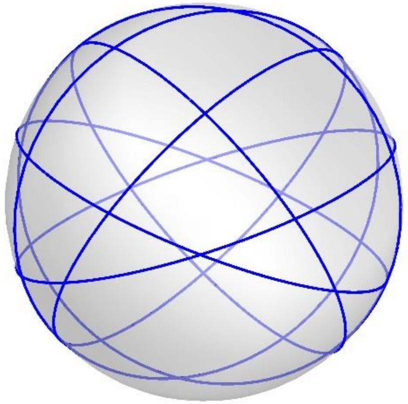


Great Circles Problem

Kha Man

04/07/2015

Outline

- Lemma 1
- Lemma 2 – The uniqueness of the special graph S_k
- Lemma 3 – The equivalent graph of S_k
- Lemma 4 – The properties of S_k
- Theorem 1 – The chromatic number of S_k
- The next steps

Lemma 1

Call n is the number of circles in the graph

1. There are $2(n-1)$ vertices and $2(n-1)$ edges on a circle
2. A pair of circles create 2 intersections. The distance between 2 intersections on a circle is $n-1$ edges

Lemma 1 - Proof

1. A circle will intersect $(n-1)$ other circles. A pair of circles will meet at 2 points. So the number of points on a circle is $2(n-1)$

   $\left|E\left(C_{2(n-1)}\right)\right| = 2(n-1)$ ➔ There are $2(n-1)$ edges on the circle

2. Assume the statement is correct with k great circles graphs which have $(2k-2)$ vertices on a circle.

Define $v_1 \to v_2 \to v_3 \to \ldots \to v_{k-1} \to \Psi(v_1) \to \Psi(v_2) \to \Psi(v_3) \to \ldots \to \Psi(v_{k-1}) \to v_1$ is the circular path that has

$$d(v_i, \Psi(v_i)) = k - 1 \; ; \; i = 1,2,3, \ldots, (k-1)$$

Lemma 1 - Proof

Now we add a new circle C_{k+1} into the graph. So on every circle C_1 to C_k, we have 2 new intersections made by C_{k+1}. Call it v_a and $\Psi(v_a)$

Without loss of generality, I consider v_a as the first vertex in my new circular path v_a -> v_1 -> v_2 -> v_3 -> … -> v_{k-1} - > $\Psi(v_a)$ -> $\Psi(v_1)$ -> $\Psi(v_2)$ -> $\Psi(v_3)$ -> …. -> $\Psi(v_{k-1})$ -> v_a

Because every vertex has O as the point symmetry, so if v_a is the first vertex that is close to $\Psi(v_{k-1})$ and v_1, $\Psi(v_a)$ must be close to v_{k-1} and $\Psi(v_1)$.

Lemma 1 - Proof

$$d\big(v_a, \Psi(v_a)\big) = d(v_a, v_1) + d\big(v_1, \Psi(v_a)\big) = 1 + \big(d(v_1, \Psi(v_1) - d(\Psi(v_1), \Psi(v_a)\big)$$
$$= 1 + (k-1) = k$$

Call v_i is the vertex in the set $\{v_1, v_2, \dots v_{k-1}\}$

$$\Rightarrow d\big(v_i, \Psi(v_i)\big) = d(v_i, v_{k-1}) + d\big(v_{k-1}, \Psi(v_a)\big) + d(\Psi(v_a), \Psi(v_i))$$
$$= t + 1 + (k - 1 - t) = k$$

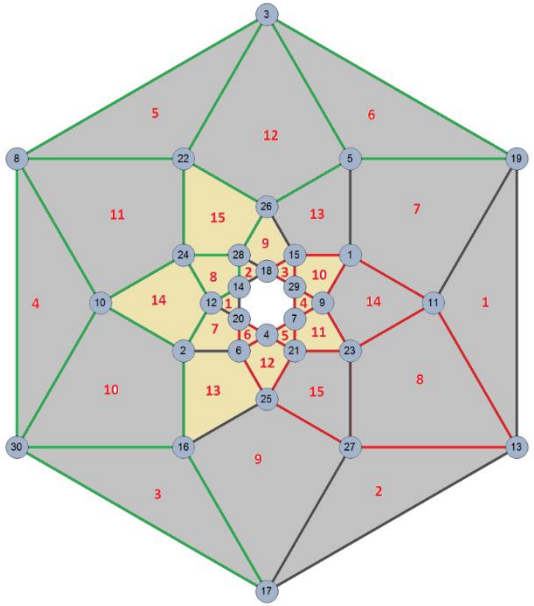
Similarly, because we have (2k-2+2) = 2k edges on the new path, the other path of $d\big(v_i, \Psi(v_i)\big)$ that contains $\Psi(v_{k-1})$ is also equal to k

➔ The induction hypothesis is correct with (k+1) circles

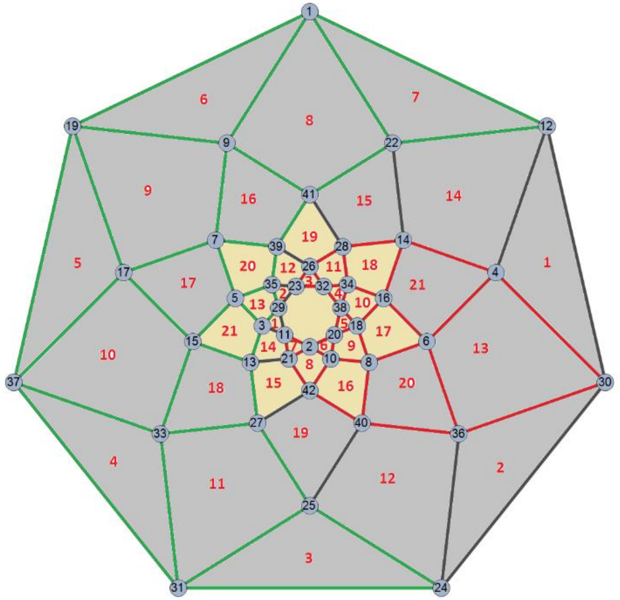
The special graph

Definition: A graph of k great circles $\mathcal{S}_k$ is **special** if it contains even number of triangles, quadrilaterals and 1 polygon that has k segments.

Lemma 2 will prove the special graph only has 1 unique structure



6 great circles



7 great circles

Lemma 2. (The uniqueness of the special graph)

1. $\mathcal{S}_{\mathrm{k+1}}$ can be made, if and only if, by $\mathcal{S}_{\mathrm{k}}$

2. $\mathcal{S}_{\mathrm{k}}$ is unique.

k = 3,4,5, ….

Lemma 2.1 – Proof

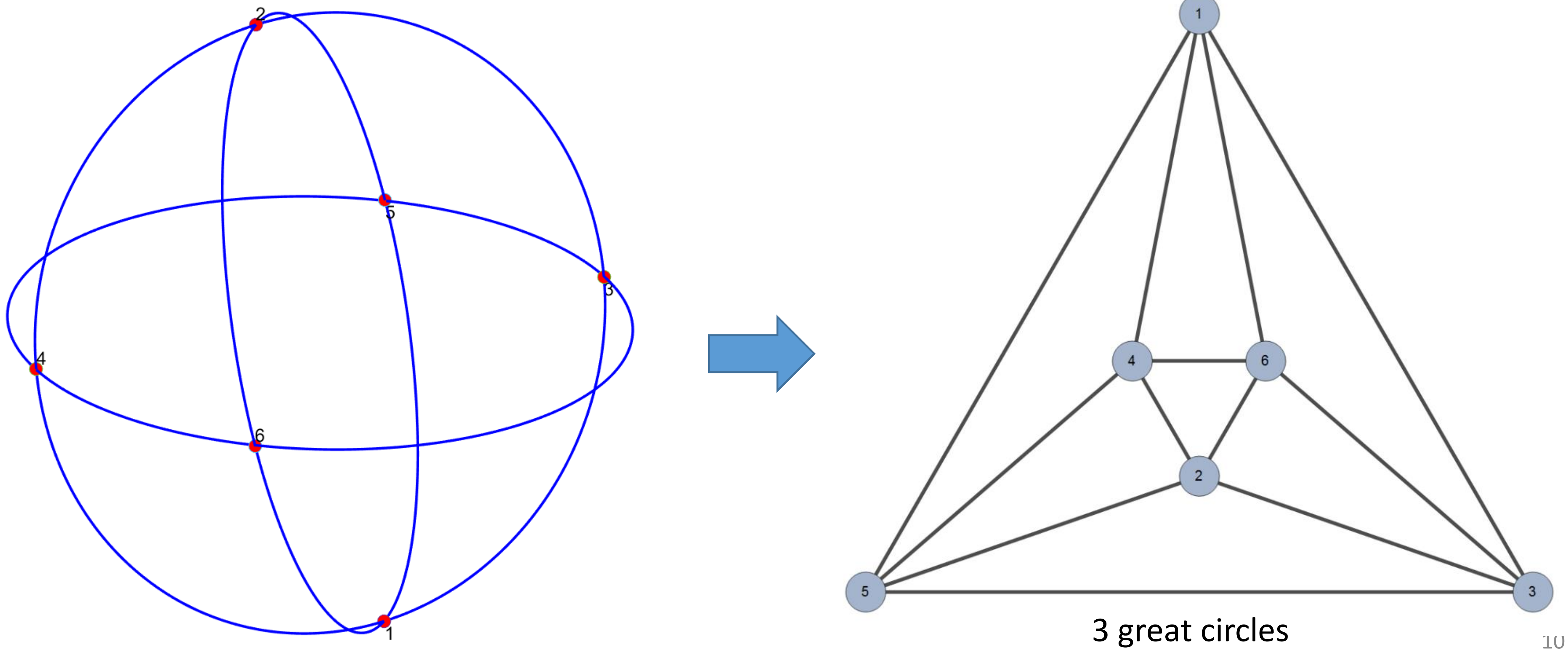
By definition, $\mathcal{S}_{\mathrm{k}}$ contains triangles, quadrilaterals and 1 polygon that has k segments.

Obviously, non-special graphs of k great circles cannot generate S_{k+1} because they must contain at least 2 polygons that have more segments than quadrilaterals and there is no way to magically reduce these polygons to quadrilaterals. It means if we add more circles into that, that polygons can have more segments.

Therefore, $\mathcal{S}_{\mathrm{k+1}}$ can be made, if and only if, by $\mathcal{S}_{\mathrm{k}}$

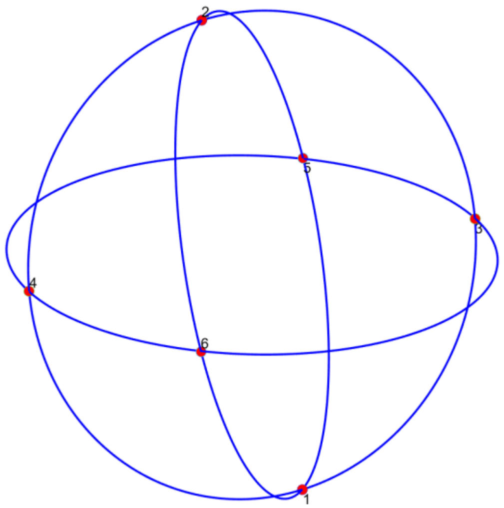
Lemma 2.2 – Proof - $\mathcal{S}_{\mathrm{k}}$ is unique

- 3 great circles has 1 non-isomorphic graph and it’s special (We can verify it easily by hand)

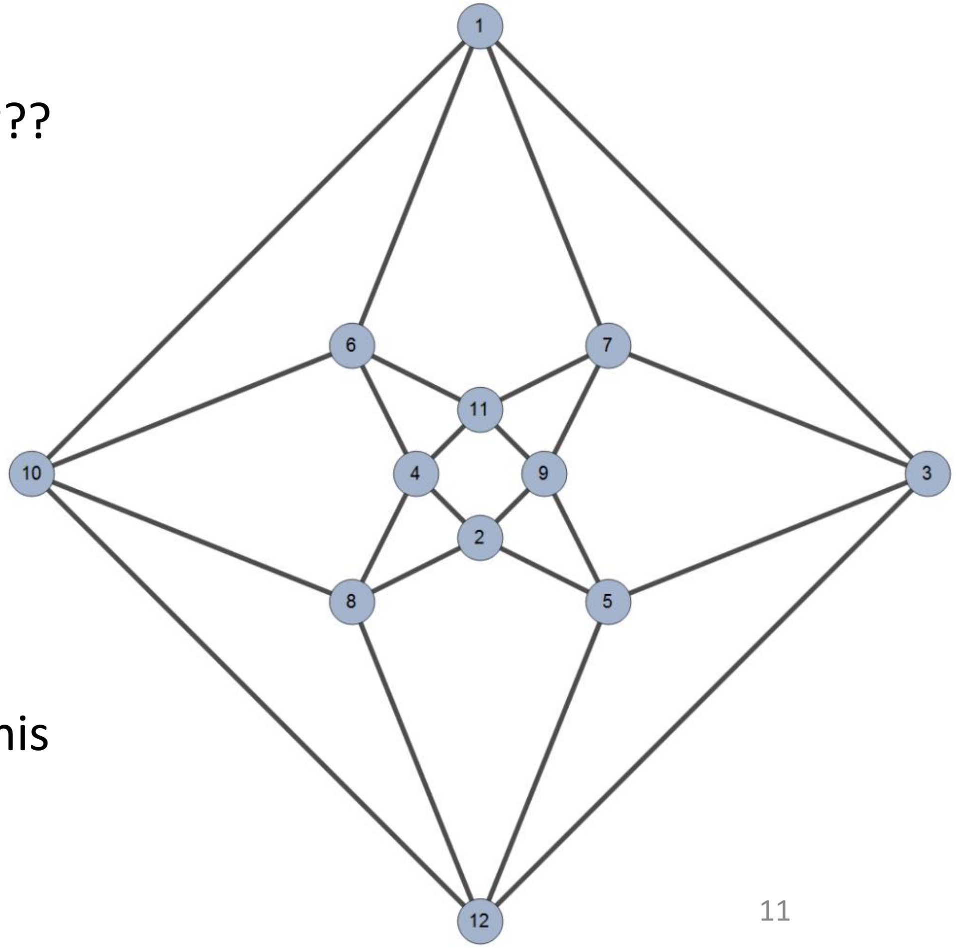


3 great circles

Lemma 2.2 – Proof



+ 1 great circle = ???



There are 3 cases happen when the 4th great circle is added:

- It cuts arc(1,5), arc(1,3), arc(2,6), arc(2,4), arc(3,6) and arc(4,5)
- It cuts arc(1,3), arc(3,5), arc(2,4), arc(4,6), arc(1,6) and arc(2,5)
- It cuts arc(1,5), arc(3,5), arc(2,6), arc(4,6), arc(2,6) and arc(1,5)

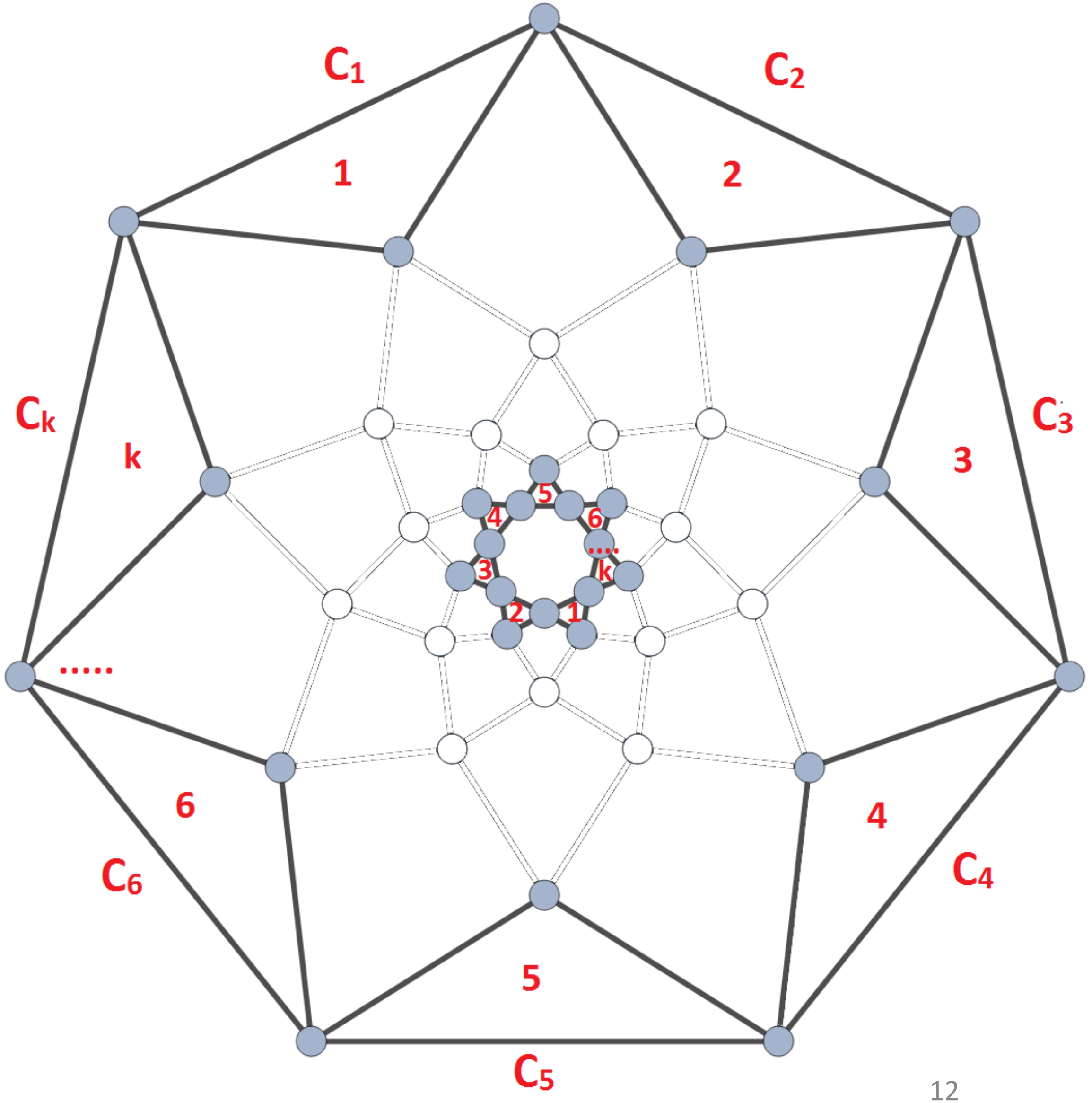
The outcome of 3 cases is the same as the figure on the right. This graph is special and unique

Lemma 2.2 – Proof

So, we have $\mathcal{S}_3$ and $\mathcal{S}_4$ are unique

By induction hypothesis, suppose $\mathcal{S}_\mathrm{k}$ has a 1 form that contains:

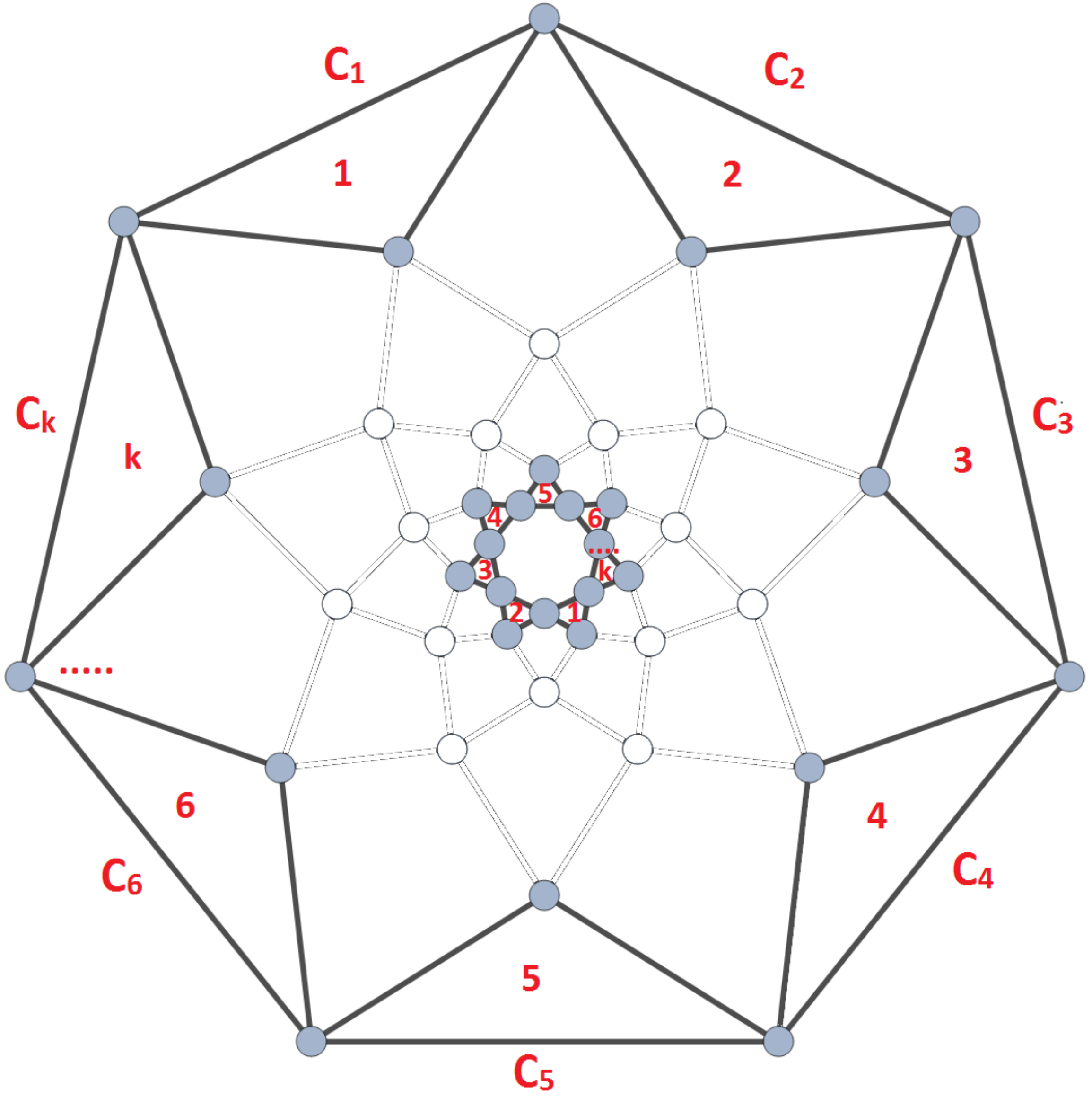
- k triangles that separate the graph and unbounded face
- Another k triangles are made by the reflection and 1 polygon has k segments.
- The other polygons are quadrilaterals



Lemma 2.2 – Proof

Is $\mathcal{S}_{\mathrm{k}}$ unique or still have other arrangements for triangles and quadrilaterals?

Answer: By lemma 2.1, only special graphs make special graphs. So if there are many non-isomorphic special graphs of **m** great circles, this suggested special graph can also guarantee it.



Lemma 2.2 – Proof

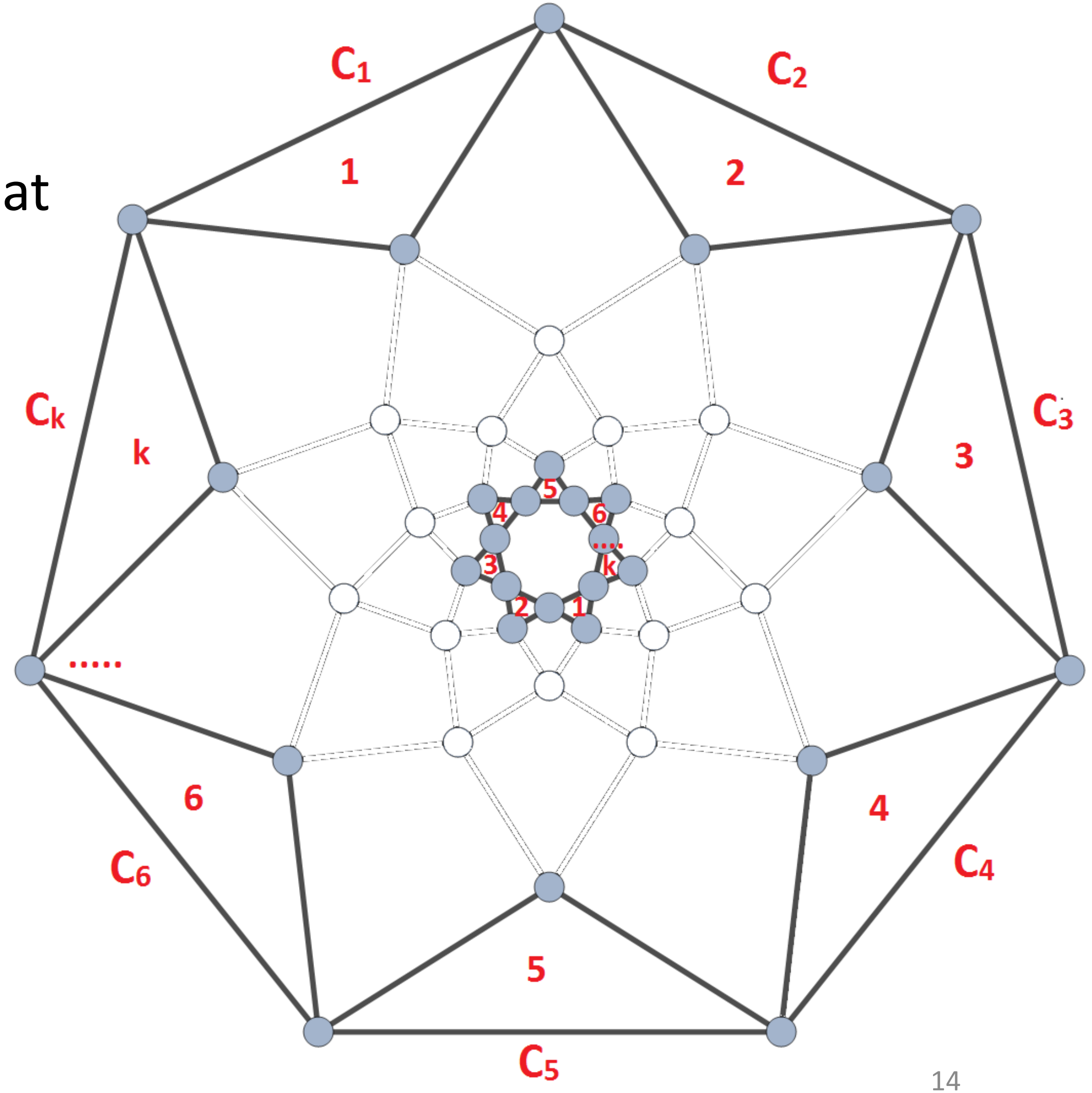
I define $\mathfrak{C}(C_i)$ is the set that will intersect C_i at first. Apparently, $\mathfrak{C}(C_i)$ contains 2 circles.

Without loss of generality, assume $\mathcal{S}_{\mathrm{k}}$ has:

- $\mathfrak{C}(C_1) = \{C_k, C_2\}$
- $\mathfrak{C}(C_2) = \{C_1, C_3\}$
- $\mathfrak{C}(C_3) = \{C_2, C_4\}$

…………………

- $\mathfrak{C}(C_{n-1}) = \{C_{k-2}, C_k\}$
- $\mathfrak{C}(C_k) = \{C_{k-1}, C_1\}$



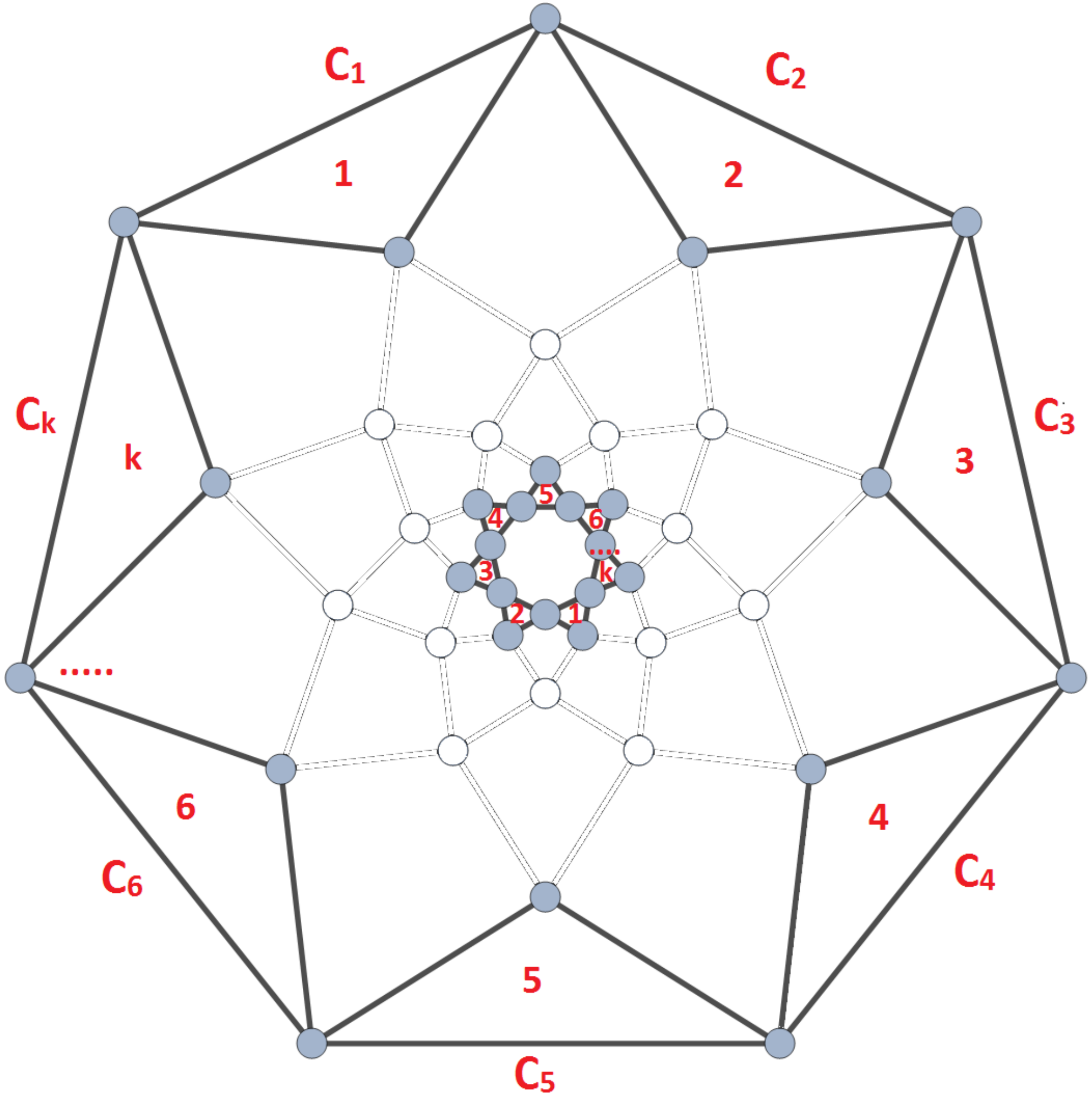
Lemma 2.2 – Proof

Now we try to add $(k+1)^{th}$ circle to $\mathcal{S}_k$. There are 2 cases happen:

1. $\mathfrak{C}(C_{k+1}) = \{C_1, C_2\}$ or $\{C_2, C_3\}$ or … $\{C_i, C_{i+1}\}$ or … $\{C_k, C_1\}$
2. $\mathfrak{C}(C_{k+1}) \neq \{C_1, C_2\}$ and $\{C_2, C_3\}$ and … $\{C_i, C_{i+1}\}$ and … $\{C_k, C_1\}$

   In other words, $\mathfrak{C}(C_{k+1})$ can be $\{C_1, C_3\}$ or $\{C_2, C_4\}$ or ……

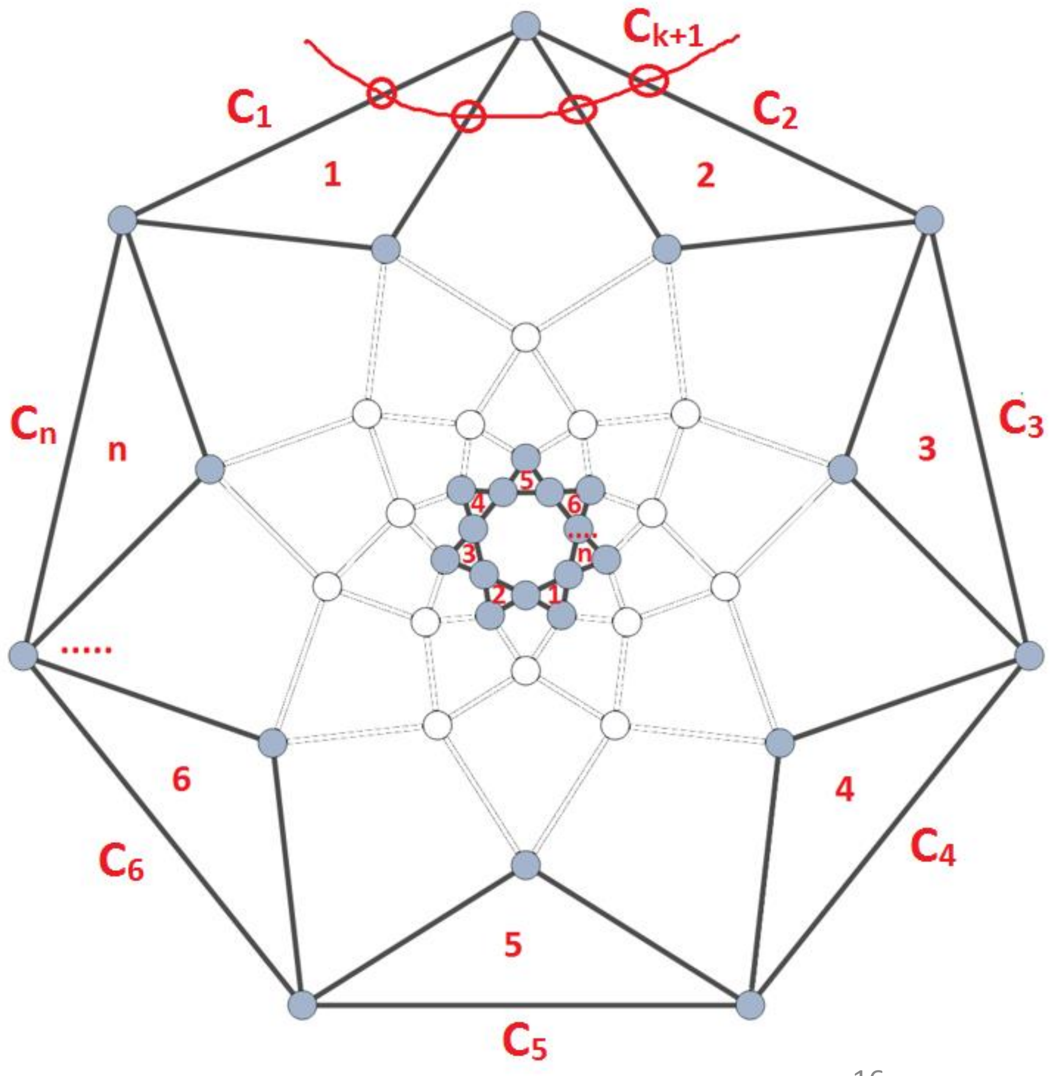
Goal: S_{k+1} is the extended version of S_k and hopefully there is no other type of graph for S_{k+1}



Lemma 2.2 – Proof – 1st case

For the 1st case, I start with $\mathfrak{C}(C_{k+1}) = \{C_1, C_2\}$.

C_{k+1} firstly intersects C_1 and C_2. If C_{k+1} cuts the quadrilateral created by (C_n, C_1, C_2, C_3), it will be closed and there are 4 intersections on C_{k+1}. By lemma 1.1, C_{k+1} must have 2k ➔ There is no case like this

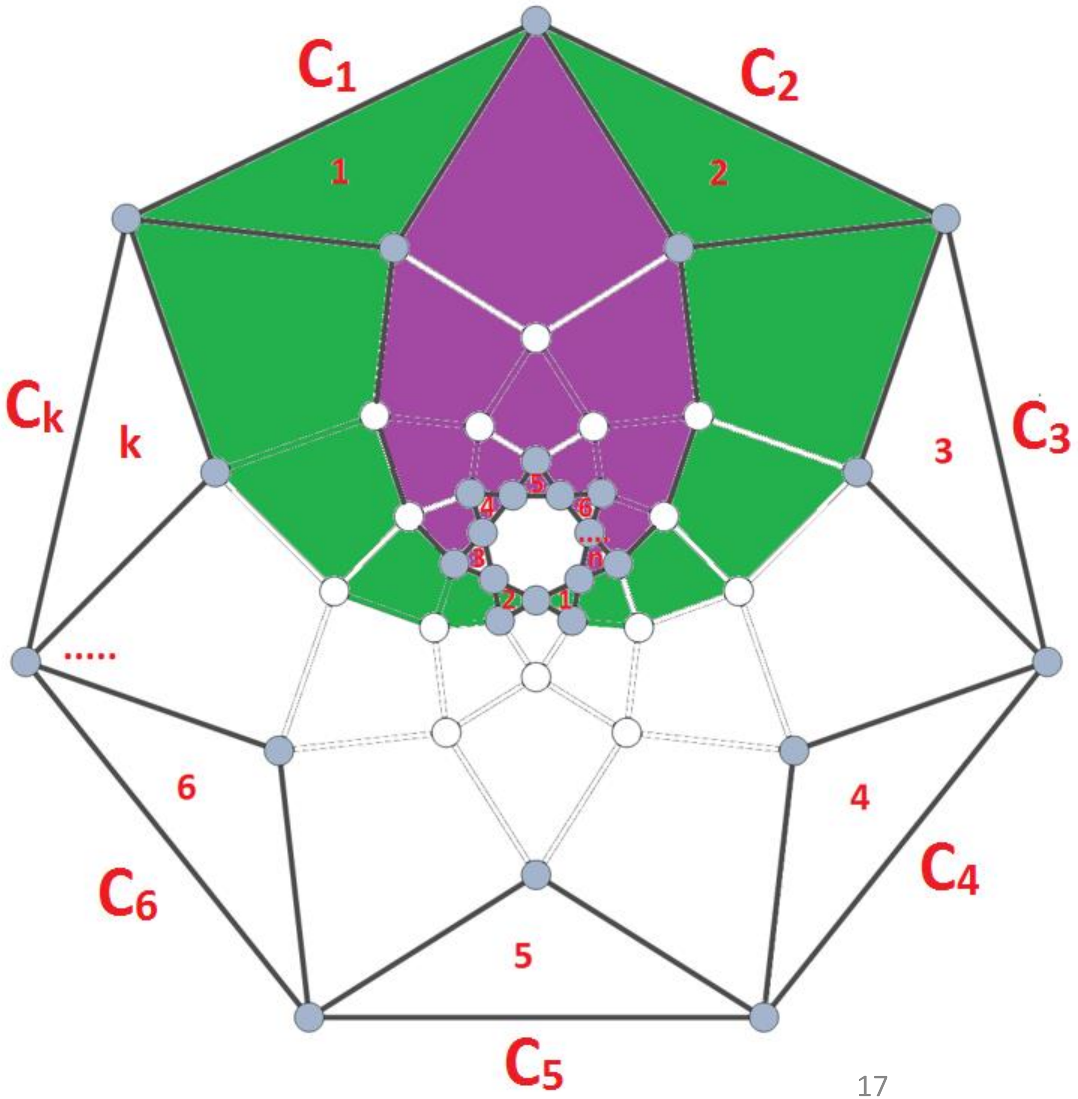


Lemma 2.2 – Proof – 1st case

Similarly, C_{k+1} cannot intersect the purpil region made by

$$\mathcal{R}_{purpil} = \{\mathfrak{C}(C_i)_1 \cap \mathfrak{C}(C_i)_2\} \setminus (C_1, C_2, \dots, C_n)$$

The reason is C_{k+1} will make pentagonal !!!



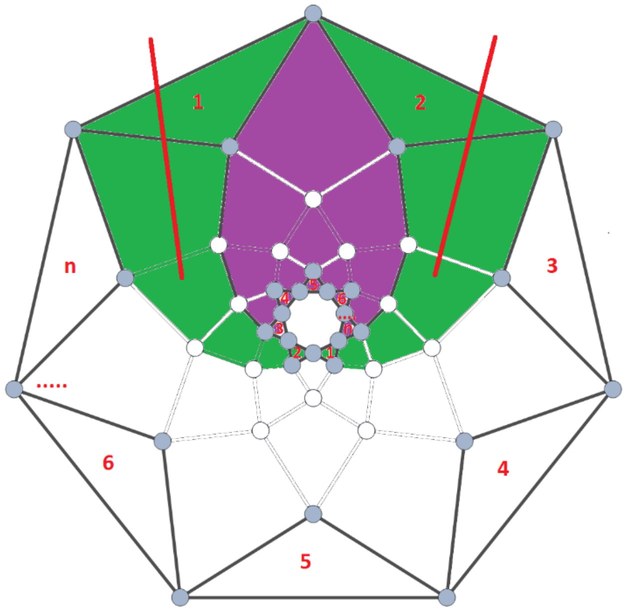
Lemma 2.2 – Proof – 1^{st} case

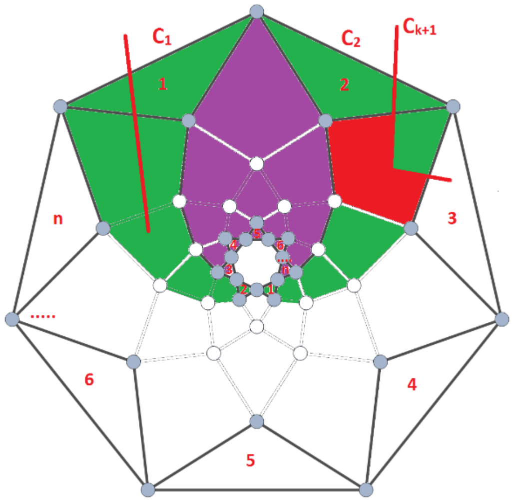
Therefore, C_{k+1} must intersect other circles by following the green region

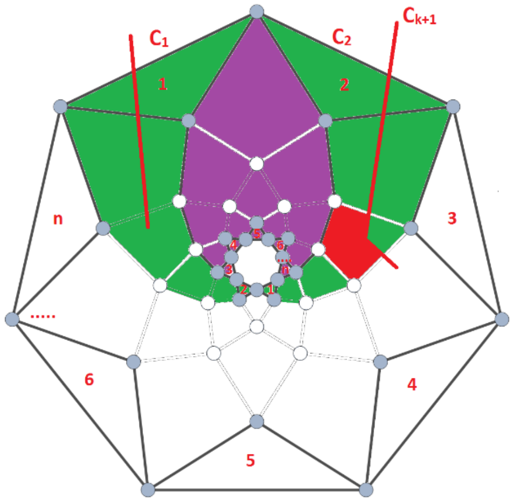
$$\mathcal{R}_{green} = \{\mathfrak{C}(C_i)_1 + \mathfrak{C}(C_i)_2\} \setminus \mathcal{R}_{purpil}$$

If there is such a counter-example such that C_{k+1} doesn't follow the green region but can create another special graph, it would be the cases shown in the figures on the right. That is when it jumped out to another quadrilateral or triangle not in the green region. However, these cases will make pentagonal.

➔Contradiction





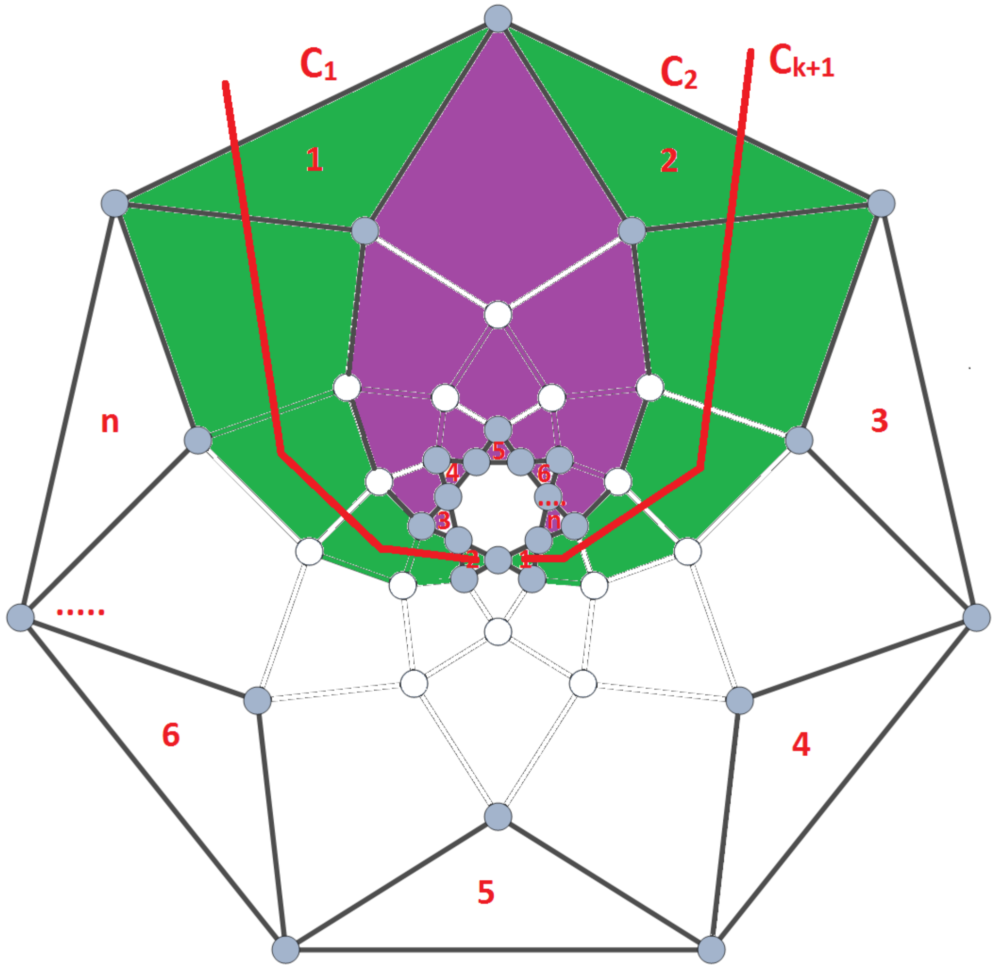


Lemma 2.2 – Proof – 1st case

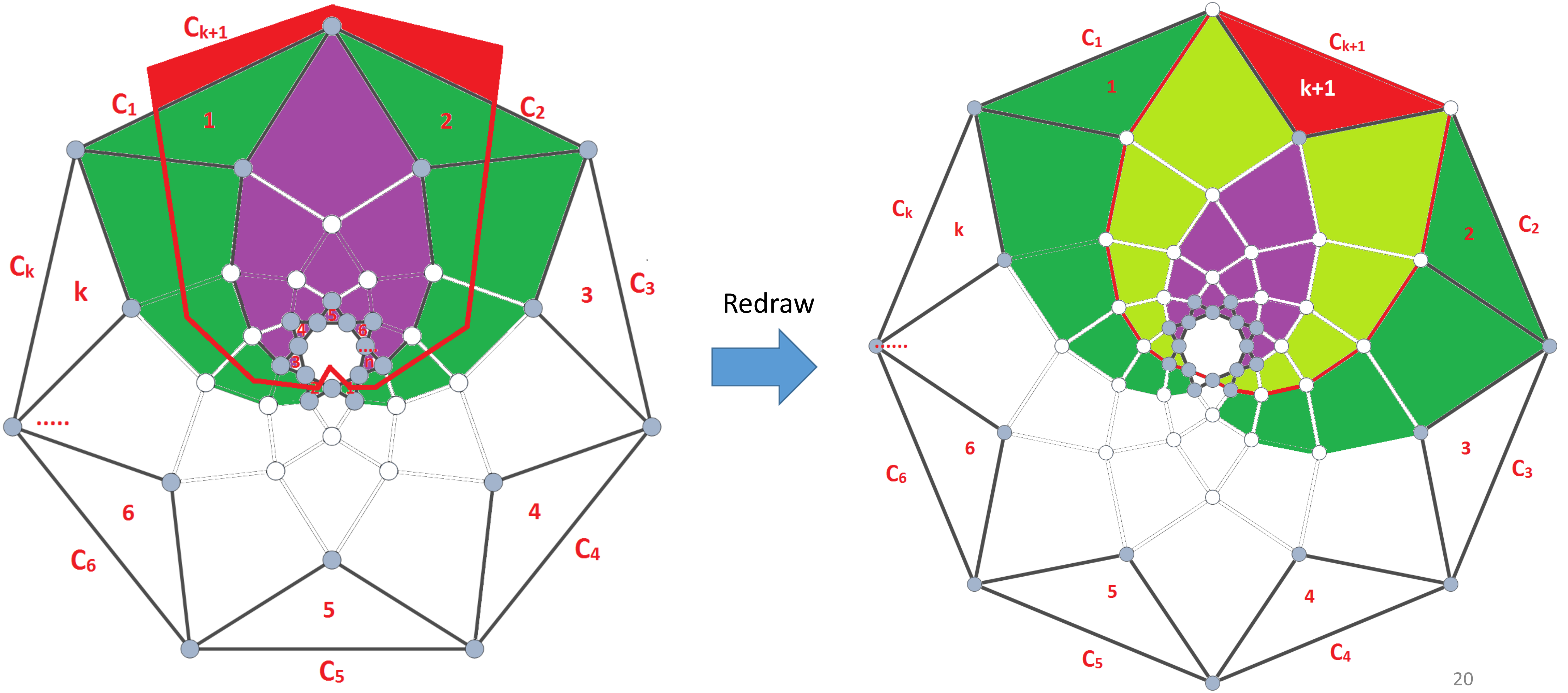
The final step is C_{k+1} will intersect the polygon of k segments or a quadrilateral to make it become a cycle. Apparently, it will make a pentagonal if it intersects the quadrilateral. However, intersecting the polygon of k segments is acceptable and it creates a special graph because

- No pentagonal is created.
- The k-segments polygon becomes (k+1)-segments polygon
- C_{k+1} increases k to (k+1) bounded triangles by the triangle of C_{k+1} and its closest set
- It's unique in the 1st case because this is the only special graph can be made by $\mathcal{S}_{\mathrm{k}}$

➔ The induction hypothesis is still correct in S_{k+1}



Lemma 2.2 – Proof – 1st case



Lemma 2.2 – Proof – 1st case

- So we have proved that when $\mathfrak{C}(\mathrm{C}_{k+1}) = \{C_1, C_2\}$, the final graph is special and unique in the 1st case
- Similarly, when $\mathfrak{C}(\mathrm{C}_{k+1}) = \{C_2, C_3\}$ or … $\{Ci, Ci_{+1}\}$ or … $\{C_k, C_1\}$, the way to add C_{k+1} is similar. It will create S_{k+1} and all the outcomes are the same.

Lemma 2.2 – Proof – 1st case

In the 2nd case, we have

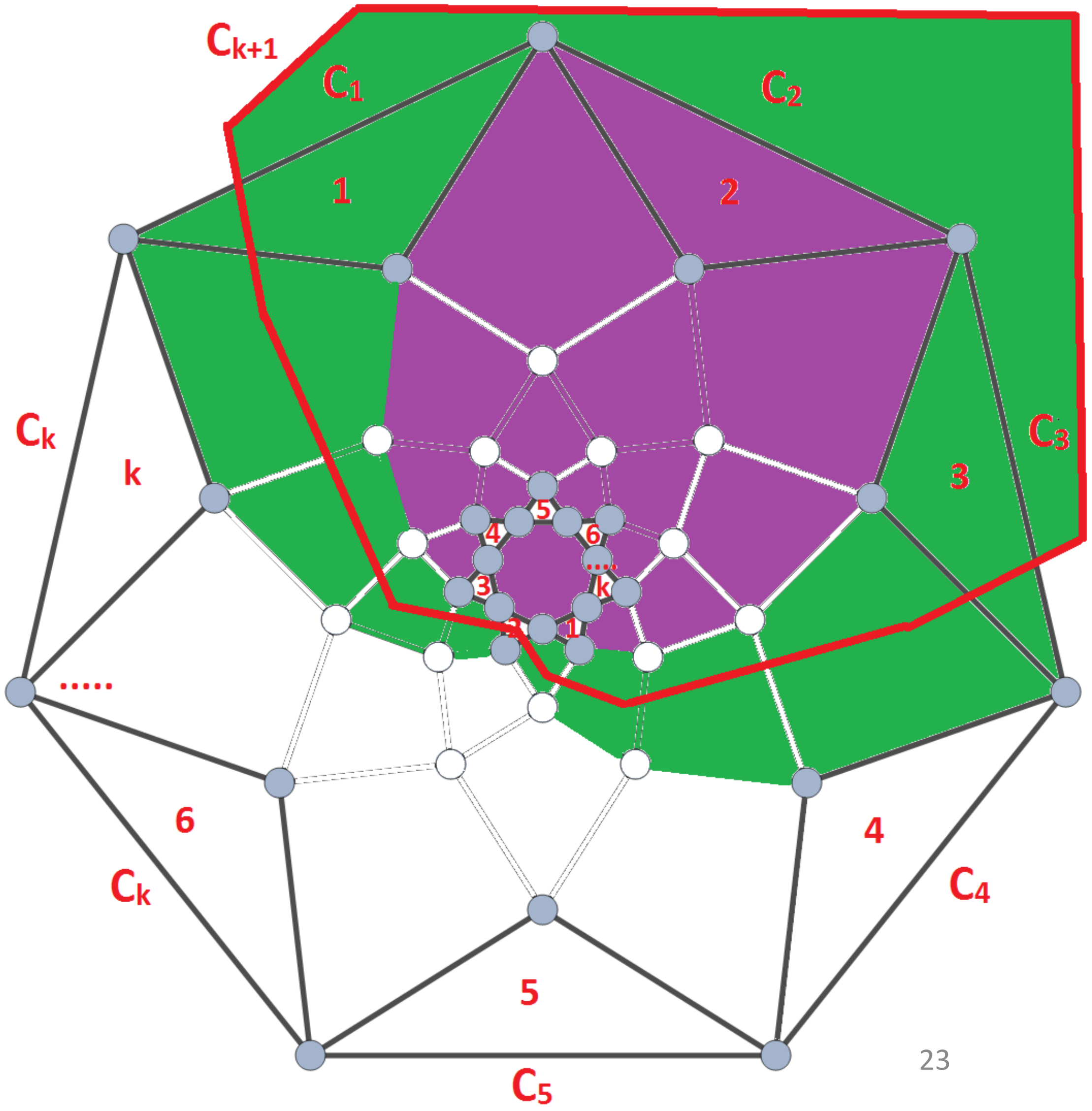
$$\mathfrak{C}(\mathrm{C}_{k+1}) \neq \{C_1, C_2\} \text{ or } \{C_2, C_3\} \text{ or } \dots \{C_i, C_{i+1}\} \text{ or } \dots \{C_k, C_1\}.$$

Now we try to prove that this case cannot generate any S_{k+1} so that yields the unique S_{k+1} in the 1st case to be unique in both cases.

Lemma 2.2 – Proof – 1^{st} case

- Assume $\mathfrak{C}(C_{k+1})$ = $\{C_1, C_3\}$
- If C_{k+1} cuts $\mathfrak{C}(C_{k+1})$ and close at the unbounded face, then there is the only cycle that doesn't make any pentagonals (shown in the figure)
- This cycle has 2k vertices but it doesn't intersect C_2 !!!

➔ There is no cycle C_{k+1} like this



Lemma 2.2 – Proof – 2nd case

- So C_{k+1} will not go to the unbounded face. It means the triangles bound will be the same

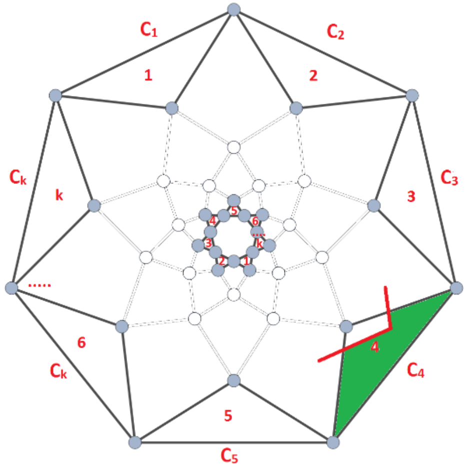
- Recall that when making intersections, C_{k+1} must split
  - 1 quadrilateral into 2 quadrilaterals
  - 1 triangle to 1 quadrilateral and 1 triangle.

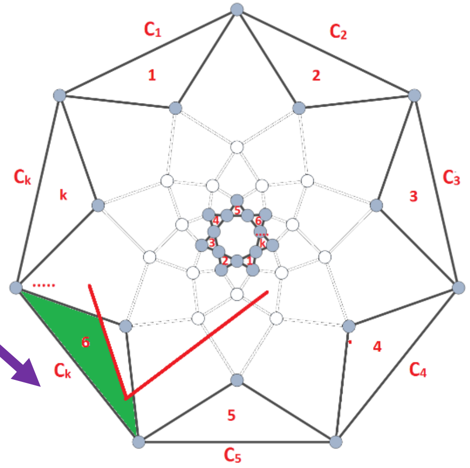
to guarantee the result as a special graph

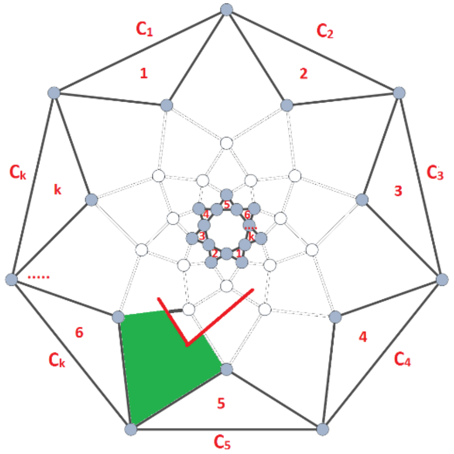
Lemma 2.2 – Proof – 2nd case

However, the rule above makes C_{k+1} follow only 1 direction that will tend towards the triangles bound.

- If C_{k+1} intersects at least 1 triangle at the bound, so the triangles bound will be broken by a new quadrilateral was made
- If C_{k+1} tries to not intersect triangle, it must change to follow other ways which then create pentagonals







Lemma 2.2 – Proof – 2^{nd} case

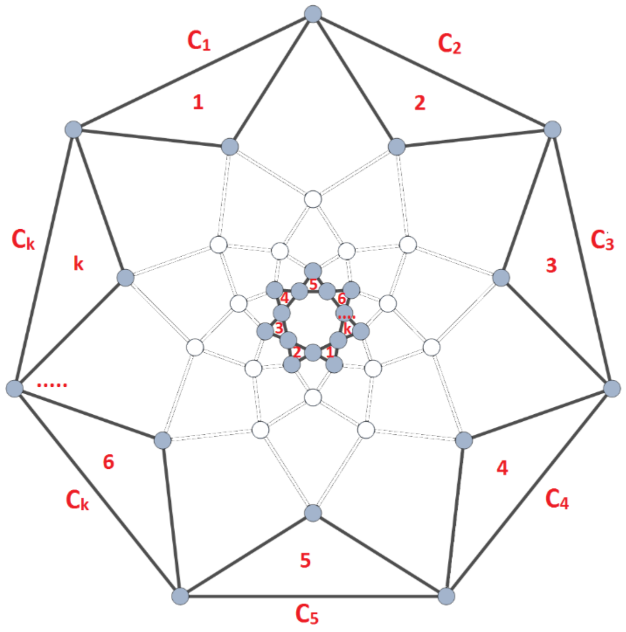
So, there is no special graph can be made in the 2^{nd} case

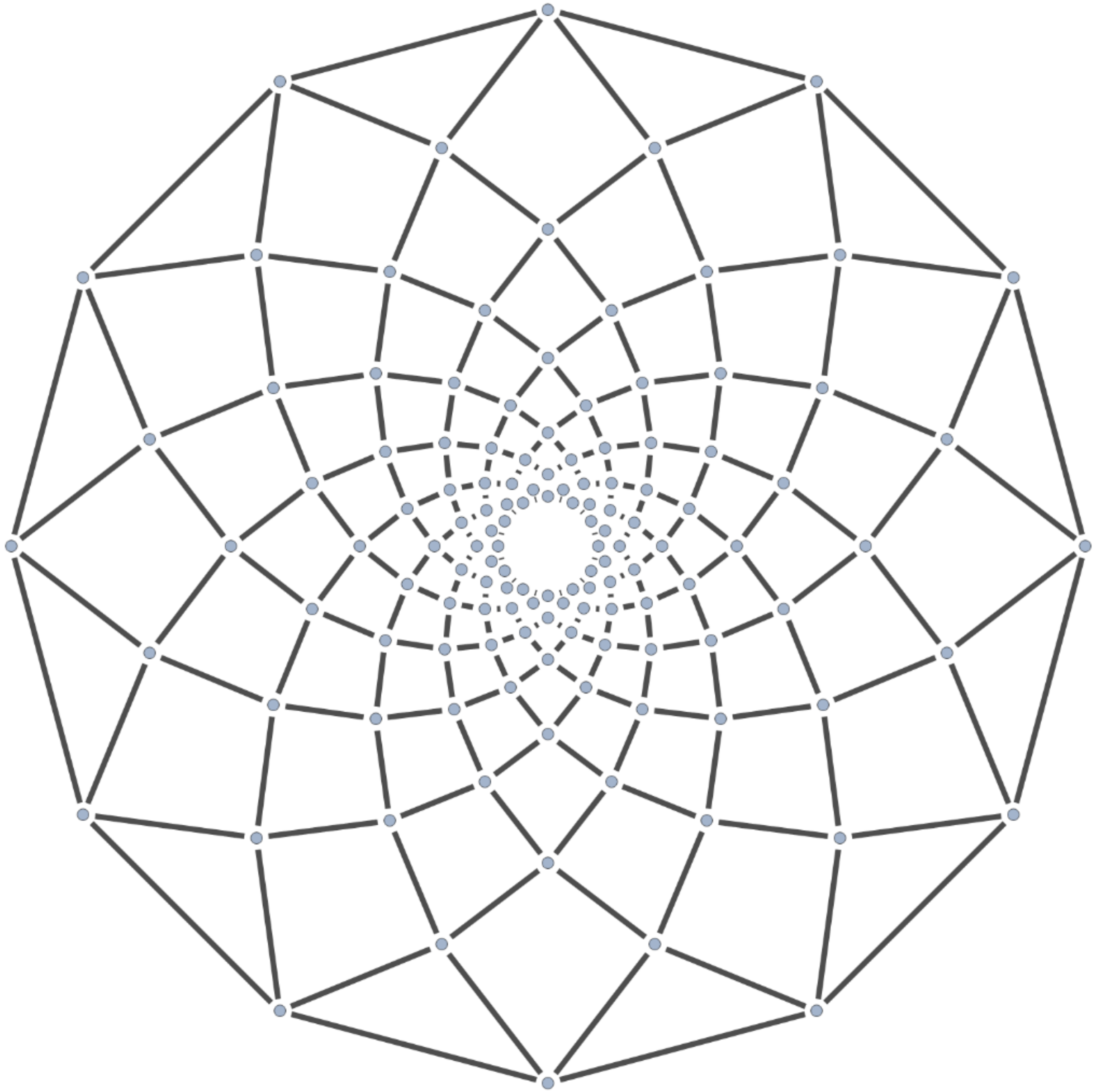
Similarly, with other closest sets $\mathfrak{C}(C_{k+1}) \neq \{C_1 , C_2\}$ or $\{C_2 , C_3\}$ or … $\{C_i, C_{i+1}\}$ or … $\{C_k, C_1\}$, they will not produce any S_{k+1}

➔ S_{k+1} in the 1^{st} case is unique after all

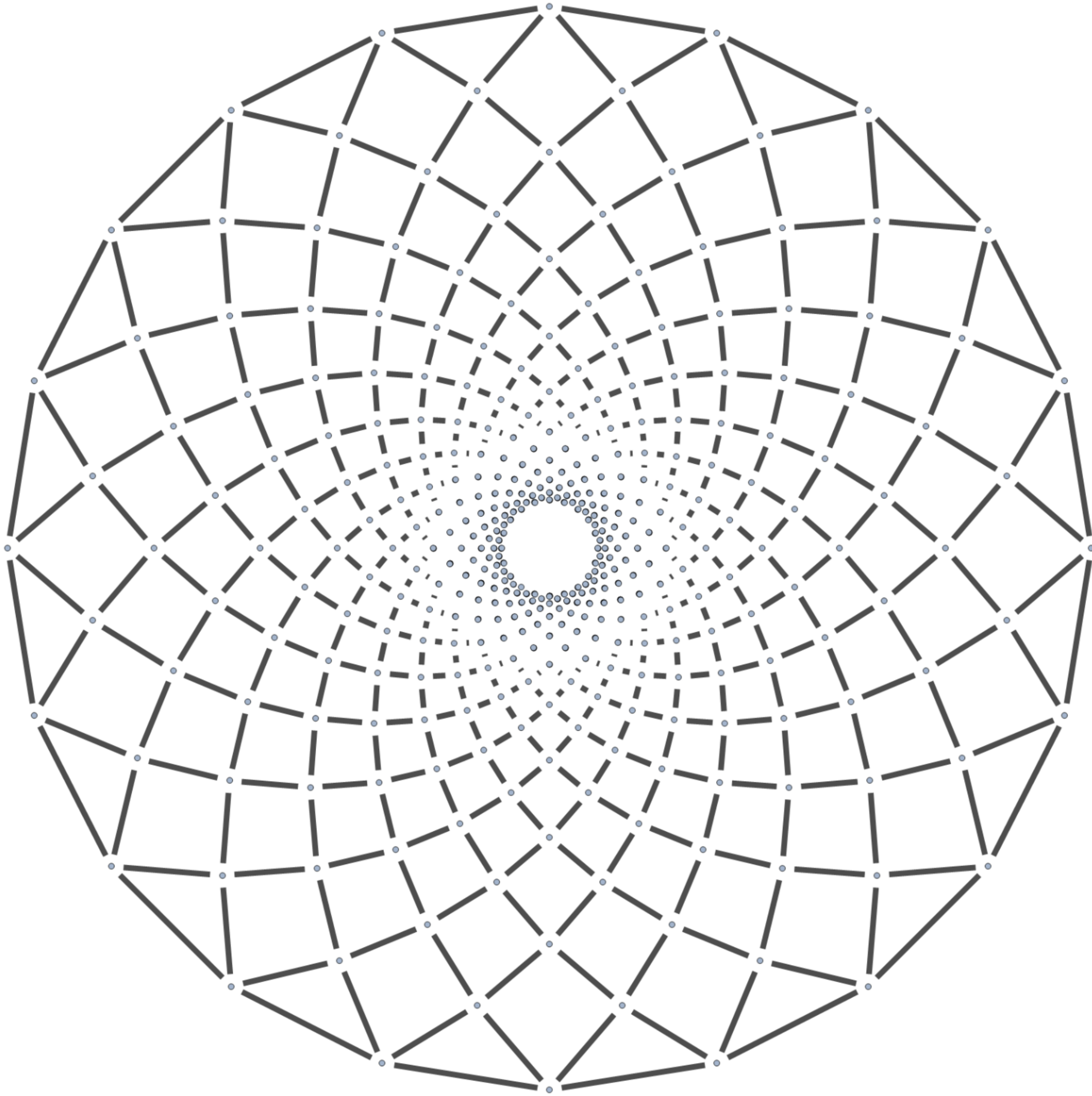
➔ The induction step is proved.

➔ S_{k+1} is unique and it has the form →





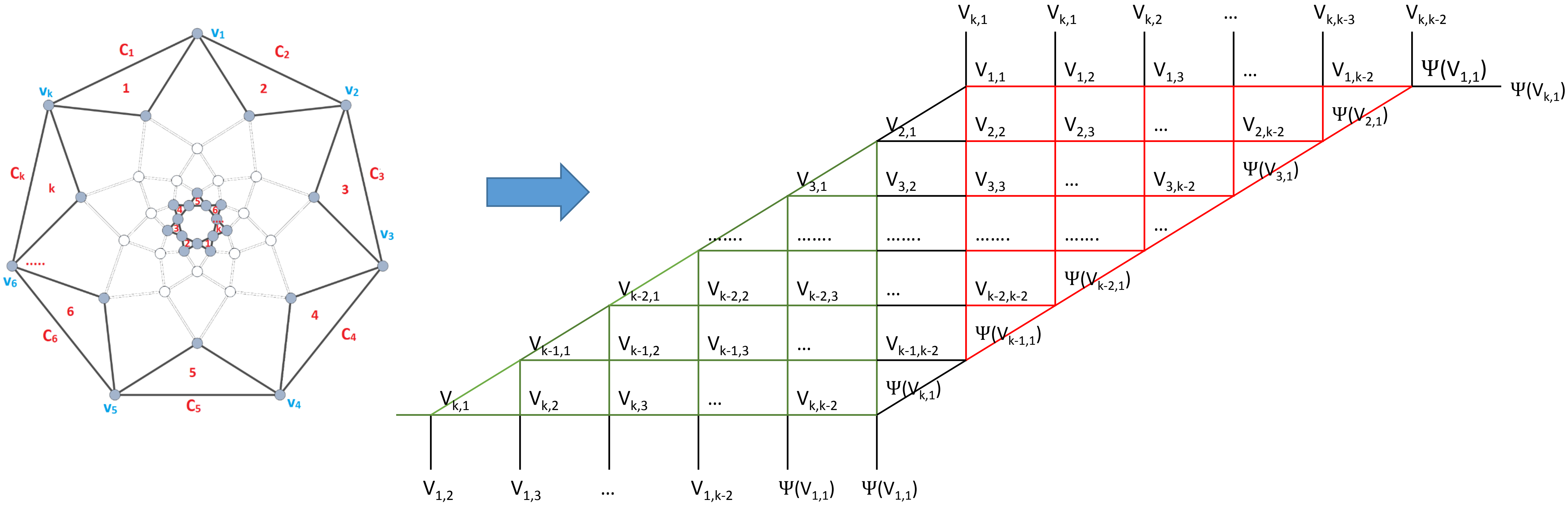
S_{12}



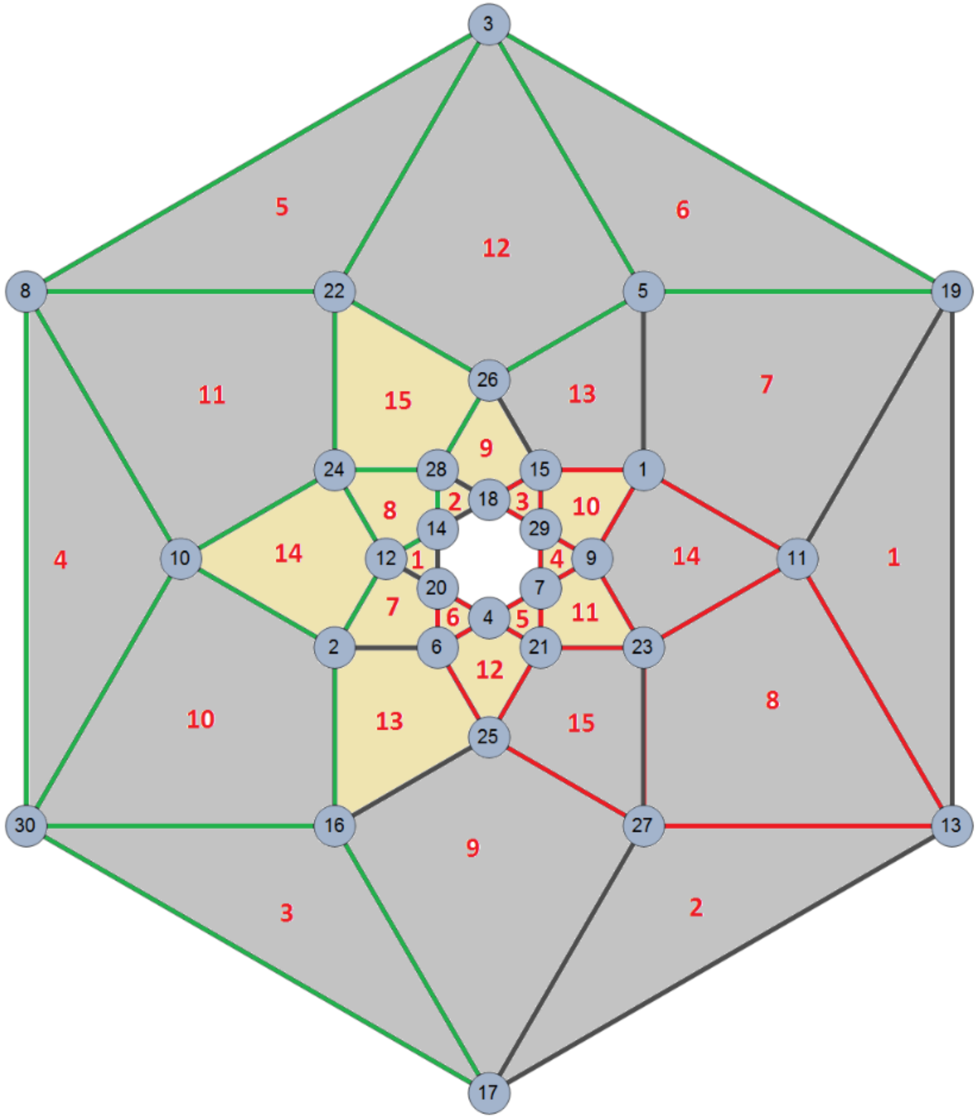
S_{20}

Lemma 3.

S_k can be transform into the following equivalent graph:



Lemma 3 – Proof – A base case with 6 circles

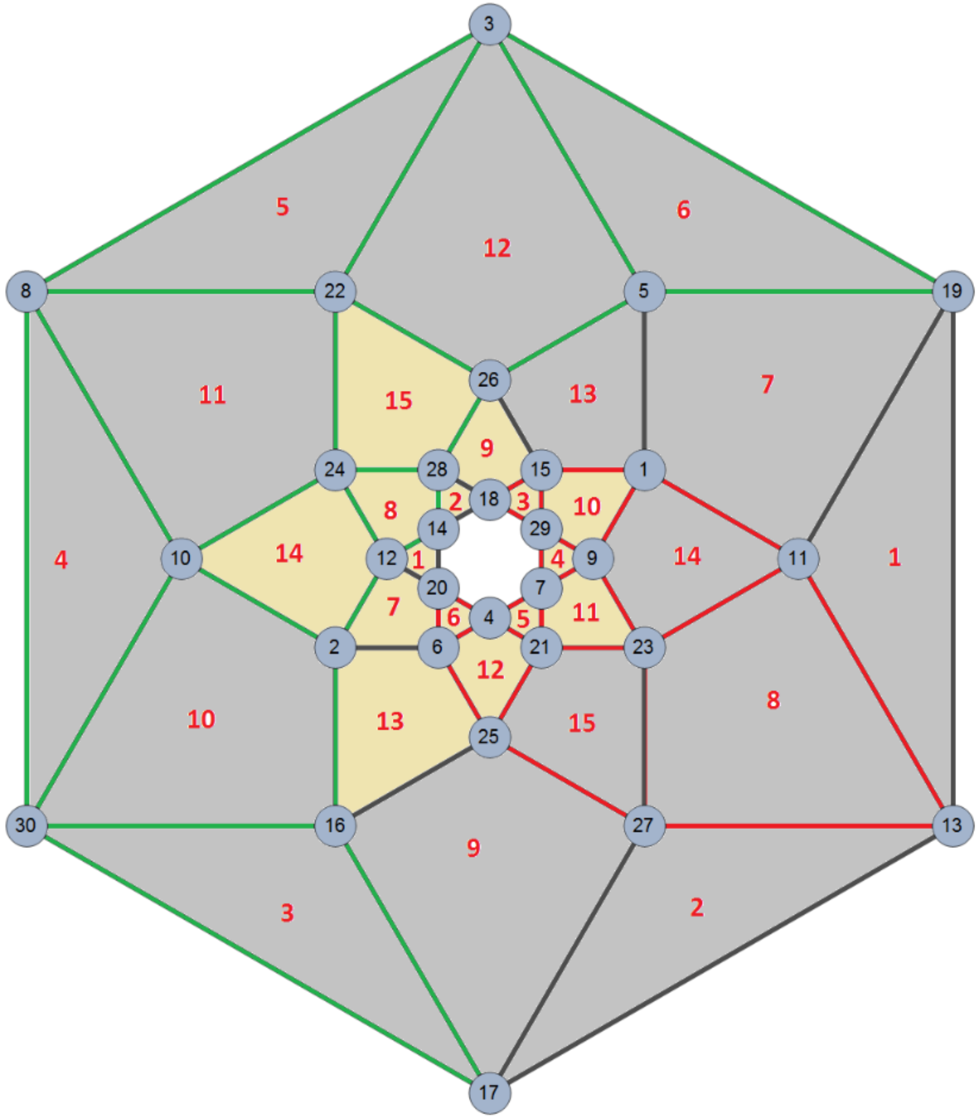


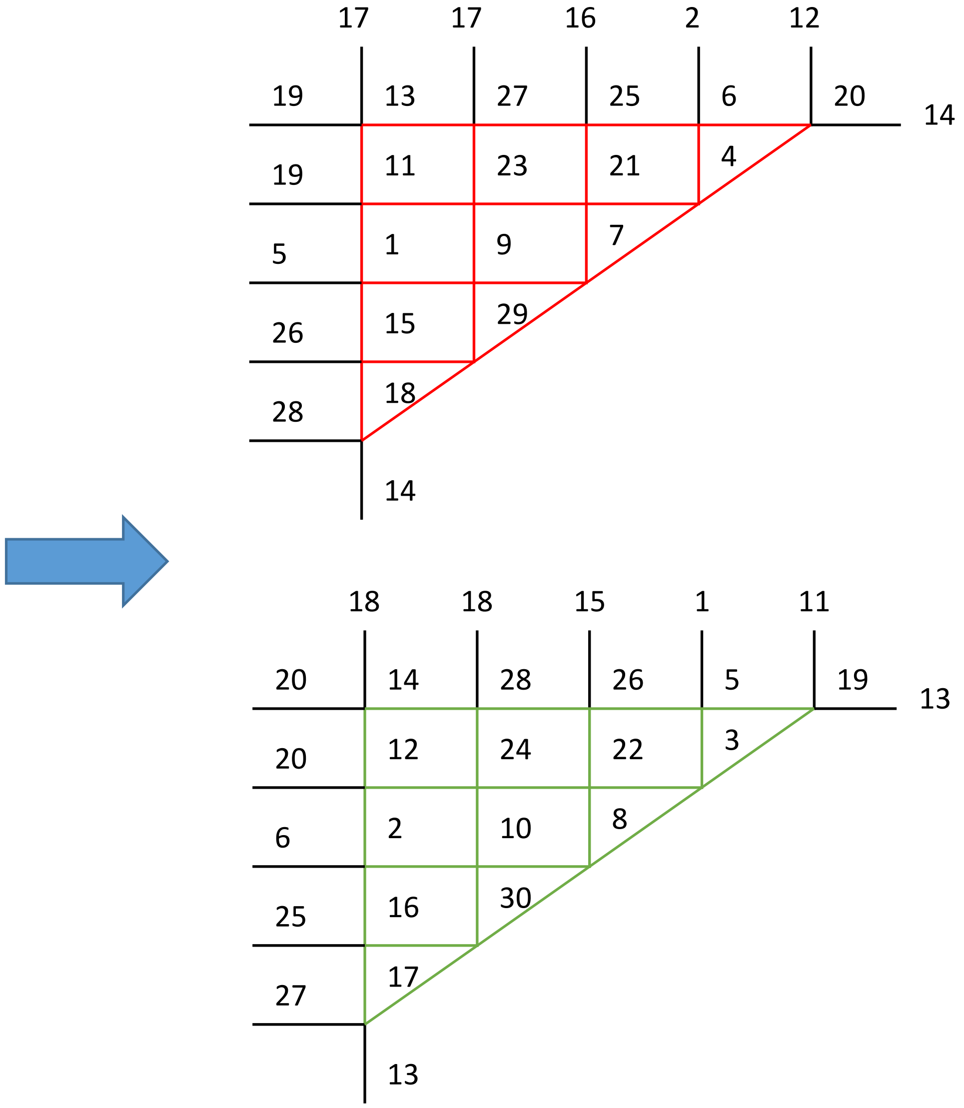
Here is the special graph of **6** great circles.

Annotation:

- Red edges ∈ 1st side
- Green edges ∈ 2nd side
- **Black** edges are the external links of the 1st side and the 2nd side
- Every region has a number in the middle
- There are 2 regions have the same number. One has 1 set of vertices while the other has the reflection of that set via O
- Grey/Yellow regions contain numbers distinctly

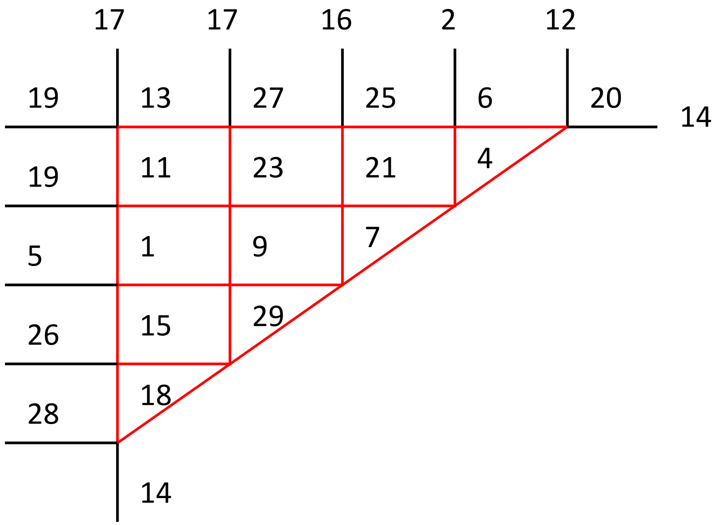
Lemma 3 – Proof – A base case with 6 circles

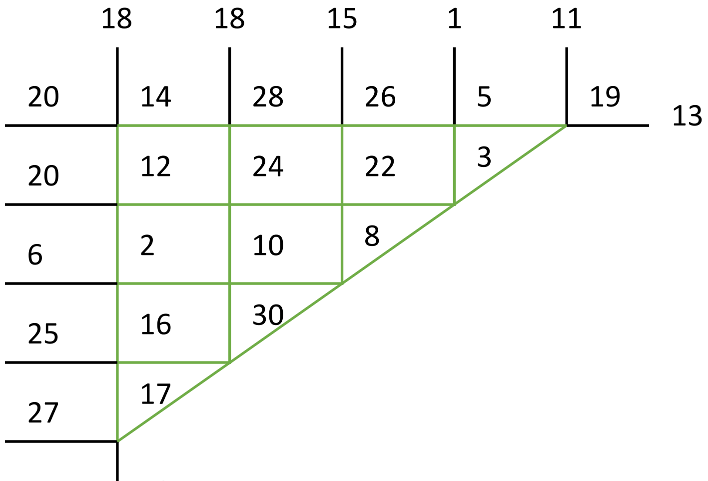


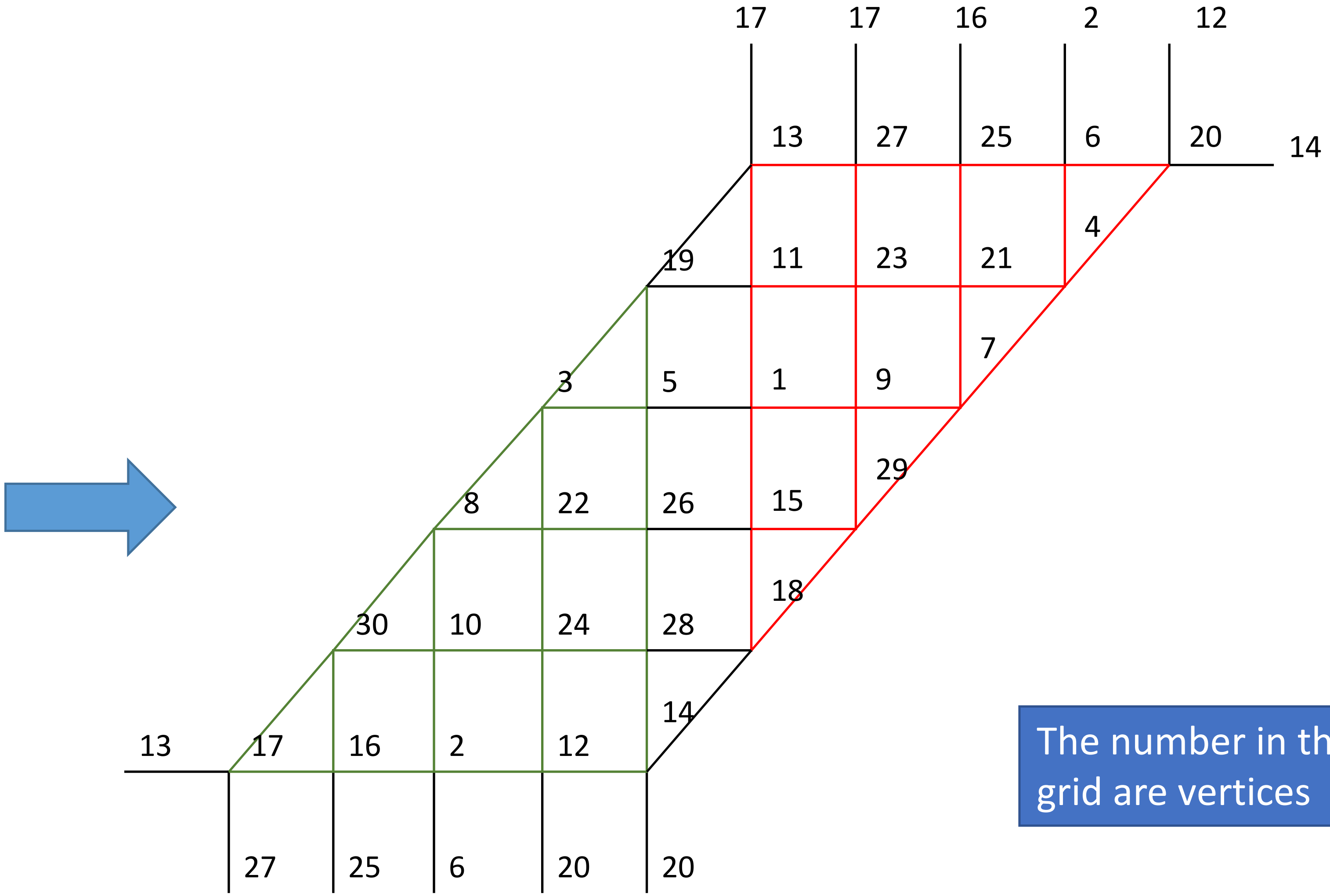


The number in this grid are vertices

Lemma 3 – Proof – A base case with 6 circles

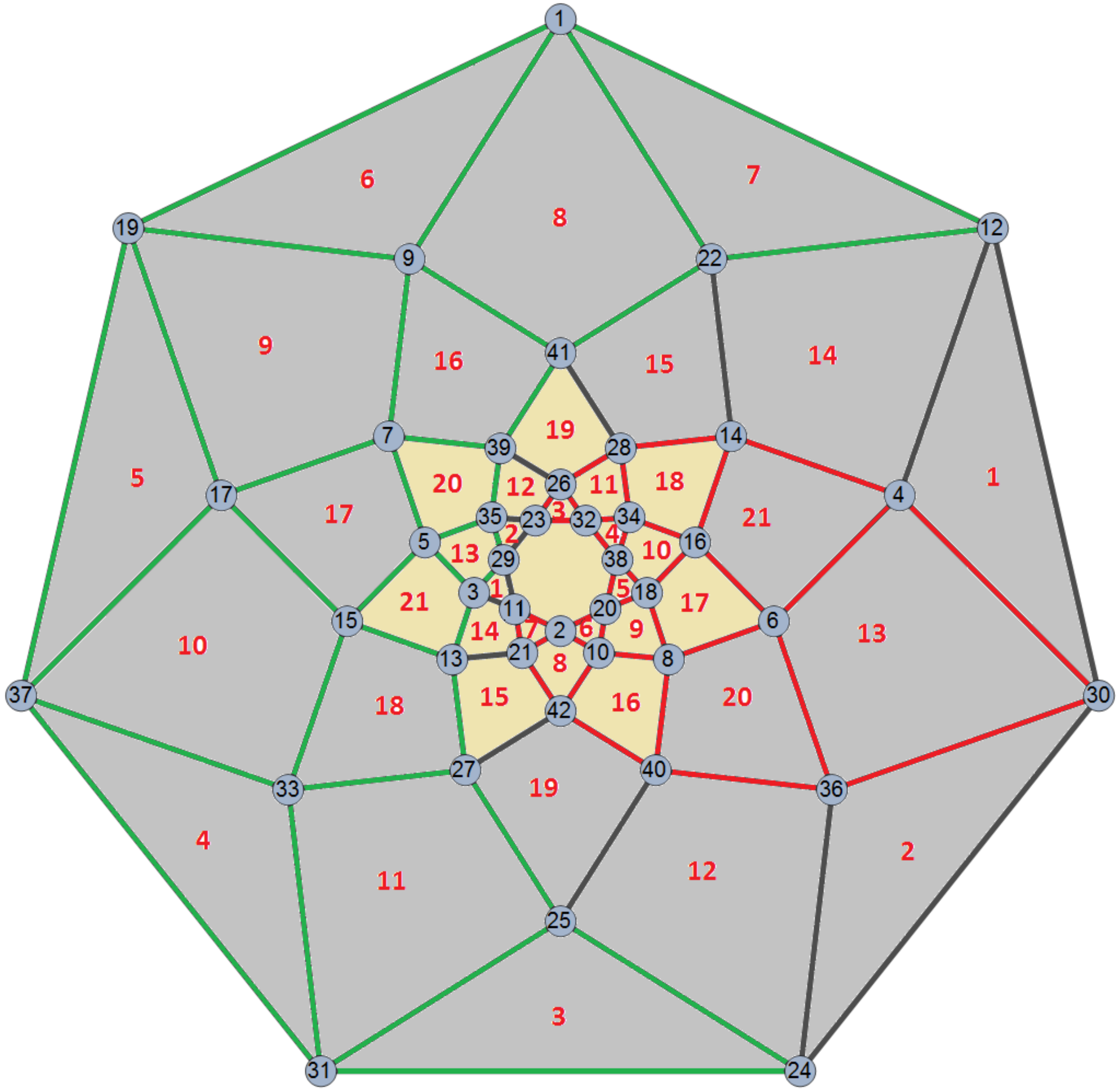






The number in this grid are vertices

Lemma 3 – Proof – A base case with 7 circles

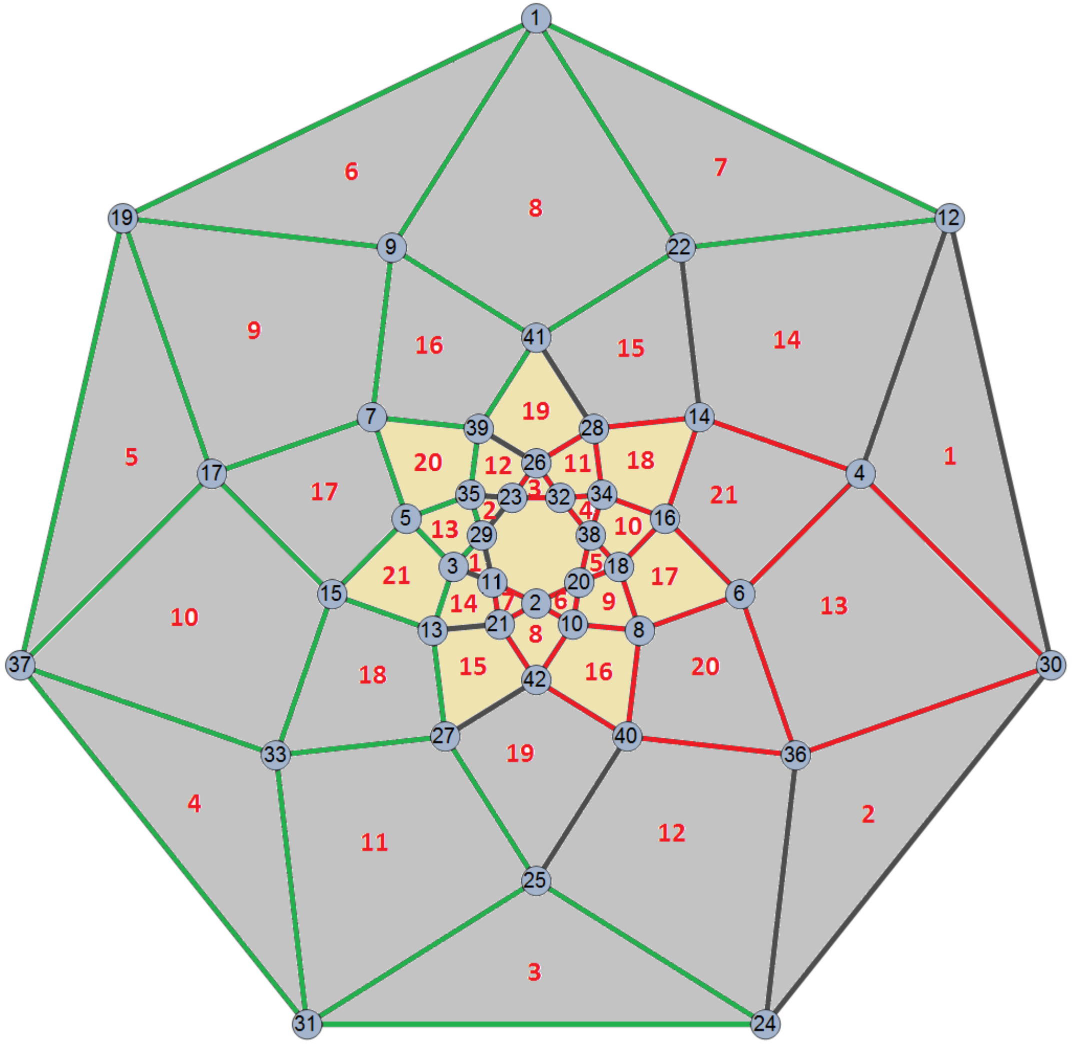


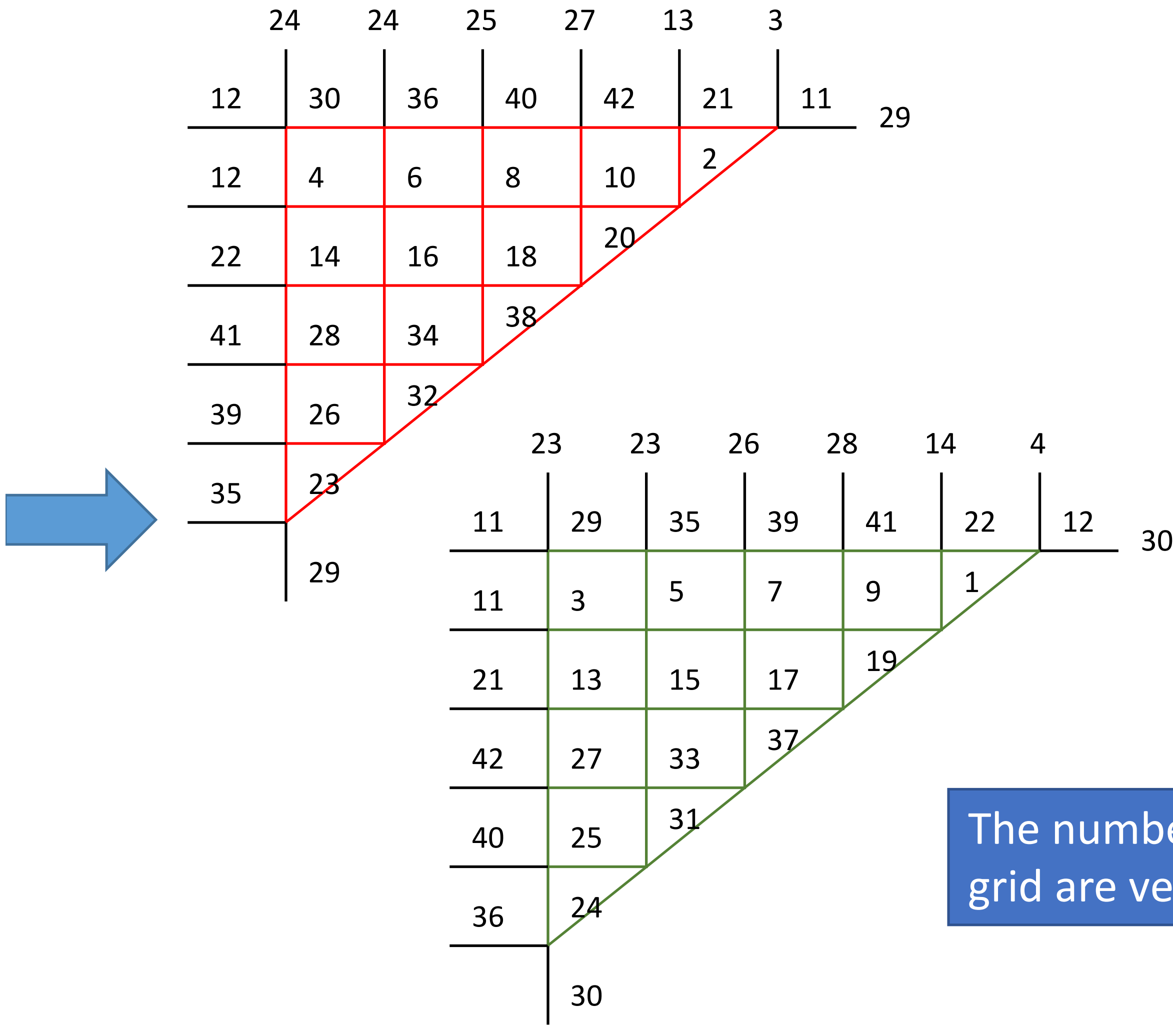
Here is the special graph of **7** great circles.

Annotation:

- Red edges ∈ 1st side
- Green edges ∈ 2nd side
- **Black** edges are the external links of the 1st side and the 2nd side
- Every region has a number in the middle
- There are 2 regions have the same number. One has 1 set of vertices while the other has the reflection of that set via O
- Grey/Yellow regions contain numbers distinctly

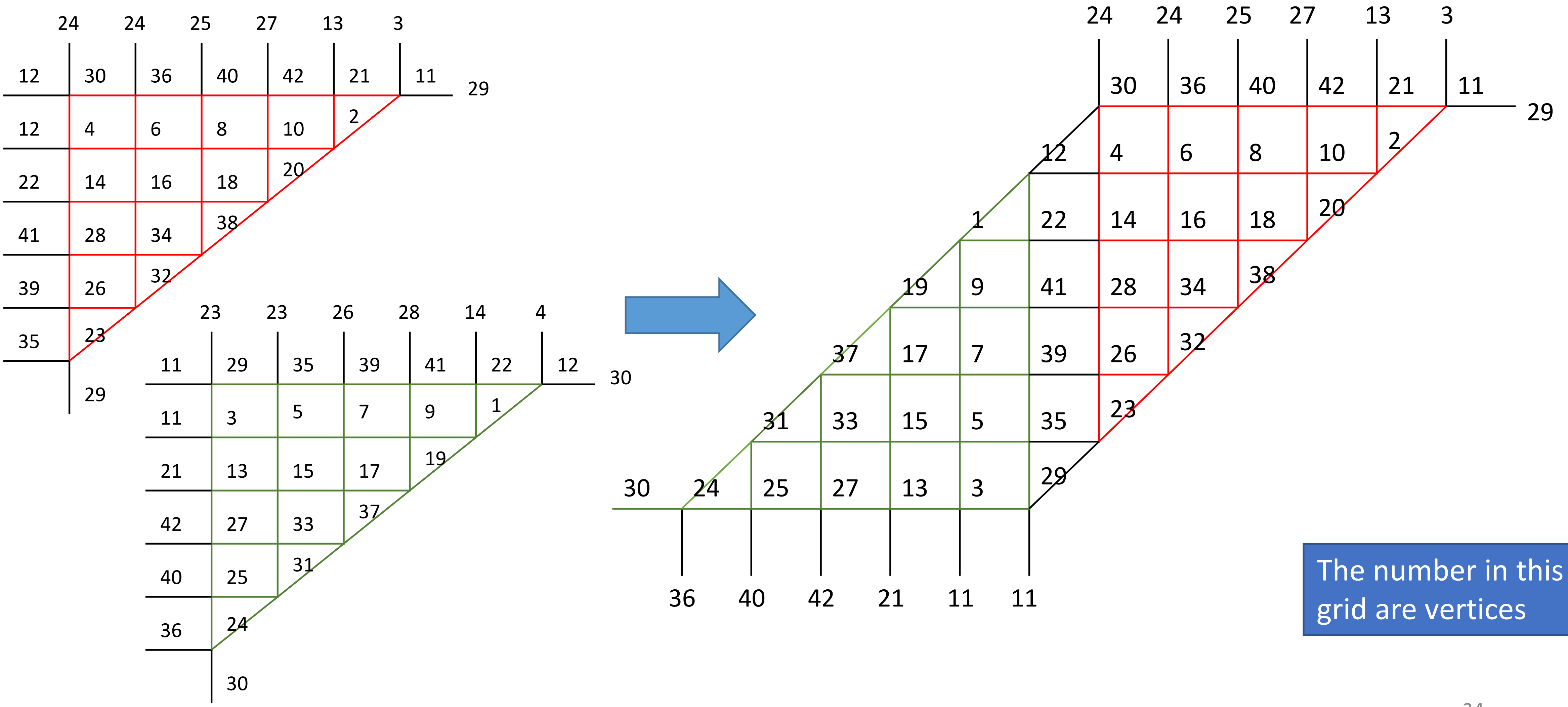
Lemma 3 – Proof – A base case with 7 circles





The number in this grid are vertices

Lemma 3 – Proof – A base case with 7 circles

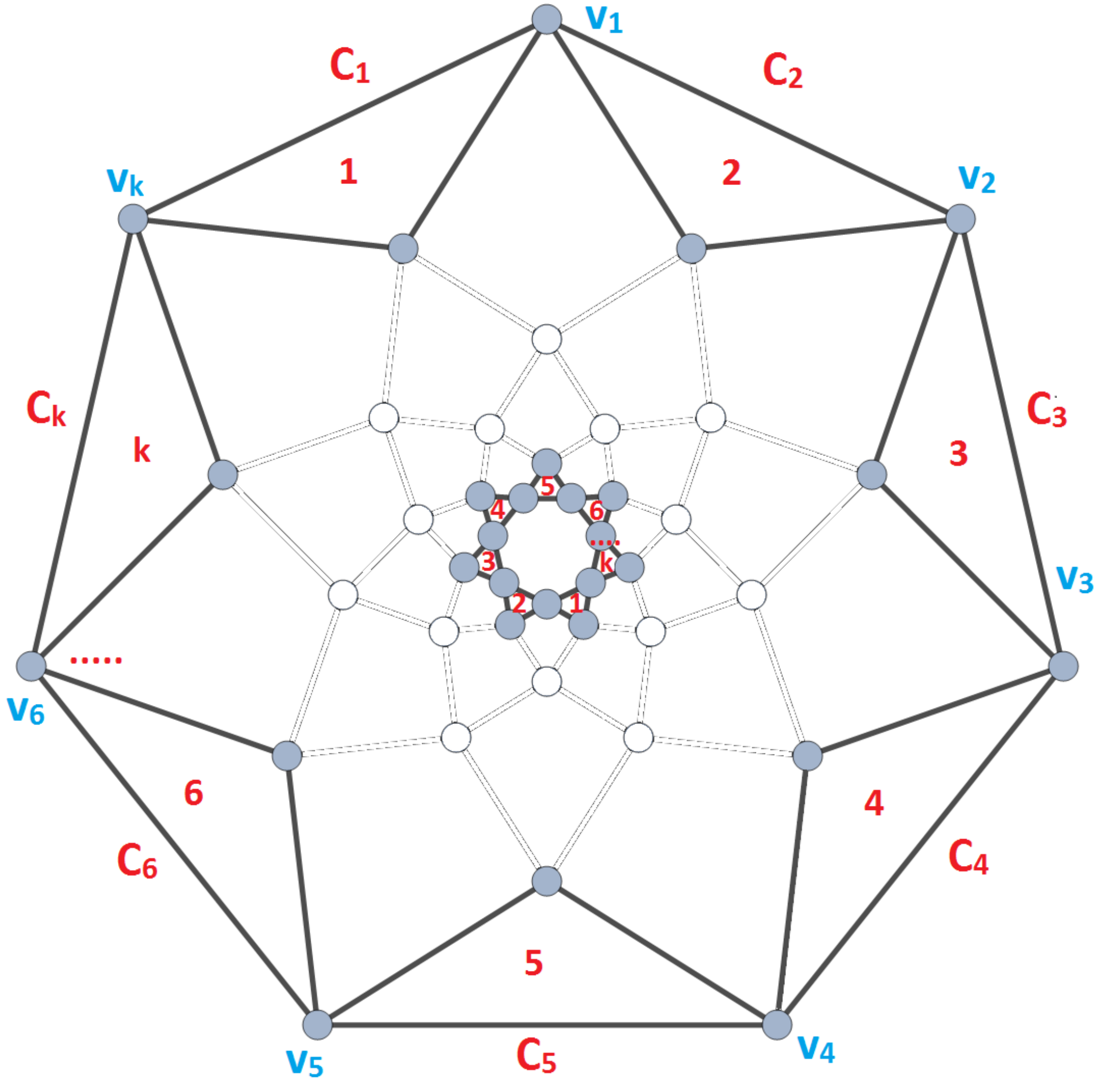


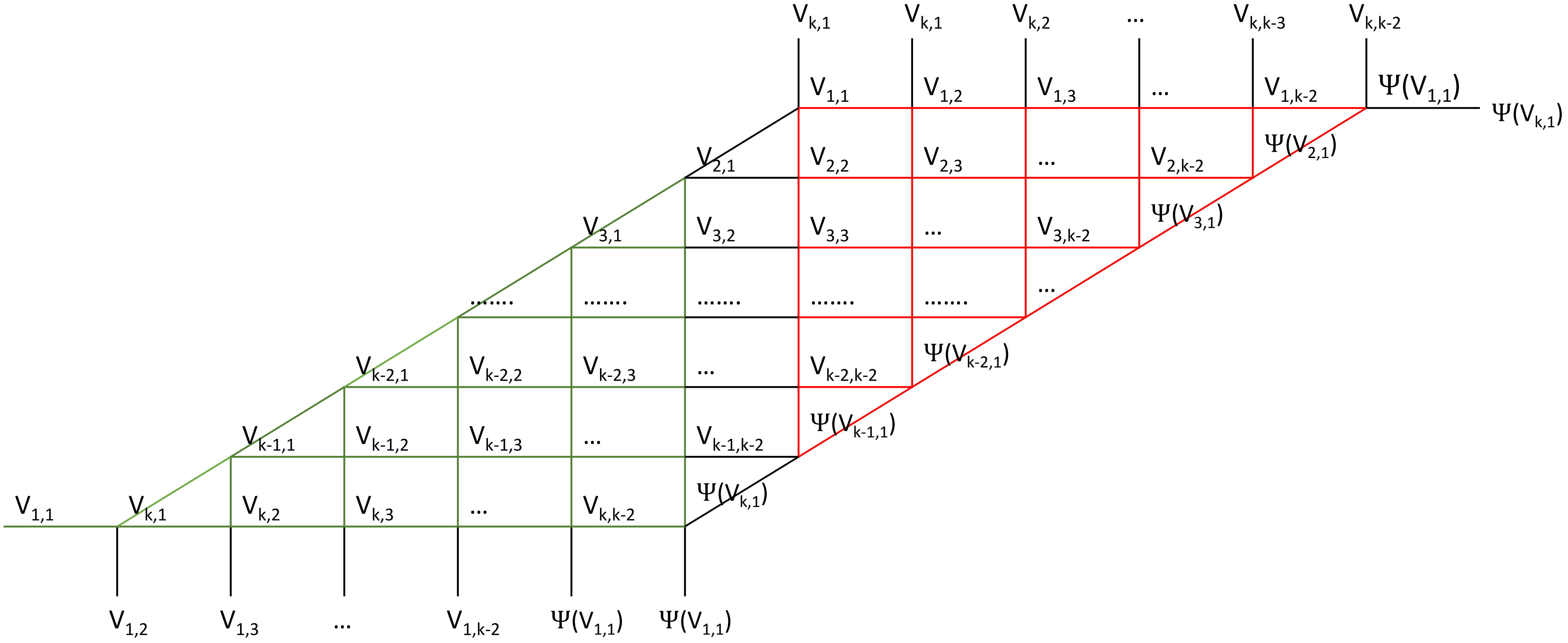
The number in this grid are vertices

Lemma 3

Call

- $V_{i,1}$ are the vertices made by C_i and C_{i+1} on the unbounded cycle.
- $V_{i,1}$, $V_{i,2}$, $V_{i,3}$, … $V_{i,2k-2}$ are the vertices on the circle C_i in the order that ($V_{i,1}$, $V_{i,2}$, $V_{i+1,1}$) is a triangle





$V_{k,1}$ $V_{k,1}$ $V_{k,2}$ … $V_{k,k-3}$ $V_{k,k-2}$
$V_{1,1}$ $V_{1,2}$ $V_{1,3}$ … $V_{1,k-2}$ $\Psi(V_{1,1})$ $\Psi(V_{k,1})$
$V_{2,1}$ $V_{2,2}$ $V_{2,3}$ … $V_{2,k-2}$ $\Psi(V_{2,1})$
$V_{3,1}$ $V_{3,2}$ $V_{3,3}$ … $V_{3,k-2}$ $\Psi(V_{3,1})$
……. ……. ……. ……. ……. …
$V_{k-2,1}$ $V_{k-2,2}$ $V_{k-2,3}$ … $V_{k-2,k-2}$ $\Psi(V_{k-2,1})$
$V_{k-1,1}$ $V_{k-1,2}$ $V_{k-1,3}$ … $V_{k-1,k-2}$ $\Psi(V_{k-1,1})$
$V_{1,1}$ $V_{k,1}$ $V_{k,2}$ $V_{k,3}$ … $V_{k,k-2}$ $\Psi(V_{k,1})$
$V_{1,2}$ $V_{1,3}$ … $V_{1,k-2}$ $\Psi(V_{1,1})$ $\Psi(V_{1,1})$

Lemma 4. The properties of S_k

There are 2k triangles, (k-1)*(k-3) quadrilaterals and 1 polygon has k segments.

Proof: By Lemma 2.2 and Lemma 3, we can easily count this

Theorem 1

The chromatic number of $\mathcal{S}_{\mathrm{k}}$ is 3

Theorem 1 – Proof

- By Lemma 2.2, S_k is unique
- By Lemma 3, S_k can be transformed into an equivalent parallelogram
- The proof to prove the equivalent parallelogram is 3-colorable is in the presentation in 01-25-2015

The next steps

- Other non-special graphs are made by special graphs. They will all have form by S_k adding m new great circles that only make vertices inside of S_k
- The non-special graphs can be counted because when adding a new great circle into S_k, there are not infinite ways to do that
- Since we have S_k is 3-colorable, adding more great circles might be solved by Kempe switch