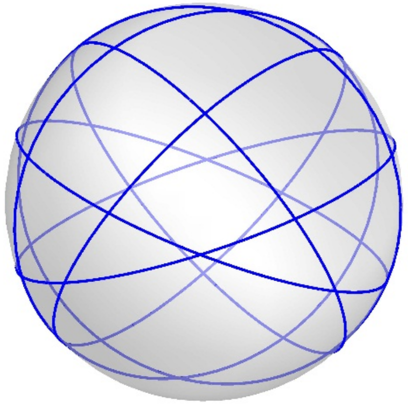


Great Circles Problem

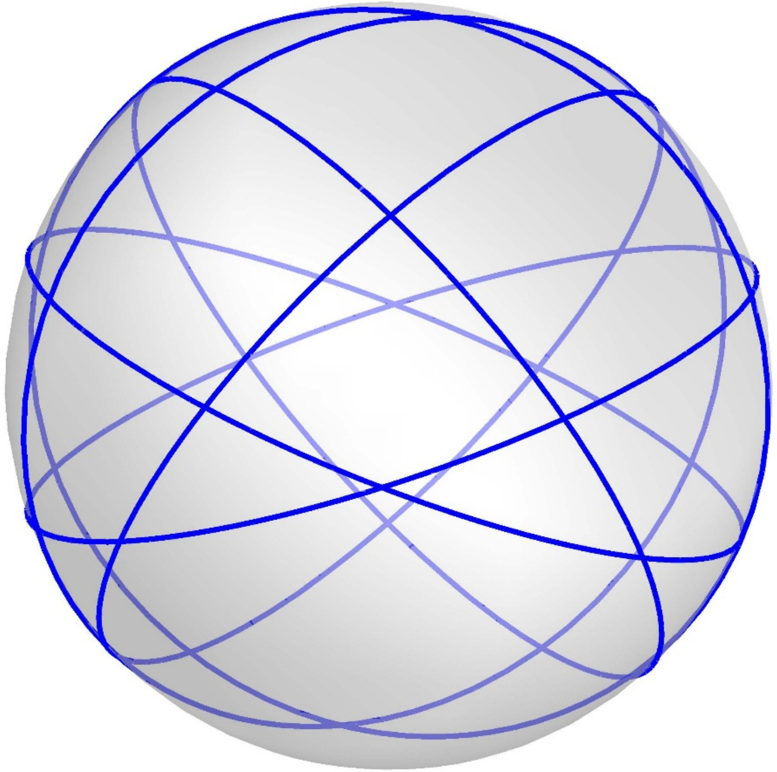
Kha Man

12-16-2014

Outline

- Problem restatement
- 2 example cases with 9 and 10 great circles
- The big picture of the problem
- The proof about 3 colourability for the problem ($\boldsymbol{\chi(G) = 3}$)

Problem Restatement

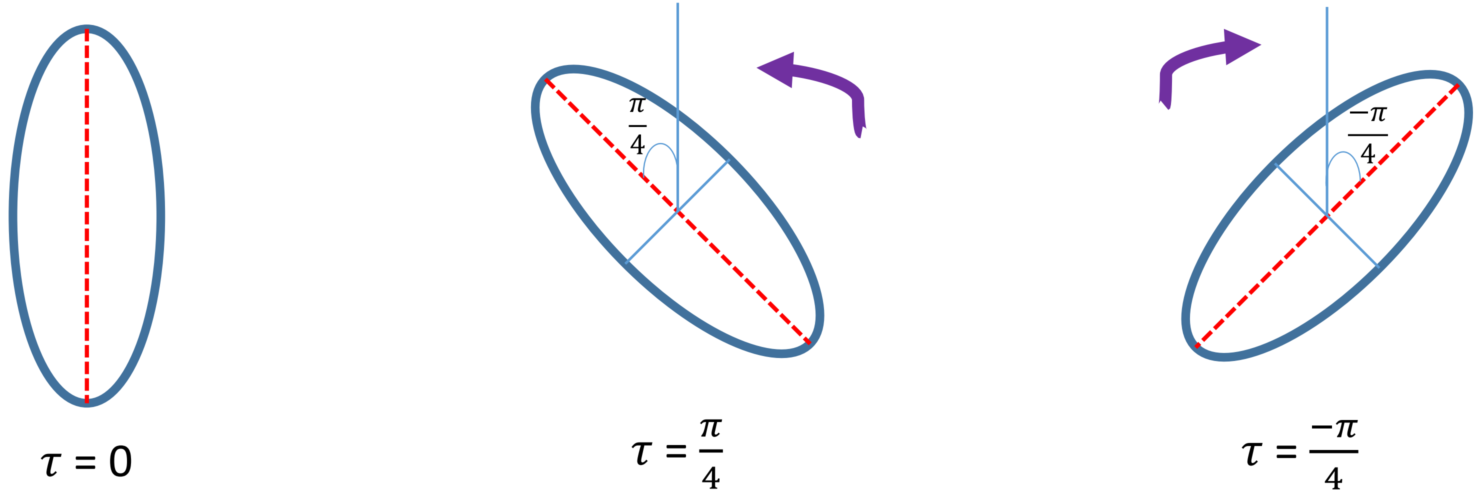


(Great Circle Problem) A great circle is any circle on a sphere whose radius is **the same as the radius of the sphere** (so it is largest possible). A circle that goes through both the North and South poles is an example of a great circle on the Earth. Given n ($n \geq 3$) great circles on a sphere, no three of which pass through a single point, form a great circle graph by making points of intersection into vertices, and connect two vertices by an edge if and only if there is an arc between them.

Problem: What is the largest chromatic number of any great circle graph?

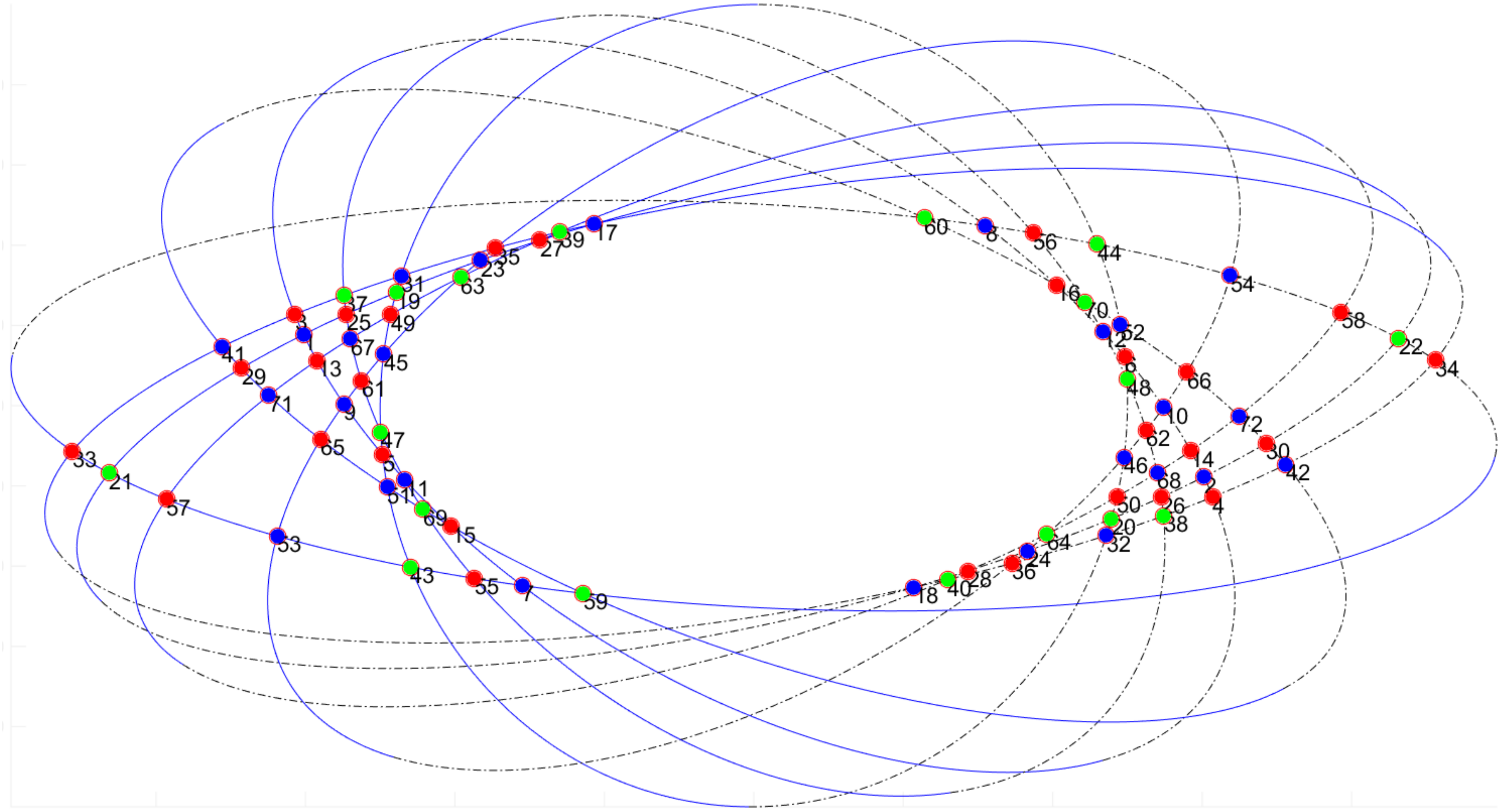
Problem Restatement

- Create ellipses with the inclination angle τ $[\frac{-\pi}{2}, \frac{\pi}{2})$ on the sphere



- Why does this definition still keep the originality of the problem?

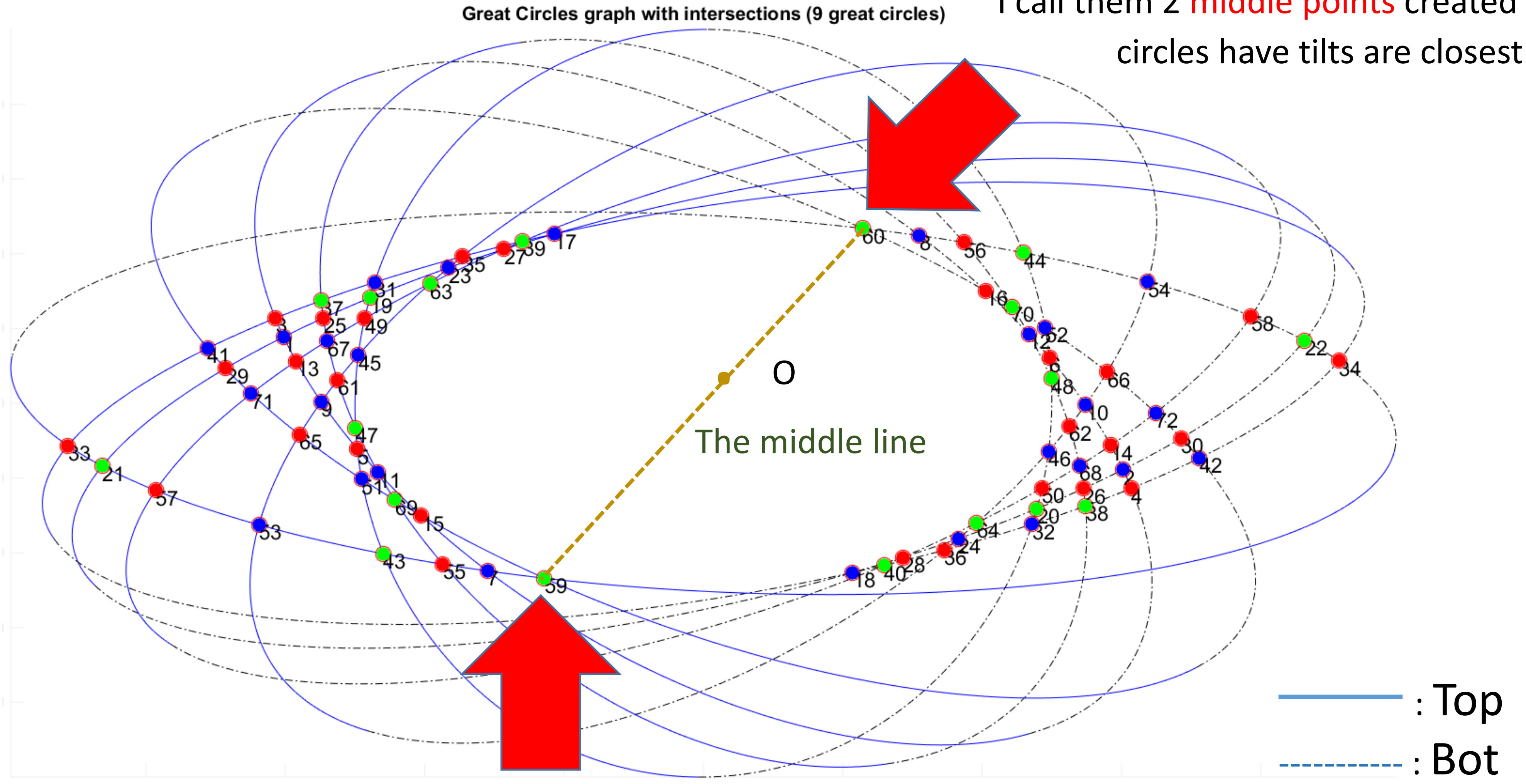
Problem Restatement



Great Circles graph with intersections (9 great circles)

An example case (9 great circles)

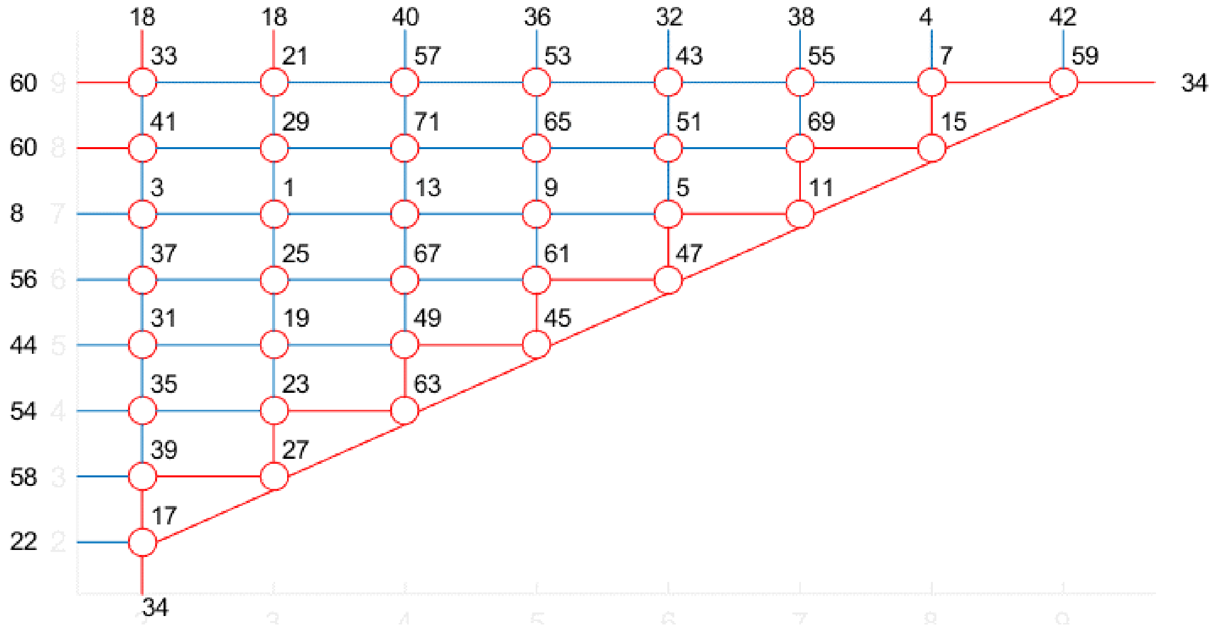
I call them 2 middle points created by 2 circles have tilts are closest to $\frac{\pi}{2}$



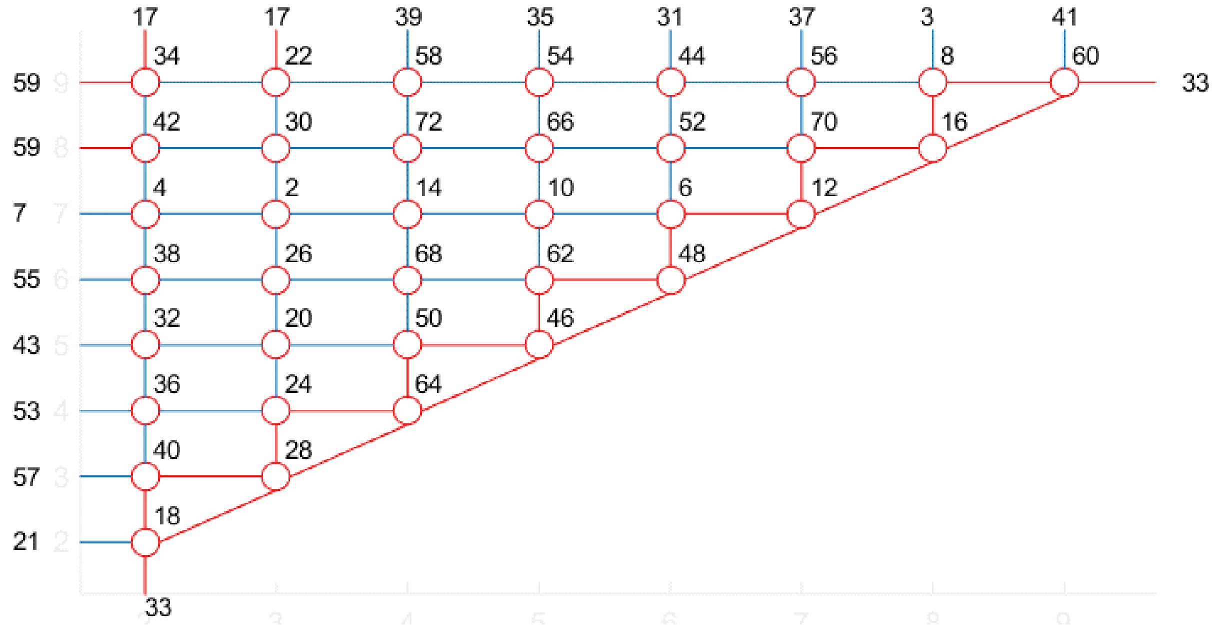
A example case (9 great circles)

- Observation:
  - The center of all circles O is the **point symmetry** of vertices on both sides of the line connected 2 middle points
    - Proof:
      - The cut of two ellipses is a line. Therefore, every couple of 2 ellipses will have 2 intersections and the center on the same line (Euclidean's theorem) or O is the point symmetry of all intersections

Redraw the graph (9 great circles)

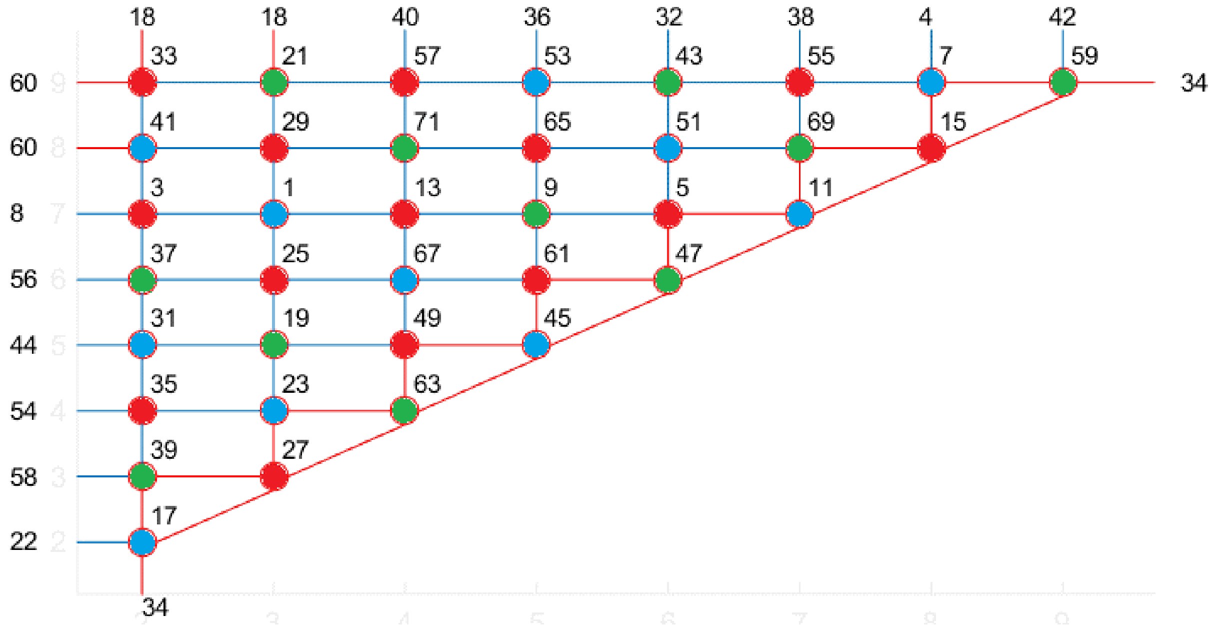


Left side

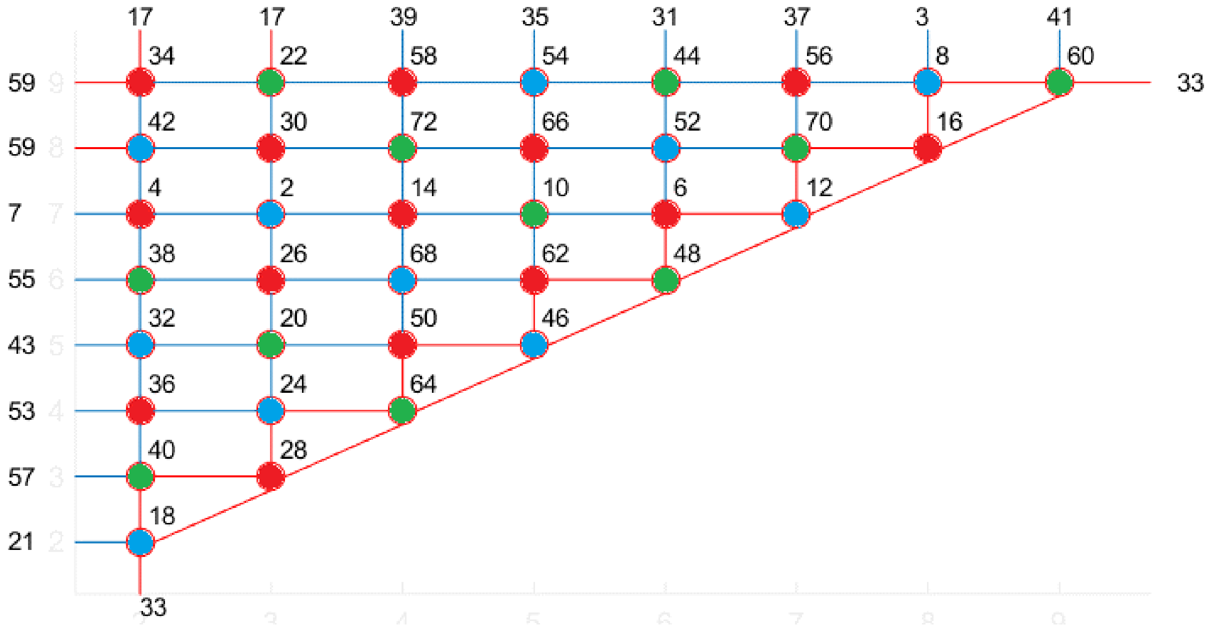


Right side

Color the graph



Left side

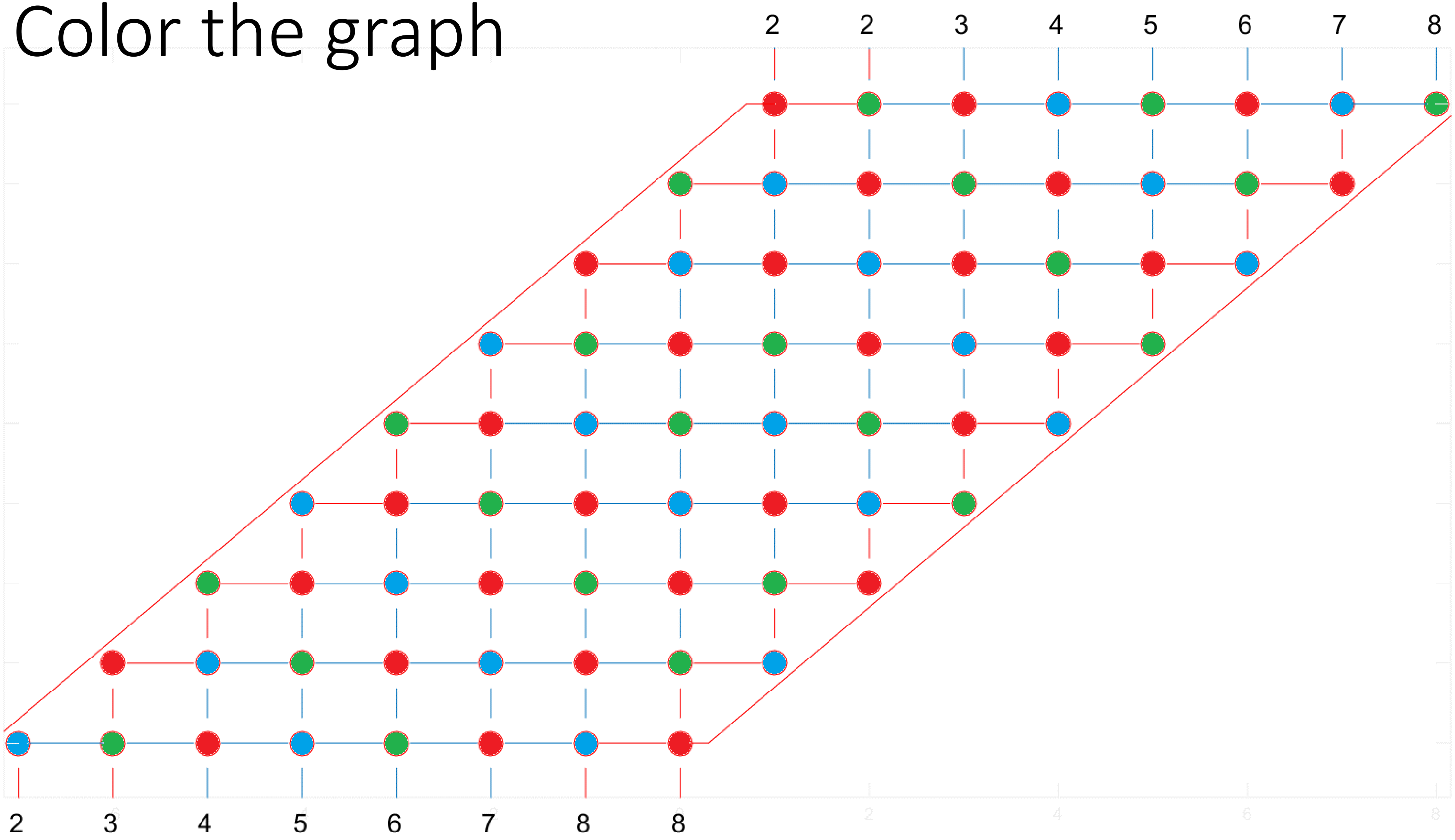


Right side

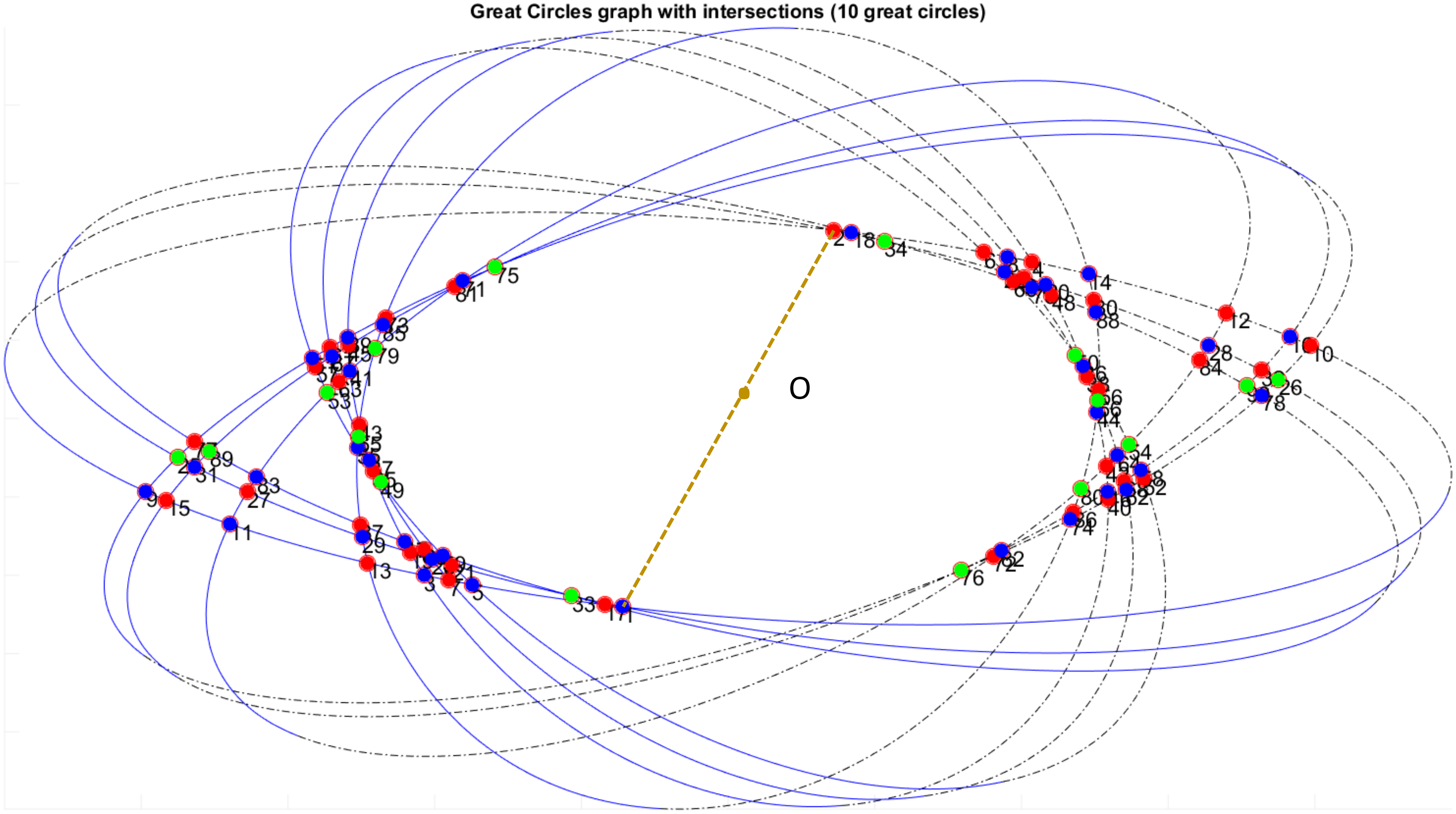
Observation: The vertices on the same diagonal usually have the same color

Color the graph

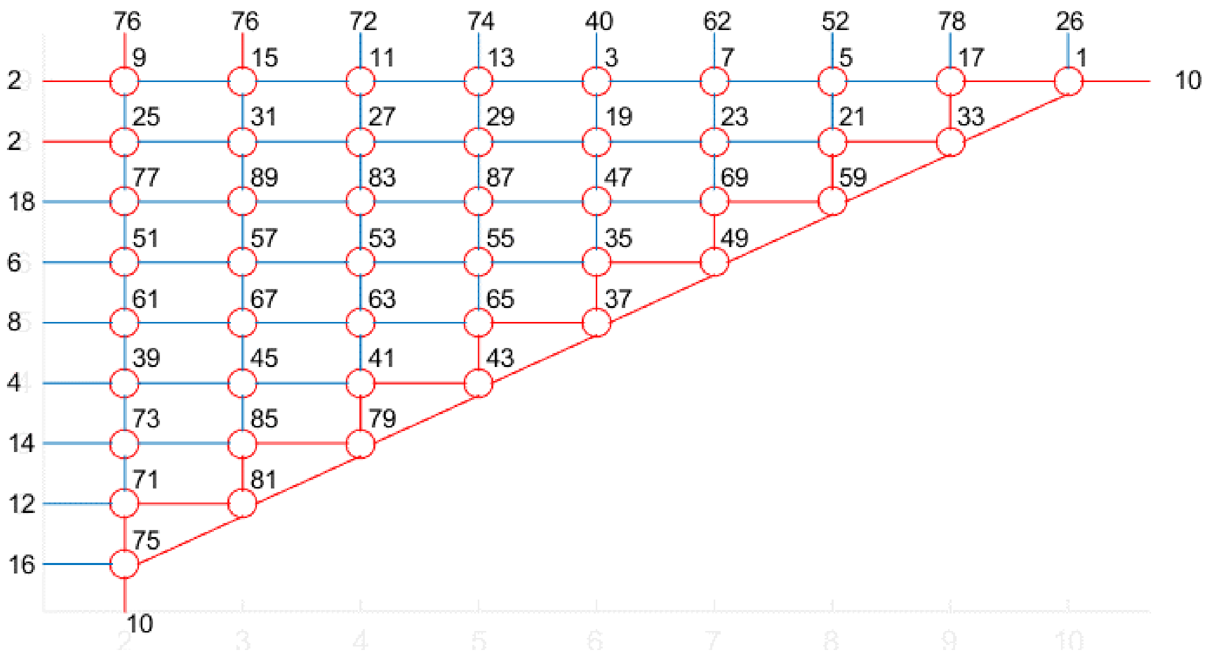
The numbers only show the connection in this graph only



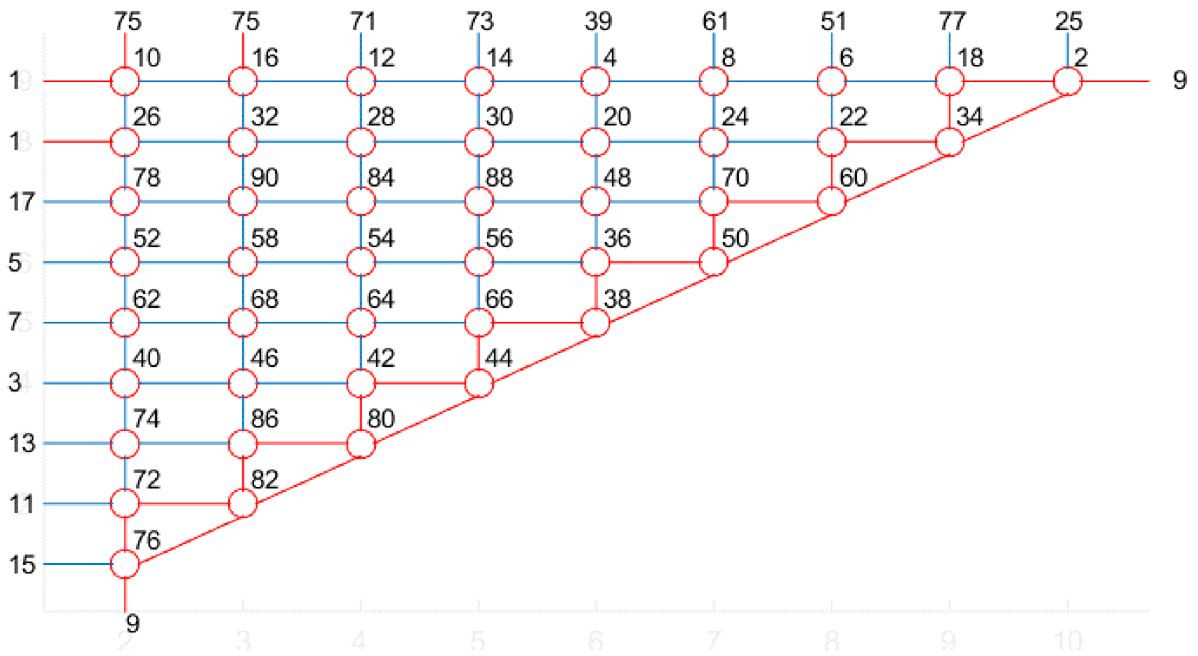
An another example (10 great circles)



Redraw the graph (10 great circles)

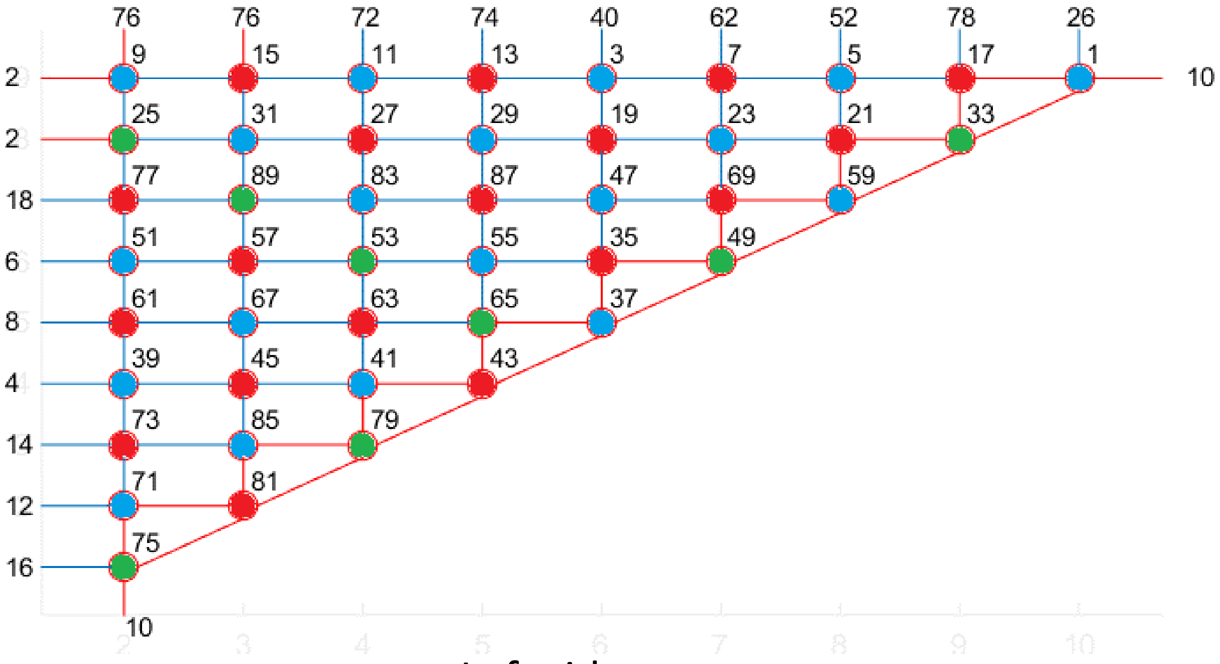


Left side

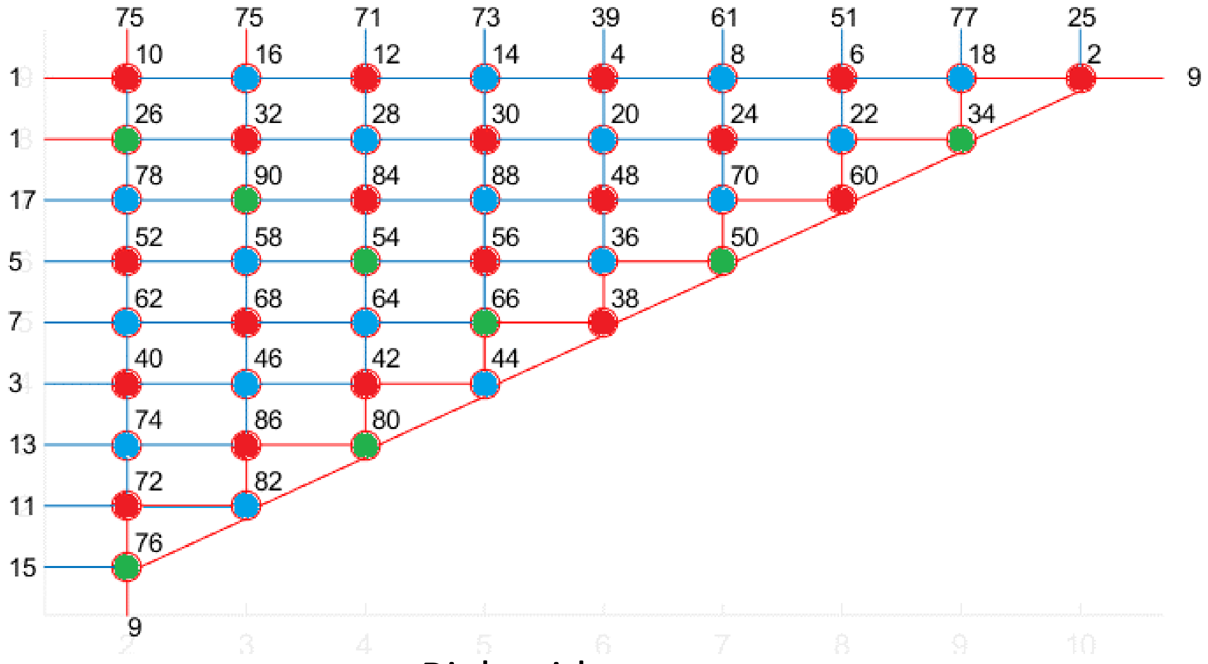


Right side

Color the graph (10 great circles)



Left side



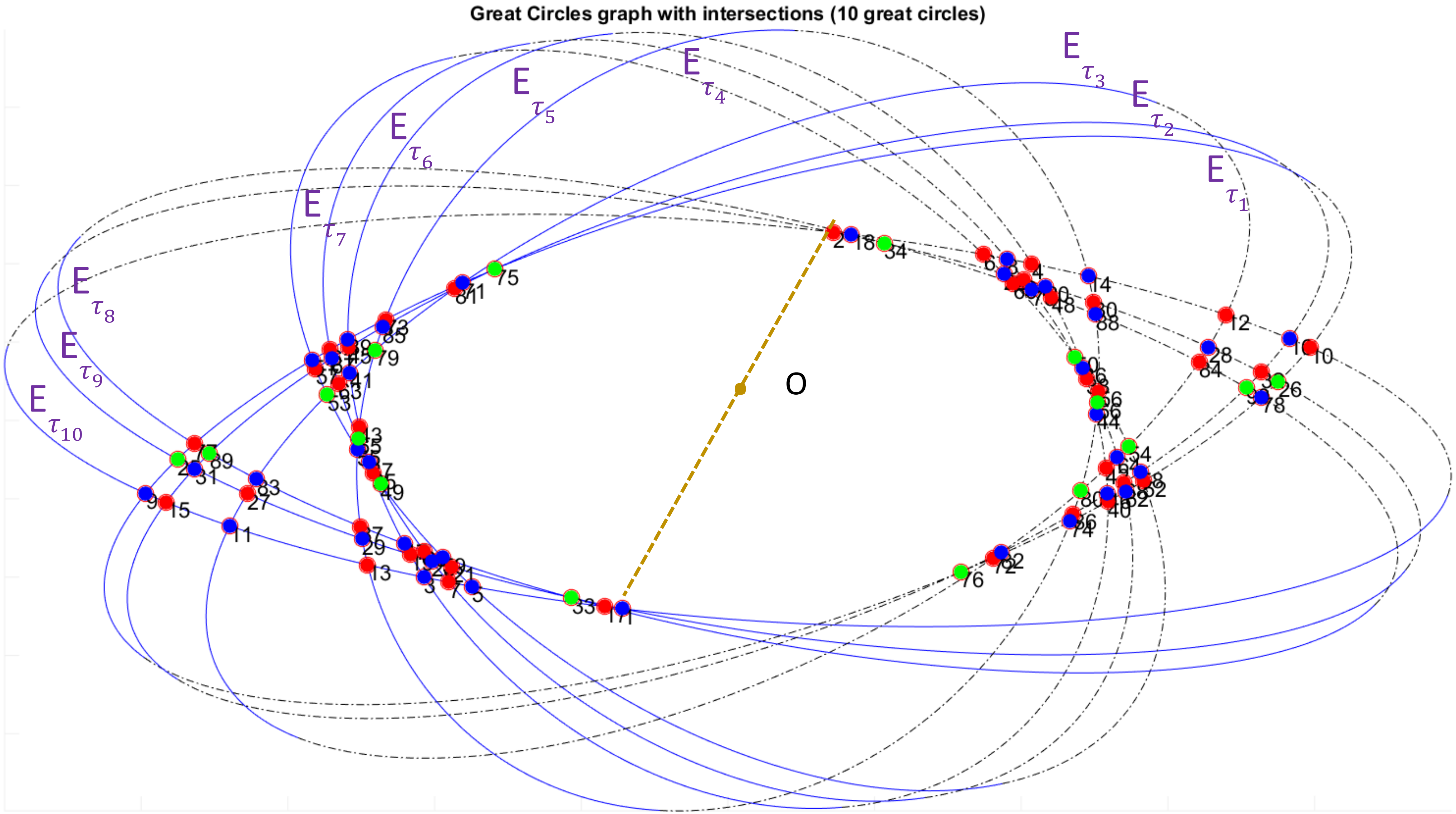
Right side

Observation: The vertices on the same diagonal usually have the same color except ones on the hypotenuse

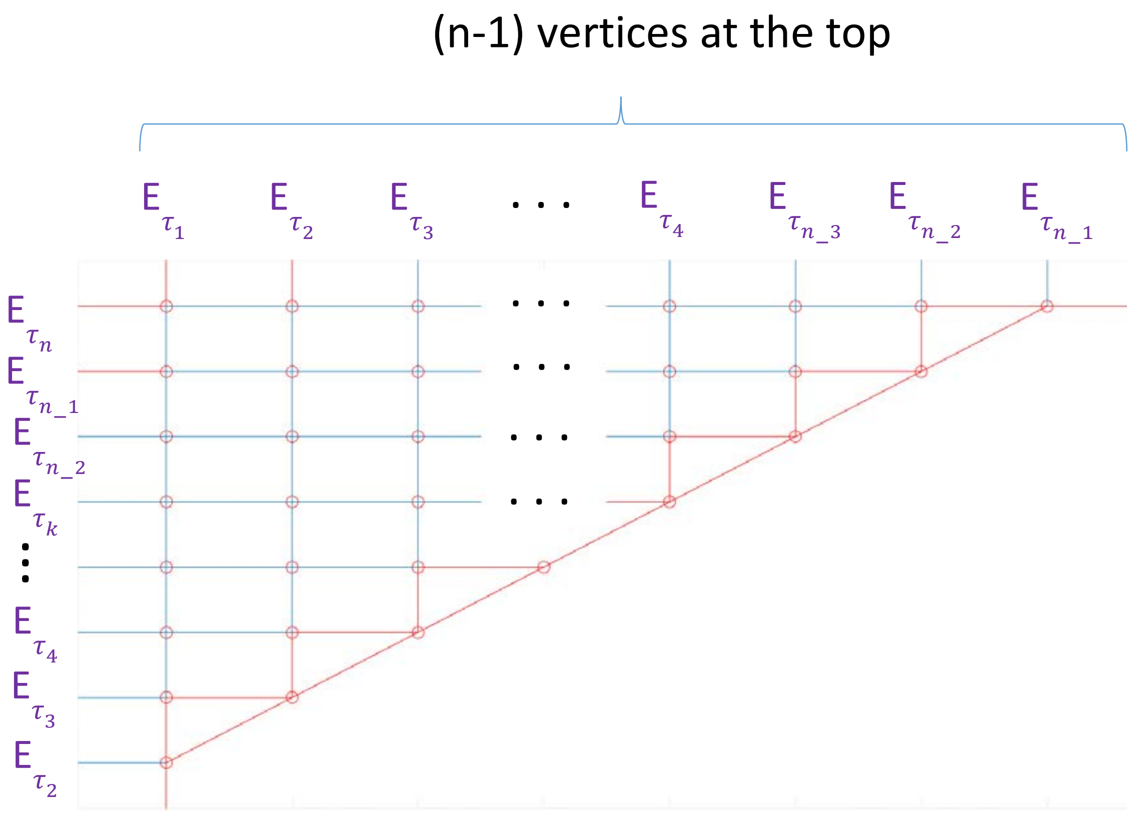
The big picture of the problem

- Call E_{τ_i} is the ellipse has the inclination angle τ_i and $\tau_1 < \tau_2 < \cdots < \tau_n$ $(\tau_i \in [\frac{-\pi}{2}, \frac{\pi}{2}))$

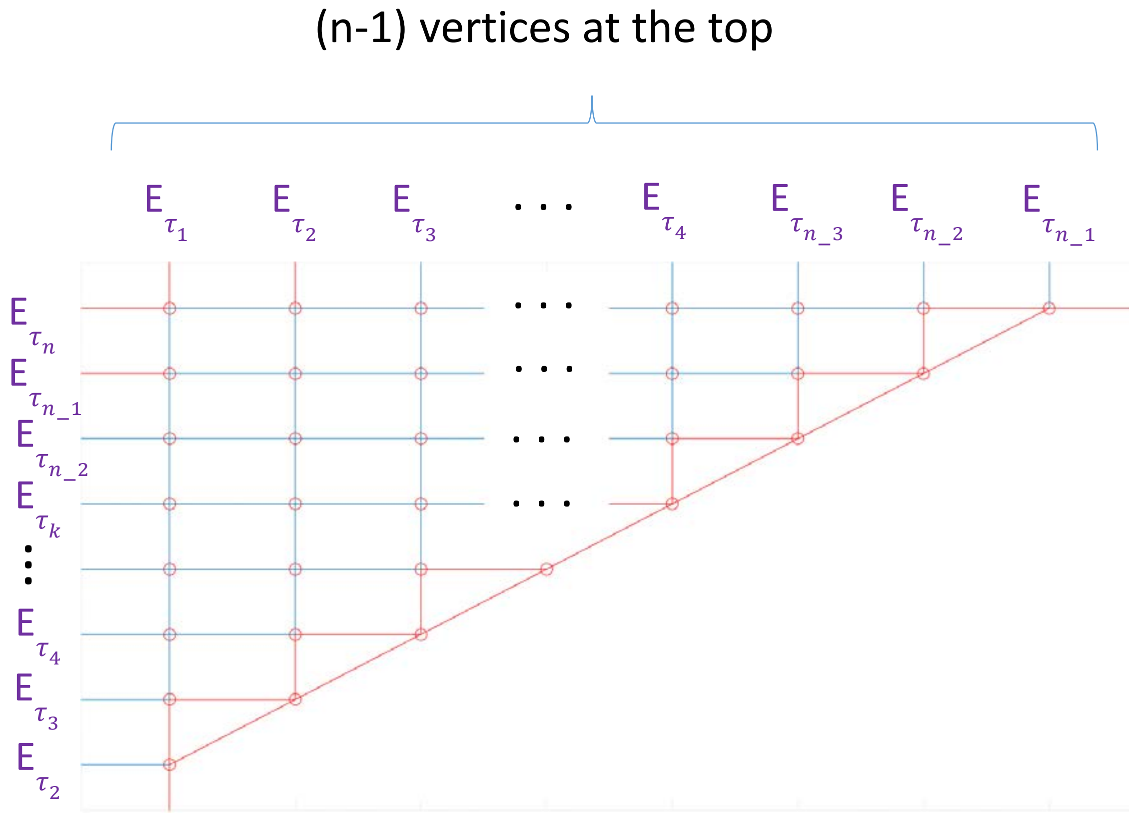
The big picture of the problem



The big picture of the problem



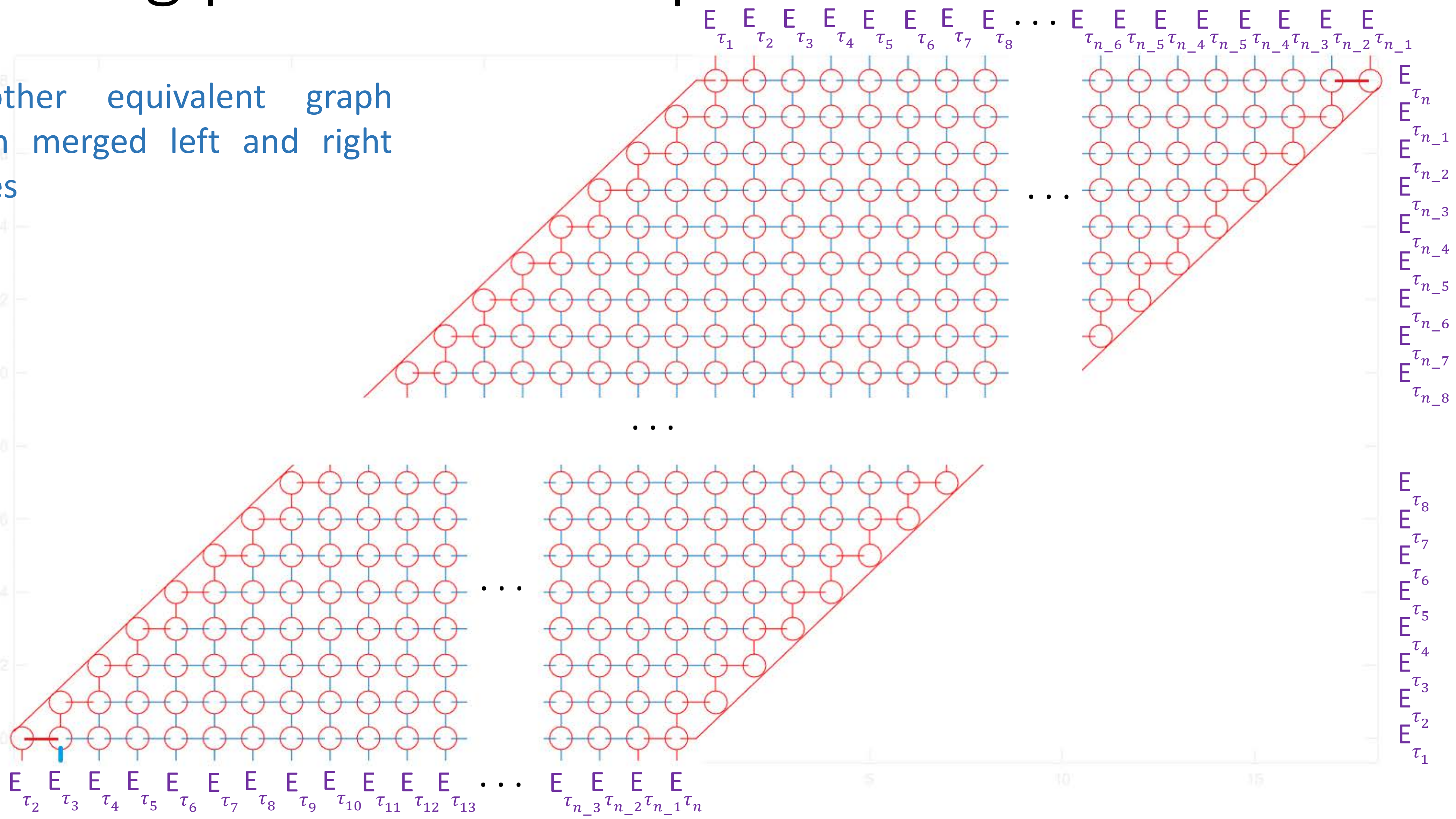
Left side



Right side

The big picture of the problem

Another equivalent graph with merged left and right sides



The big picture of the problem

- Observation (n is the number of great circles)
  - On every side, all the vertices have the degree 4
  - There are $\frac{(n-1)*n}{2}$ vertices on every side
    - Proof: a couple of great circles will create 2 intersections on the Earth, so the total vertices on both side would be $2 * C_n^2 = (n-1) * n$. Moreover, the number of vertices is split equally into 2 sides or every side will have $\frac{(n-1)*n}{2}$ vertices
  - There are (n-1)*2+2 = 2n external links on every side to connect to other side.
  - There are 2 K_3 formed by the external links
  - There are (n-1) K_3 on both sides

  ➔ 2*(n-1) + 2 = 2n K_3 (triangles) where we need 3 different colors for 3 vertices

Chromatic number

- $\deg(V_i) = 4$
  - Since a vertex only allows 2 circles to pass through it, so every vertex will have 4 neighbors which means $\deg(V_i) = 4$
- The graph G is planar
  - All the vertices are formed by the intersections of the circles. So, there is no sudden arc may cut through the connection between vertices since by contradiction, it will keep forming the vertices continuously and it makes no sense
- $3 \leq \chi(G) \leq 4$
  - According to four color theorem, a planar graph only needs 4 colors
  - A triangle can be formed by 3 random circles which means at that triangle, its vertices needs 3 different colors. Or the graph G contains a sub graph K_3
- **Prove $\boldsymbol{\chi(G) = 3}$** by providing a way to color the graph with 3 colors

Chromatic number

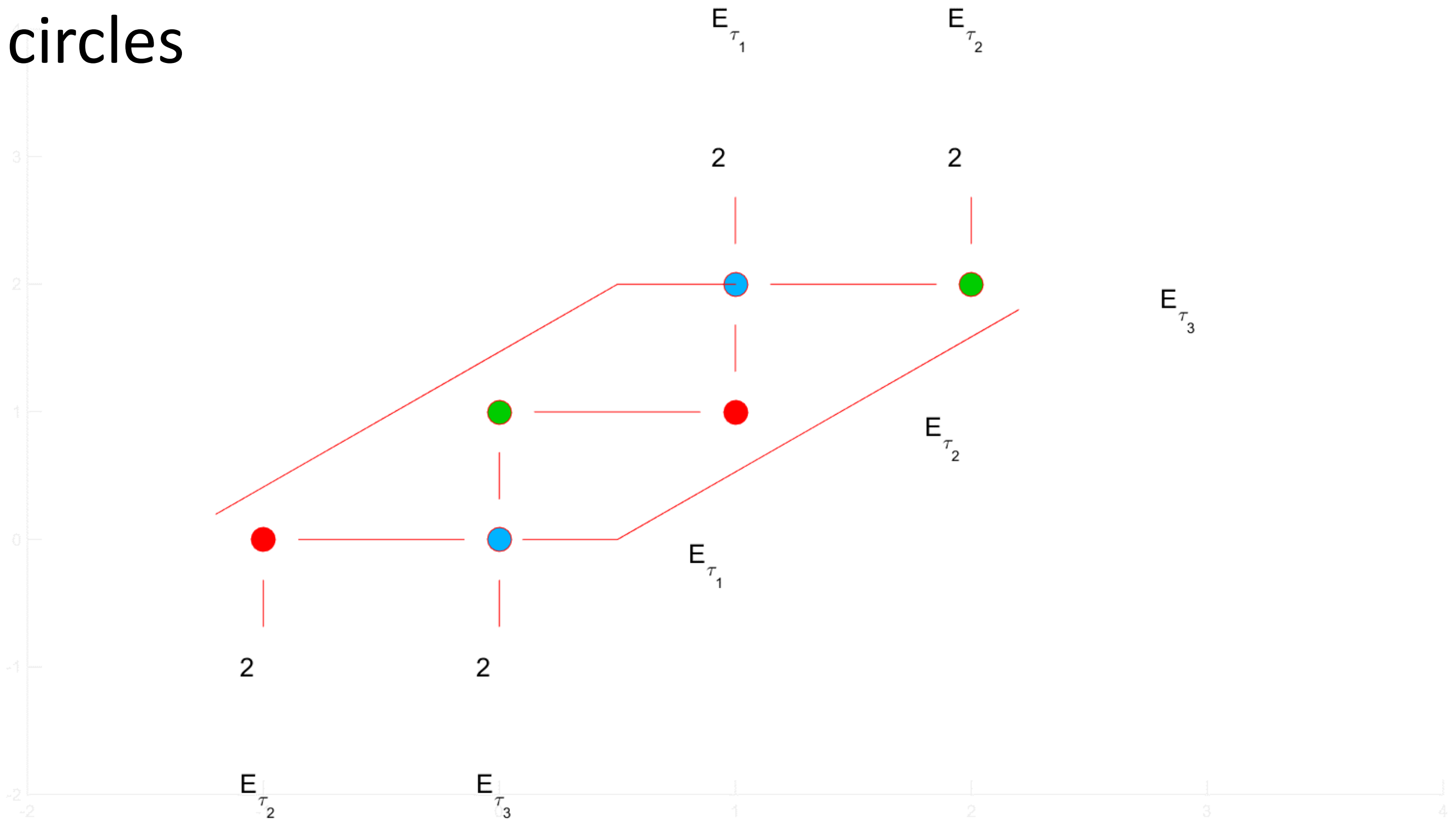
- **Prove $\boldsymbol{\chi(G) = 3}$**
  - I split the problem into 4 sub-problems that have:
    - 3k great circles (3, 6, 9, 12, 15,….) (including (6k+3))
    - 2k great circles (2, 4, 6, 8, 10, 12,…) (including 6k, (6k+2), (6k+4))
    - (6k+1) great circles (7, 13, 19, 25,…)
    - (6k+5) great circles (5, 11, 17, 23,…)

Chromatic number

- Some rules before coloring
  - Diagonal rule: The vertices on the same diagonal **should** have the same color (not a must because there are some places might have K_3 rule)
    - Proof: The vertices on the same diagonal are not connected together.
  - K_3 rule: 3 vertices that form a triangle **must** have 3 different colors

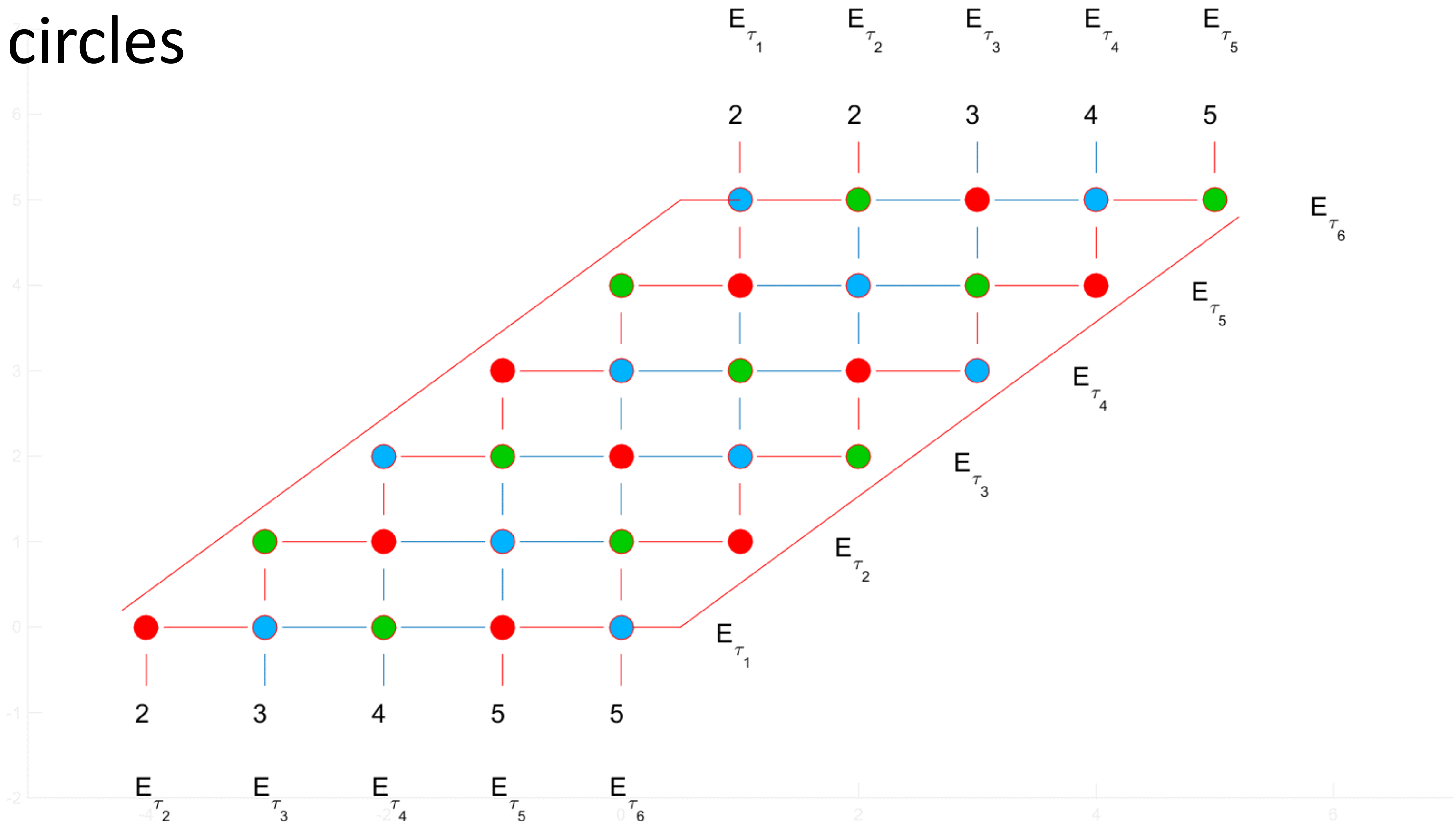
Chromatic number (with 3k circles)

- Base cases: 3 great circles



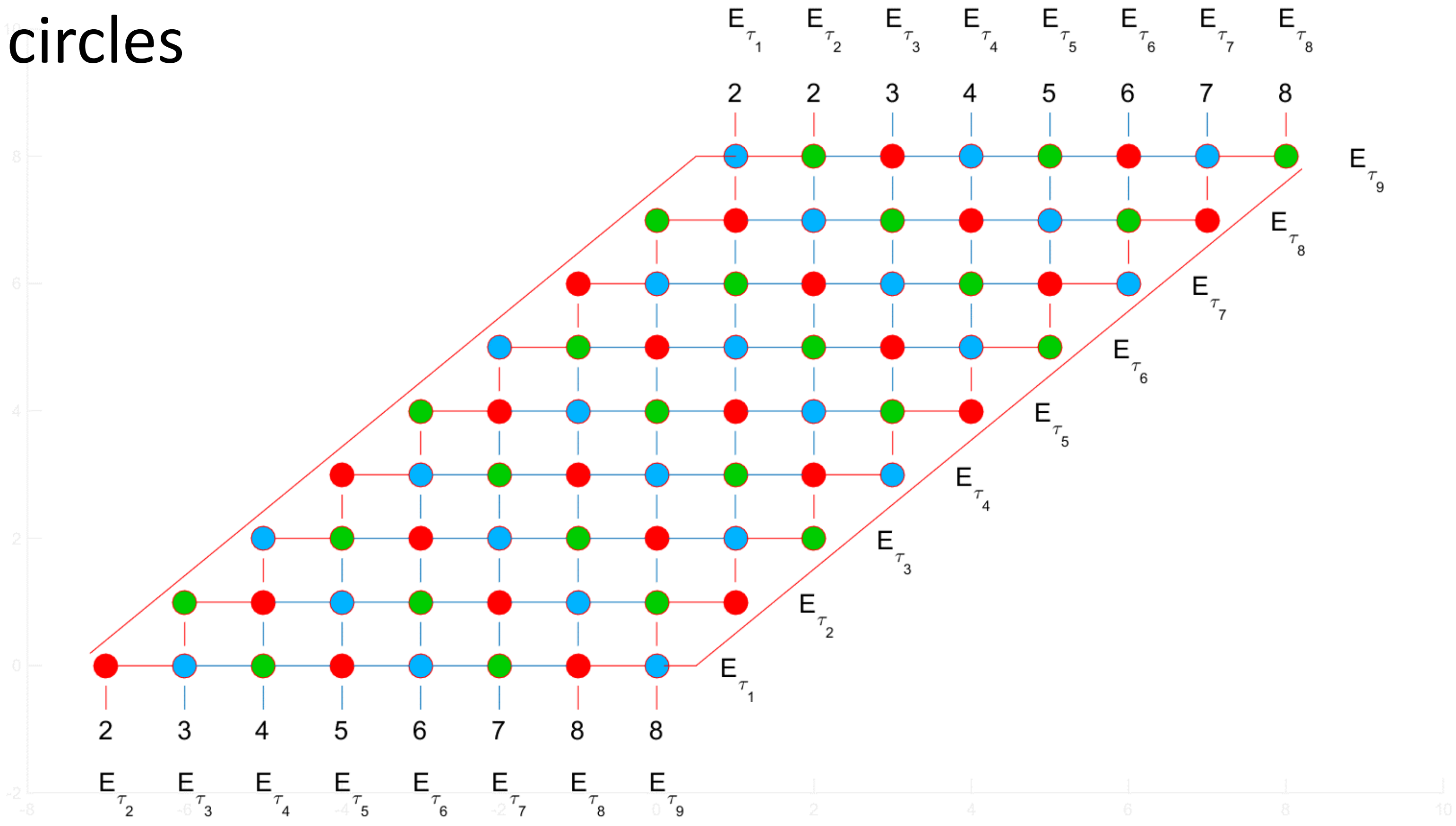
Chromatic number (with 3k circles)

- Base cases: 6 great circles



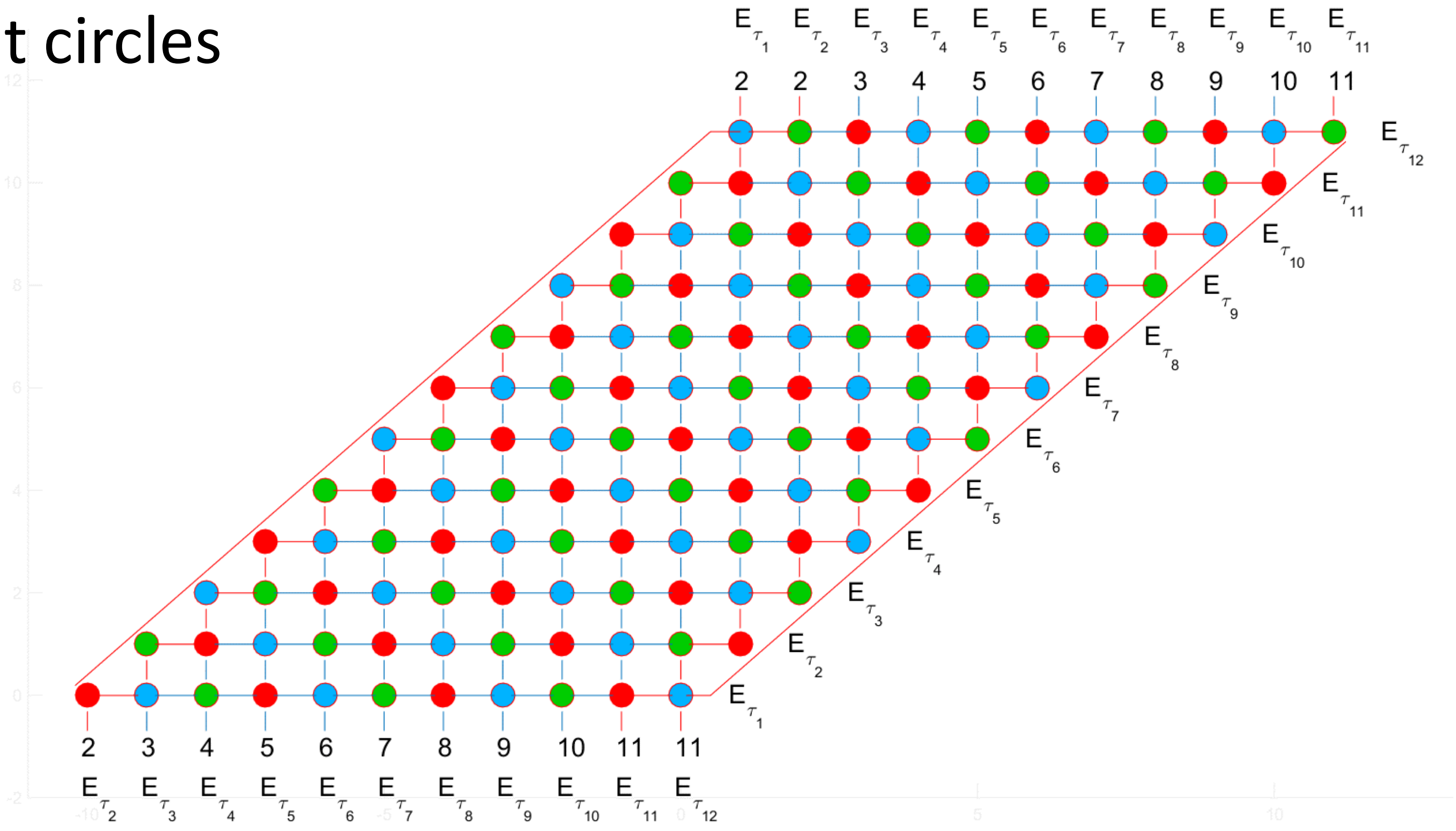
Chromatic number (with 3k circles)

- Base cases: 9 great circles



Chromatic number (with 3k circles)

- Base cases: 12 great circles

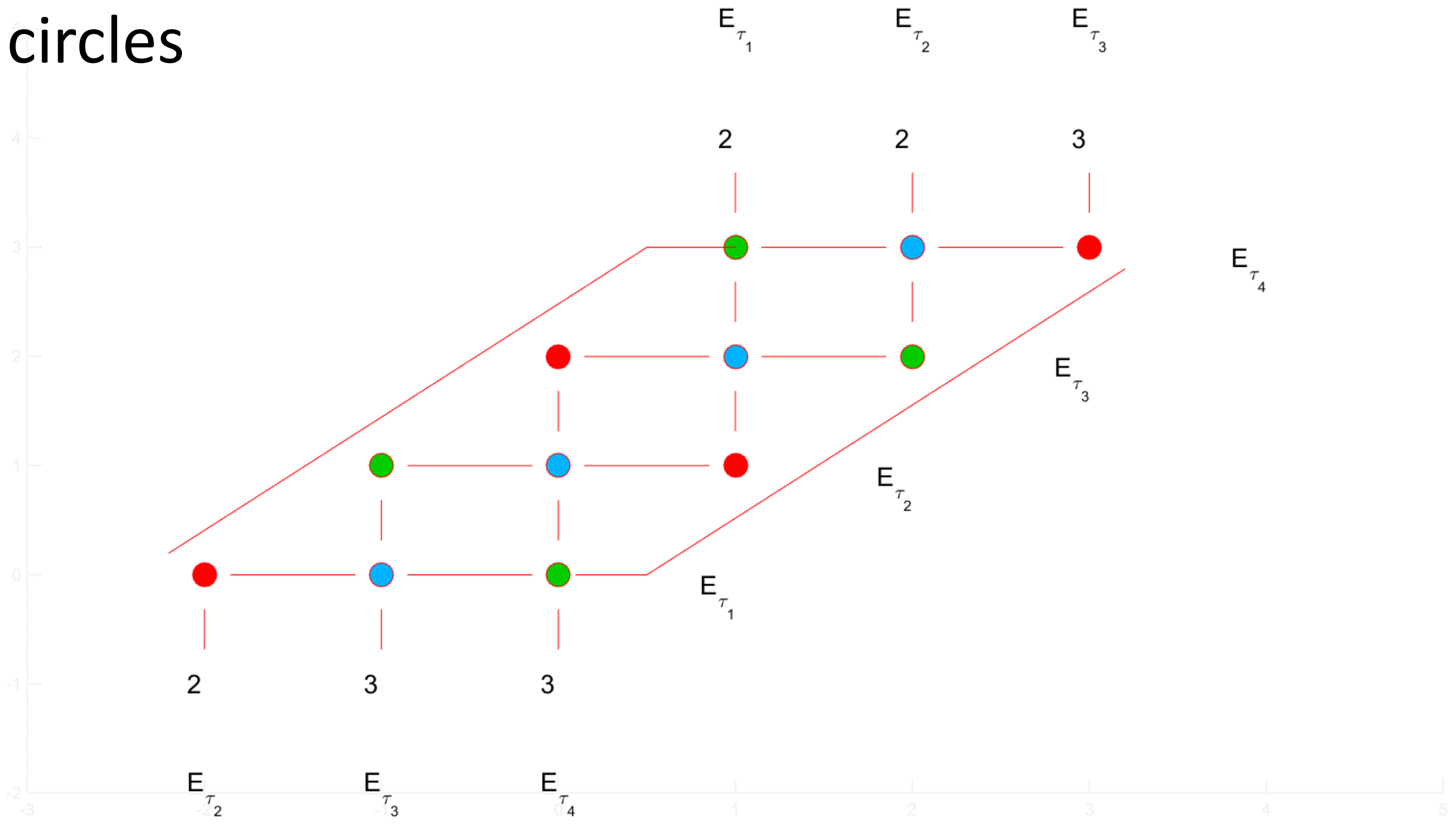


Chromatic number (with 3k circles)

- I can color a 3k circles graph with 3 colors by repeatedly spreading RED-BLUE-GREEN respectively on the diagonals
- I can prove it by induction hypothesis
  - Proof:
    - Assume I can use this technique to color a 3k circles graph, so in 3(k+1) graph, the edges on the equivalent parallelogram are extended 3 more vertices on the top right and the bottom left. Then I can use RED-BLUE-GREEN to color the vertices with diagonal rule
    - This technique also satisfies K_3 rule

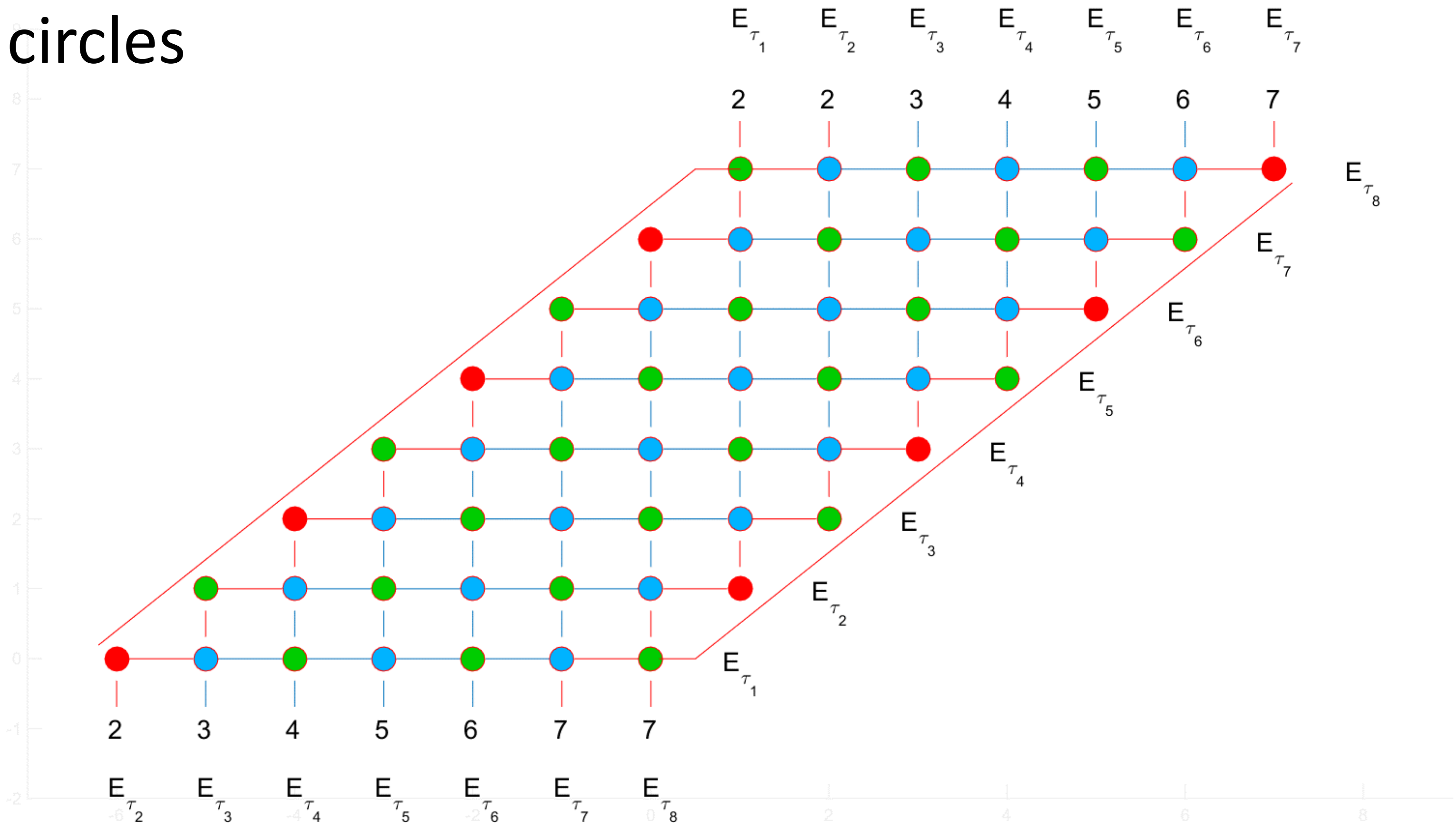
Chromatic number (with 2k circles)

- Base cases: 4 great circles



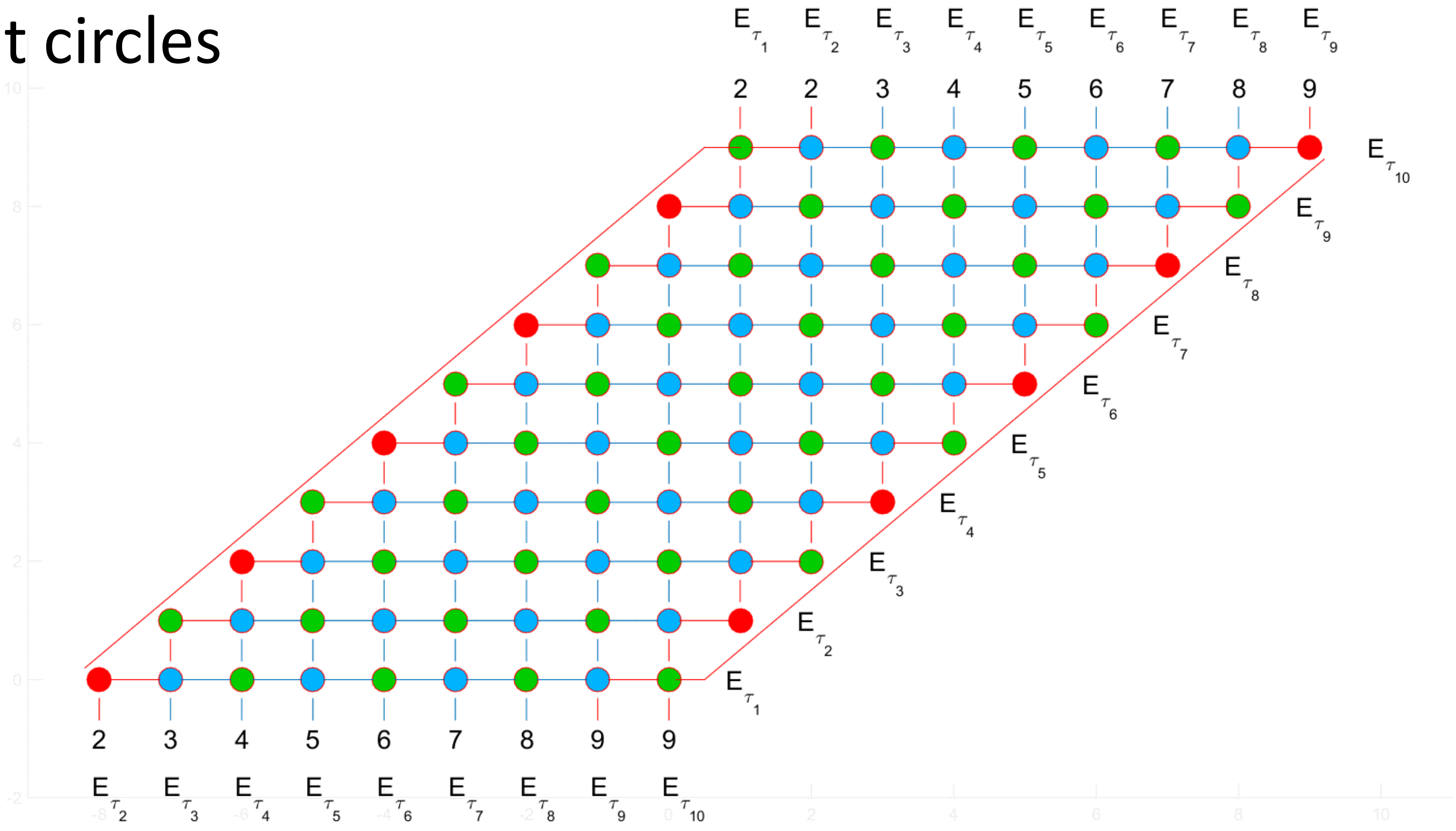
Chromatic number (with 2k circles)

- Base cases: 8 great circles



Chromatic number (with 2k circles)

- Base cases: 10 great circles

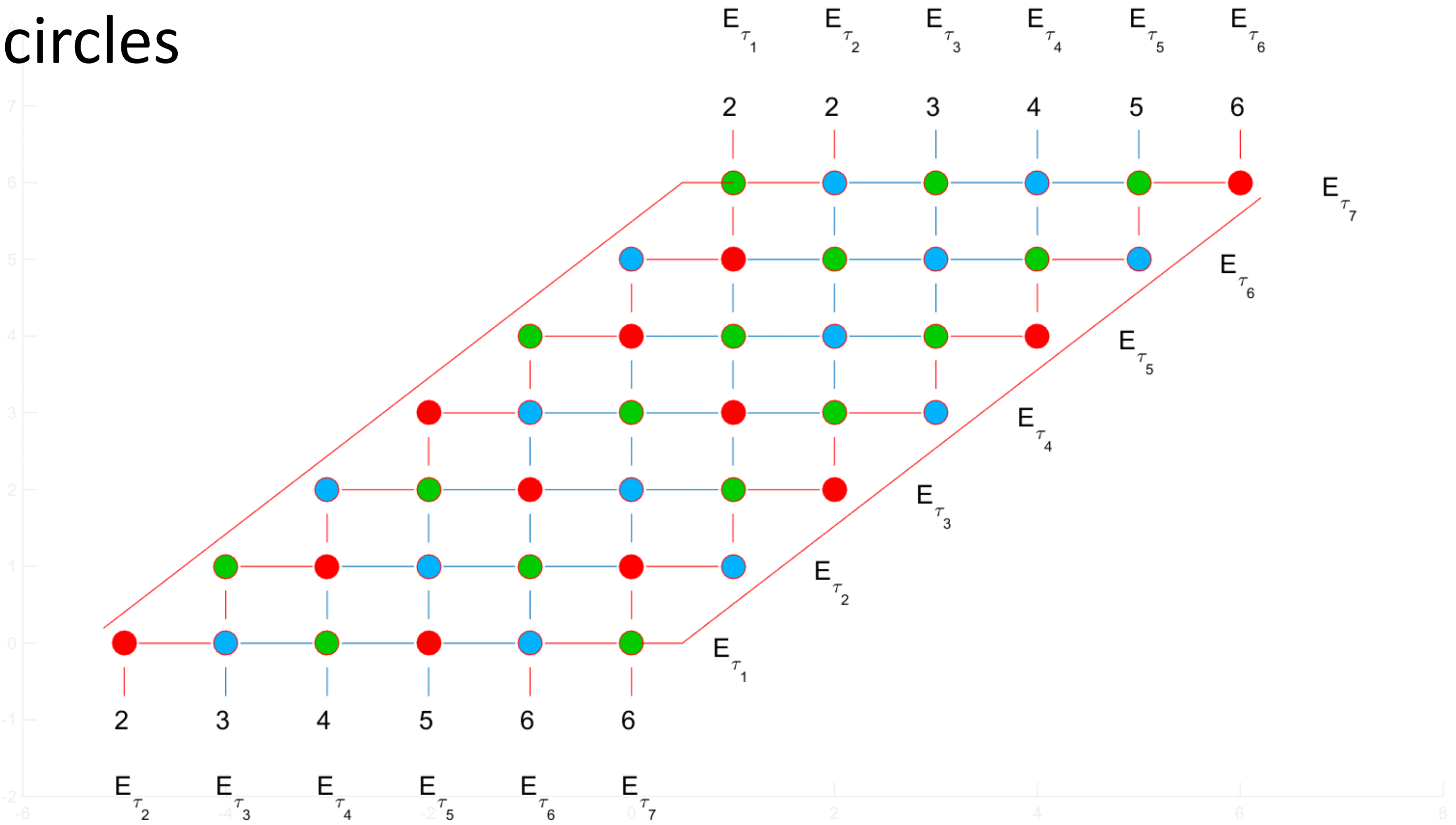


Chromatic number (with 2k circles)

- I can color a 2k circles graph with 3 colors:
  - Repeatedly spreading BLUE-GREEN respectively on the diagonals
  - Replace the red dots to the vertices on the left and right edges
    - Start with the first one at bottom left
    - Color R-G-R-G-R-G-…..-G
    - Start with the first one at bottom right
    - Color G-R-G-R-G-R-…..-R
- Why can this technique be possibly done?
  - Easily find out that at the external links, there is no problem there since if by induction hypothesis, with 2(k+1) graph, I will keep spreading BLUE-GREEN
  - Adding red dots doesn't matter the current graph with 2 colors. Moreover, it helps to satisfy 2 K_3 on the external links and other (2(k+1)-2) K_3 on the parallelogram edges

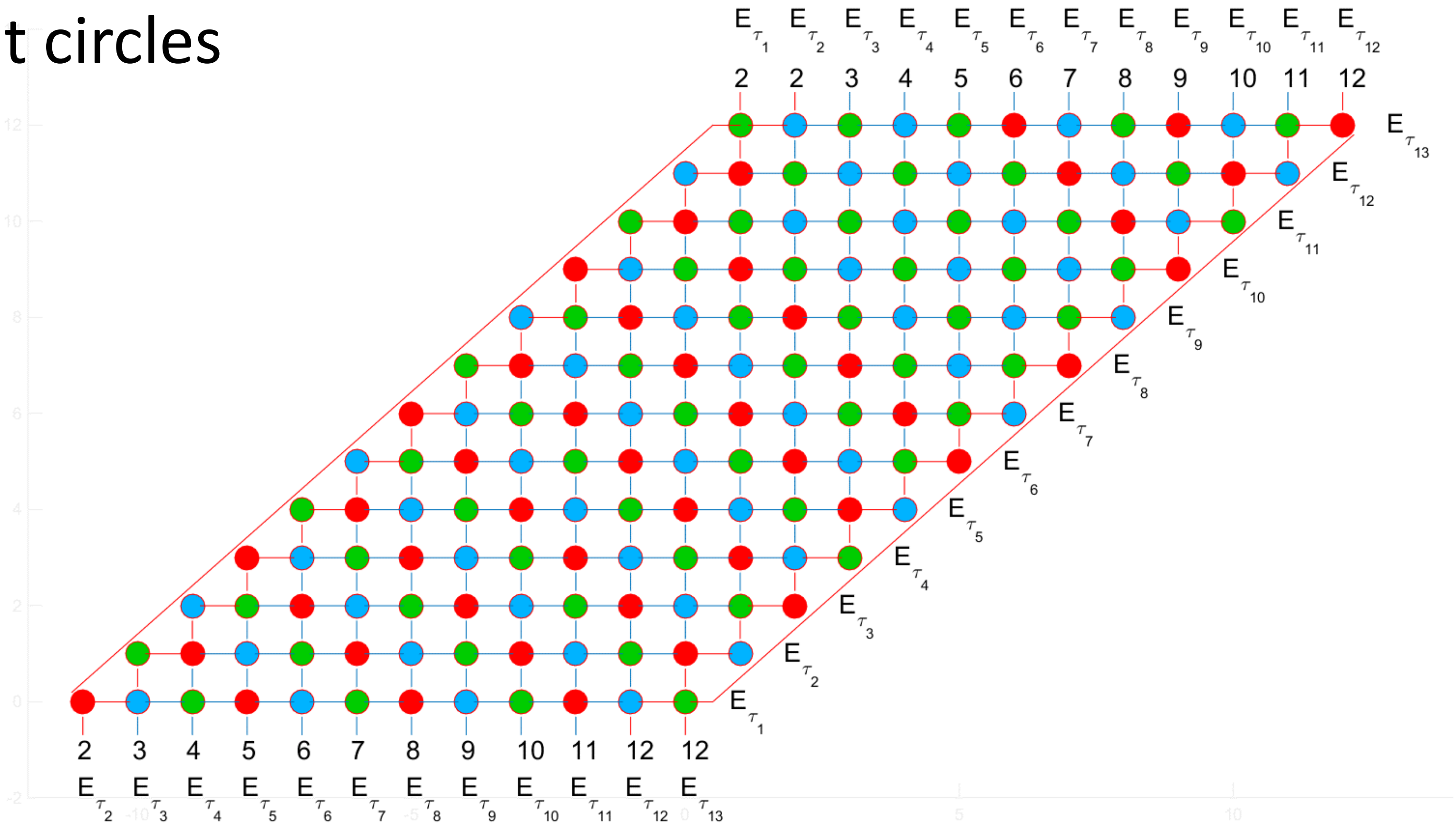
Chromatic number (with (6k+1) circles)

- Base cases: 7 great circles



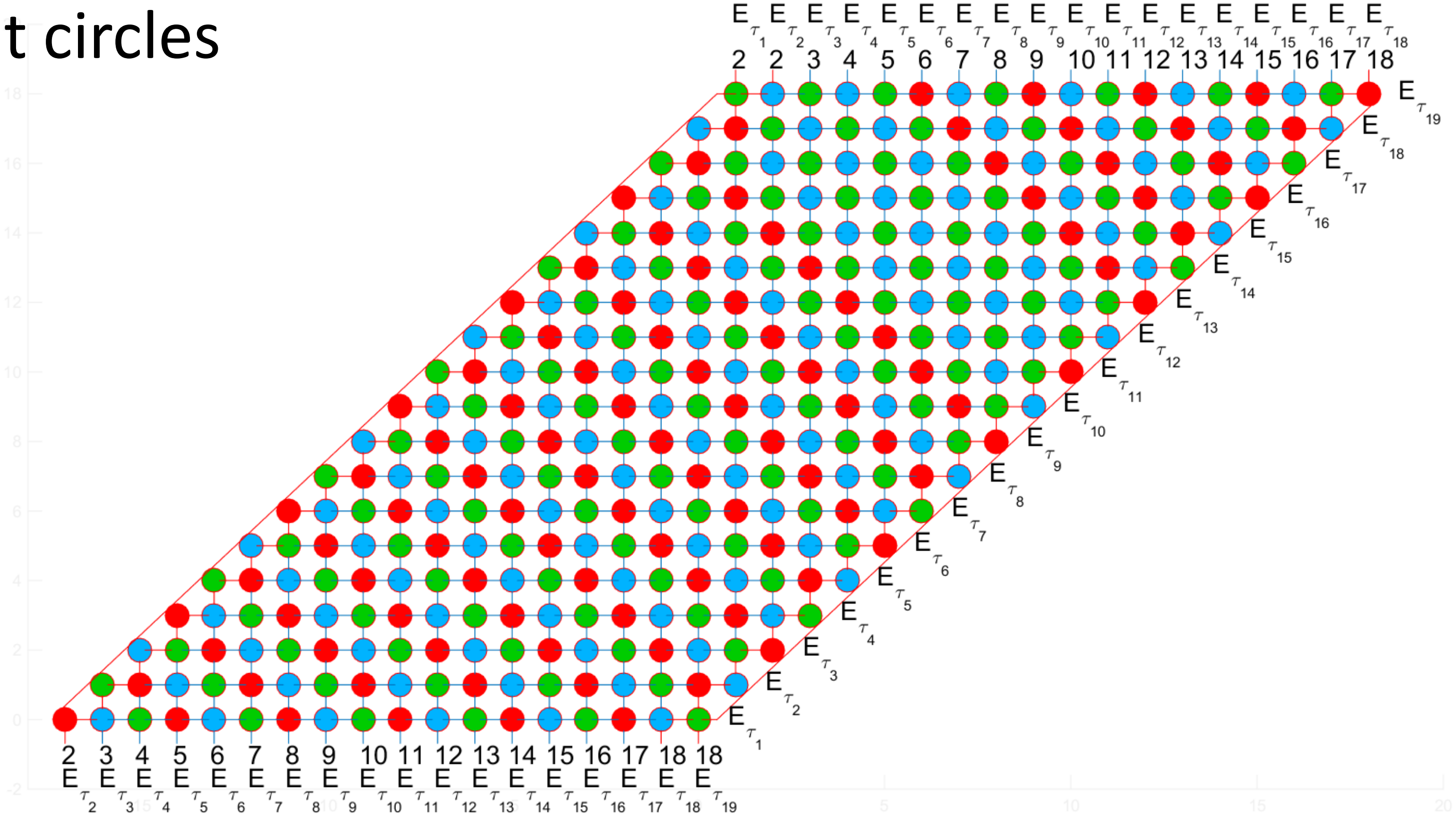
Chromatic number (with (6k+1) circles)

- Base cases: 13 great circles



Chromatic number (with (6k+1) circles)

- Base cases: 19 great circles

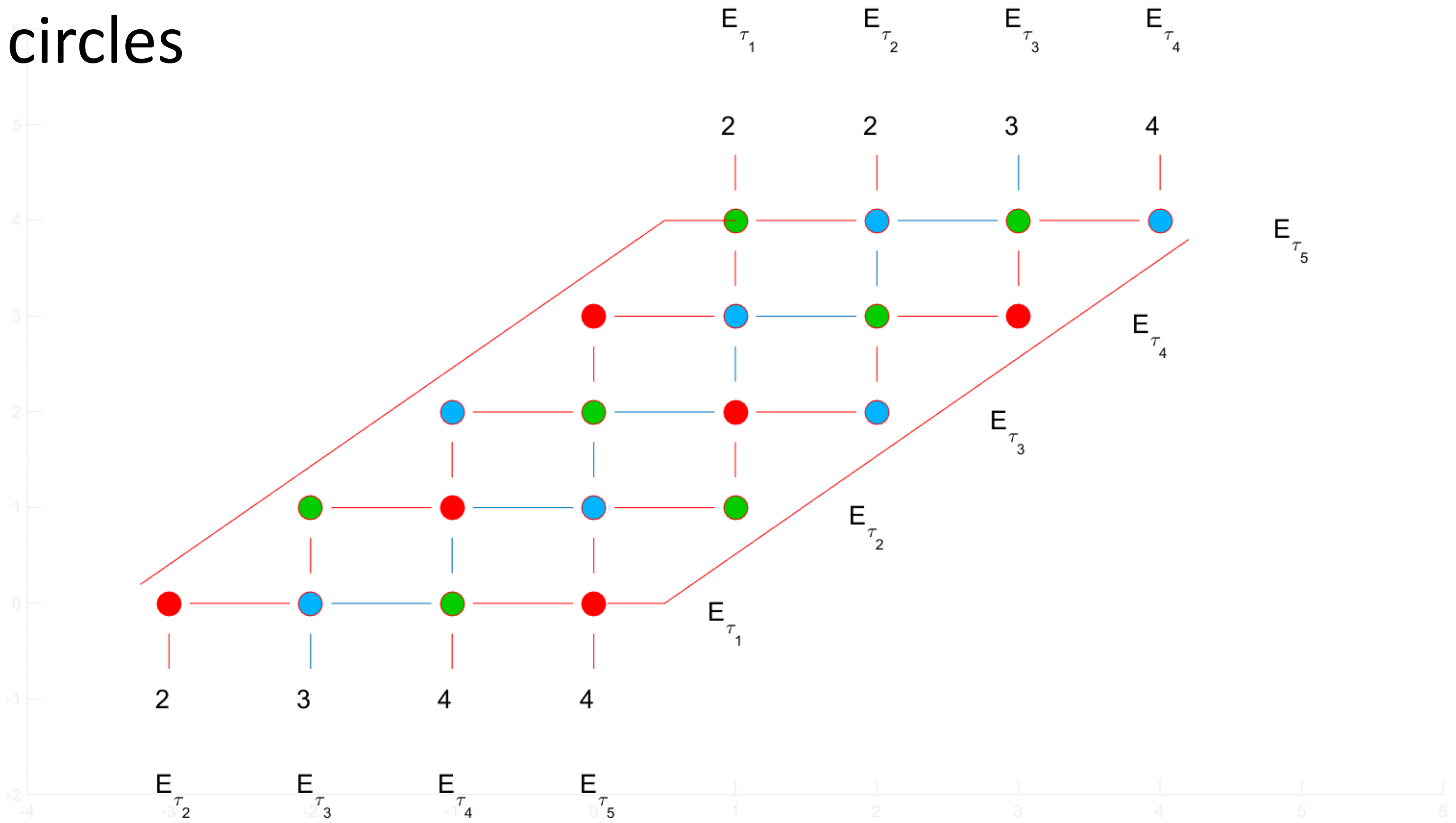


Chromatic number (with (6k+1) circles)

- With k>2, by spreading RED-BLUE-GREEN respectively on the additional diagonals, I can color all this type of graph based on the case k=2

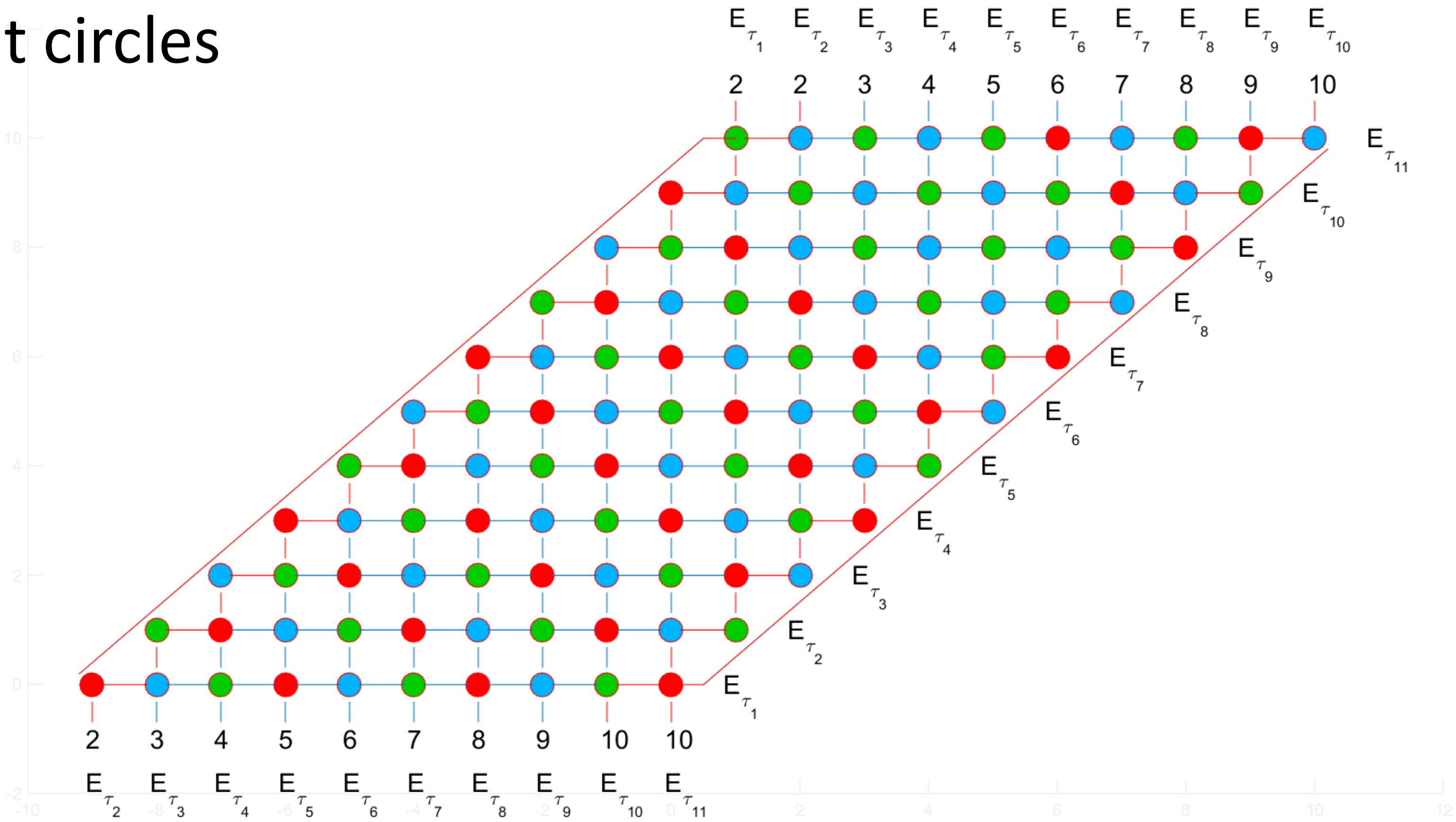
Chromatic number (with (6k+5) circles)

- Base cases: 5 great circles



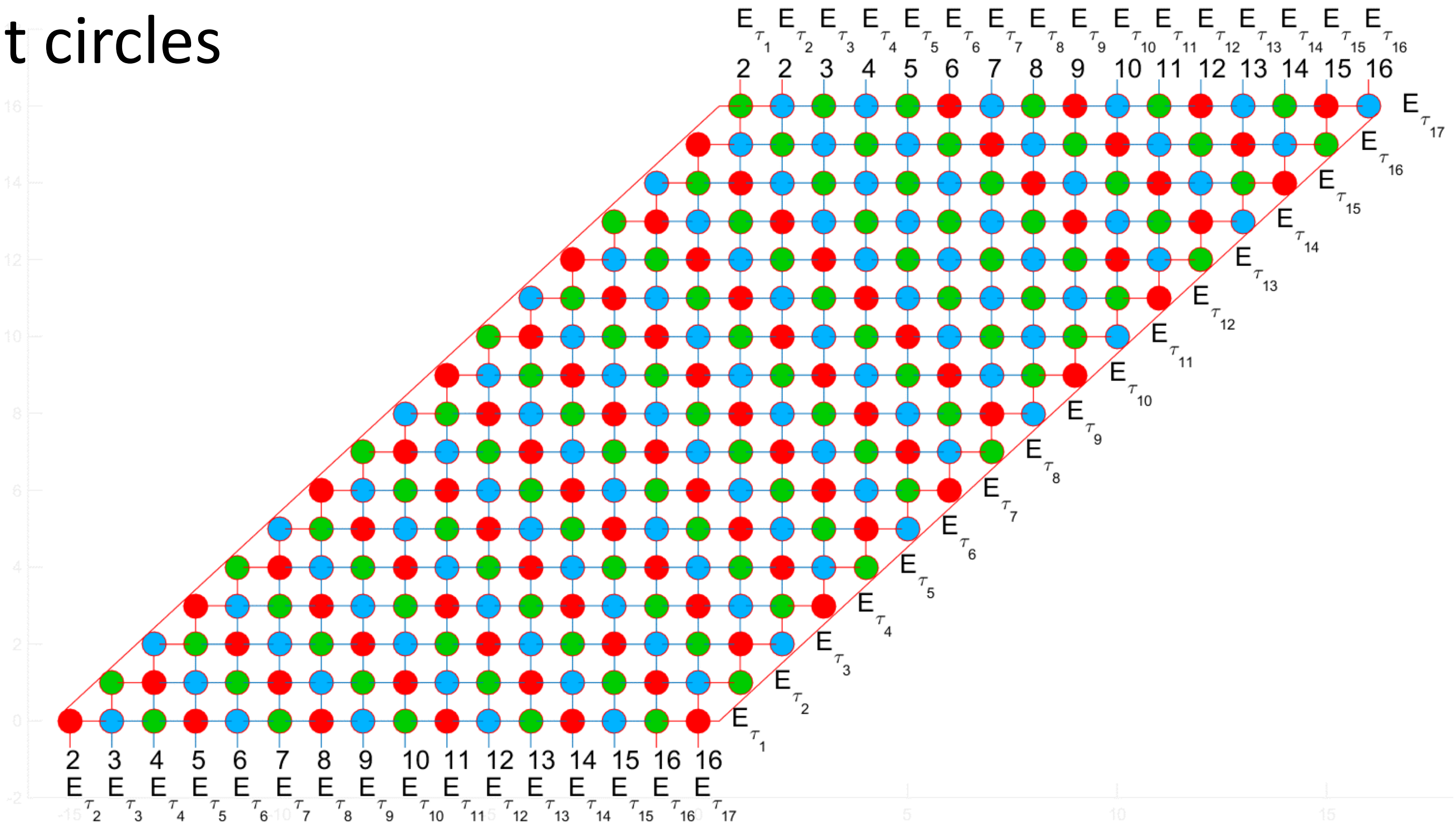
Chromatic number (with (6k+5) circles)

- Base cases: 11 great circles



Chromatic number (with (6k+5) circles)

- Base cases: 17 great circles



Chromatic number (with (6k+5) circles)

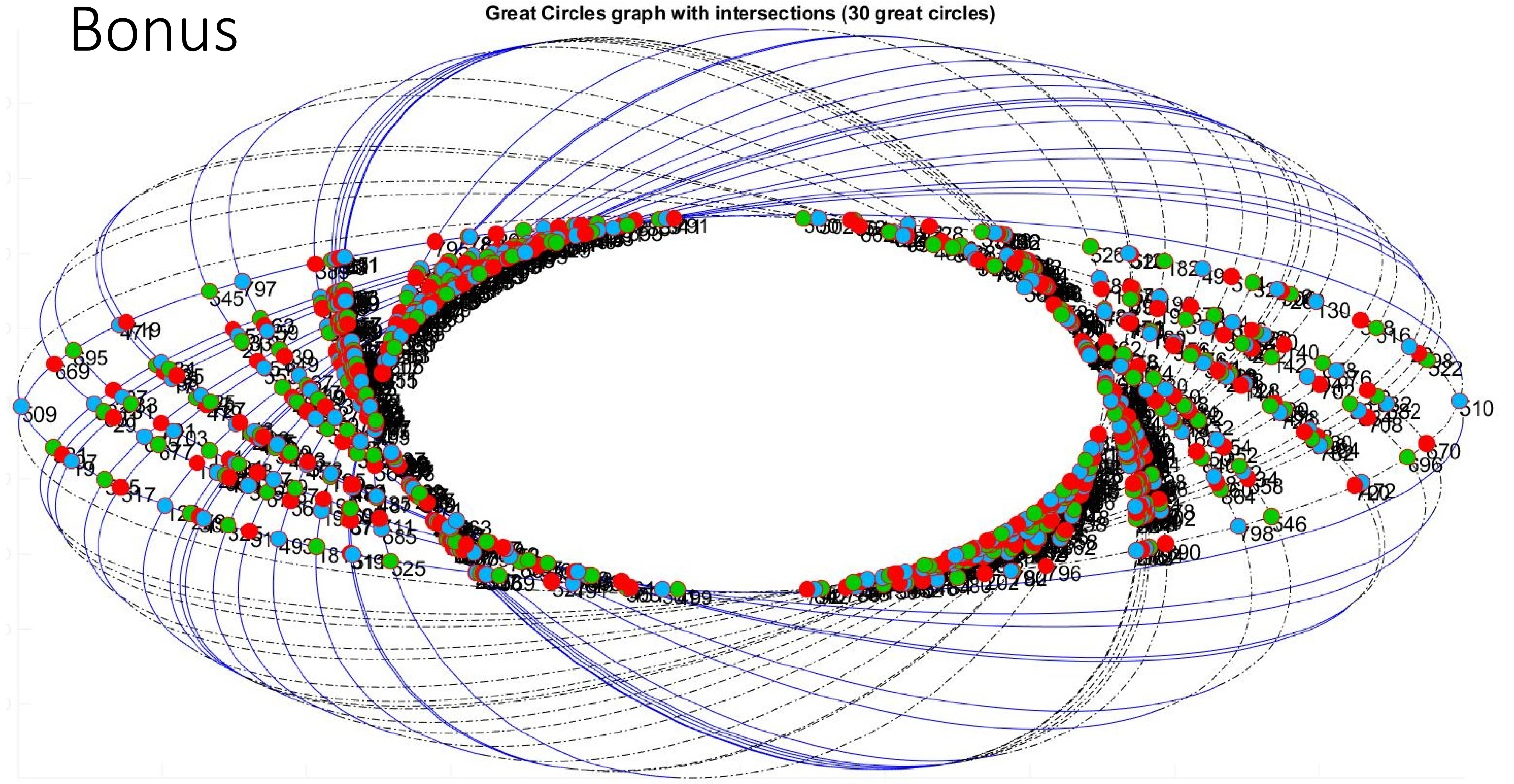
- This type of graph is similar to (6k+1) graph. With cases k>1, by spreading RED-BLUE-GREEN respectively on the diagonals, I can color all this type of graph based on the case k=1

Chromatic number

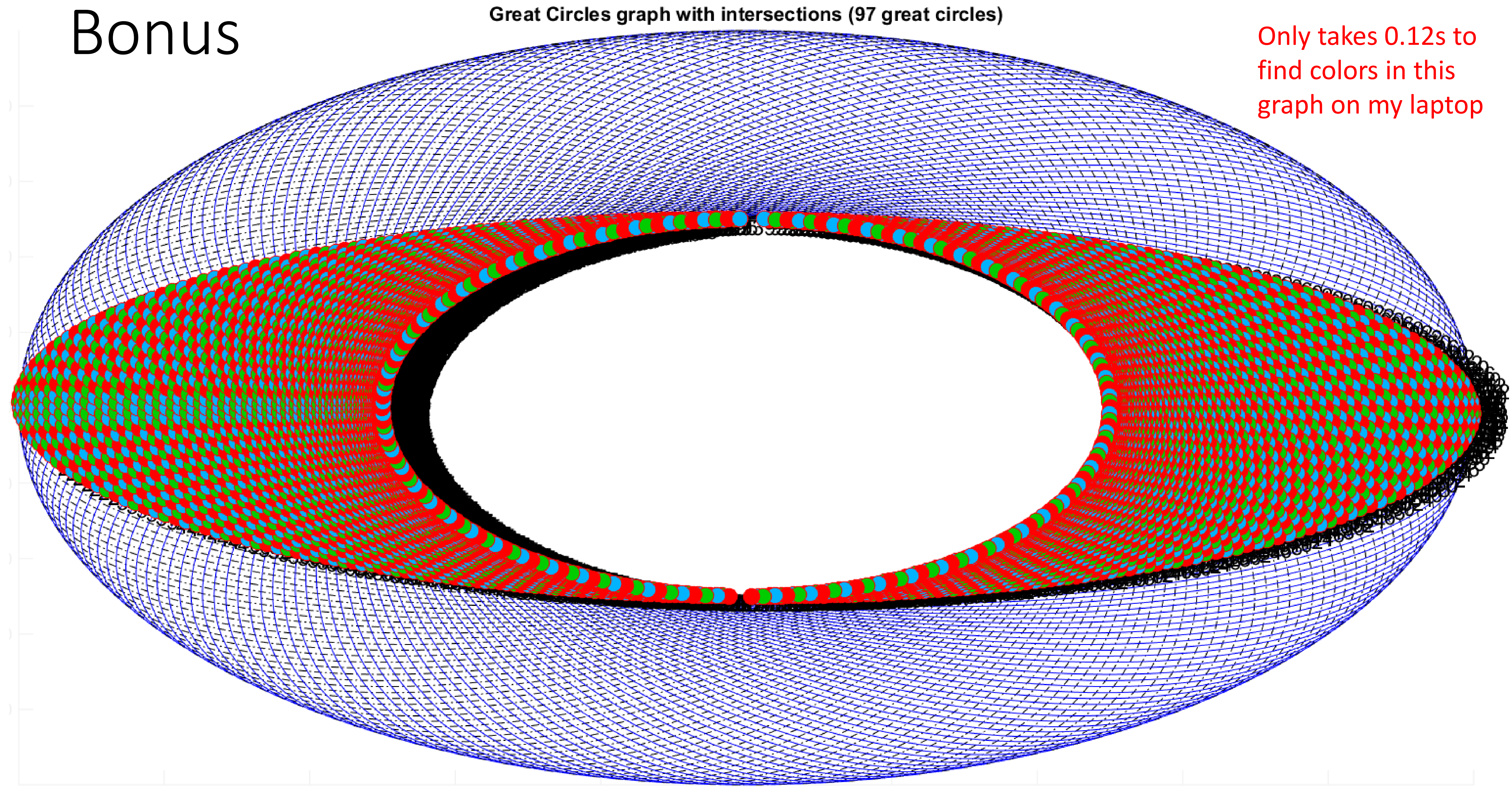
- According to the techniques to color the graphs including 3k, 2k, 6k+1 and 6k+5 great circles, there are only 3 colors used

➔ $\boldsymbol{\chi(G) = 3}$

Bonus



Bonus



Only takes 0.12s to find colors in this graph on my laptop

Thank you