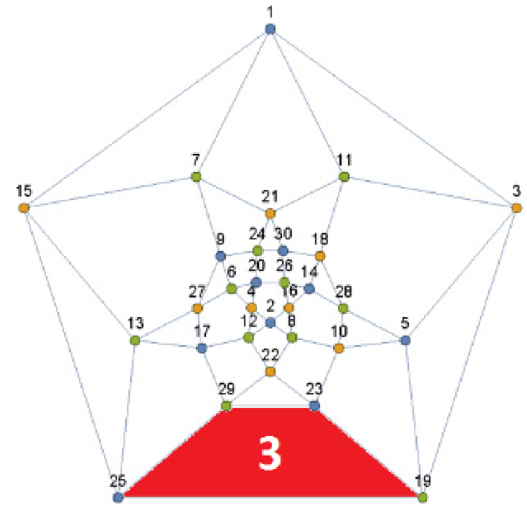
5->6 great circles



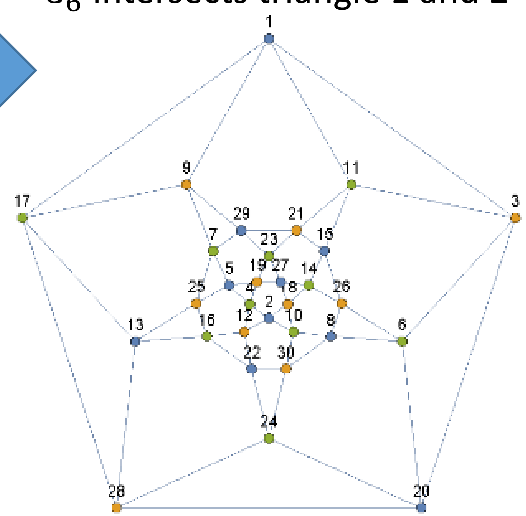
S_5



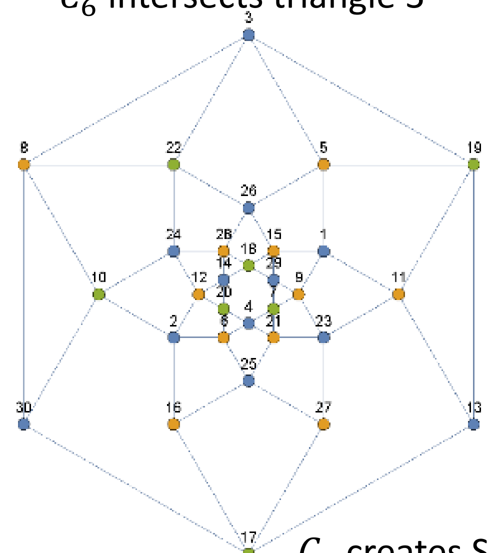
C_6 intersects triangle 1 and 2



C_6 intersects triangle 3

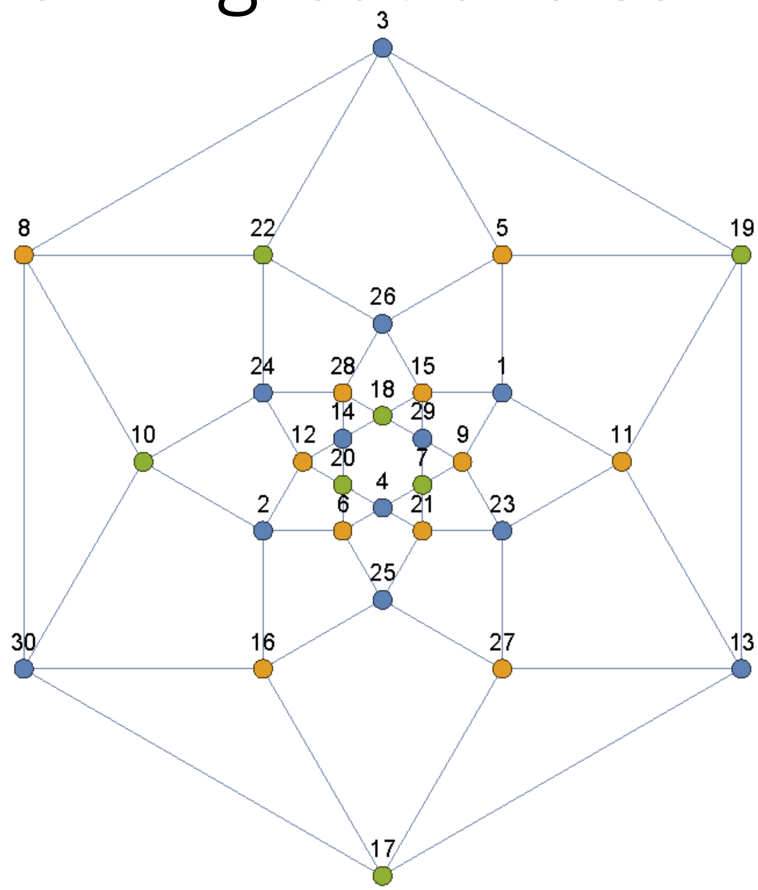


C_6 intersects no triangle

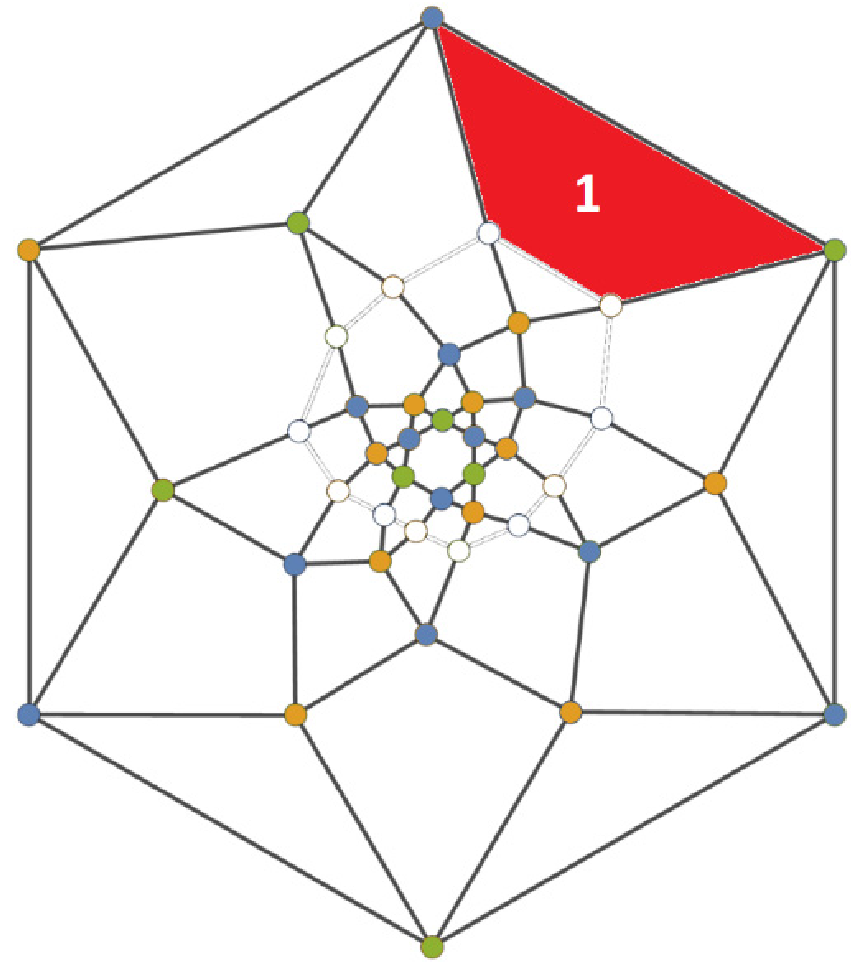


C_6 creates S_7

6->7 great circles - 1

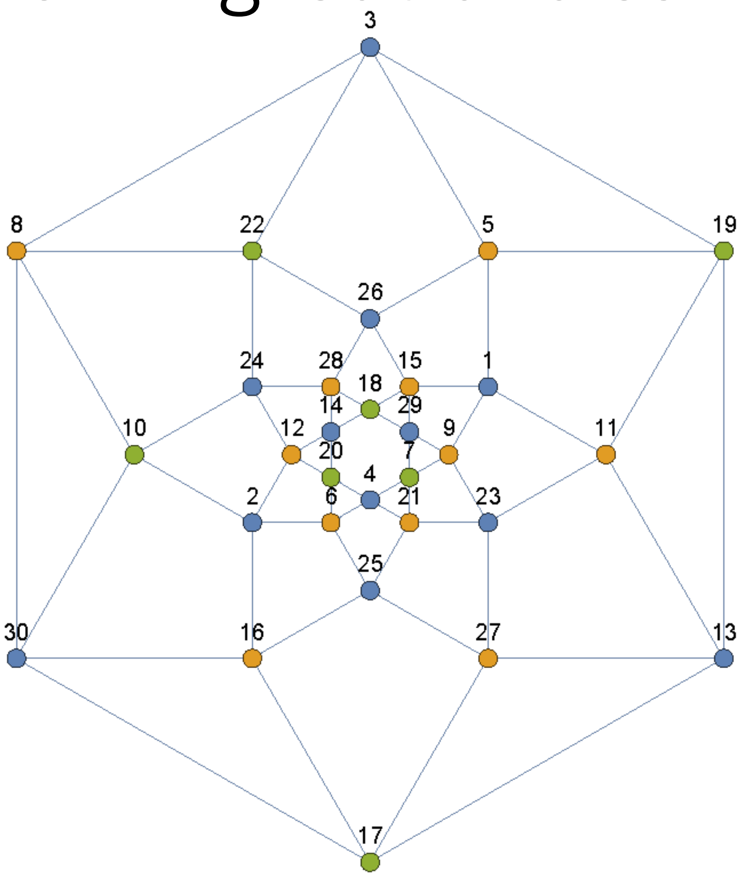


S_6

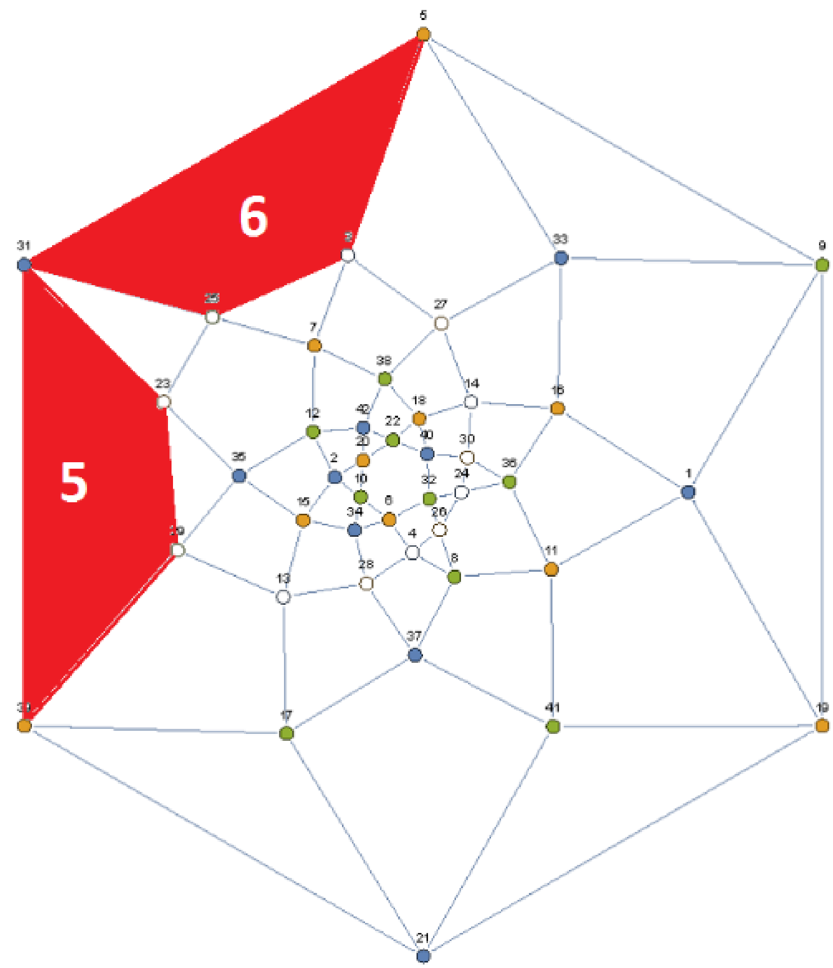


S_6 ; a circle intersecting triangle 1

6->7 great circles - 2

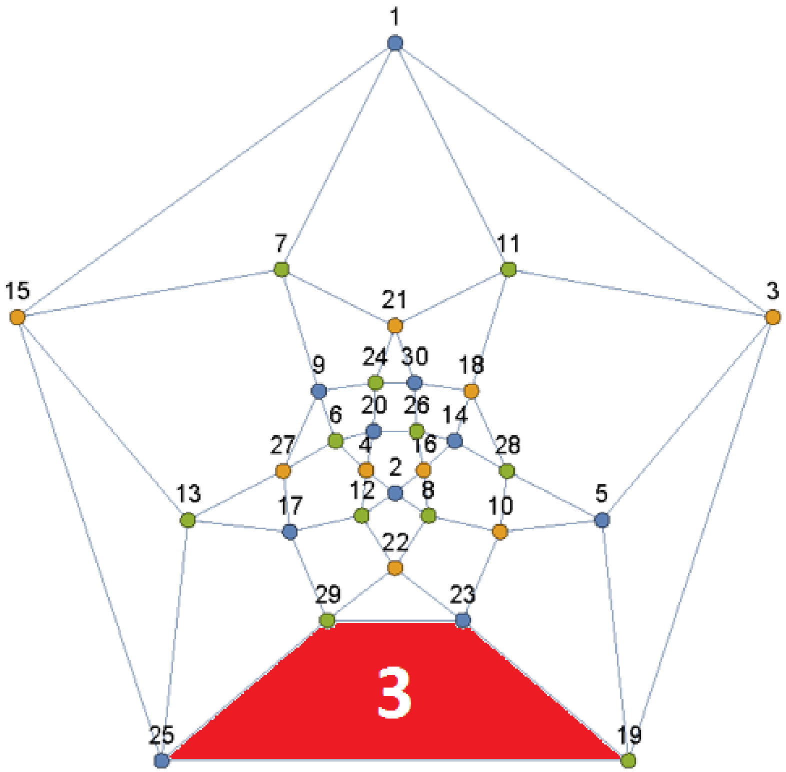


S_6

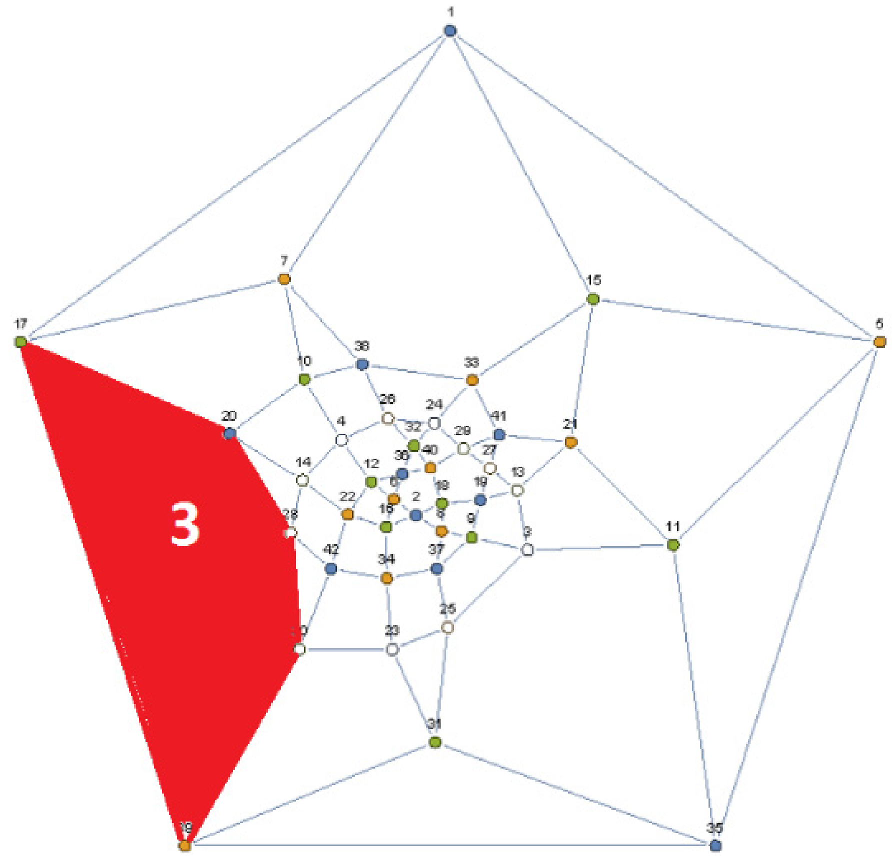


S_6 ; a circle intersecting triangle 5 and 6

6->7 great circles - 3

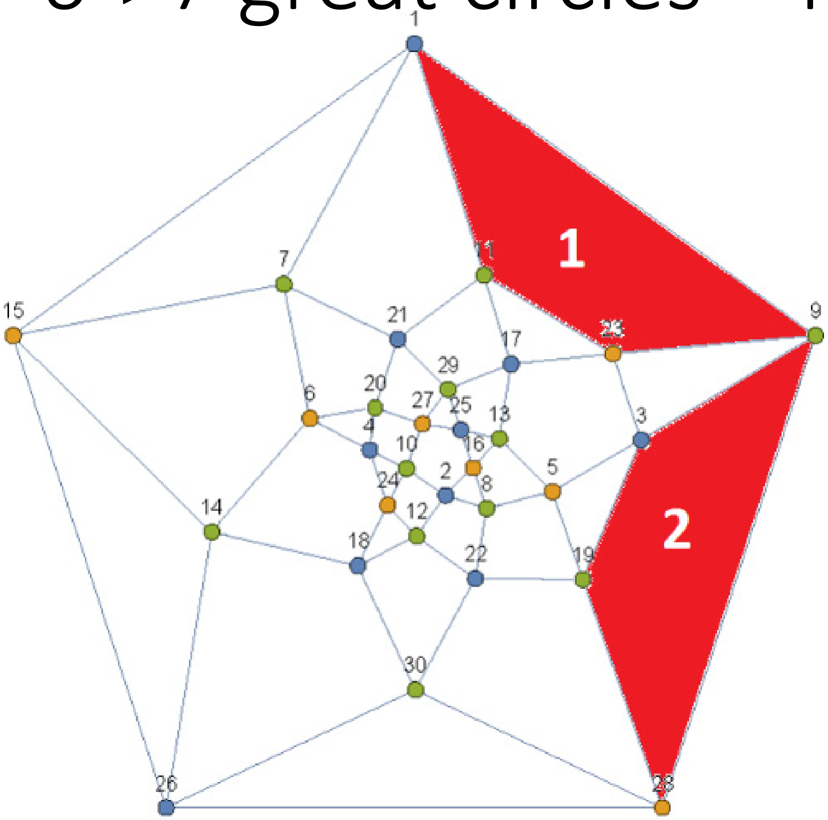


S_5 ; a circle intersecting triangle 3

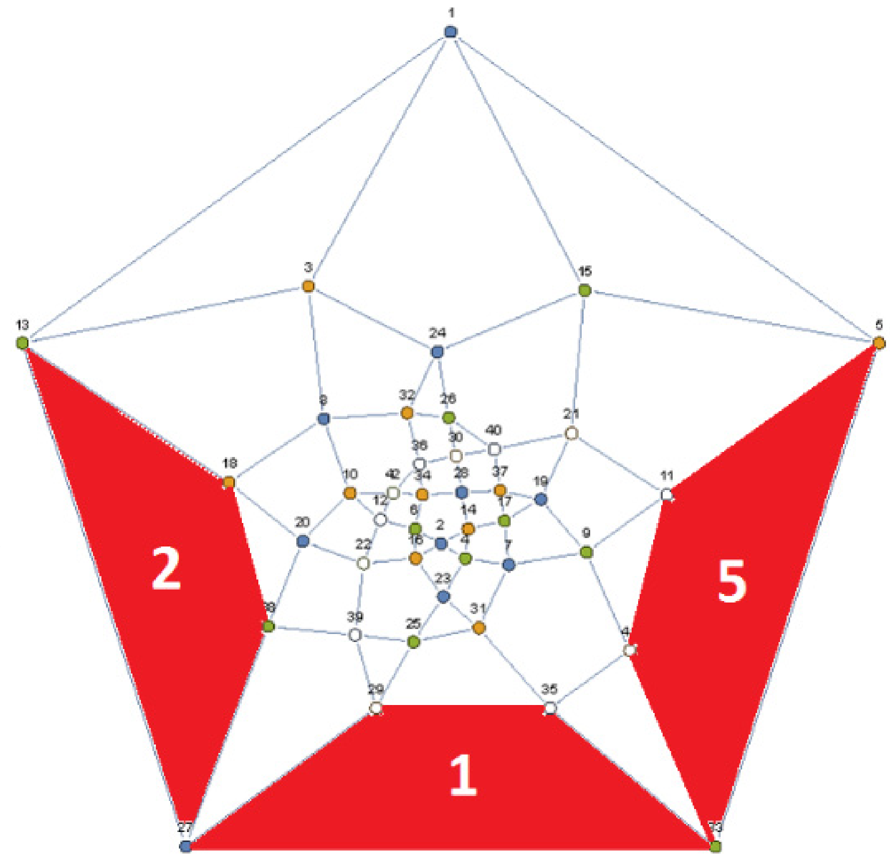


S_5 ; 2 circles intersecting triangle 3 and they also intersected each other in the triangle 3

6->7 great circles - 4

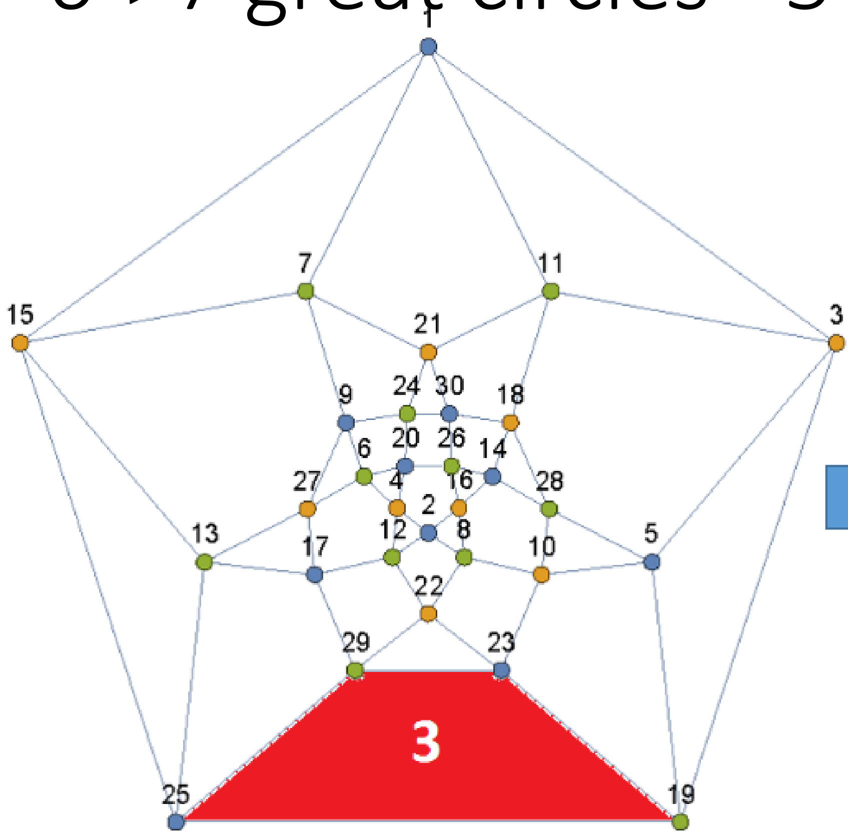


S_5 ; a circle intersecting triangle 1 and 2

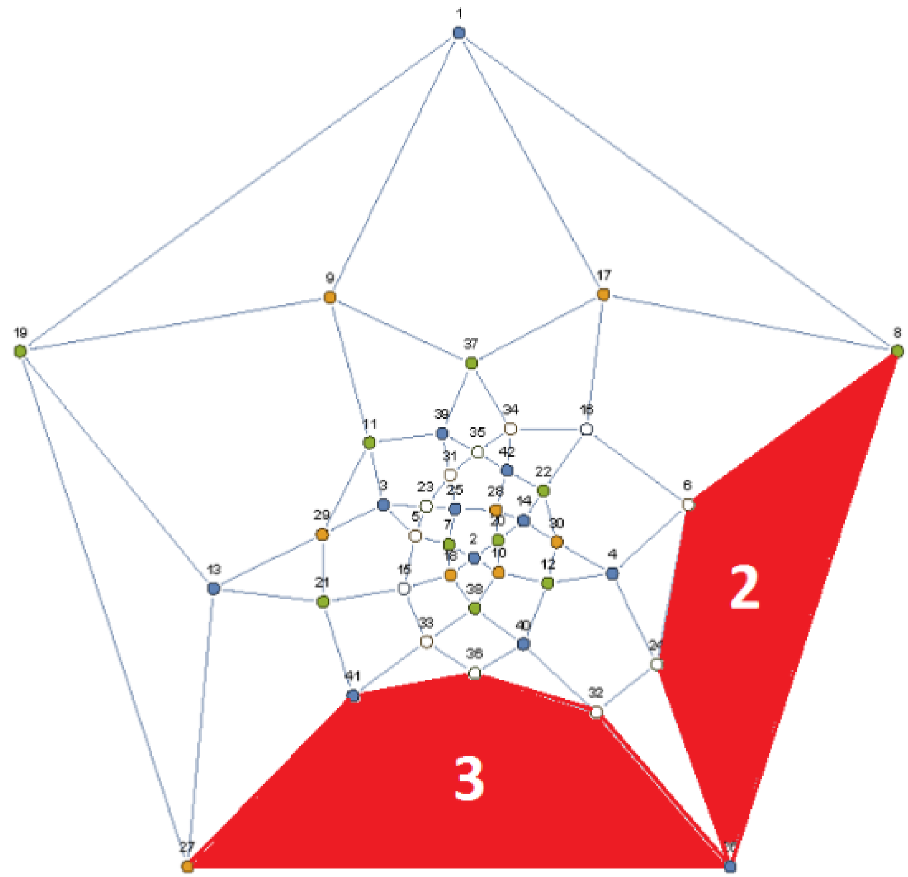


S_5 ; a circle intersecting triangle 1 and 2
; a circle intersecting triangle 1 and 5

6->7 great circles - 5



S_5 ; a circle intersecting triangle 3

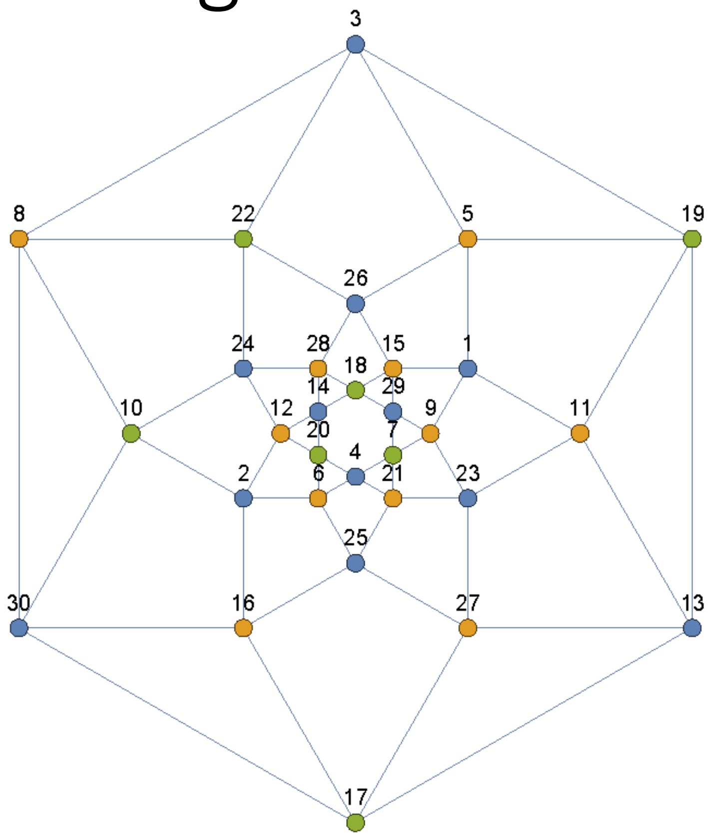


S_5 ; a circle intersecting triangle 3

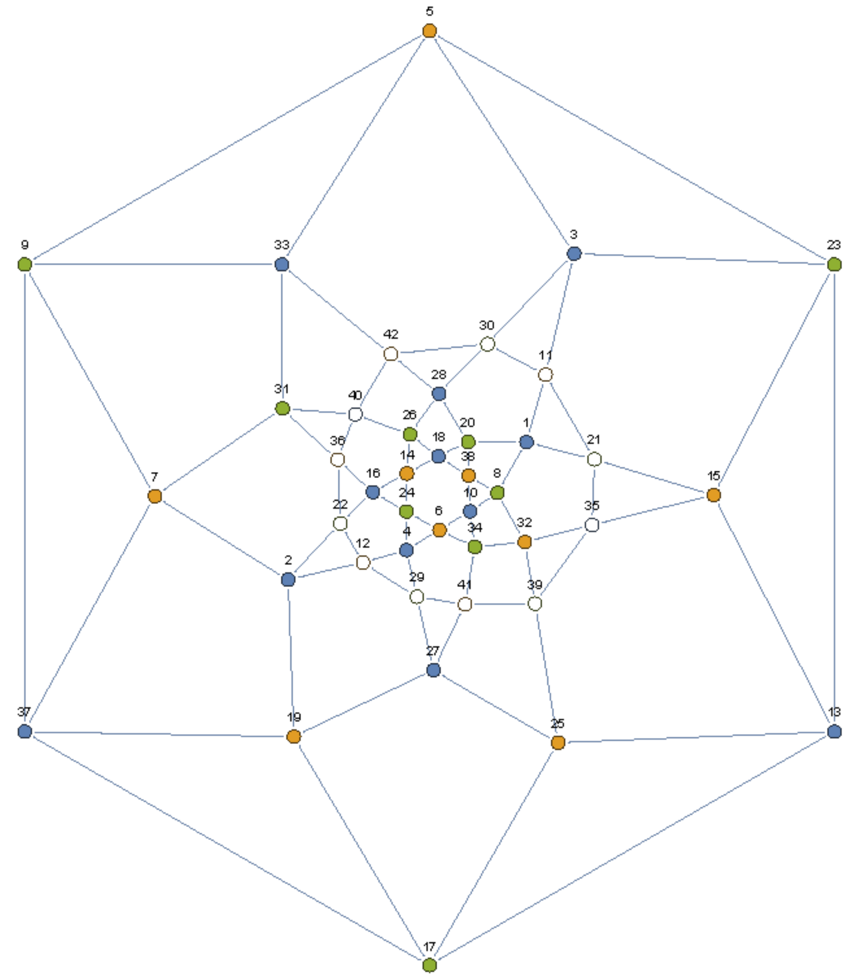
; a circle intersecting 2 and 2

; 2 circles intersected inside of triangle 3

6->7 great circles - 6

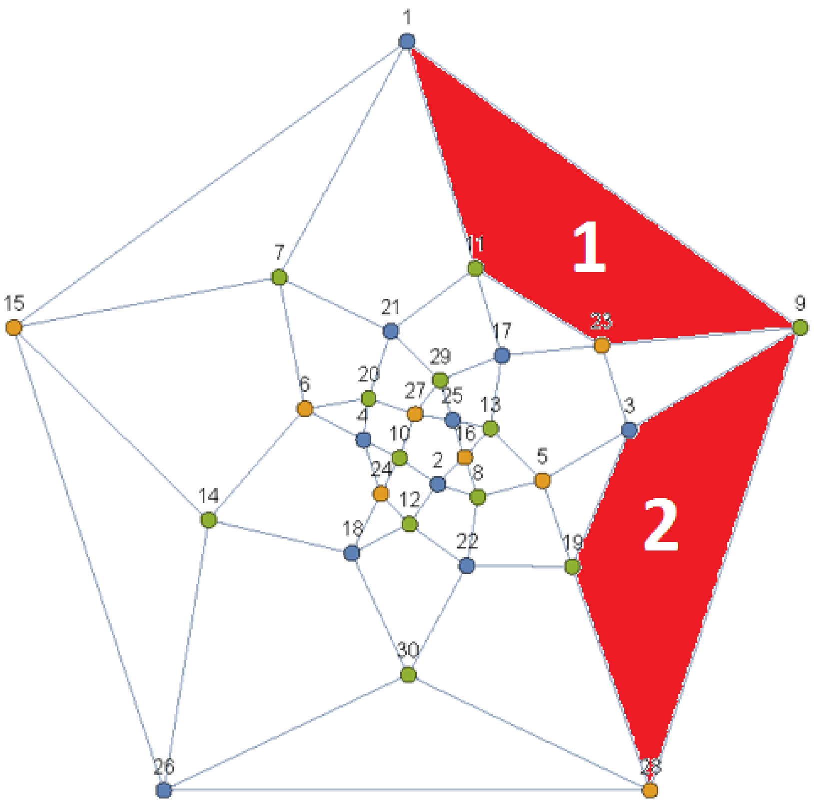


S_6

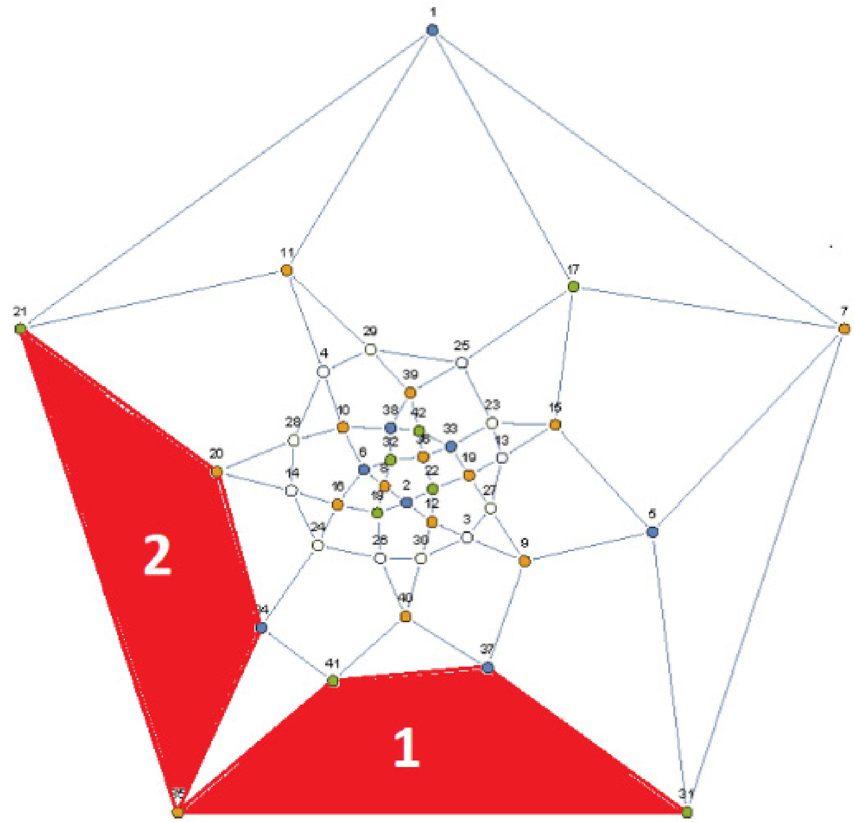


S_6 ; a circles doesn't intersect any triangle

6->7 great circles - 7



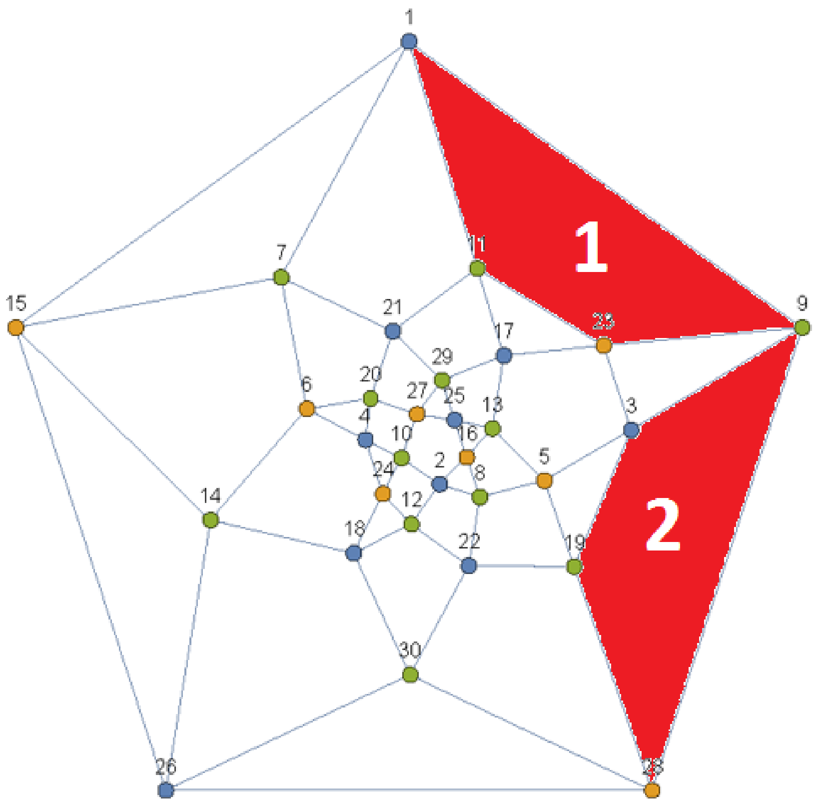
S_5 ; a circle intersecting triangle 1 and 2



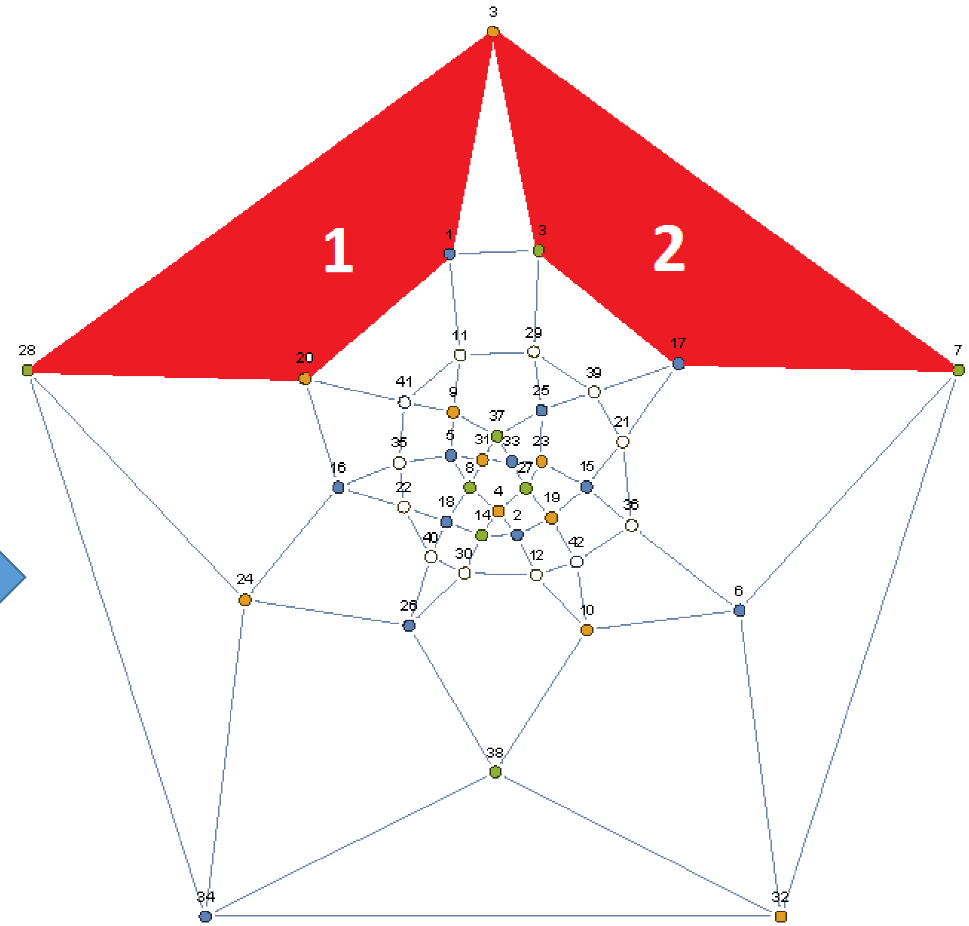
S_5 ; a circle intersecting triangle 1 and 2

; a cicle doesn't intersect any triangle

6->7 great circles - 8

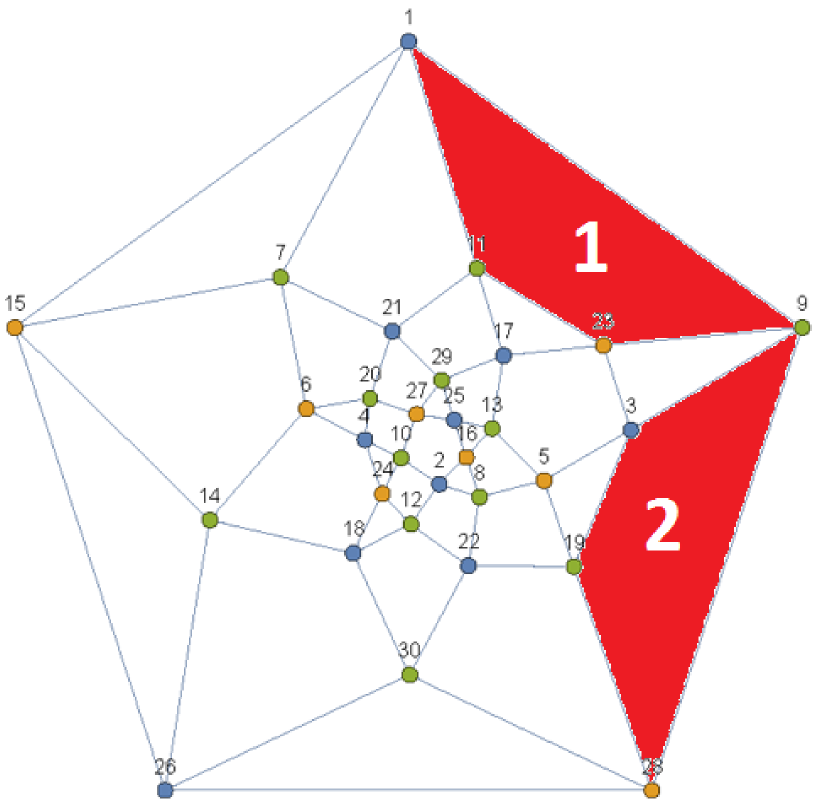


S_5 ; a circle intersecting triangle 1 and 2

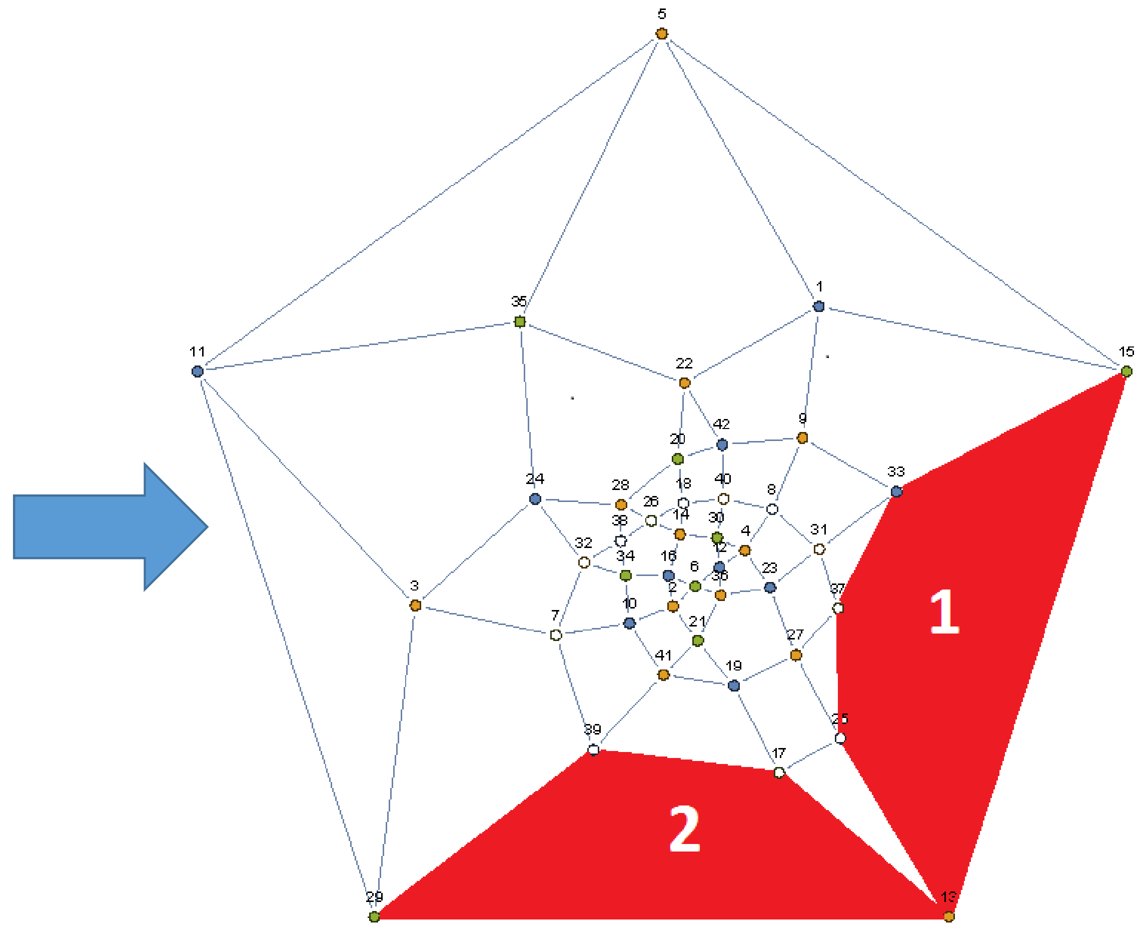


S_5 ; 2 circles intersecting triangle 1 and 2

6->7 great circles - 9



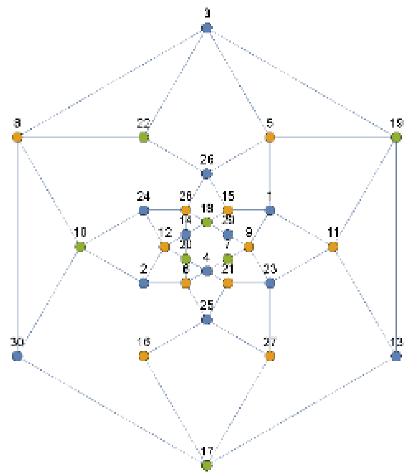
S_5 ; a circle intersecting triangle 1 and 2

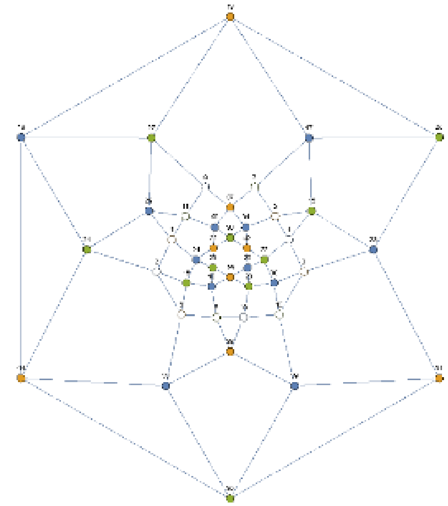


S_5 ; 2 circles intersecting triangle 1 and 2 and they also intersected each other in the triangle 1

6->7 great circles - 10

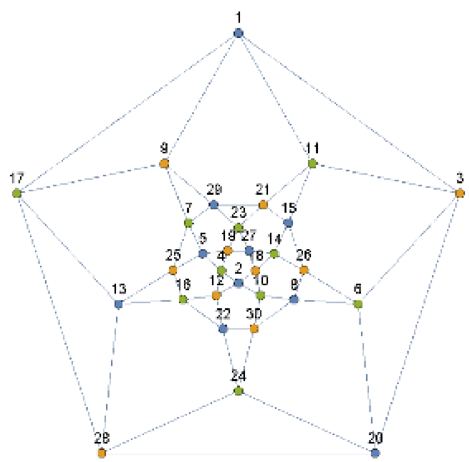
S_6

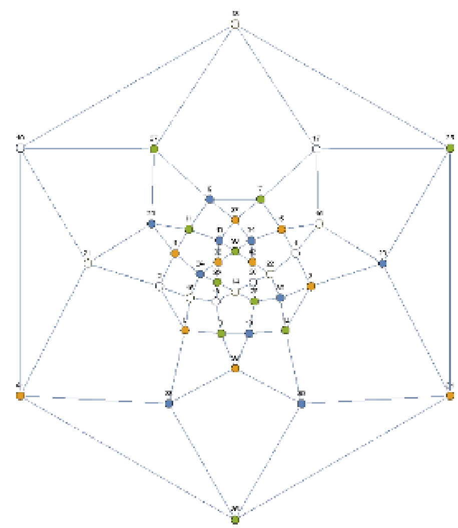




S_6; a circle doesn't intersect any triangle

S_5 ; a circle doesn't intersect any triangle





S_6; a circle doesn't intersect any triangle

6->7 great circles - 11