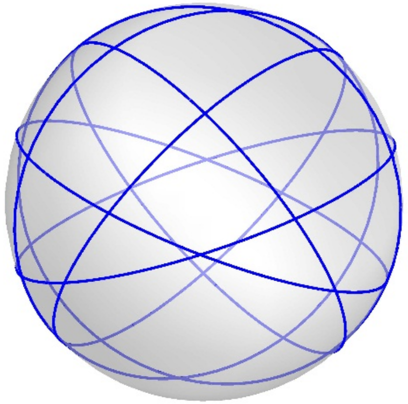


Great Circles Problem

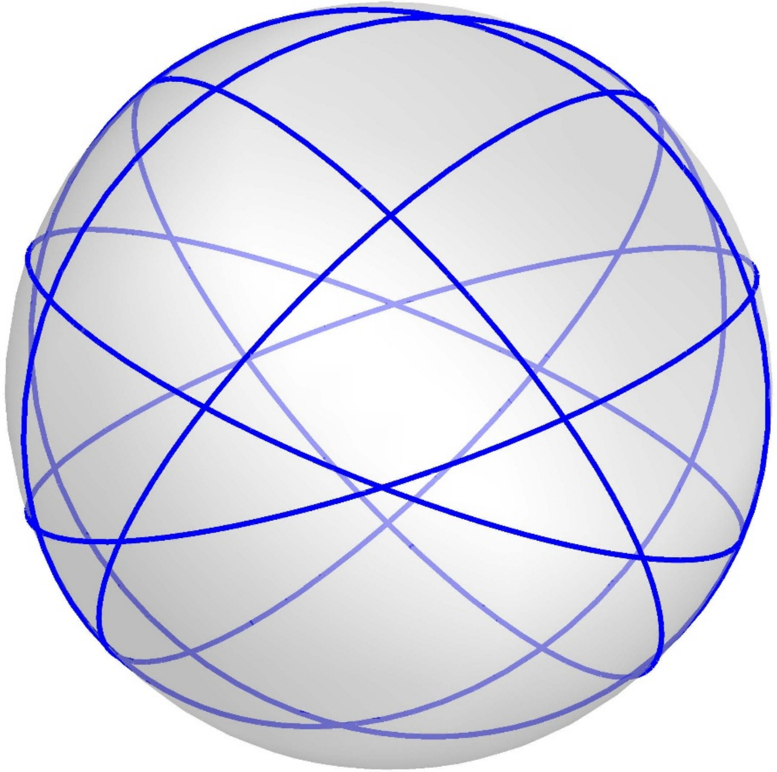
Kha Man

1-25-2015

Outline

1. Problem restatement
2. 2 example cases with 9 and 10 great circles
3. The big picture of the problem
4. The proof about 3 colourability for the problem ($\boldsymbol{\chi(G) = 3}$)
5. Visualization
   - Algorithms used in the application

Problem Restatement



(Great Circle Problem) A great circle is any circle on a sphere whose radius is **the same as the radius of the sphere** (so it is largest possible). A circle that goes through both the North and South poles is an example of a great circle on the Earth. Given n ($n \geq 3$) great circles on a sphere, no three of which pass through a single point, form a great circle graph by making points of intersection into vertices, and connect two vertices by an edge if and only if there is an arc between them.

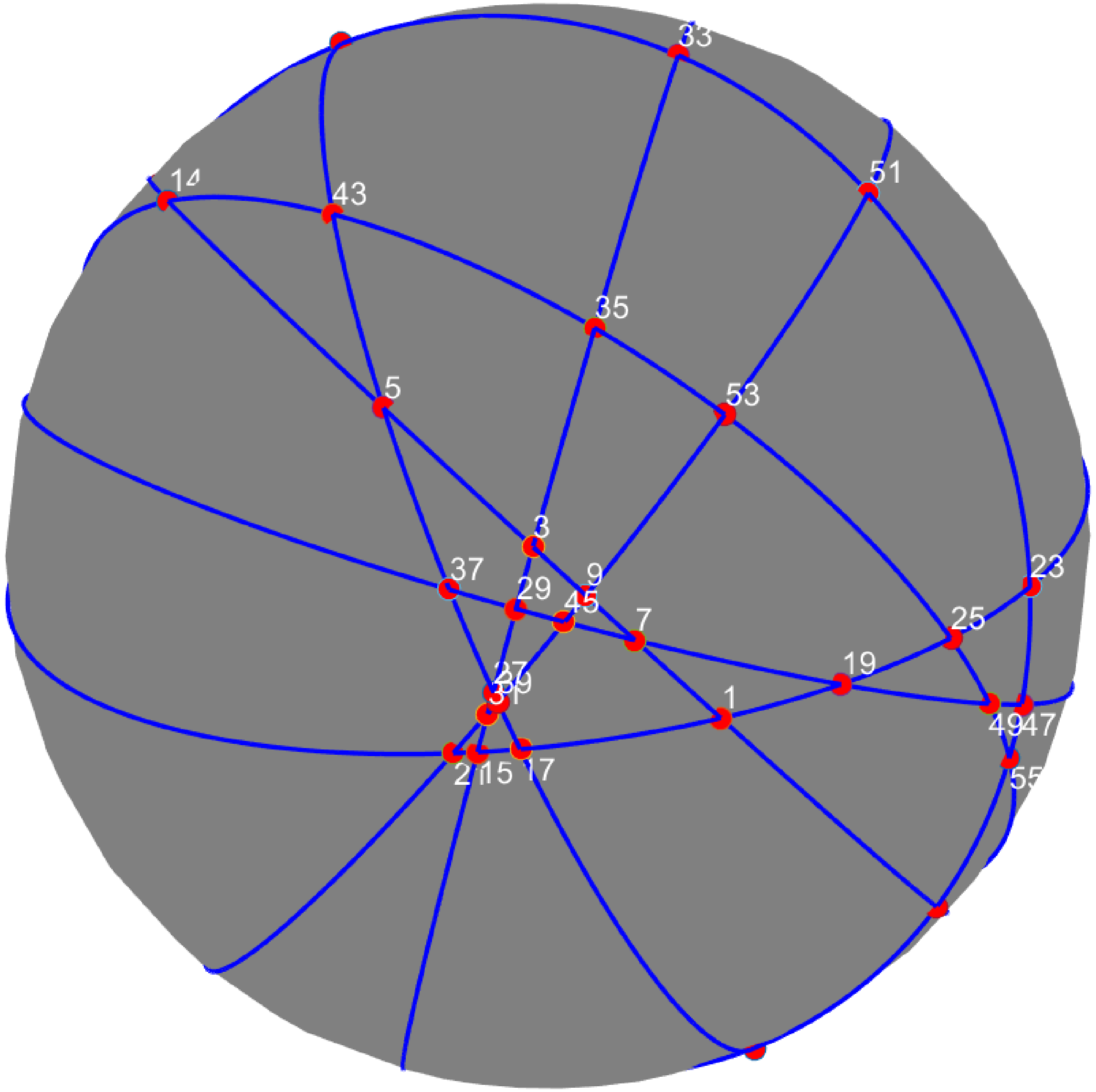
Problem: What is the largest chromatic number of any great circle graph?

Observation

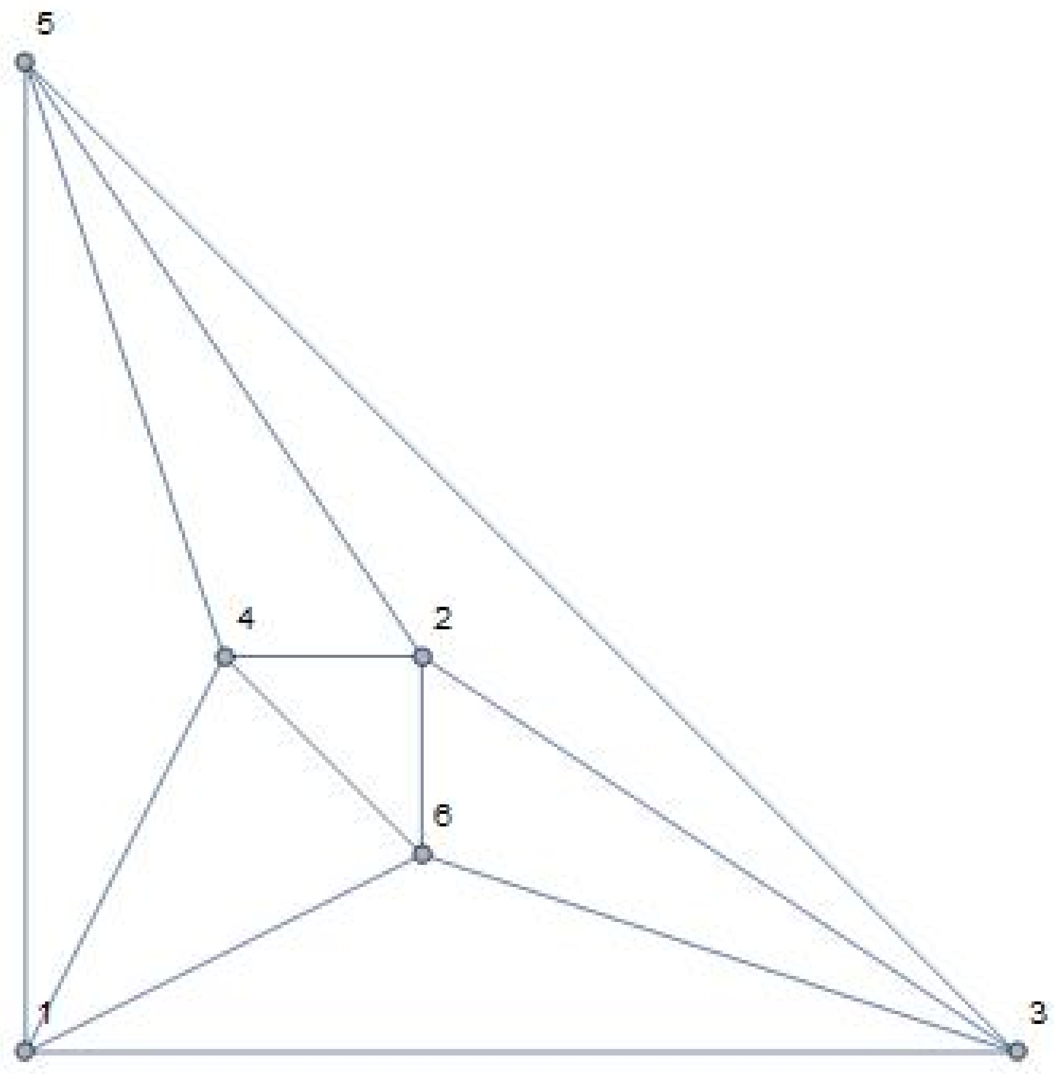
Great circles problem has been visualized on MATLAB which helps to have an abstract view about the problem. Moreover, I tried to plot out the planar embedding of some graphs.

One significant meaning from the planar embedding graphs plot out is with a specific number of great circles, there exists a way to rearrange vertices (intersections) to form an unique graph. For example, all 5 great circles can have the same planar embedding graph. It means we can reduce the problem into coloring the unique graphs

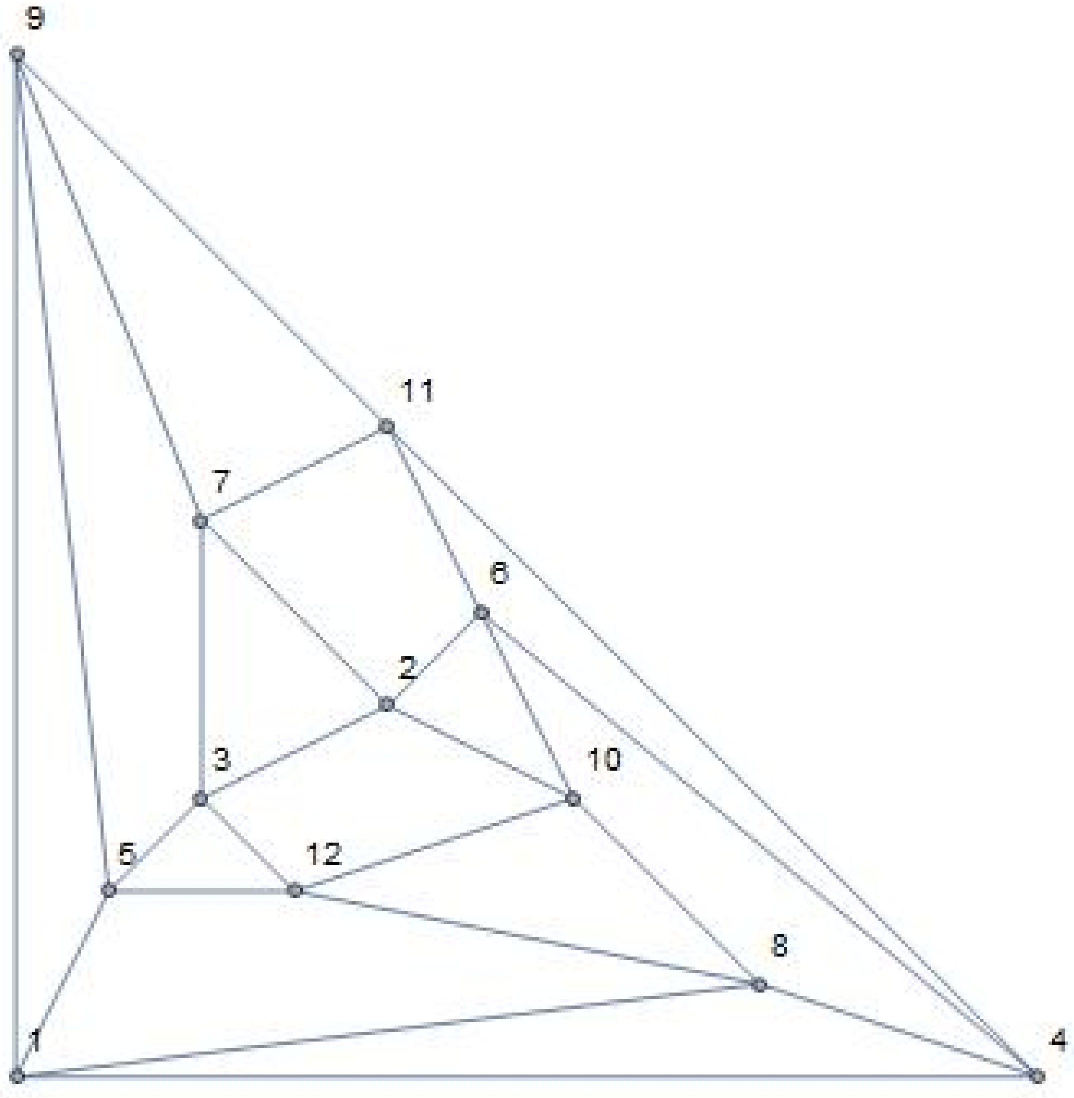
We can know more about this work in the section 5.



Planar embedding (3,4)

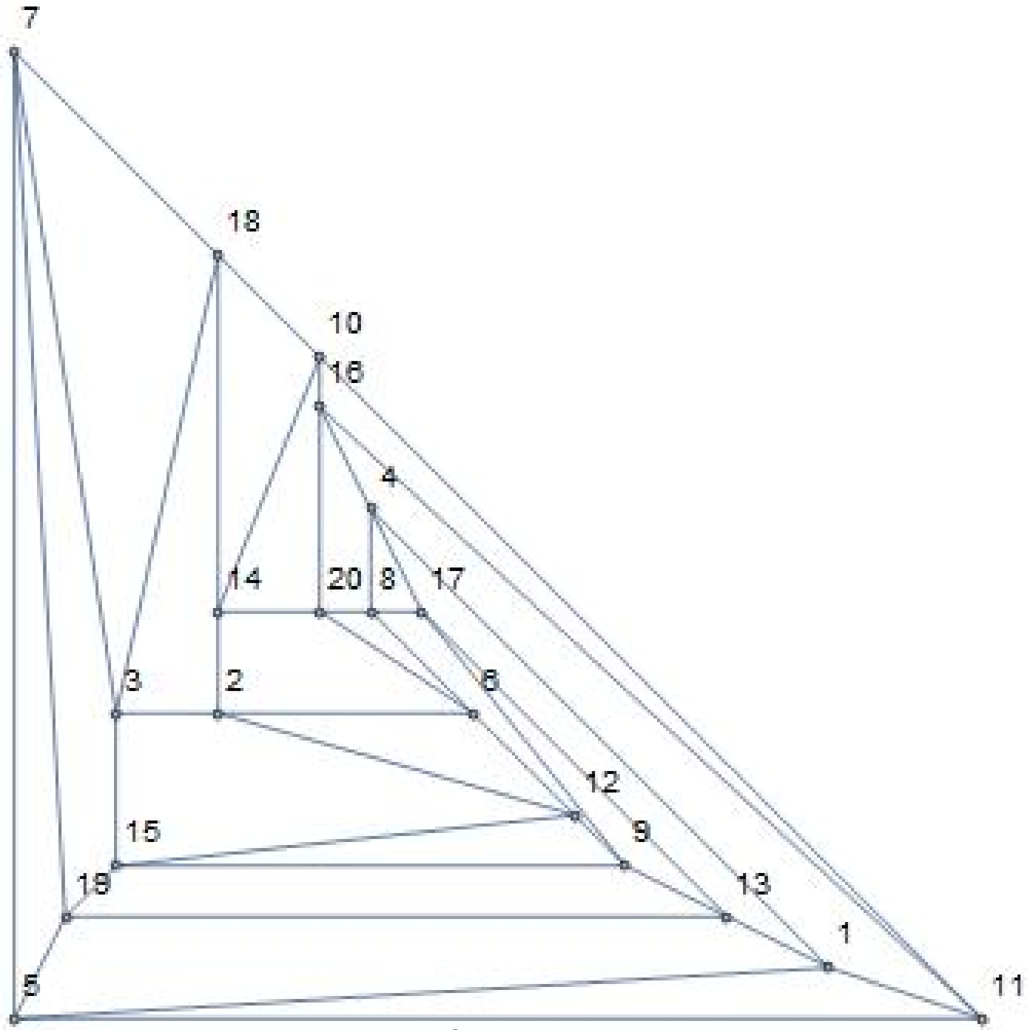


3 great circles

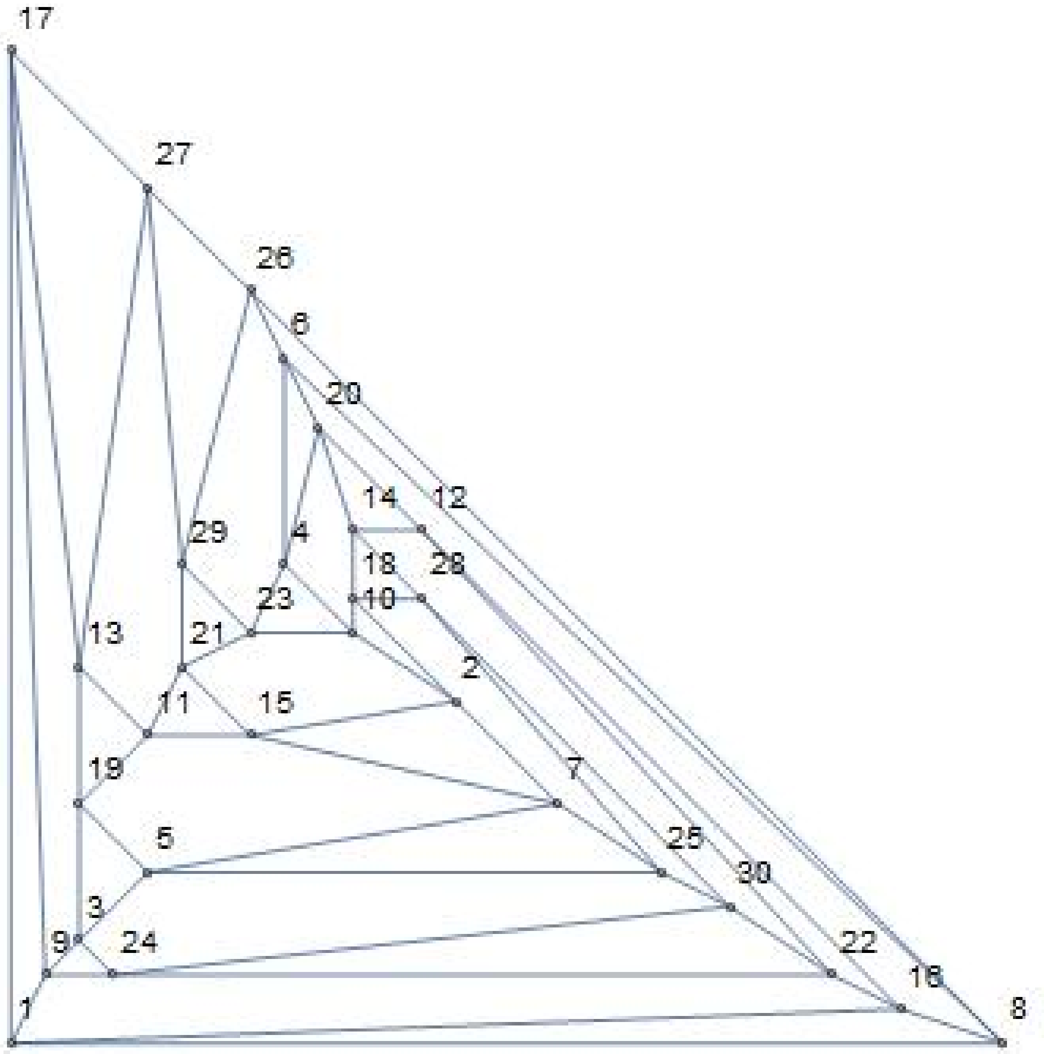


4 great circles

Planar embedding (5,6)

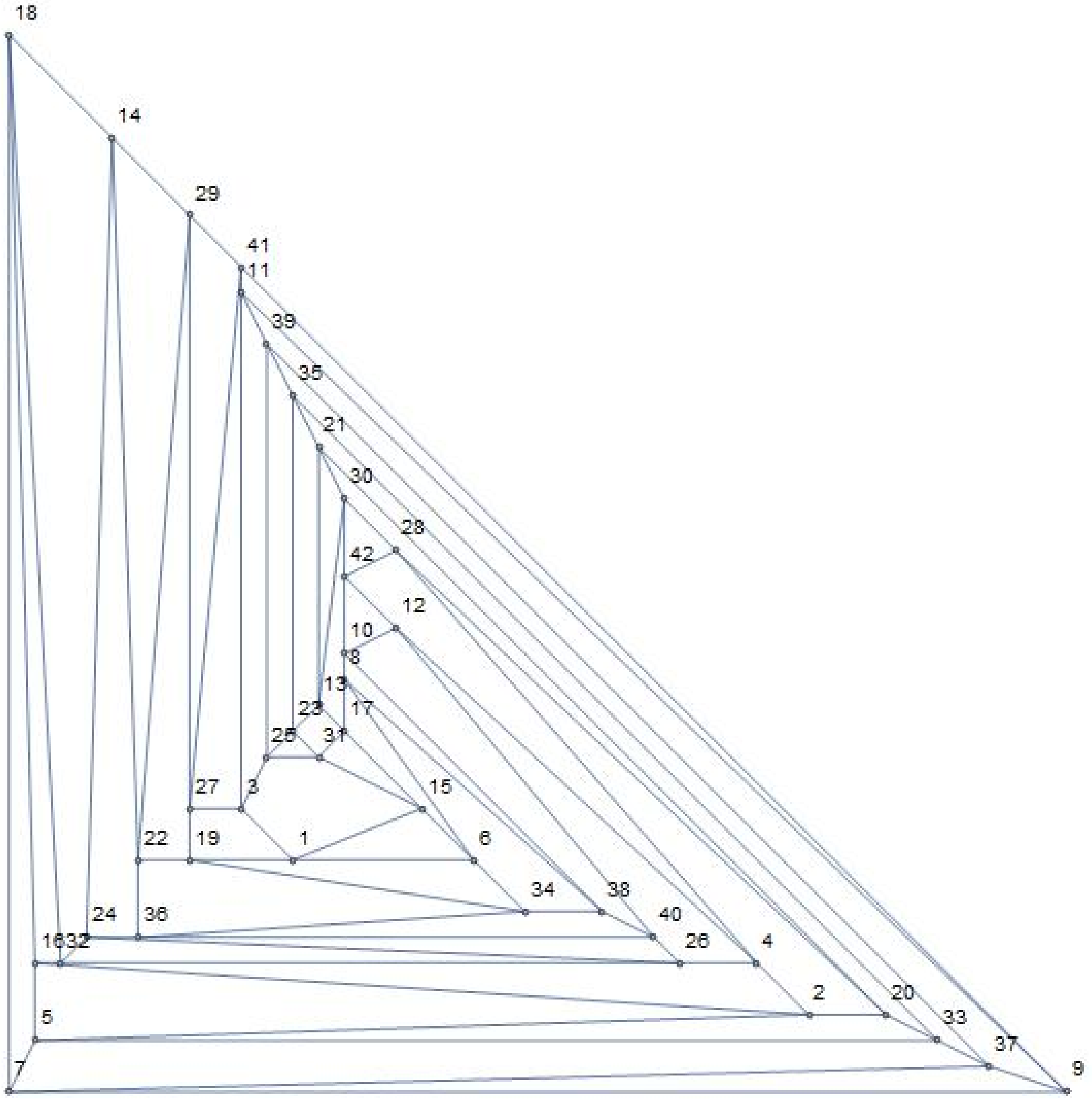


5 great circles

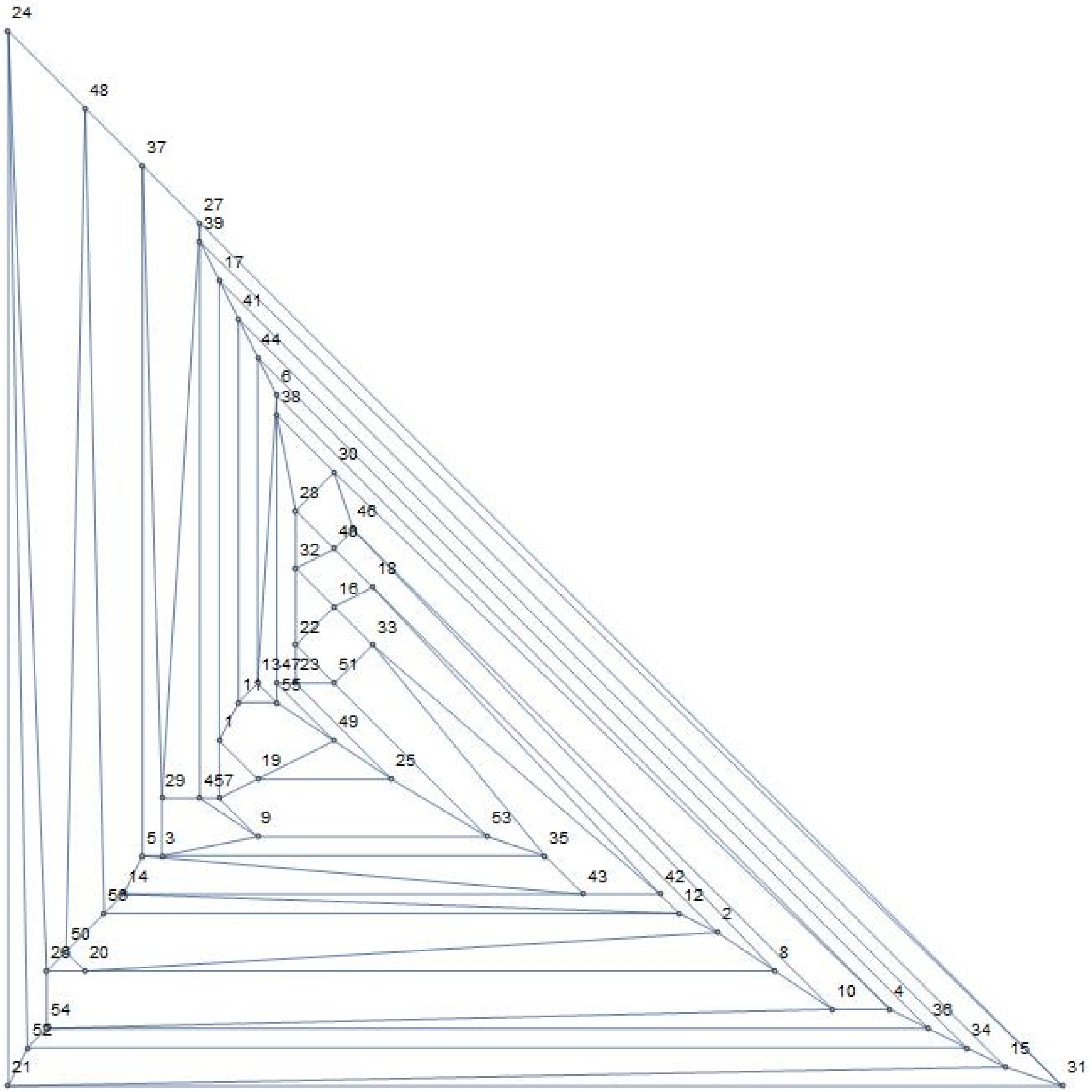


6 great circles

Planar embedding (7,8)

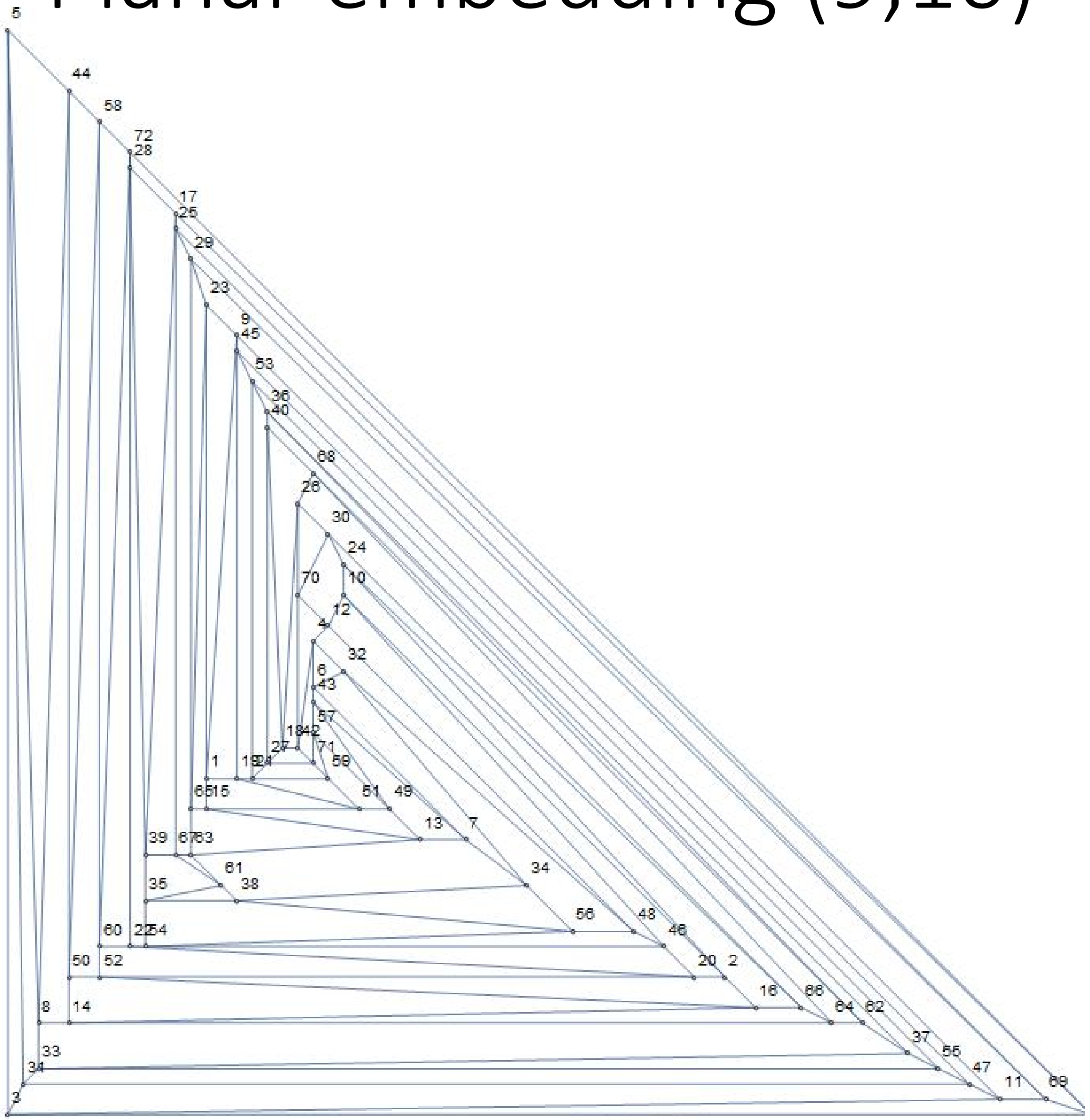


7 great circles

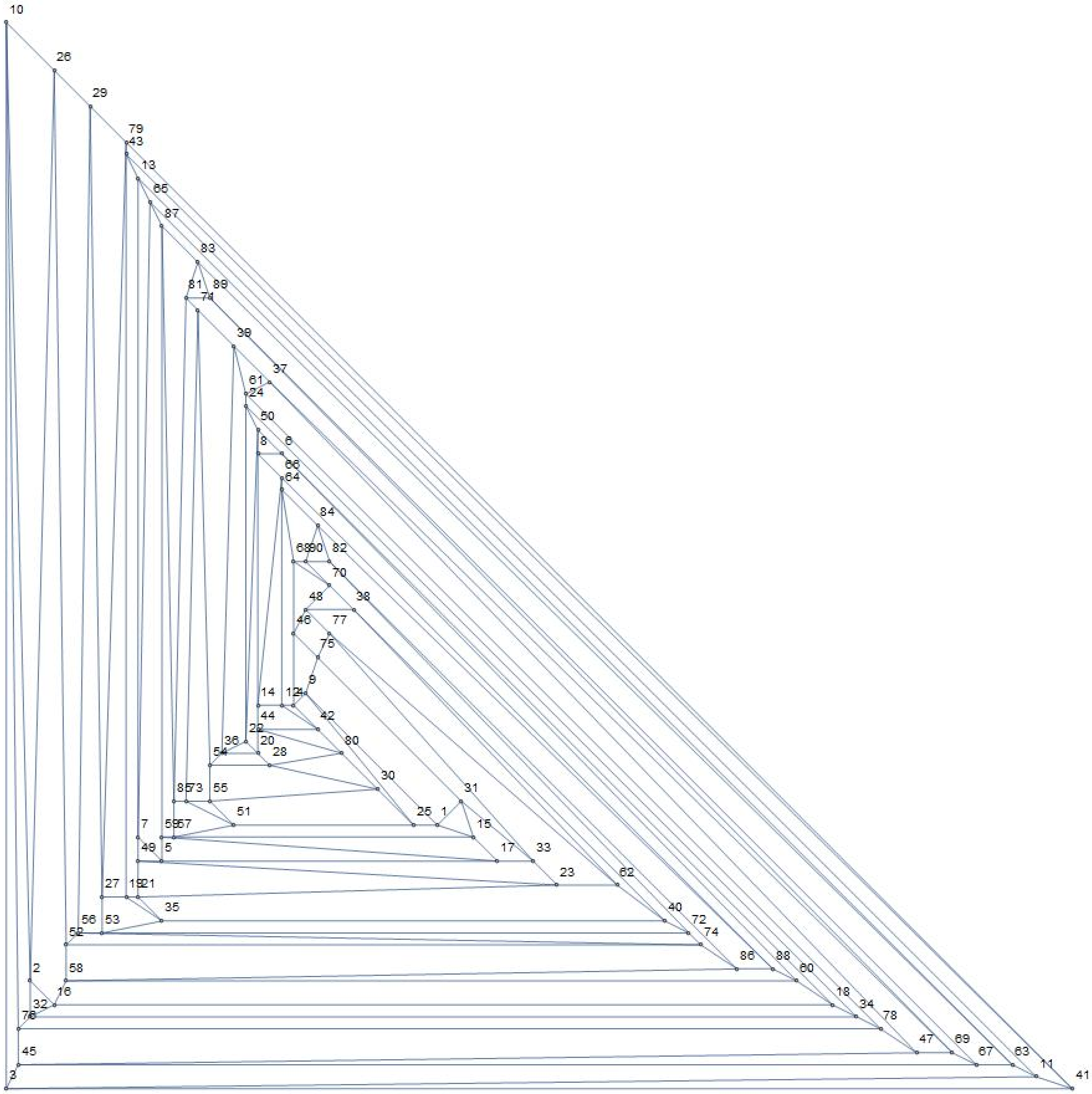


8 great circles

Planar embedding (9,10)

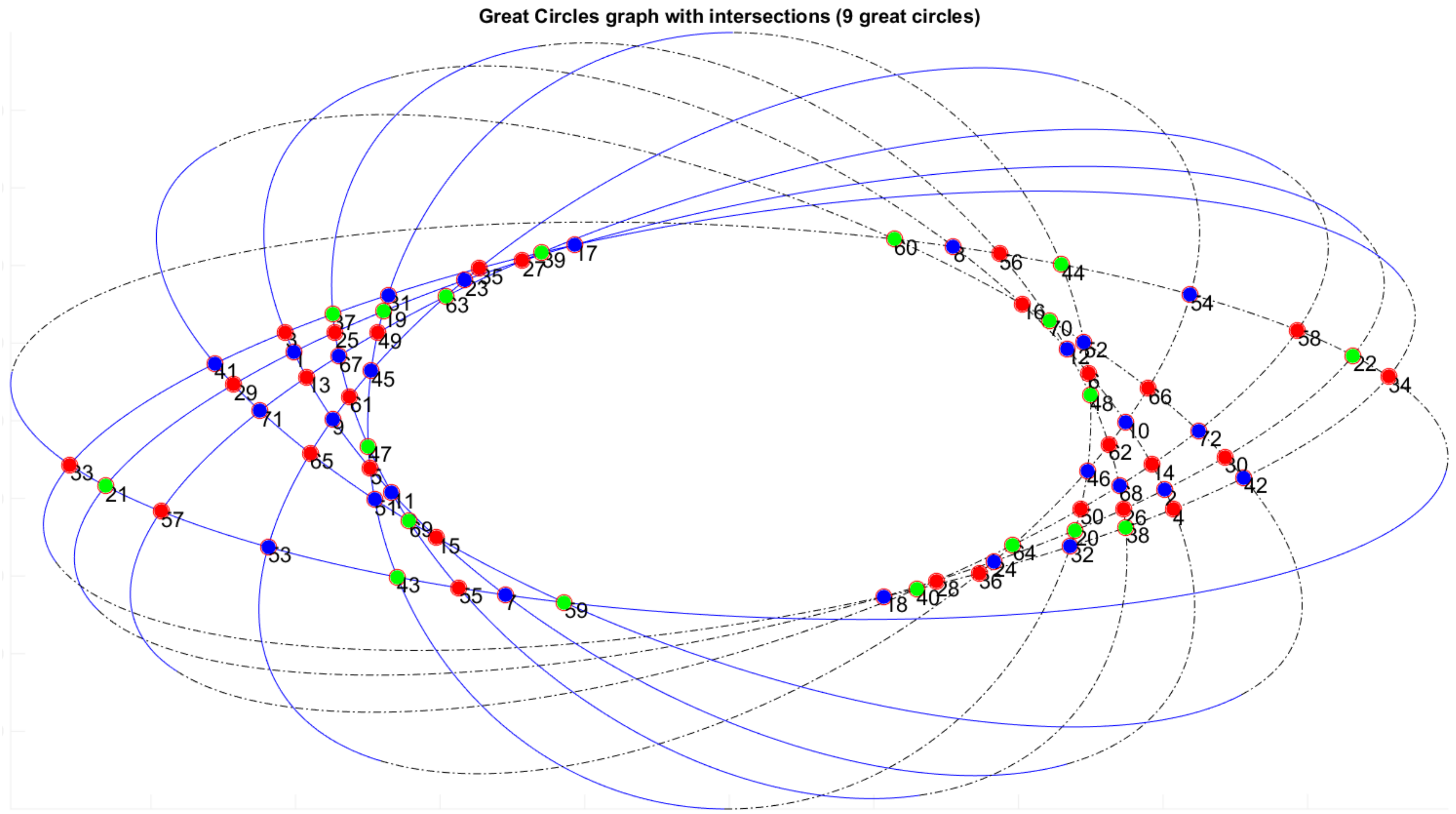


9 great circles



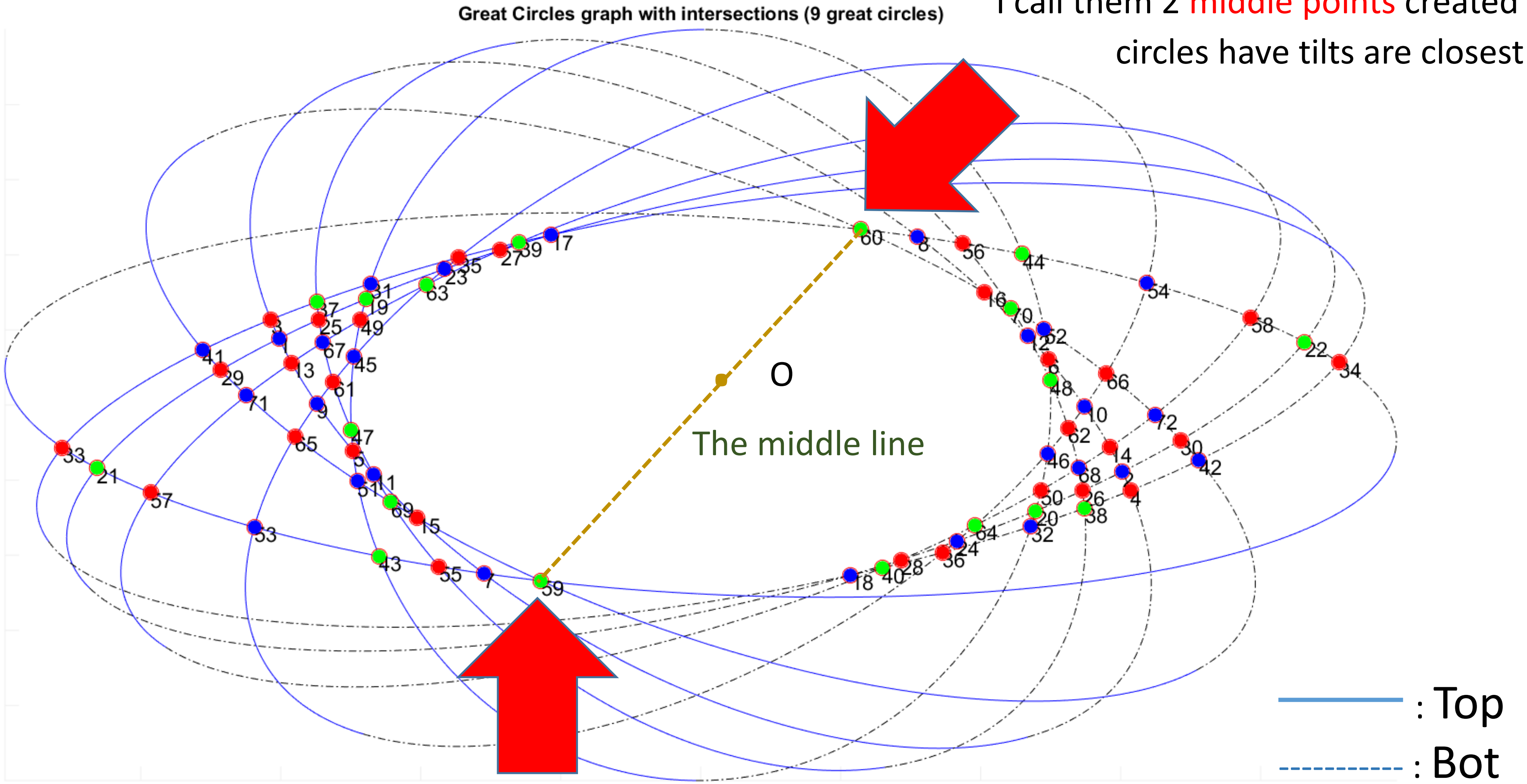
8 great circles

Problem Restatement



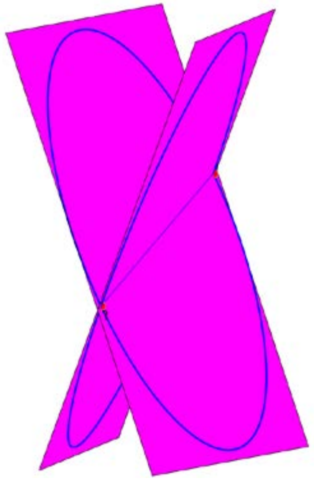
An example case (9 great circles)

I call them 2 middle points created by 2 circles have tilts are closest to $\frac{\pi}{2}$

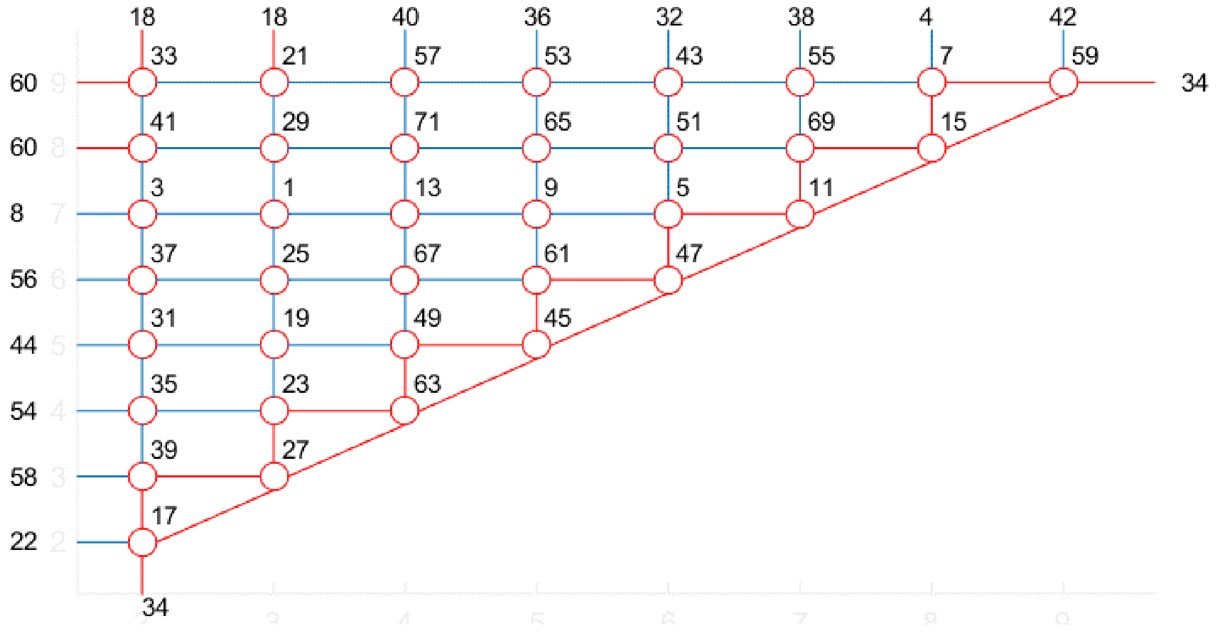


A example case (9 great circles)

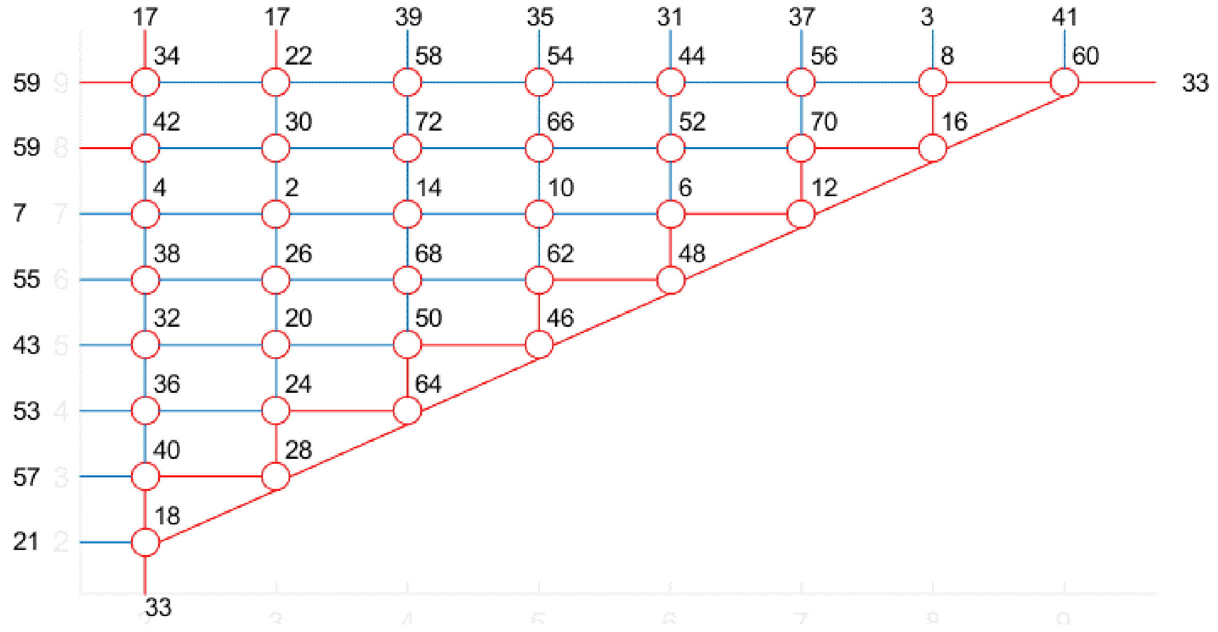
- Observation:
  - The center of all circles O is the **point symmetry** of vertices on both sides of the line connected 2 middle points
    - Proof:
      - 2 circles in 3D can be considered as 2 circle planes.
      - The intersection of 2 planes is a line
      - Obviously all possible intersections of 2 planes are on that line including O since both great circles have the same center
      - The line intersects a circle at 2 points
      - Therefore, all the intersections created by pairs of circles and O are on a line.



Redraw the graph (9 great circles)

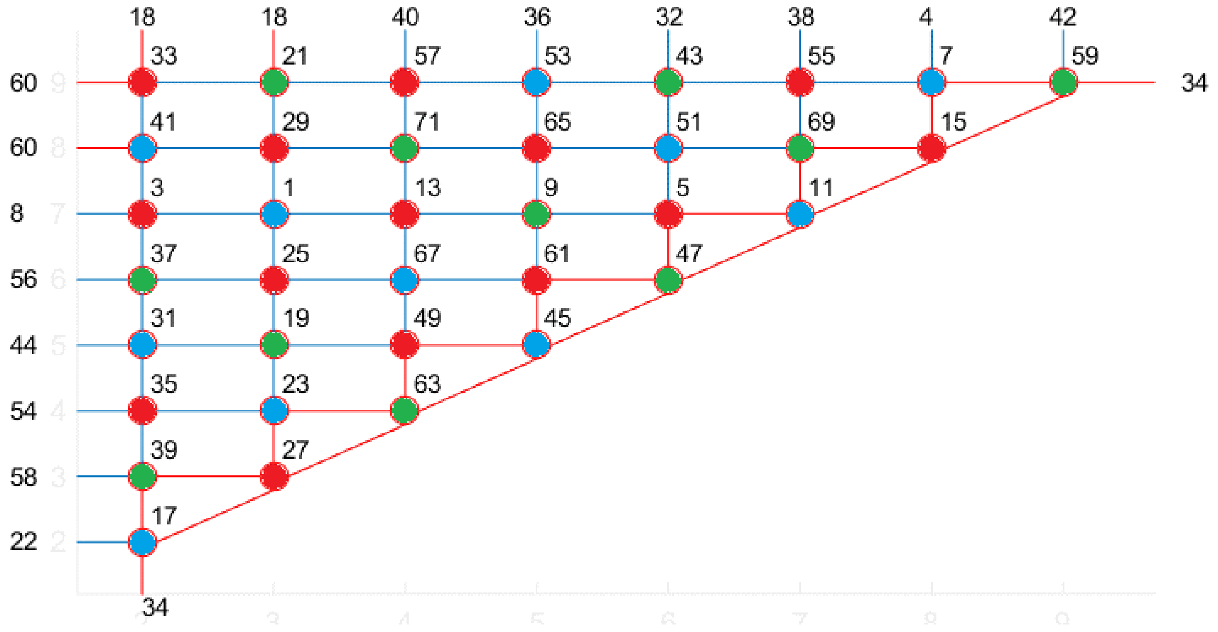


Left side

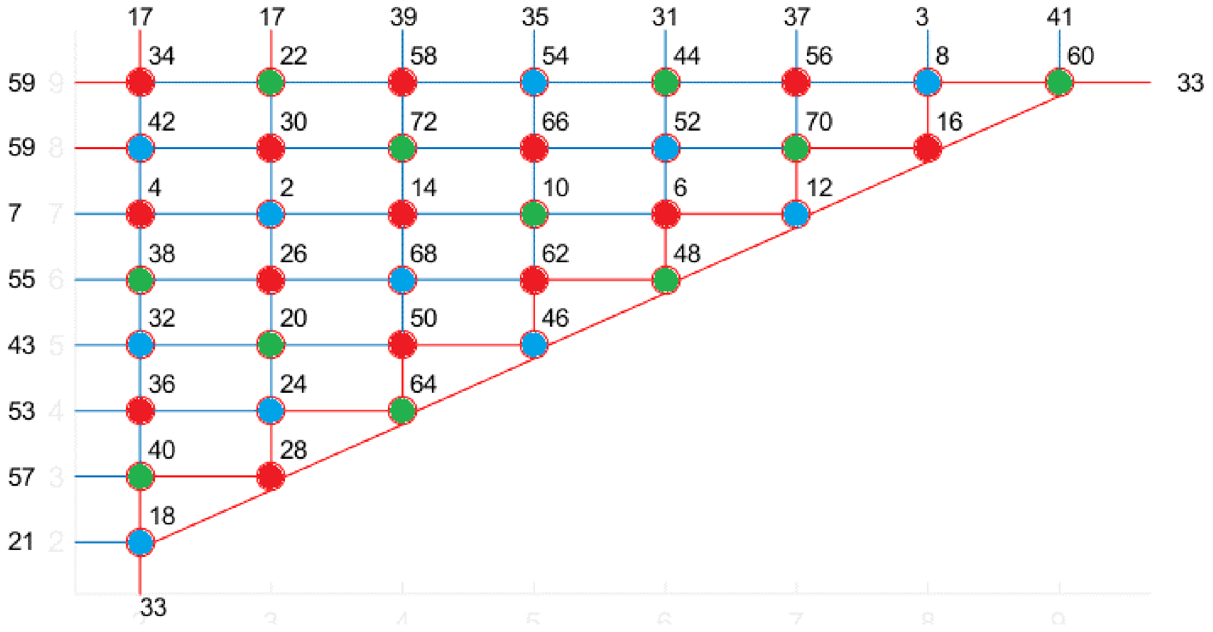


Right side

Color the graph



Left side

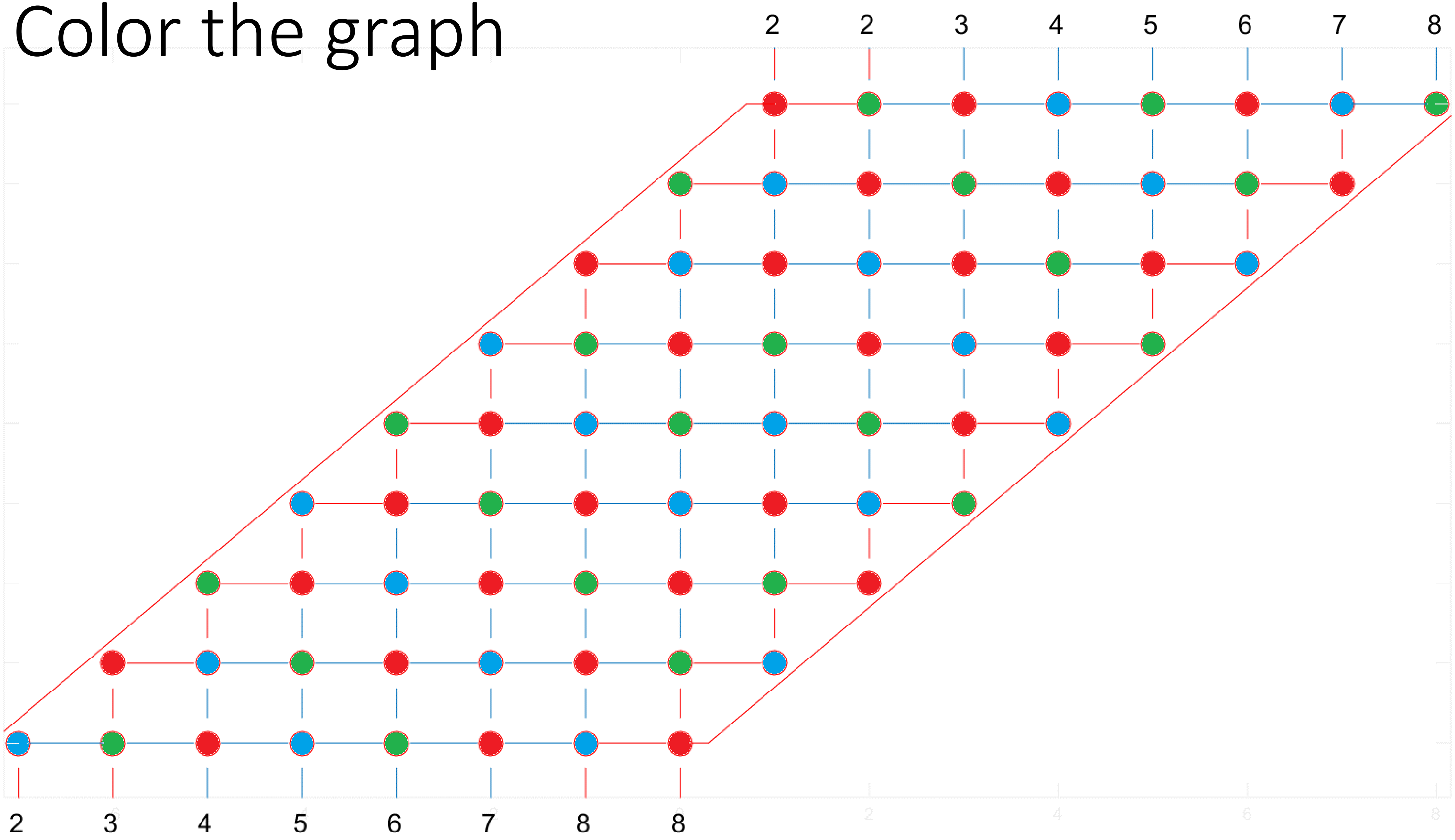


Right side

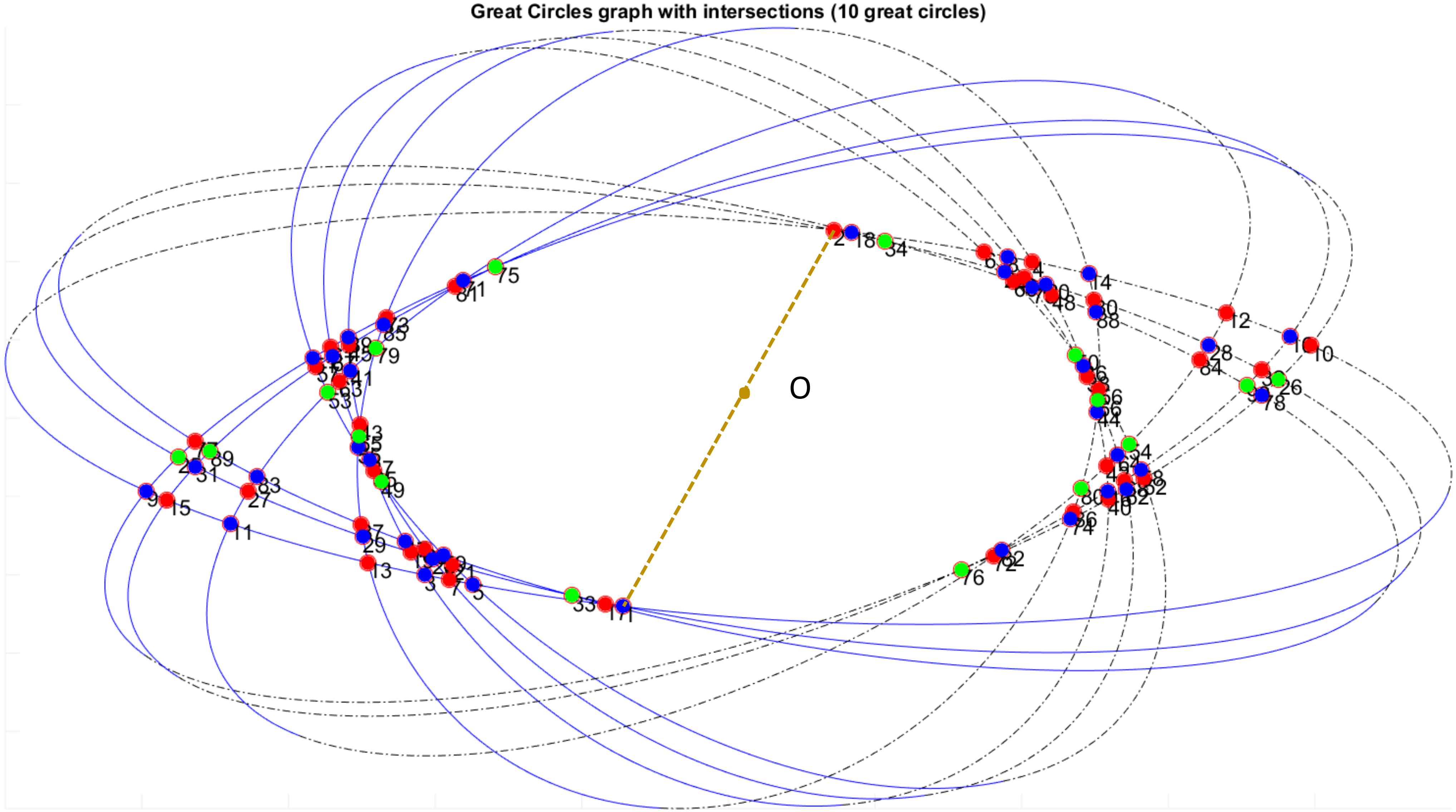
Observation: The vertices on the same diagonal usually have the same color

Color the graph

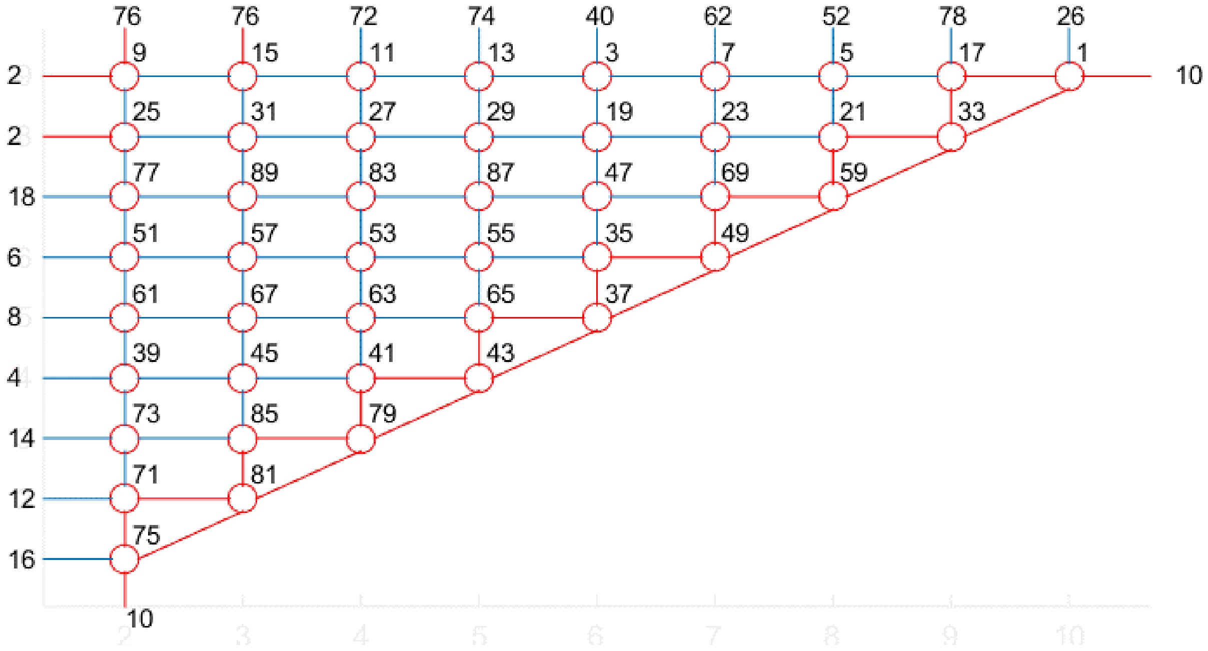
The numbers only show the connection in this graph only



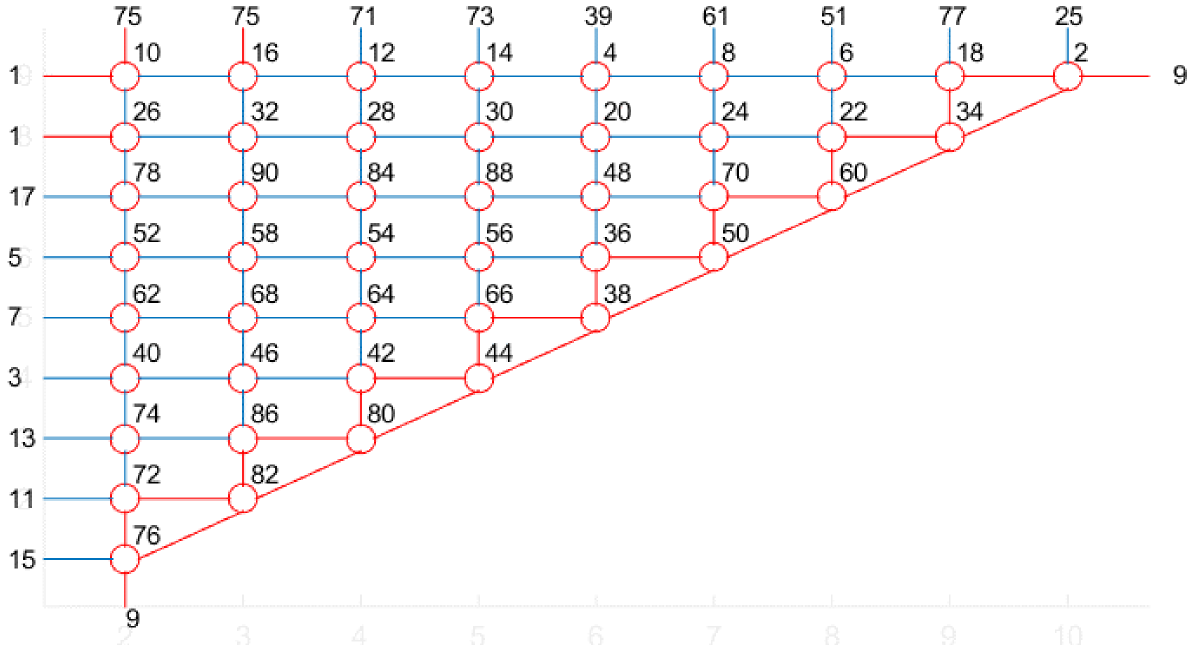
An another example (10 great circles)



Redraw the graph (10 great circles)

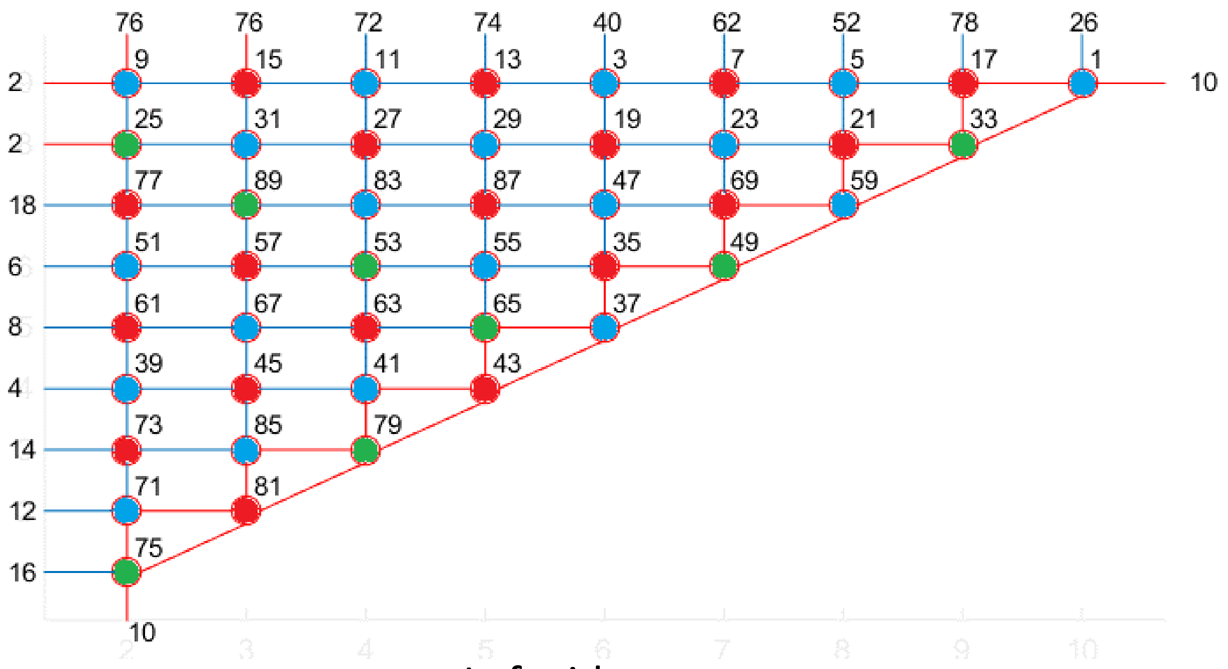


Left side

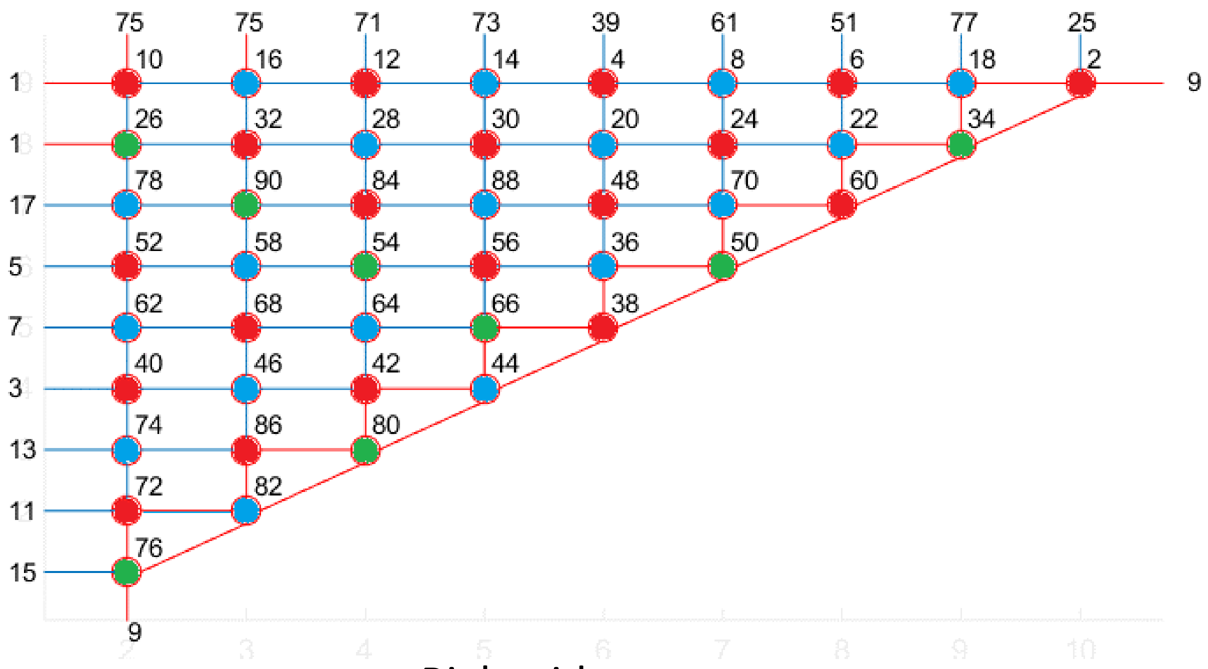


Right side

Color the graph (10 great circles)



Left side



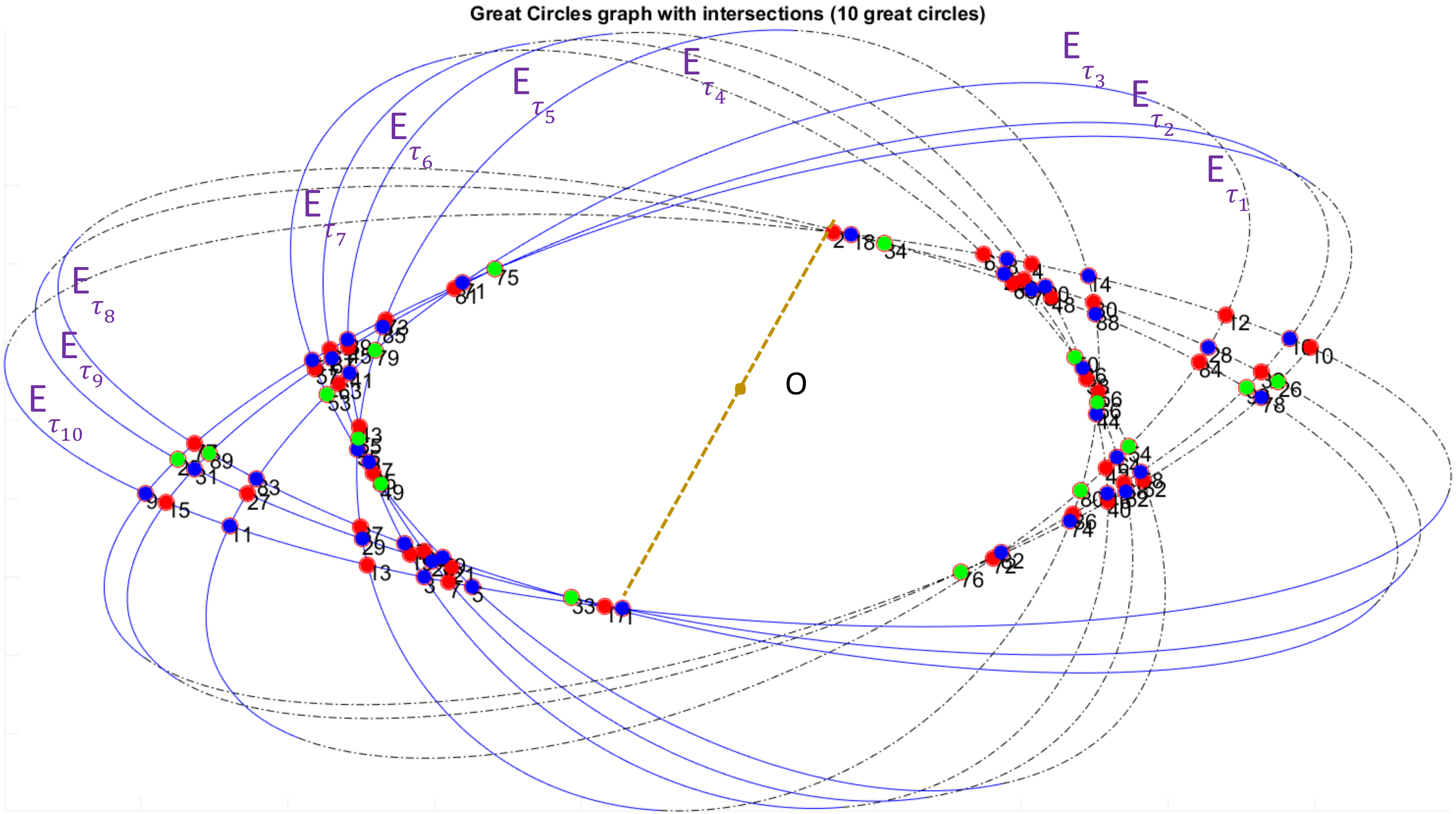
Right side

Observation: The vertices on the same diagonal usually have the same color except ones on the hypotenuse

The big picture of the problem

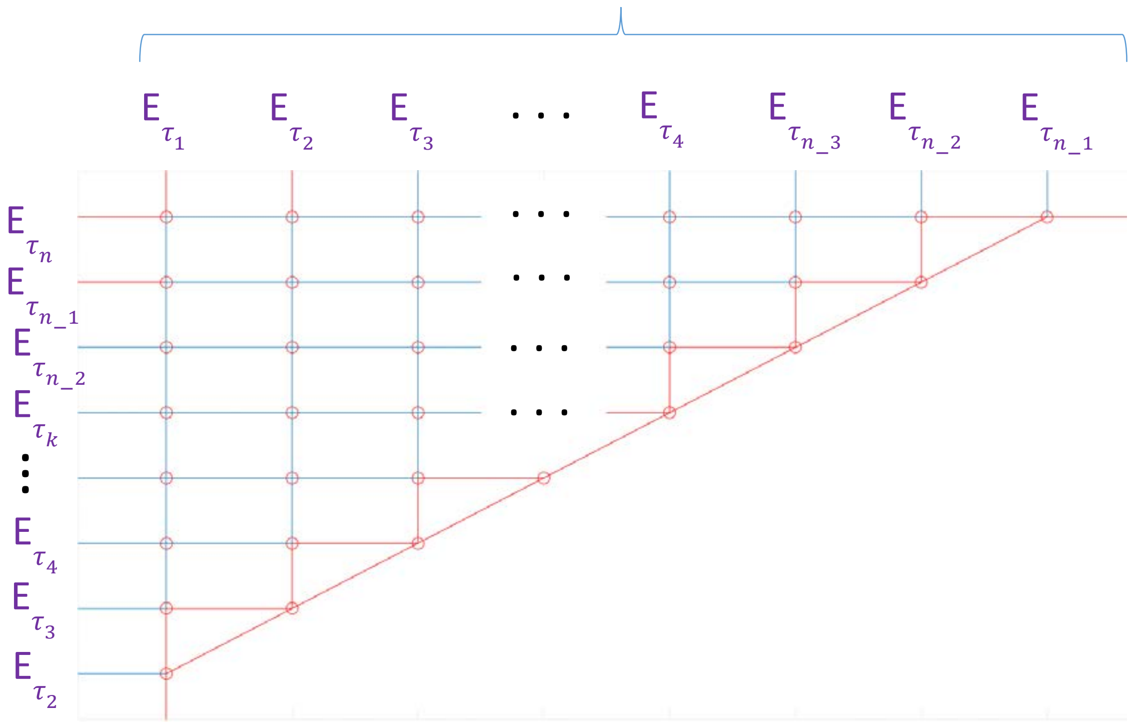
- Call E_{τ_i} is the ellipse has the inclination angle τ_i and $\tau_1 < \tau_2 < \cdots < \tau_n$ $(\tau_i \in [\frac{-\pi}{2}, \frac{\pi}{2}))$

The big picture of the problem



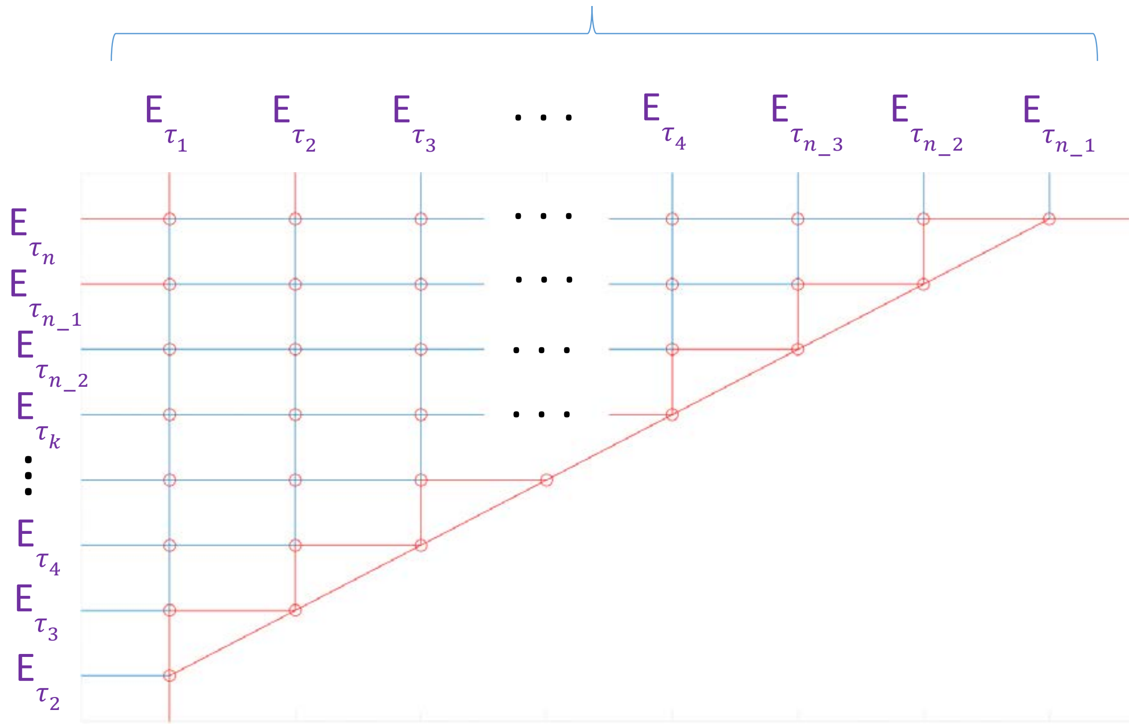
The big picture of the problem

(n-1) vertices at the top



Left side

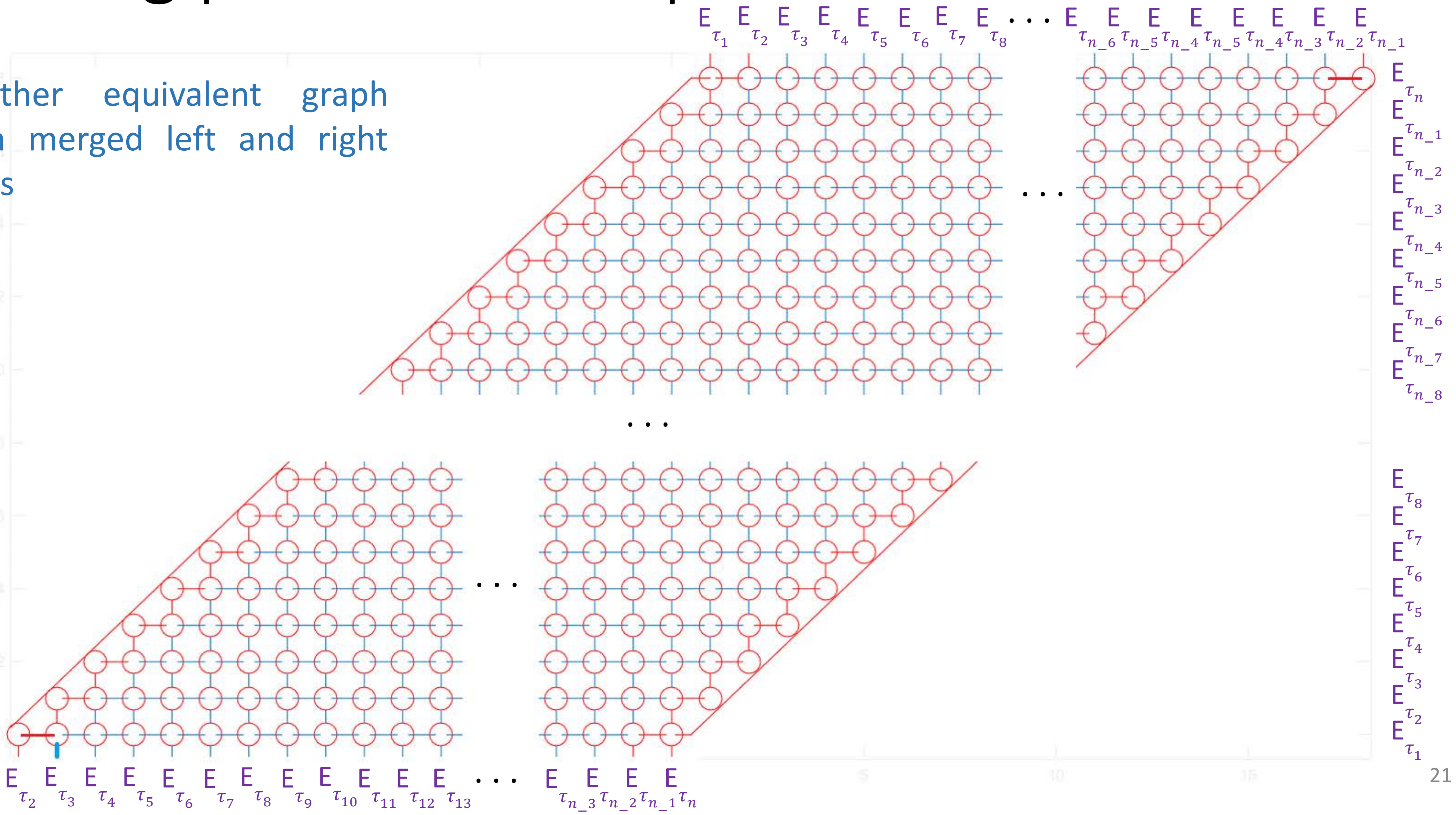
(n-1) vertices at the top



Right side

The big picture of the problem

Another equivalent graph with merged left and right sides



The big picture of the problem

- Observation (n is the number of great circles)
  - On every side, all the vertices have the degree 4
  - There are $\frac{(n-1)*n}{2}$ vertices on every side
    - Proof: a couple of great circles will create 2 intersections on the Earth, so the total vertices on both side would be $2 * C_n^2 = (n-1) * n$. Moreover, the number of vertices is split equally into 2 sides or every side will have $\frac{(n-1)*n}{2}$ vertices
  - There are (n-1)*2+2 = 2n external links on every side to connect to other side.
  - There are 2 K_3 formed by the external links
  - There are (n-2) K_3 on both sides

➔ 2*(n-2) + 2 = 2(n-1) K_3 (triangles) where we need 3 different colors for 3 vertices

Chromatic number

- $\deg(V_i) = 4$
  - Since a vertex only allows 2 circles to pass through it, so every vertex will have 4 neighbors which means $\deg(V_i) = 4$
- The graph G is planar
  - All the vertices are formed by the intersections of the circles. So, there is no sudden arc may cut through the connection between vertices since by contradiction, it will keep forming the vertices continuously and it makes no sense
- $3 \leq \chi(G) \leq 4$
  - According to four color theorem, a planar graph only needs 4 colors
  - A triangle can be formed by 3 random circles which means at that triangle, its vertices needs 3 different colors. Or the graph G contains a sub graph K_3
- **Prove $\boldsymbol{\chi(G) = 3}$** by providing a way to color the graph with 3 colors

Chromatic number

- **Prove $\boldsymbol{\chi(G) = 3}$**
  - I split the problem into 4 sub-problems that have:
    - 3k great circles (3, 6, 9, 12, 15,….) (including (6k+3))
    - 2k great circles (4, 6, 8, 10, 12,…) (including 6k, (6k+2), (6k+4))
    - (6k+1) great circles (7, 13, 19, 25,…)
    - (6k+5) great circles (5, 11, 17, 23,…)

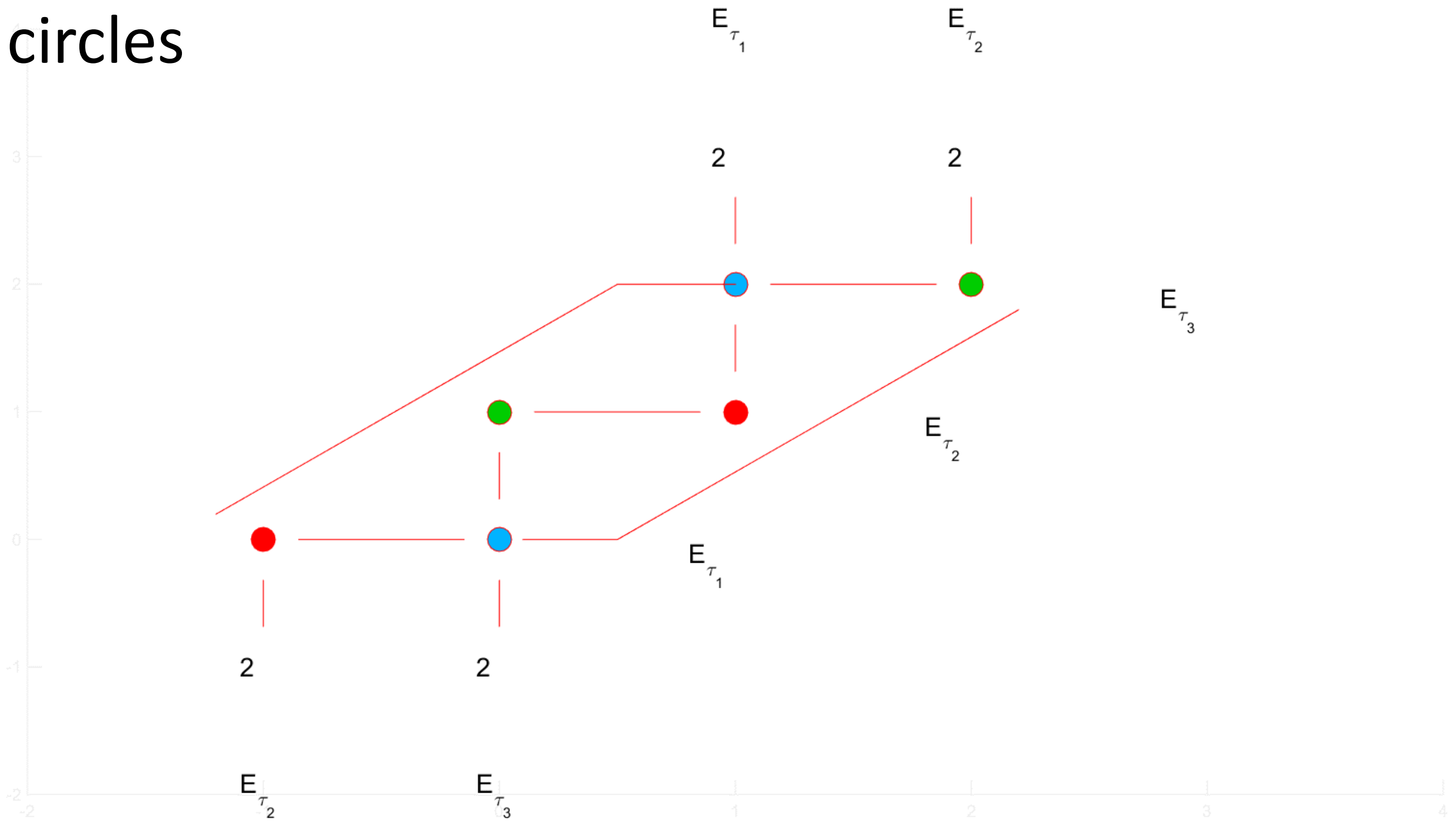
Chromatic number

- Some rules before coloring
  - Diagonal rule: The vertices on the same diagonal **should** have the same color (not a must because there are some places might have K_3 rule)
    - Proof: The vertices on the same diagonal are not connected together.
  - K_3 rule: 3 vertices that form a triangle **must** have 3 different colors

Chromatic number of 3k circles

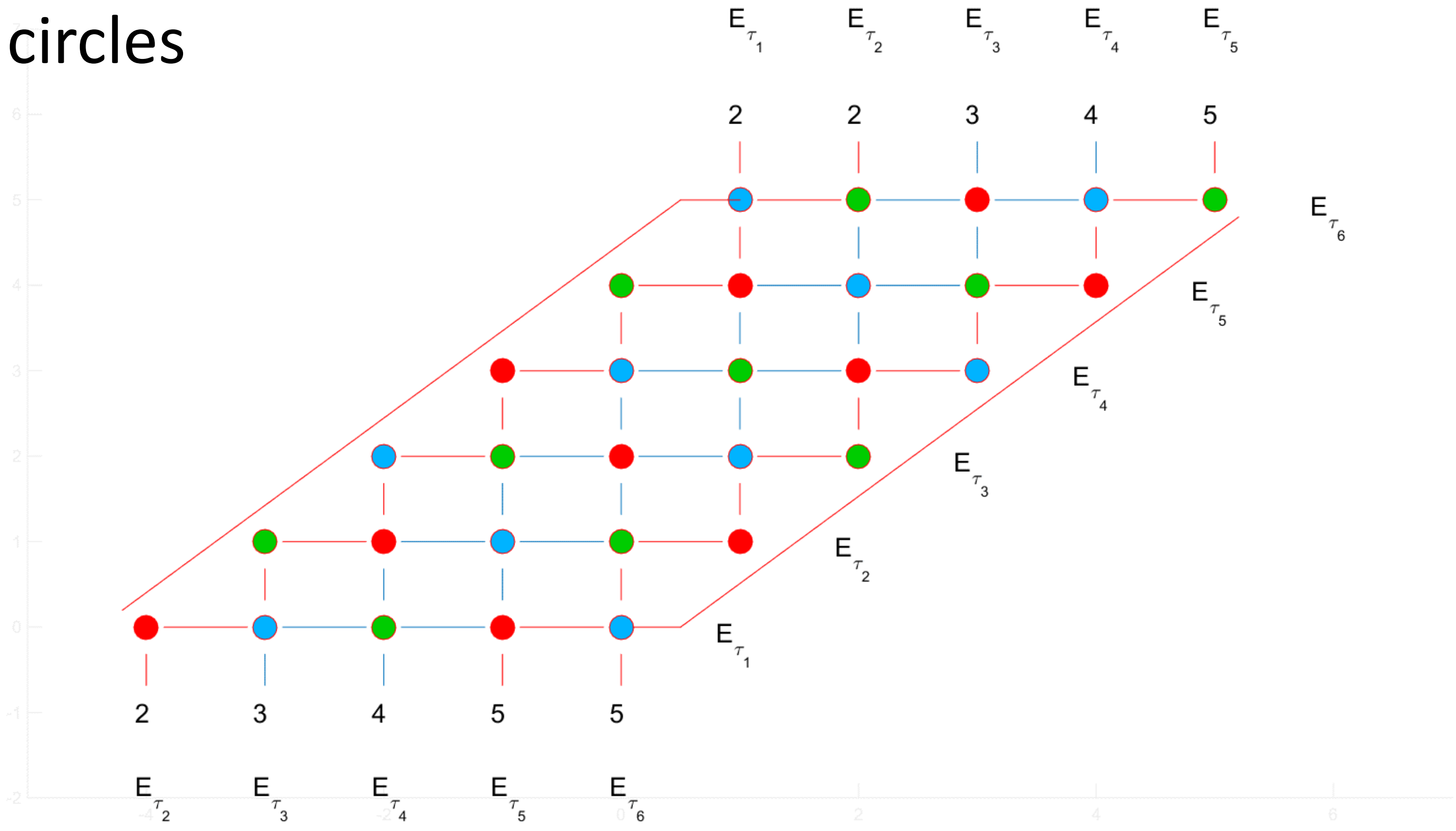
Chromatic number of 3k circles

- Base cases: 3 great circles



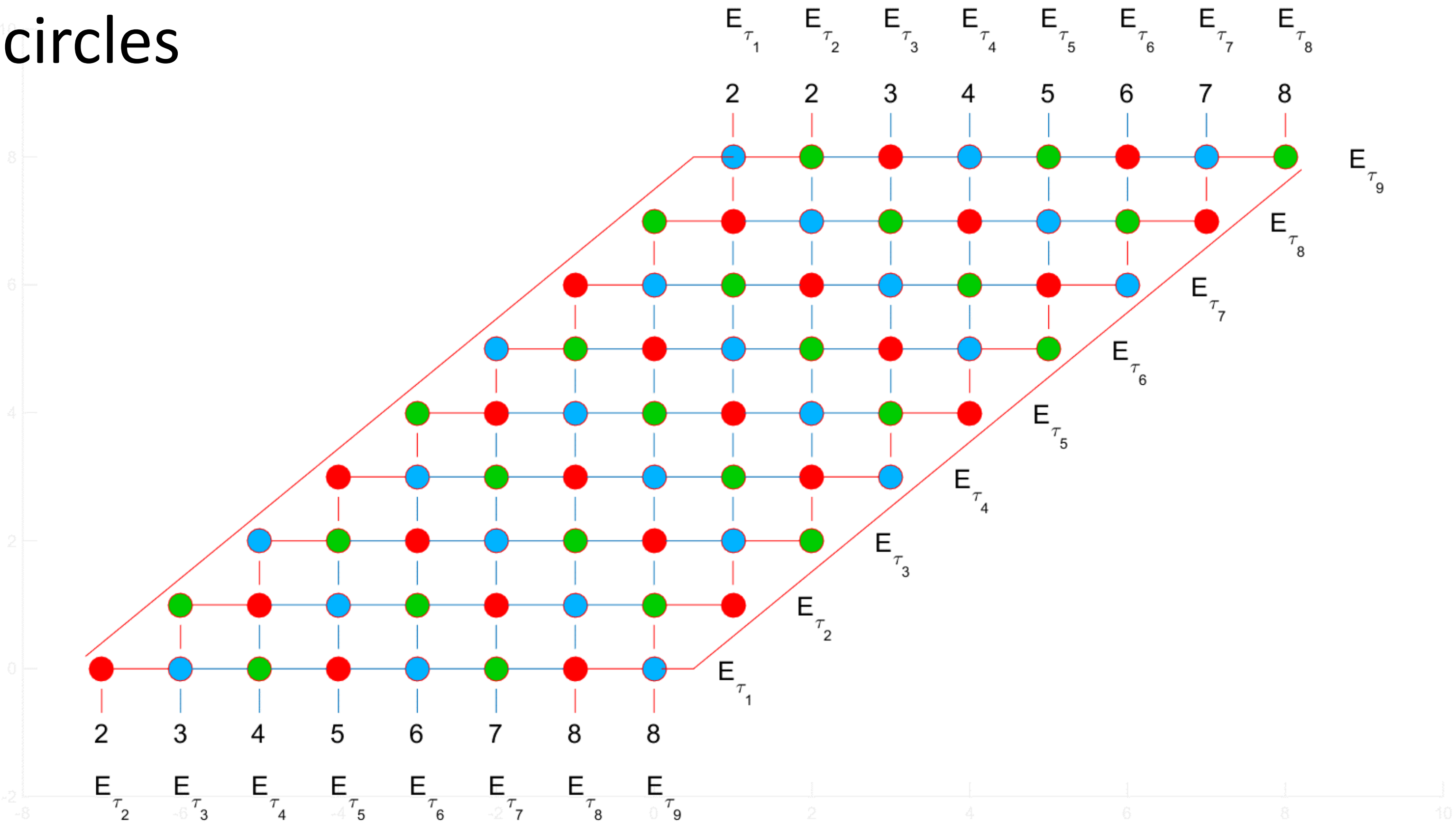
Chromatic number of 3k circles

- Base cases: 6 great circles



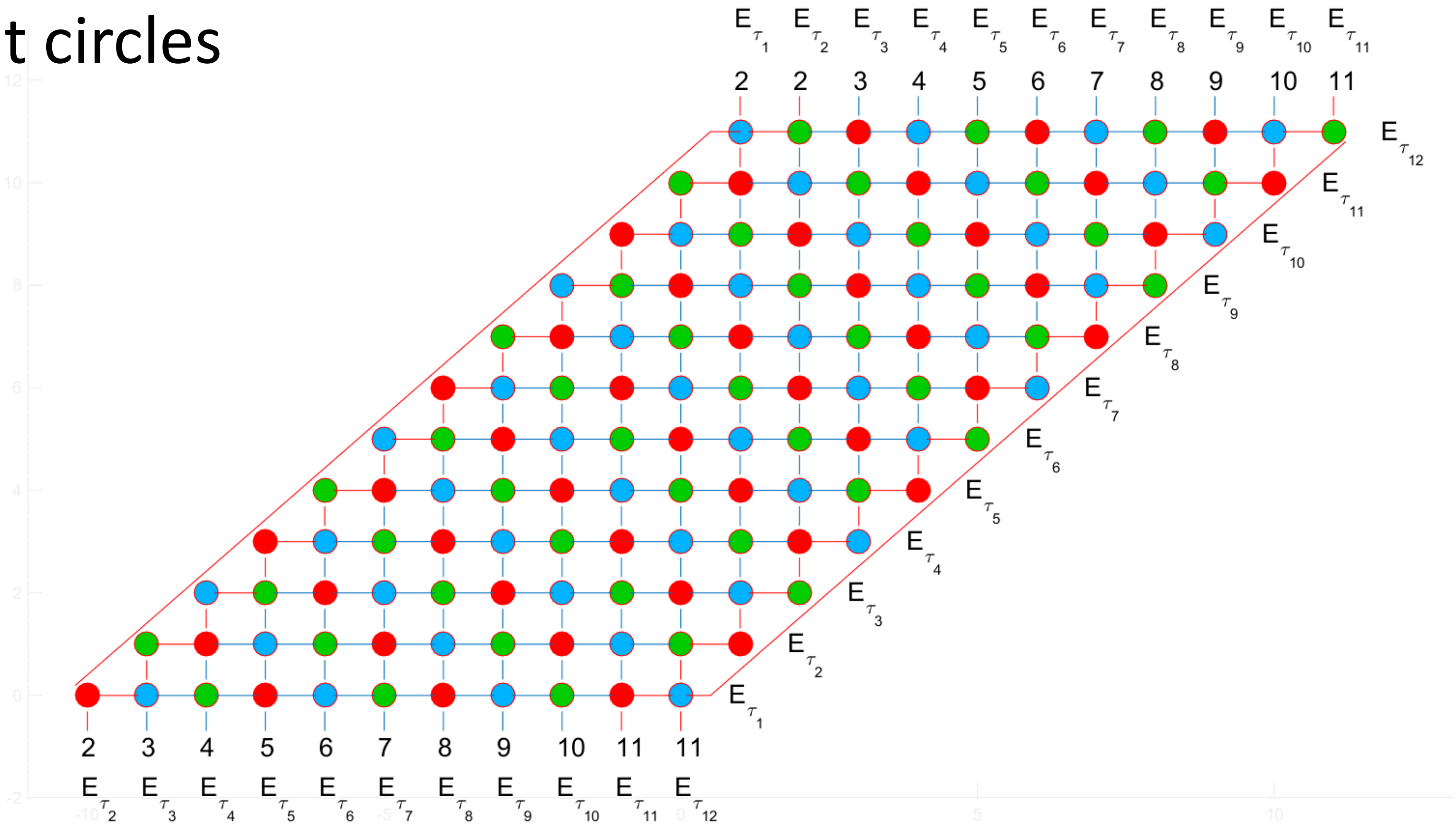
Chromatic number of 3k circles

- Base cases: 9 great circles



Chromatic number of 3k circles

- Base cases: 12 great circles



Chromatic number of 3k circles

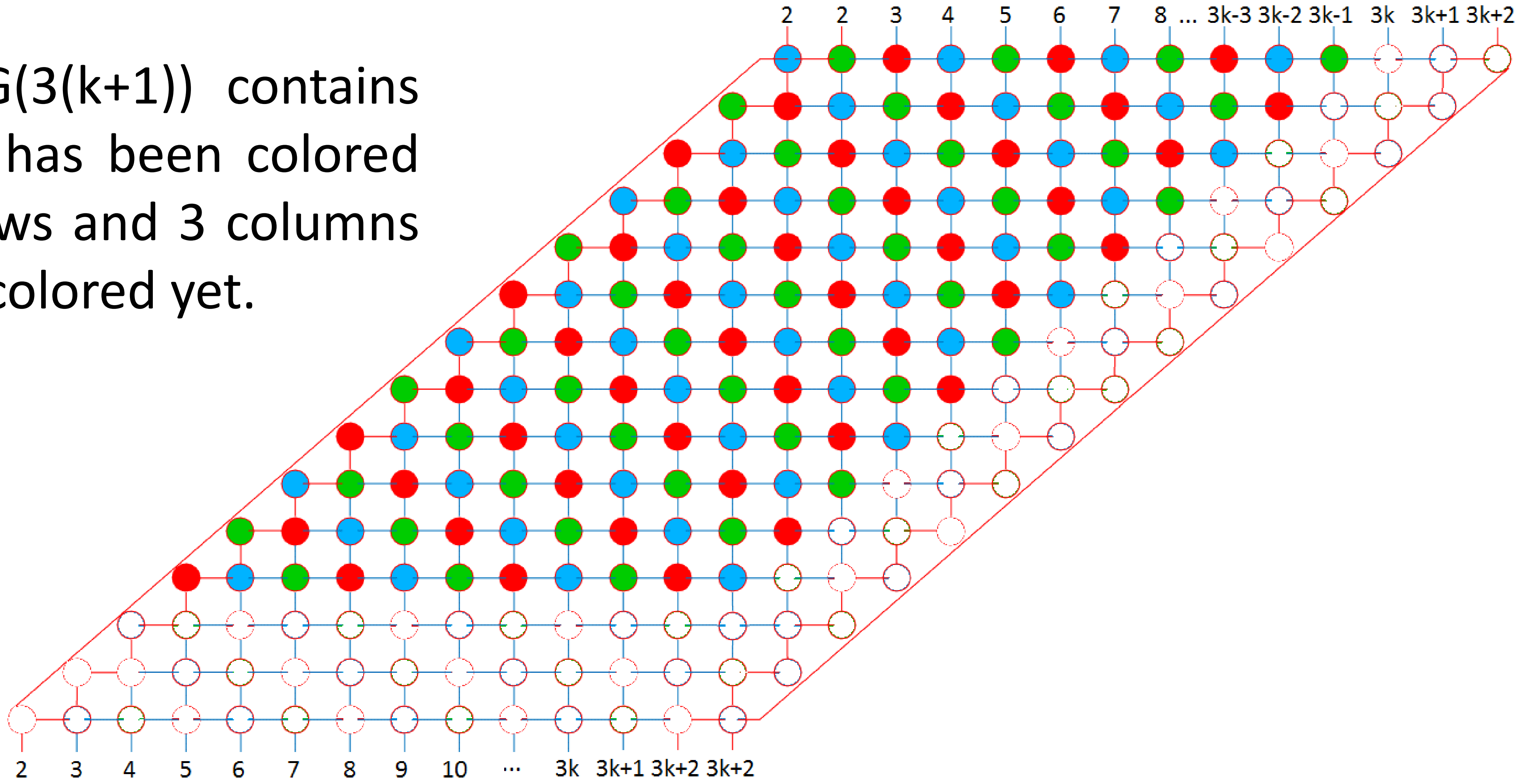
- According to the base cases with 3,6,9,12 great circles, the chromatic number is 3

➔Induction hypothesis: $\chi(G(3K)) = 3; (k > 0, k \in N)$ that has been correct with k=1,2,3,4 by spreading 3 colors from the bottom left to the top right of the equivalent graph.

➔Induction step: Prove that $\chi(G(3(K+1))) = 3$

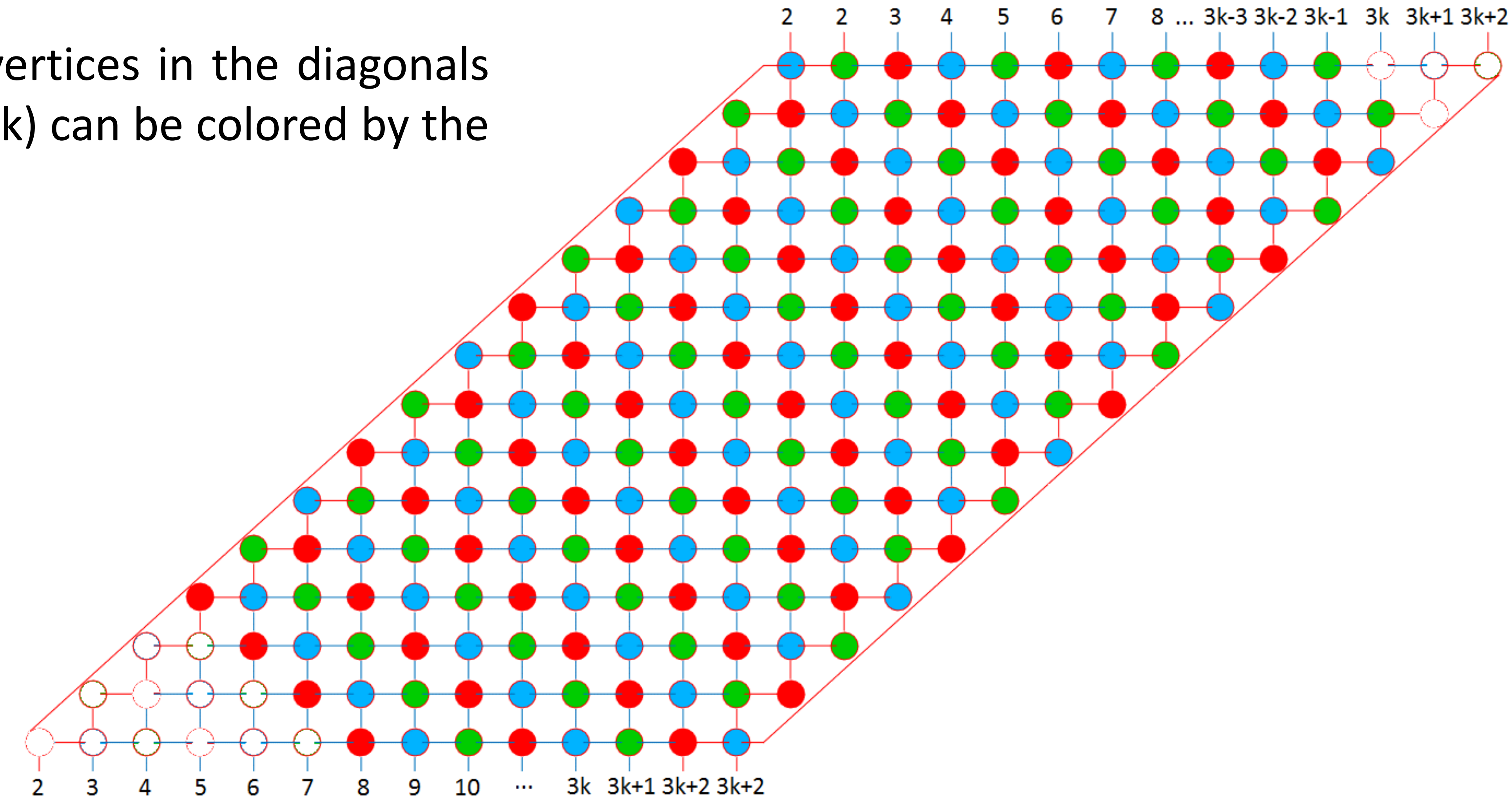
Chromatic number of 3k circles

The graph G(3(k+1)) contains G(3k) which has been colored already, 3 rows and 3 columns that are not colored yet.



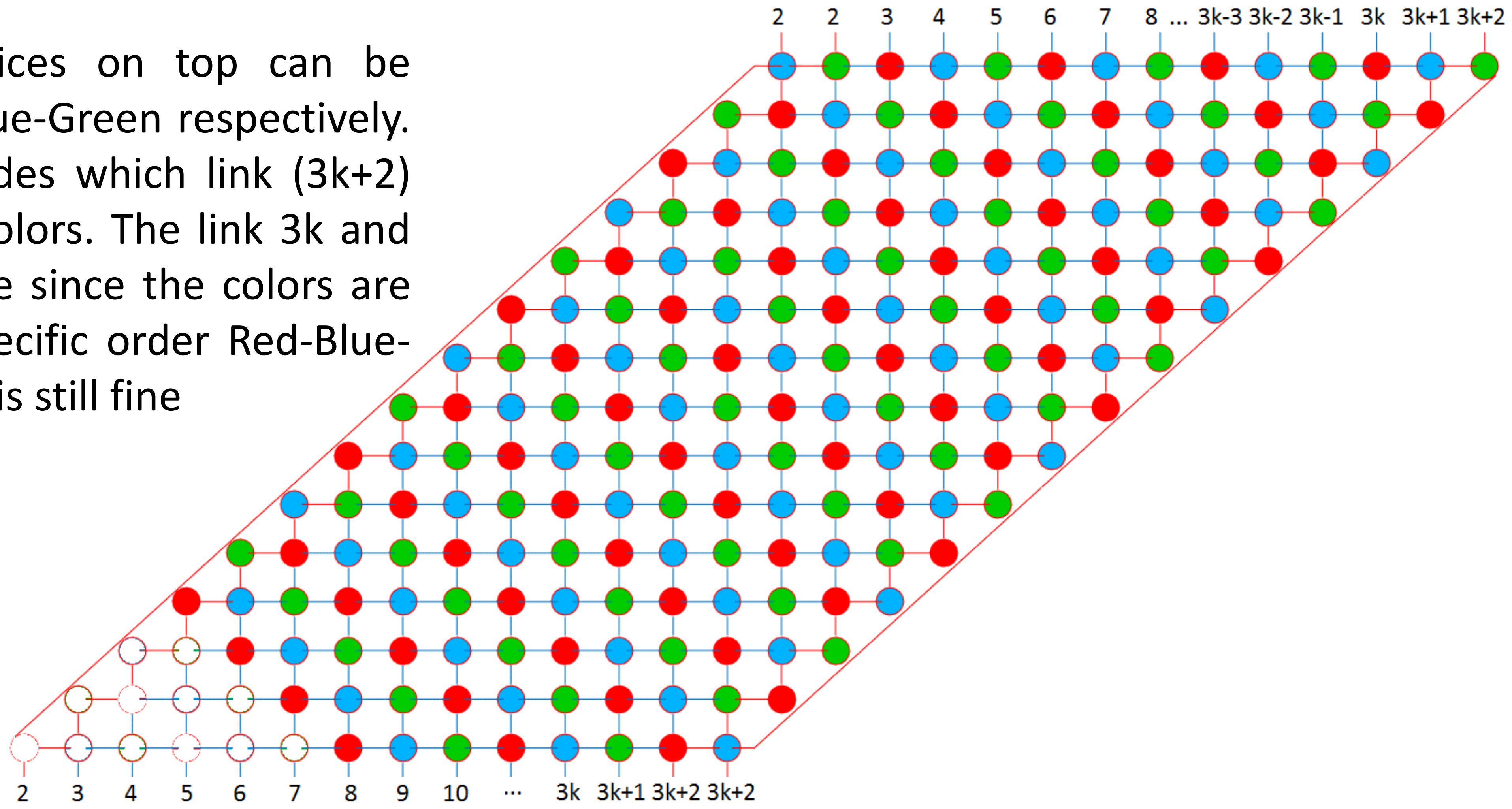
Chromatic number of 3k circles

The uncolored vertices in the diagonals of the graph G(3k) can be colored by the diagonal rule



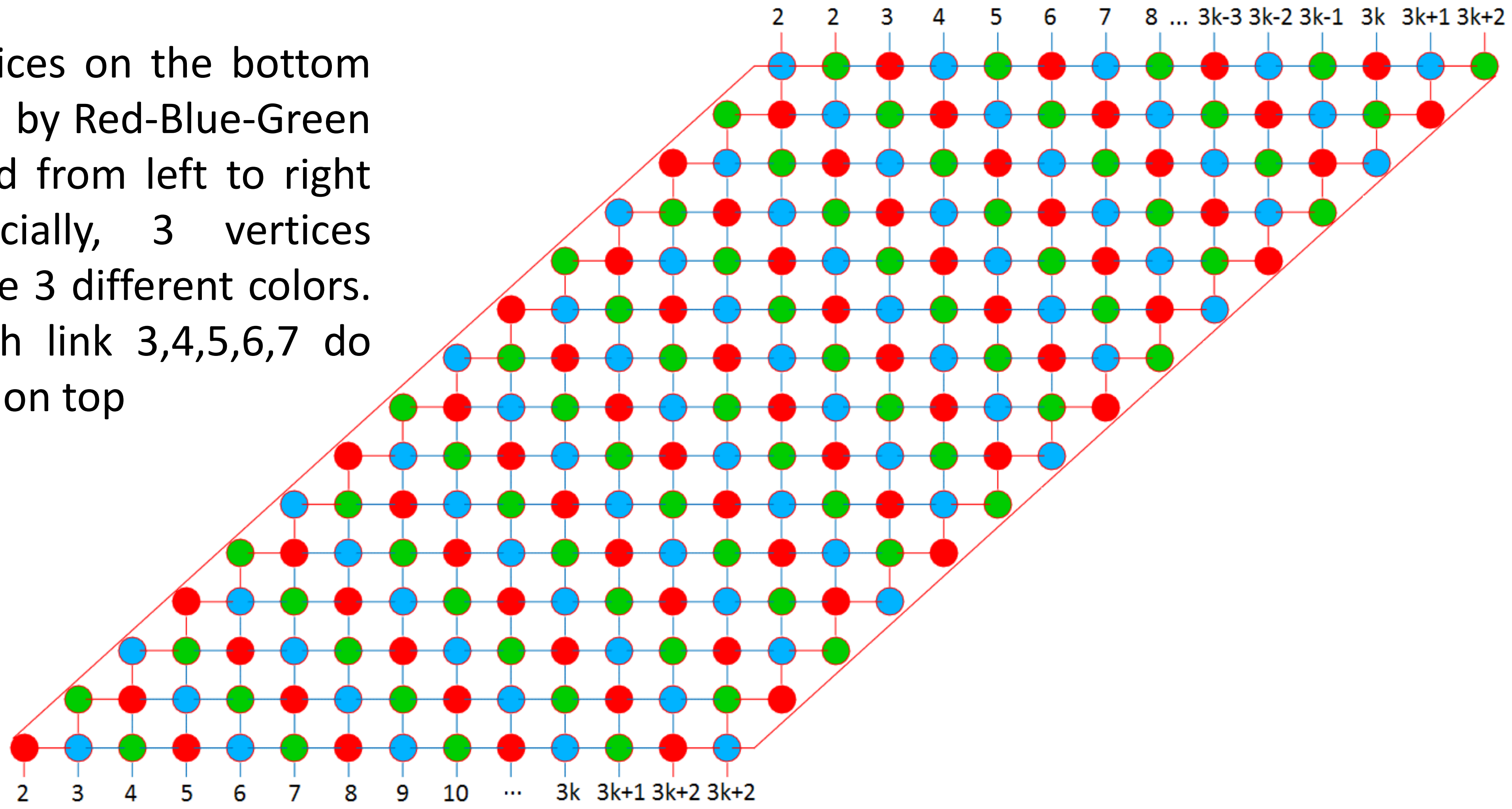
Chromatic number of 3k circles

4 uncolored vertices on top can be colored by Red-Blue-Green respectively. We can see 3 nodes which link (3k+2) have 3 different colors. The link 3k and (3k+1) are still fine since the colors are generated in a specific order Red-Blue-Green. Everything is still fine



Chromatic number of 3k circles

12 uncolored vertices on the bottom left can be colored by Red-Blue-Green respectively spread from left to right respectively. Specially, 3 vertices whose link (2) have 3 different colors. The vertices which link 3,4,5,6,7 do not conflict others on top



Chromatic number of 3k circles

➔ $\chi(G(3(K+1))) = 3$

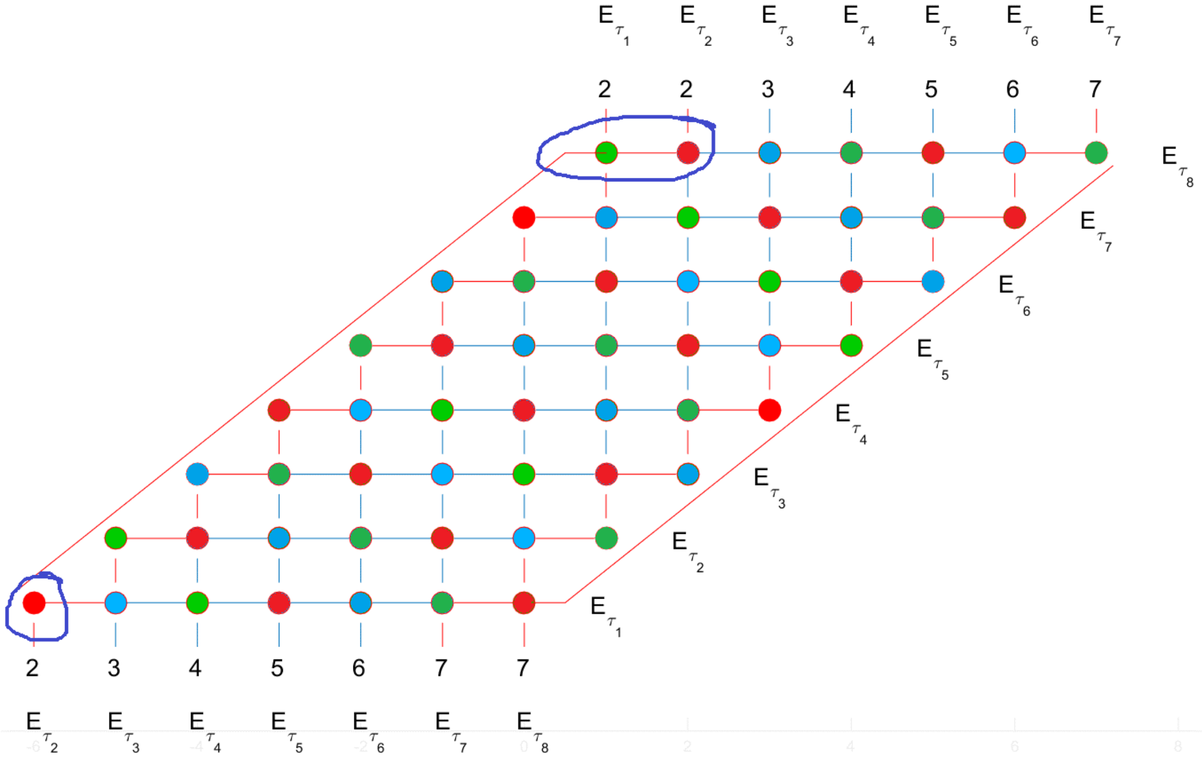
➔ The induction hypothesis is correct

➔ $\chi(G(3k)) = 3\,; (k > 0, k \in N)$

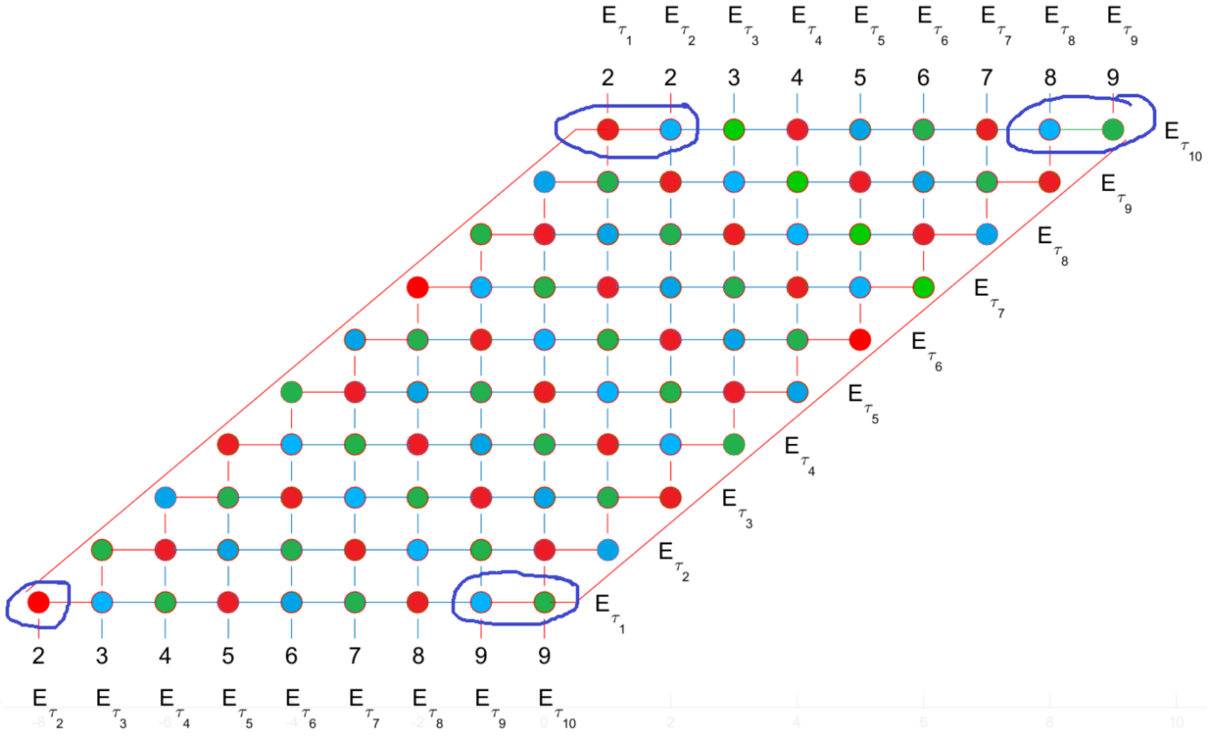
Chromatic number of 2k circles

Chromatic number of 2k circles

- With the graph 2k, I tried the same way to color the graph but it didn't work out



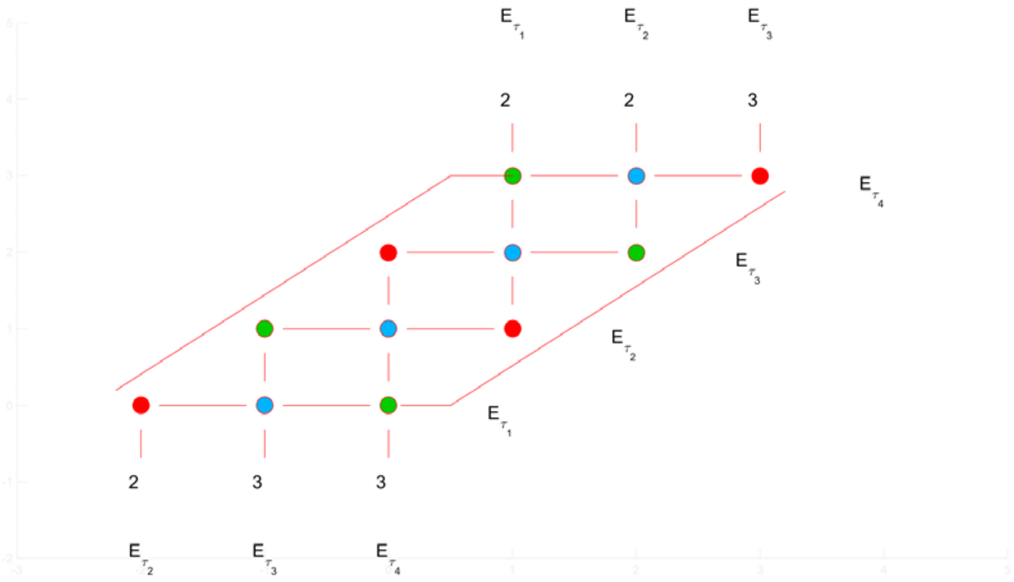
8 great circles



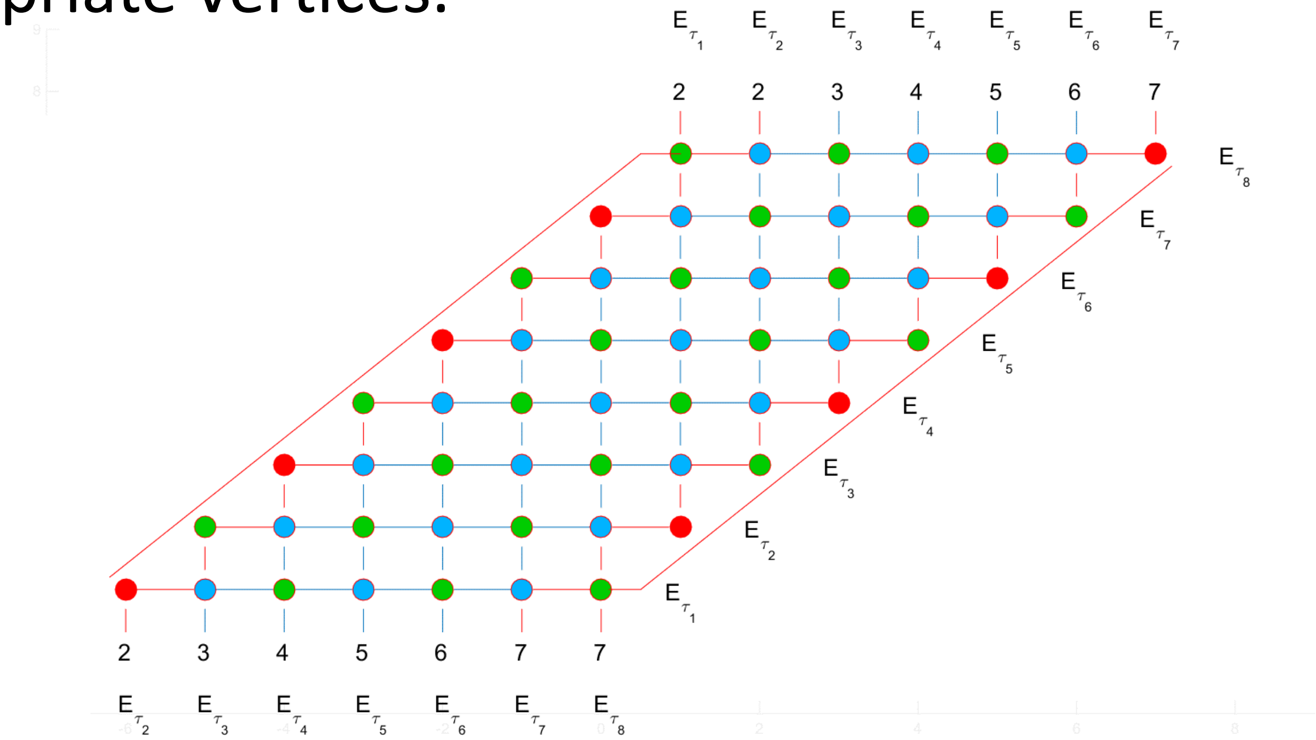
10 great circles

Chromatic number of 2k circles

- I tried to color it into a different way is to spread 2 colors Blue-Green from the bottom left to the top right of the equivalent graph and then put the color Red into the appropriate vertices.



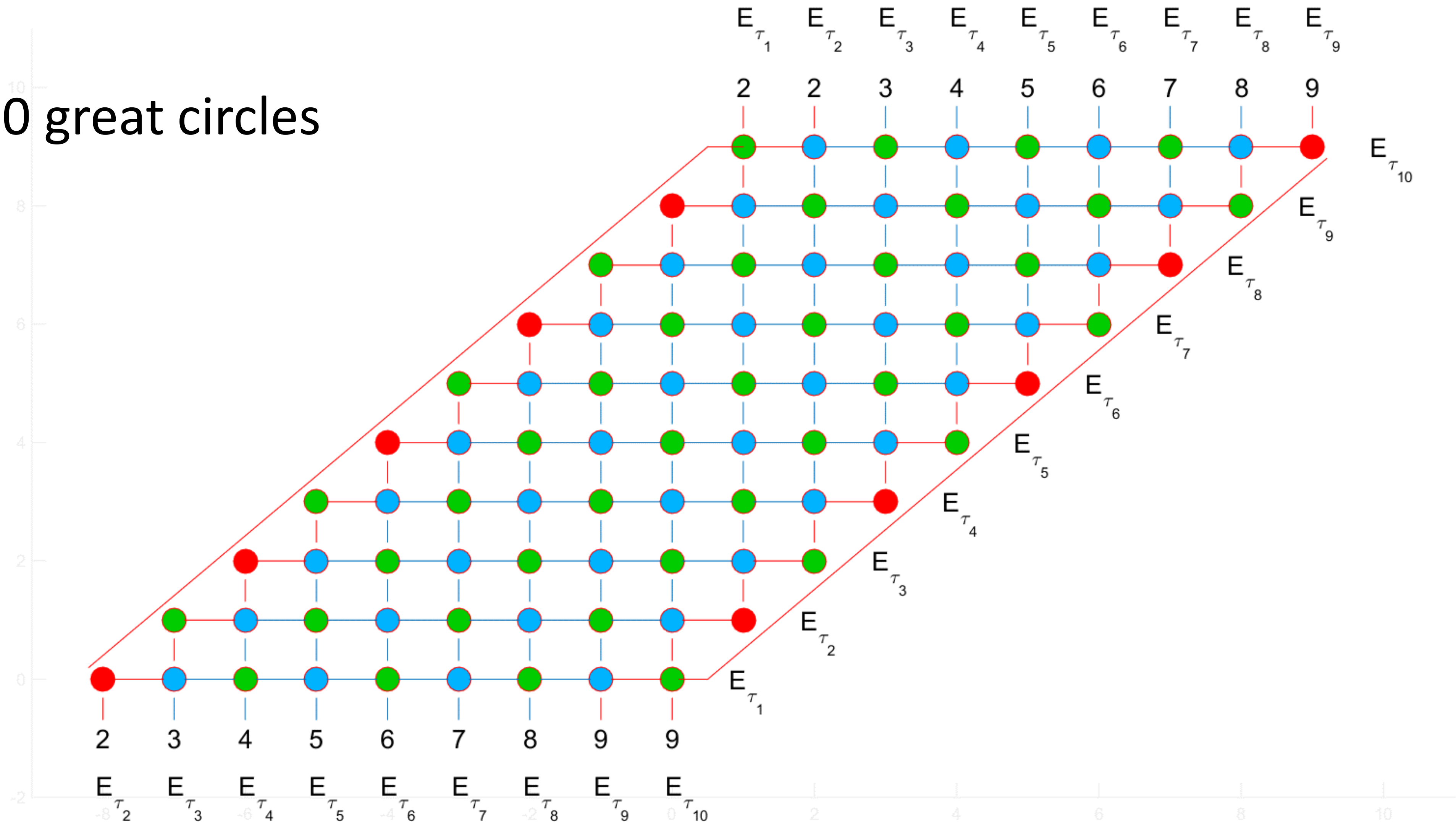
6 great circles



8 great circles

Chromatic number of 2k circles

- Base cases: 10 great circles



Chromatic number of 2k circles

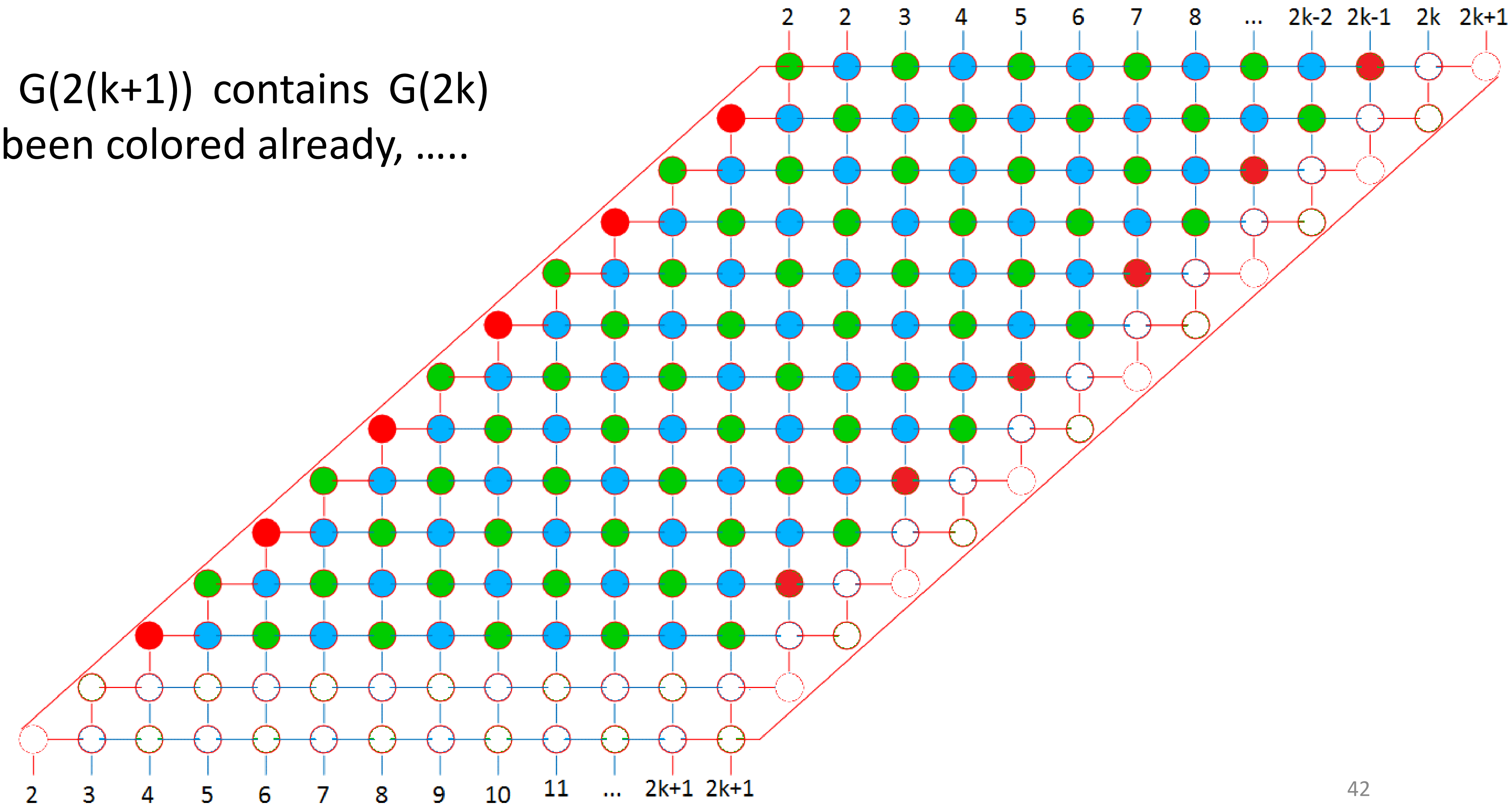
- According to the base cases with 4,8,10 great circles, the chromatic number is 3

➔Induction hypothesis: $\chi(G(2K)) = 3$; $(k > 1, k \in N)$ that has been correct with k=2,4,5 by spreading 2 colors Blue-Green from the bottom left to the top right of the equivalent graph and then putting the color Red into the appropriate vertices.

➔Induction step: Prove that $\chi(G(2(K+1))) = 3$

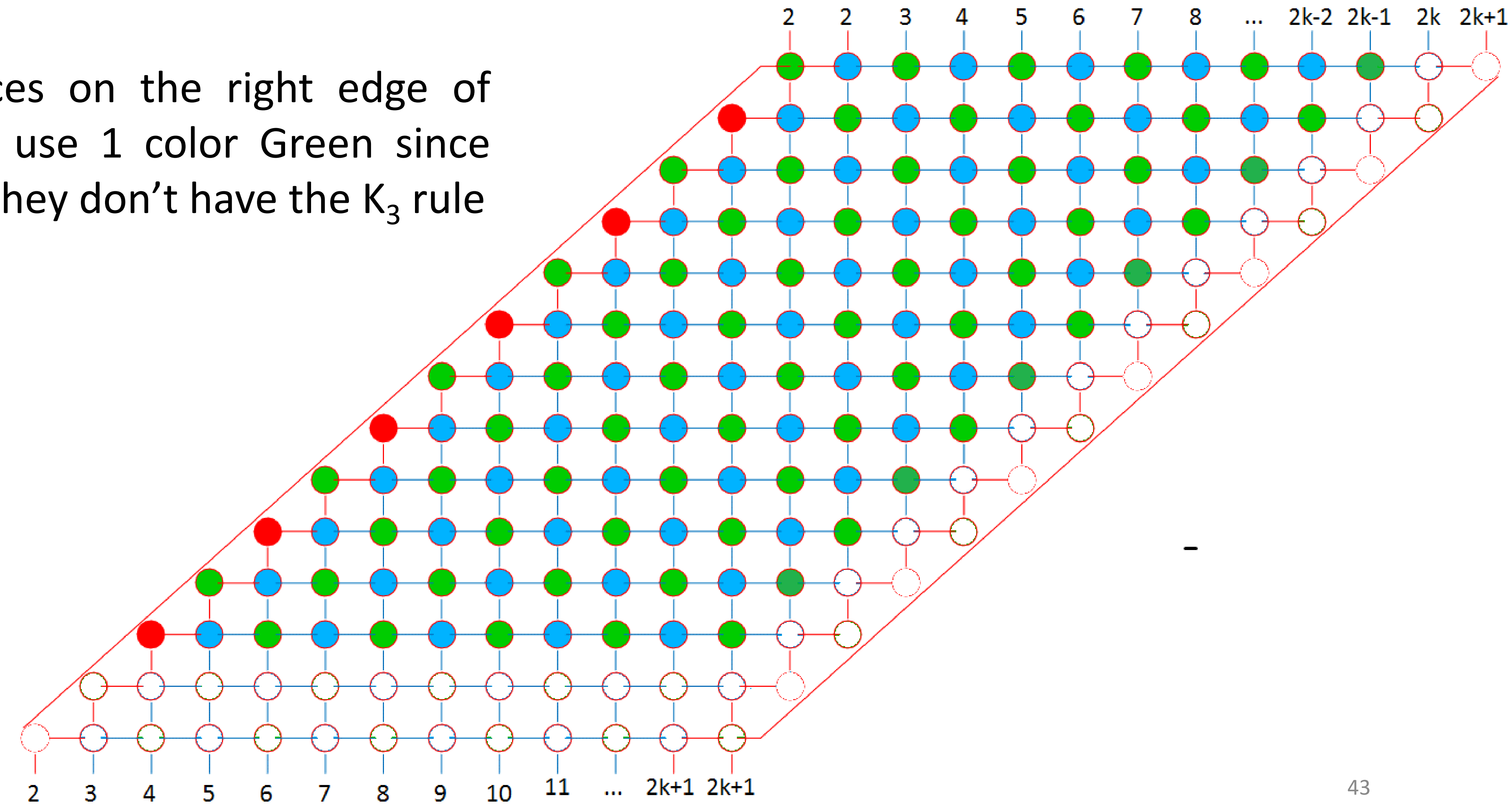
Chromatic number of 2k circles

The graph G(2(k+1)) contains G(2k) which has been colored already, …..



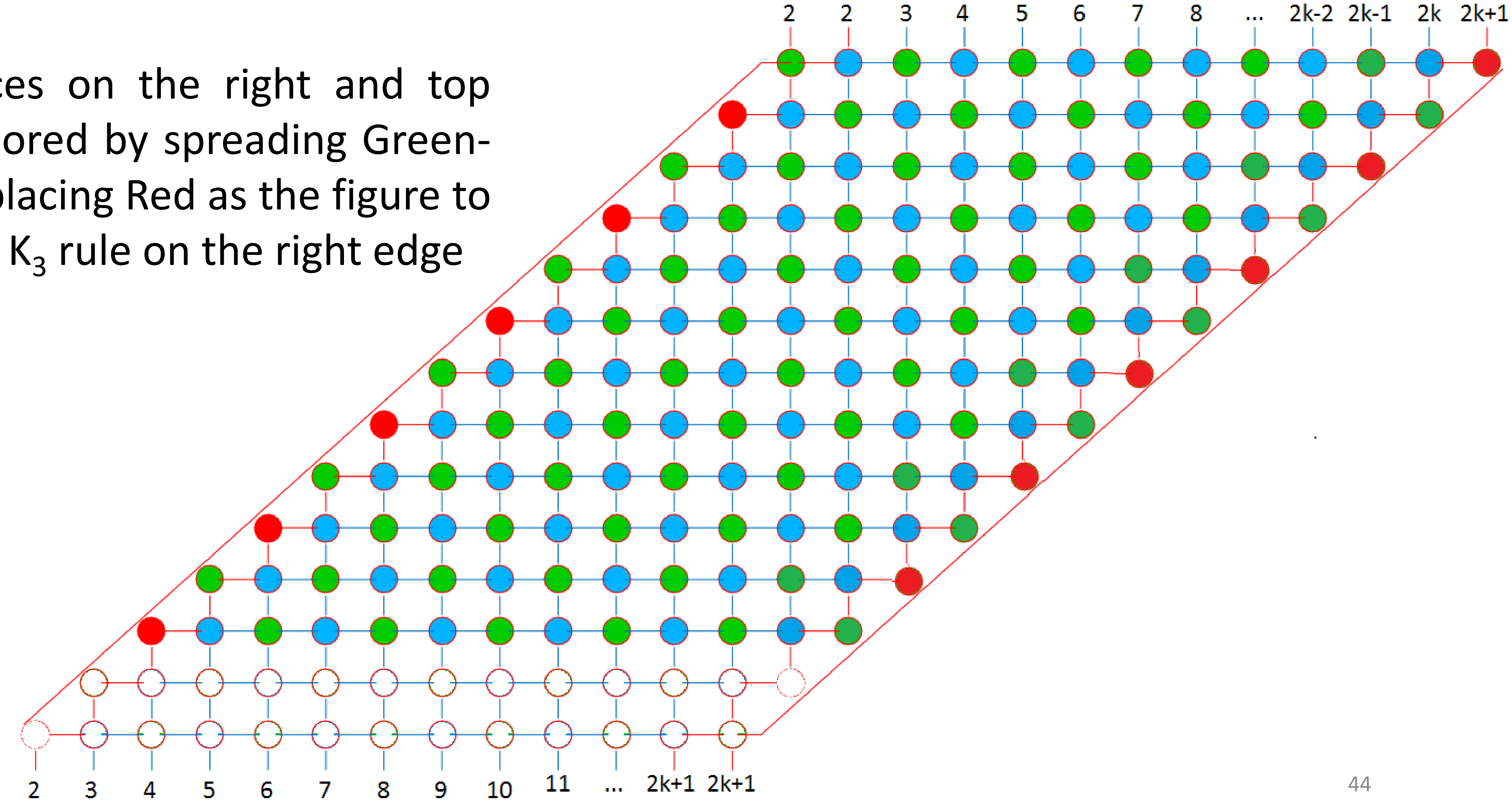
Chromatic number of 2k circles

The vertices on the right edge of G(2k) can use 1 color Green since currently they don't have the K_3 rule



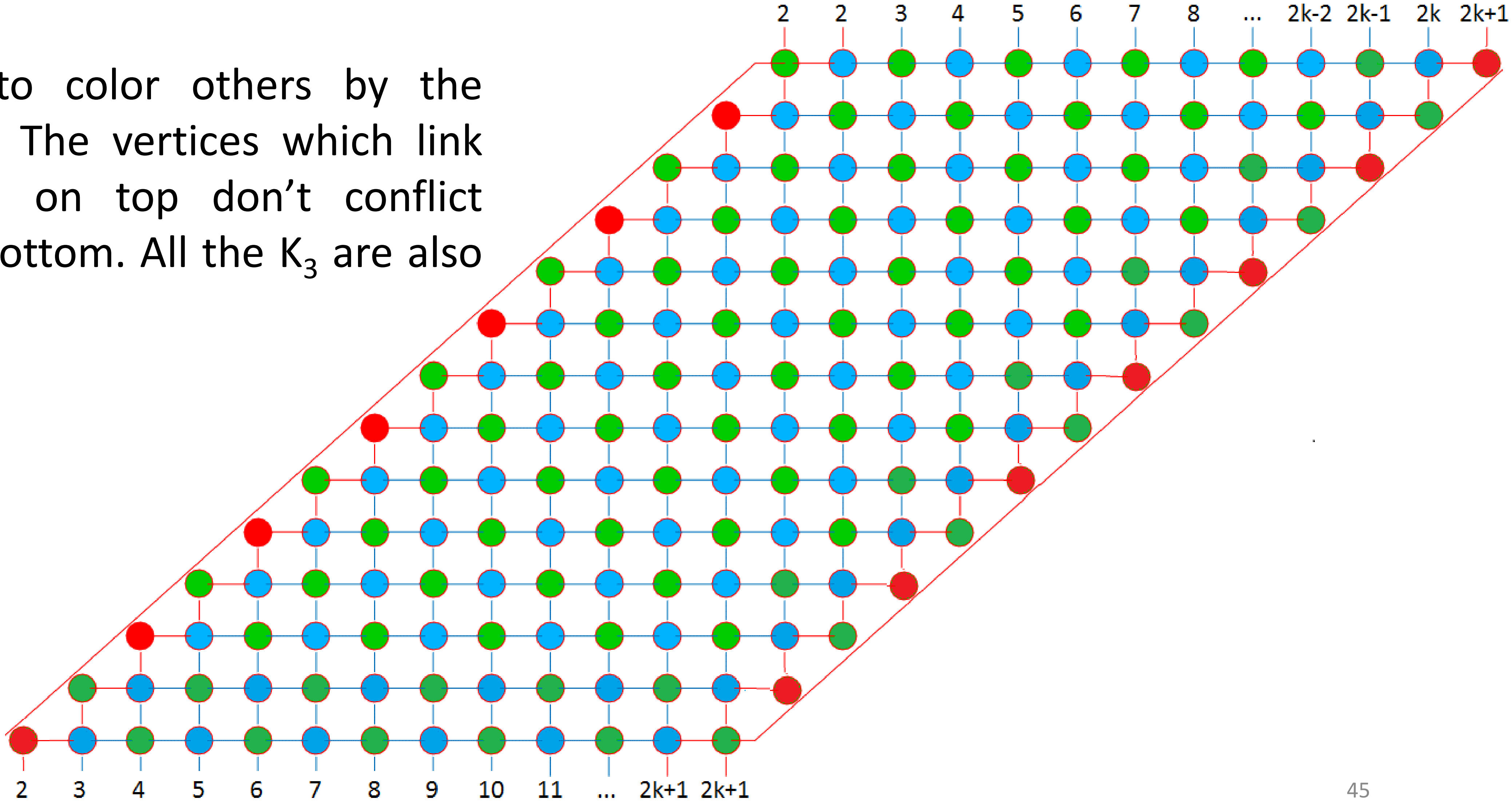
Chromatic number of 2k circles

The vertices on the right and top can be colored by spreading Green-Blue and placing Red as the figure to satisfy the K_3 rule on the right edge



Chromatic number of 2k circles

Continue to color others by the same way. The vertices which link (3,4,5,…2k) on top don't conflict those on bottom. All the K_3 are also satisfied



Chromatic number of 2k circles

➔ $\chi(G(2(K+1))) = 3$

➔ The induction hypothesis is correct

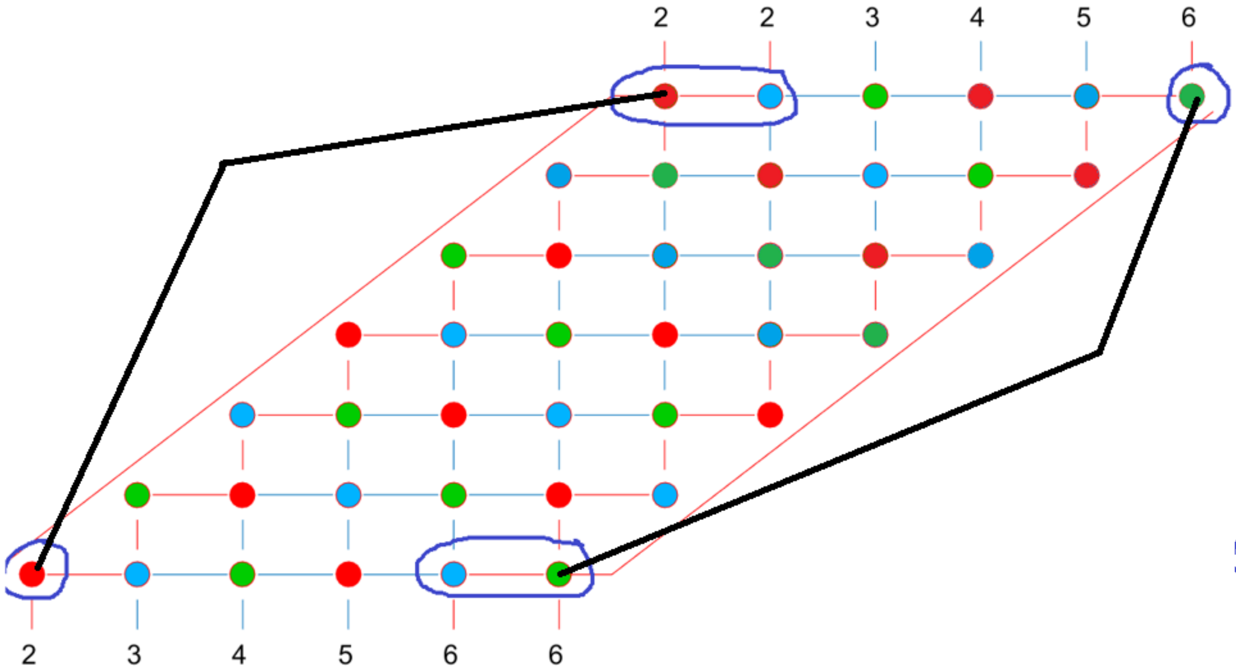
➔ $\chi(G(2k)) = 3$; $(k > 1, k \in N)$

Chromatic number of (6k+1) circles

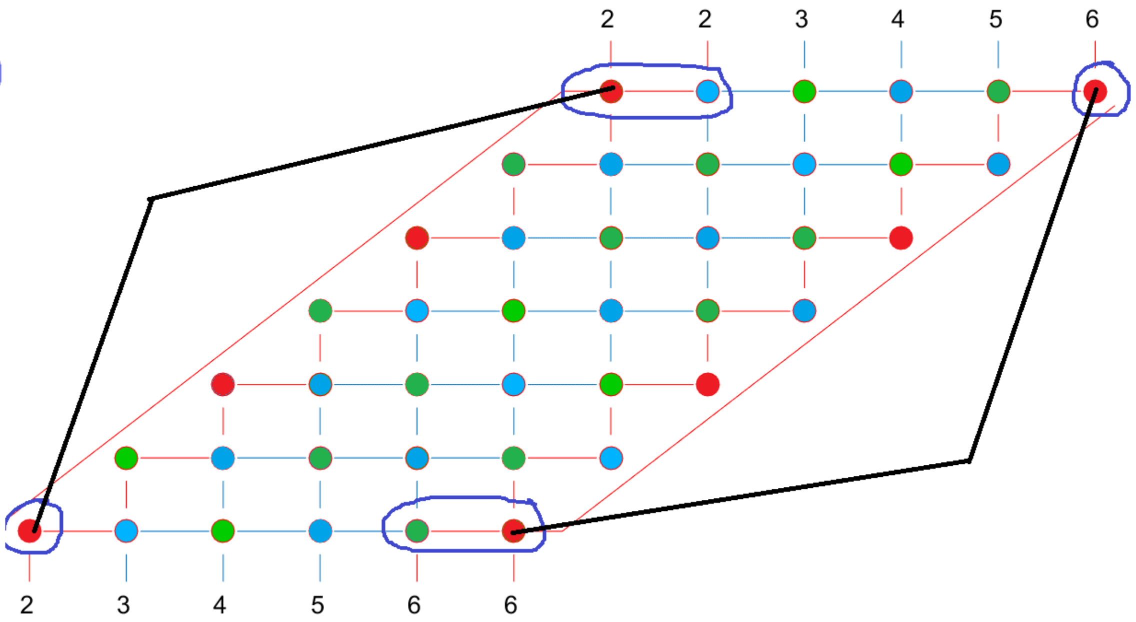
Chromatic number of (6k+1) circles

This type of graph doesn't suit 2 previous coloring ways because:

- 3k: There is always a K_3 (2) that contains 2 vertices have the same color because $(6k+1) \equiv 1(mod\ 3)$
- 2k: $(6k+1) \equiv 1(mod\ 2)$, so K_3 (contains link 2) always has 2 vertices have the same color



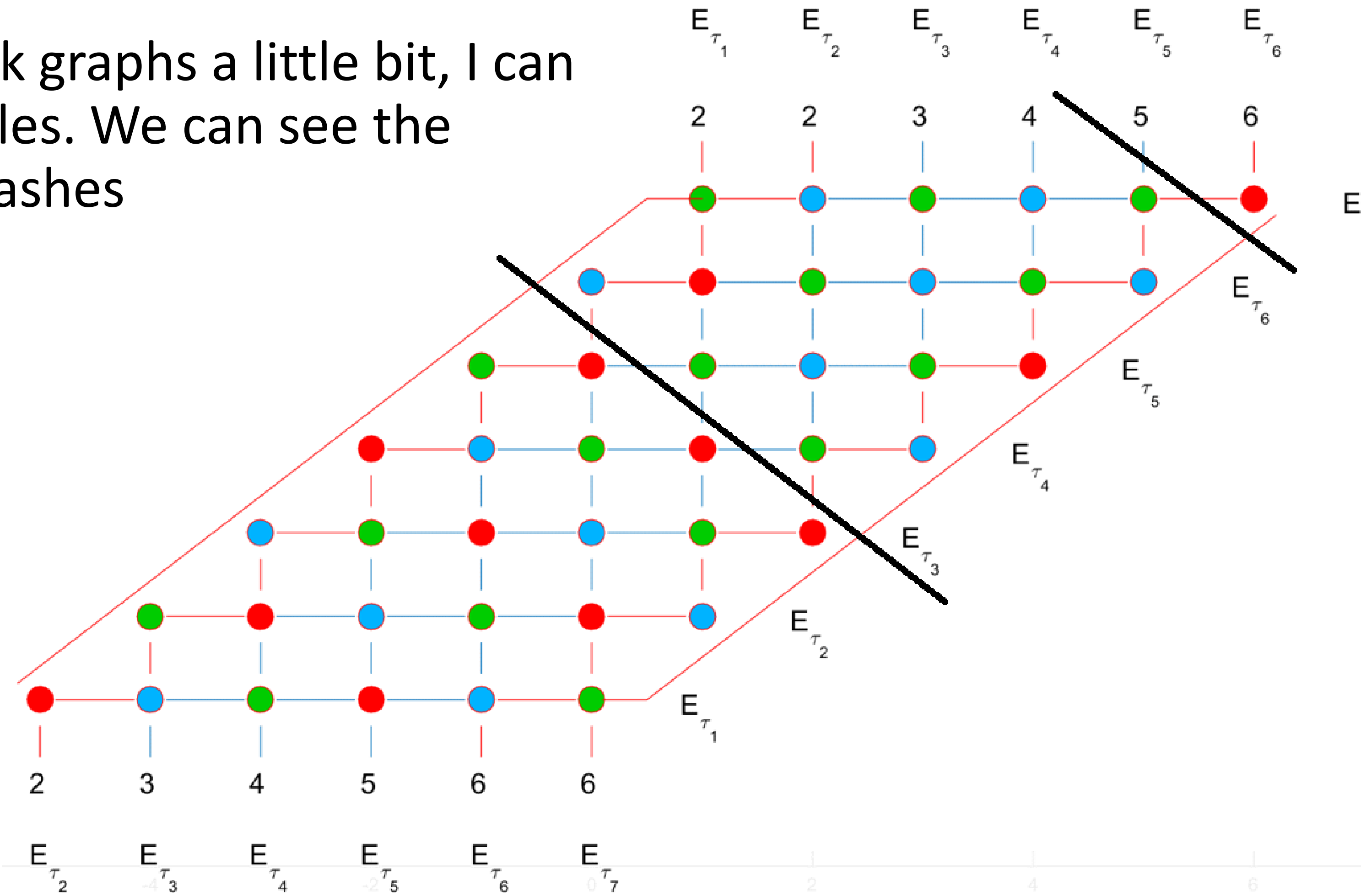
Color the graph by the way used on 3k graphs



Color the graph by the way used on 2k graphs

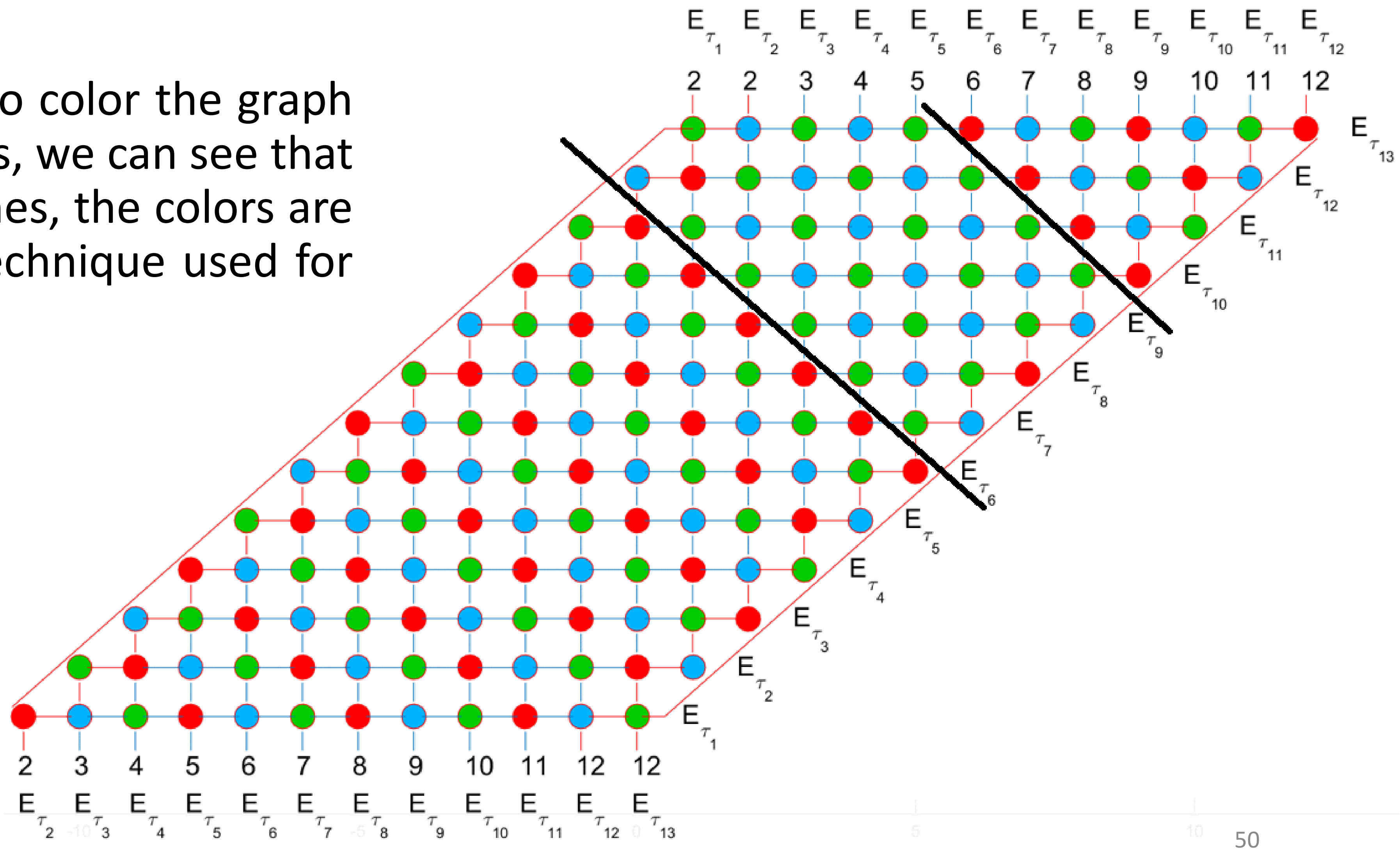
Chromatic number of (6k+1) circles

- By modifying the technique of 3k graphs a little bit, I can color the graph with 7 great circles. We can see the difference in between 2 black slashes



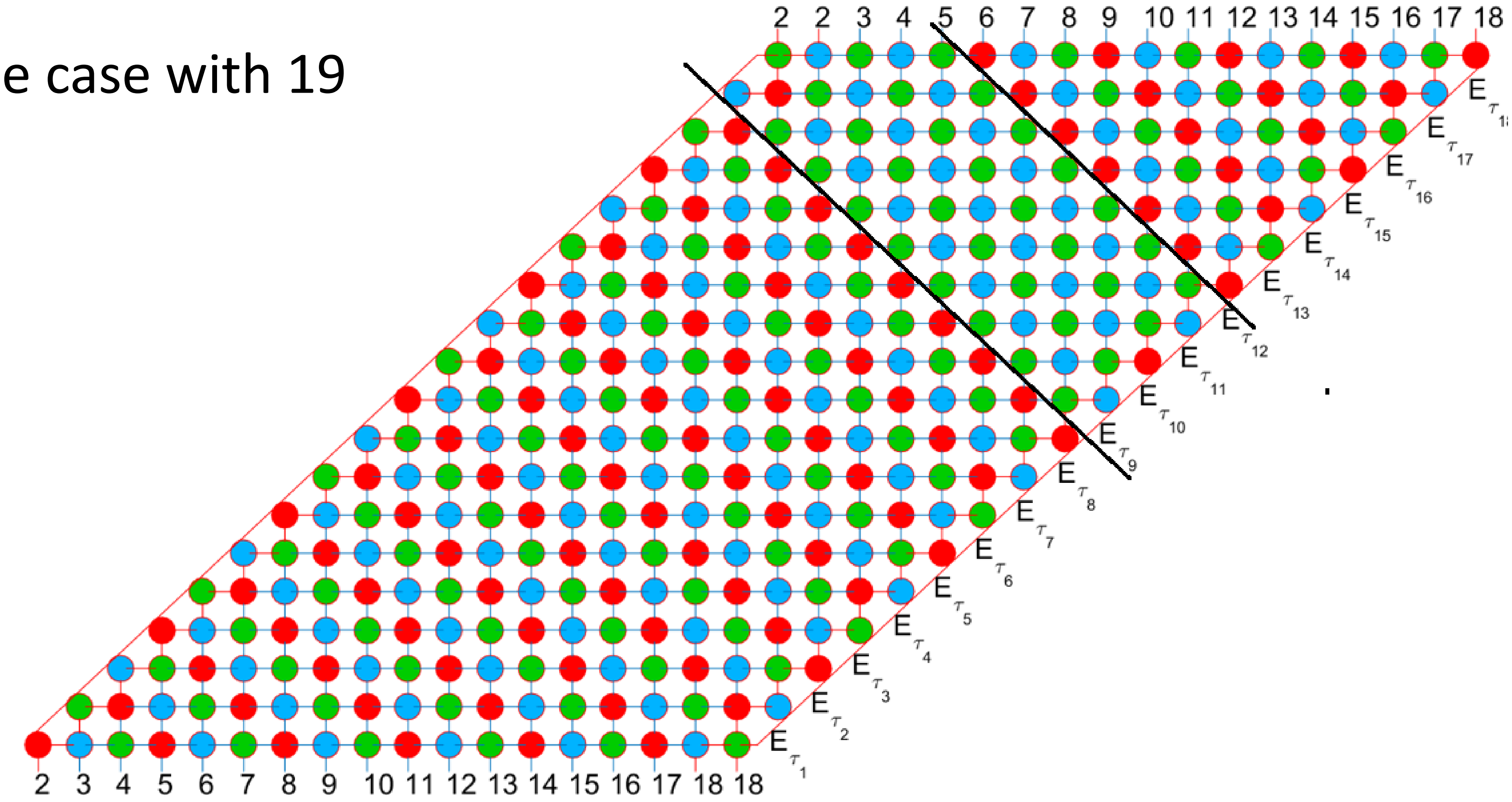
Chromatic number of (6k+1) circles

- Greedily continue to color the graph with 13 great circles, we can see that outside of the slashes, the colors are similar to the 1st technique used for 3k graphs



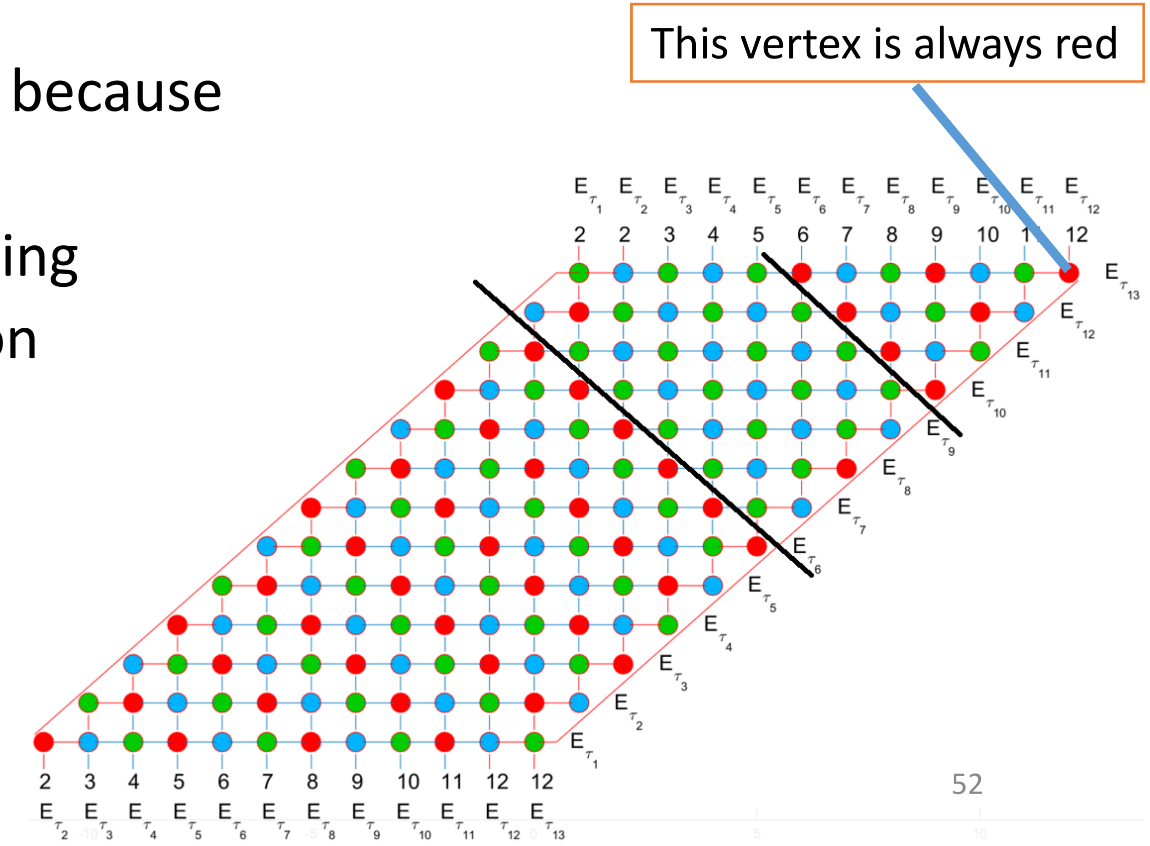
Chromatic number of (6k+1) circles

- Another base case with 19 great circles



Chromatic number of (6k+1) circles

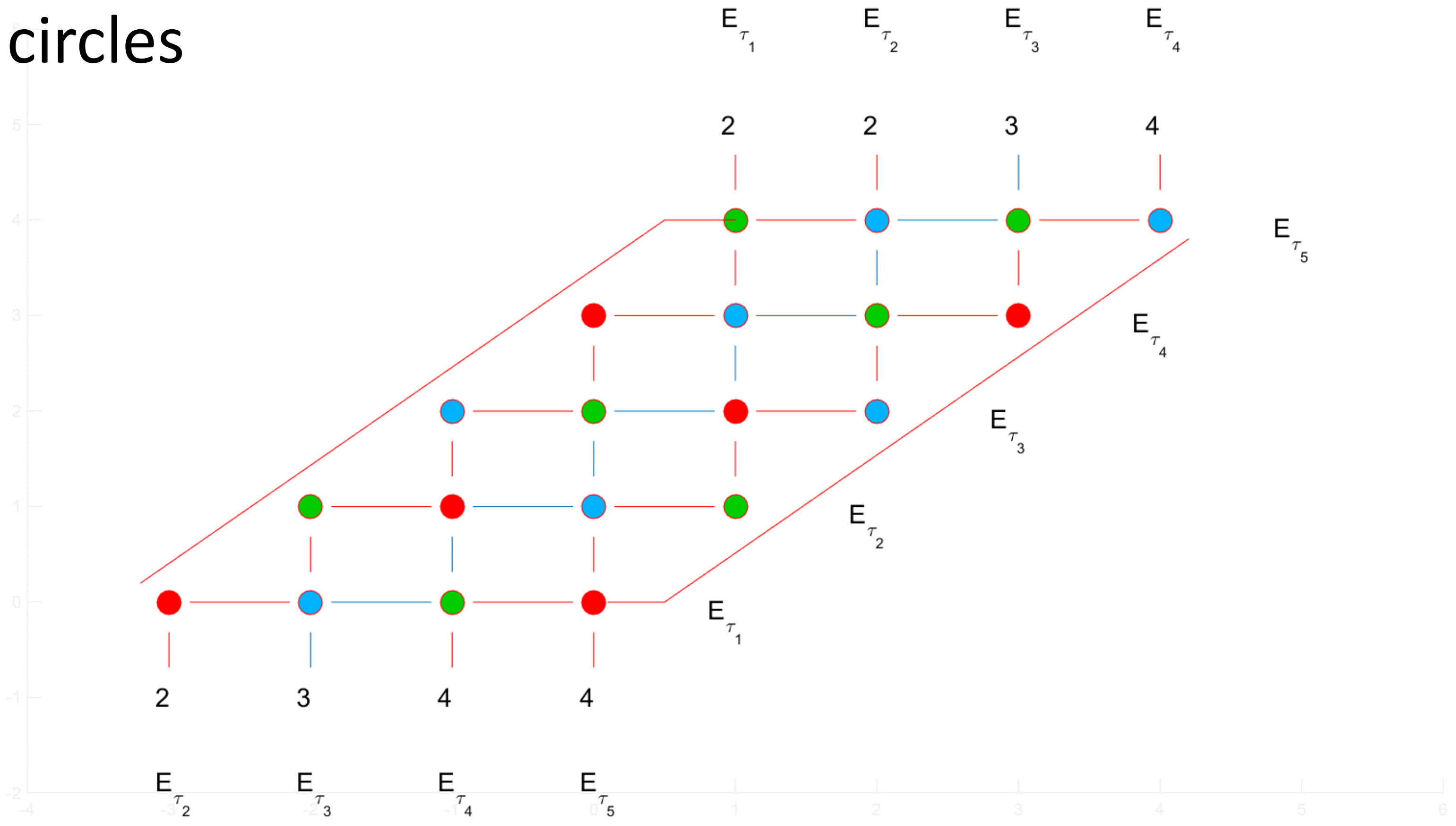
- With k>2, by spreading RED-BLUE-GREEN respectively on the additional diagonals, I can color all this type of graph based on the case k=2
  - The vertex on the top right is always left because step $6k \equiv 0(mod 3)$
  - The other additional vertices use spreading technique which doesn't create conflicts on the links at edges
  - Similar to the coloring way of 3k graphs, all additional K_3 are also satisfied



Chromatic number of (6k+5) circles

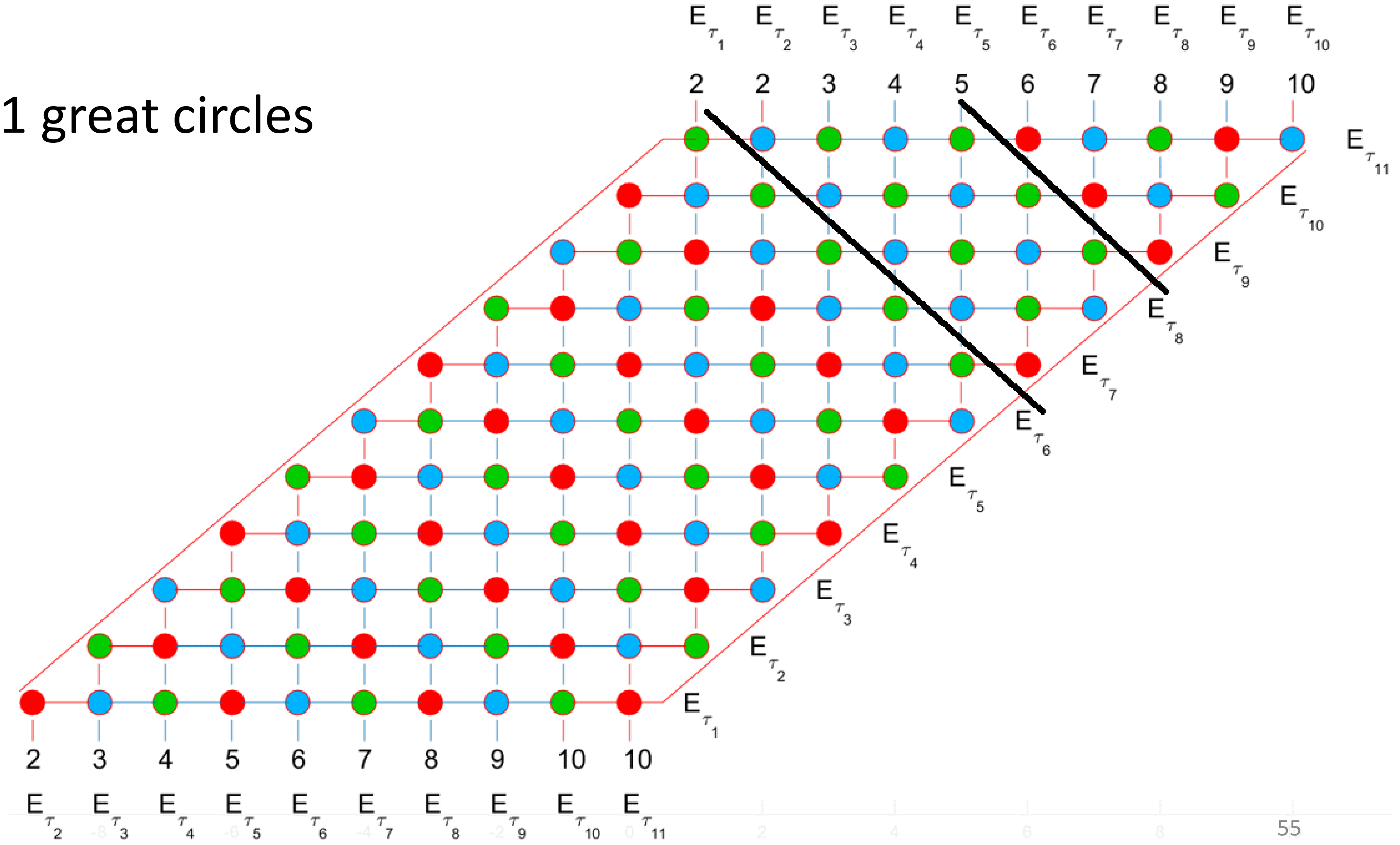
Chromatic number of (6k+5) circles

- Base cases: 5 great circles



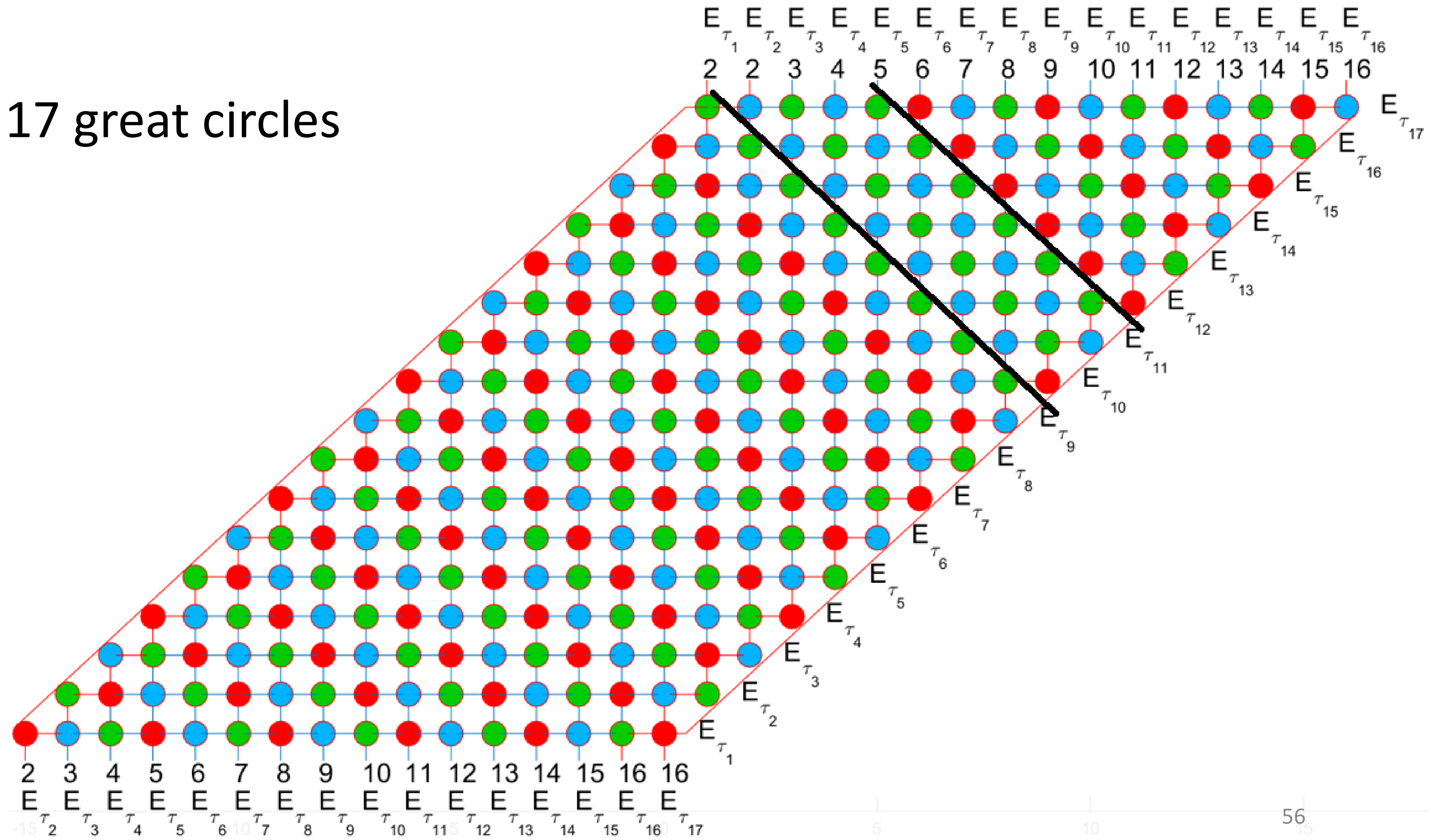
Chromatic number of (6k+5) circles

- Base cases: 11 great circles



Chromatic number of (6k+5) circles

- Base cases: 17 great circles



Chromatic number of (6k+5) circles

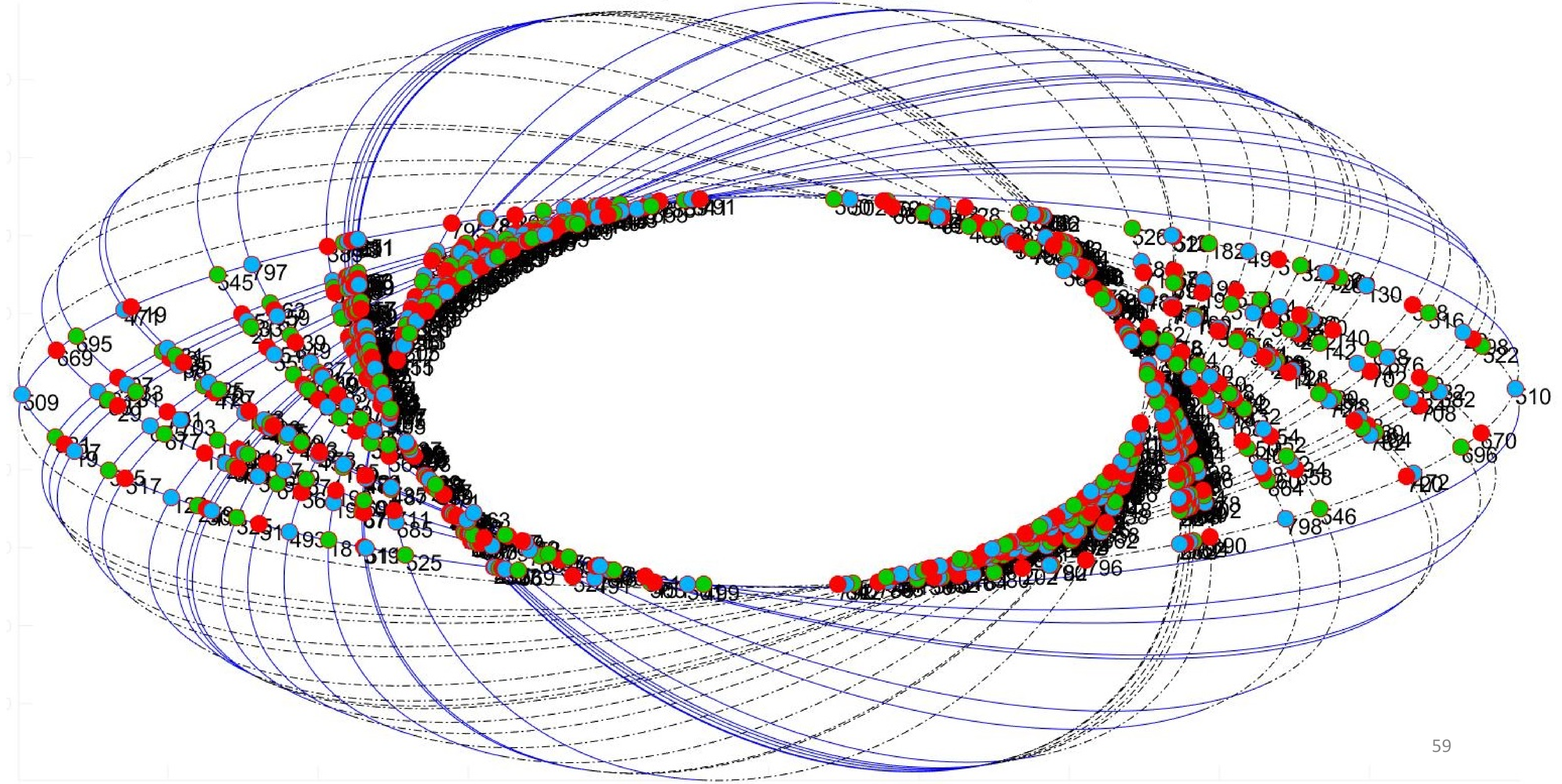
- This type of graph is similar to (6k+1) graph. With cases k>1, by spreading RED-BLUE-GREEN respectively on the diagonals, I can color all this type of graph based on the case k=1

Chromatic number of (6k+5) circles

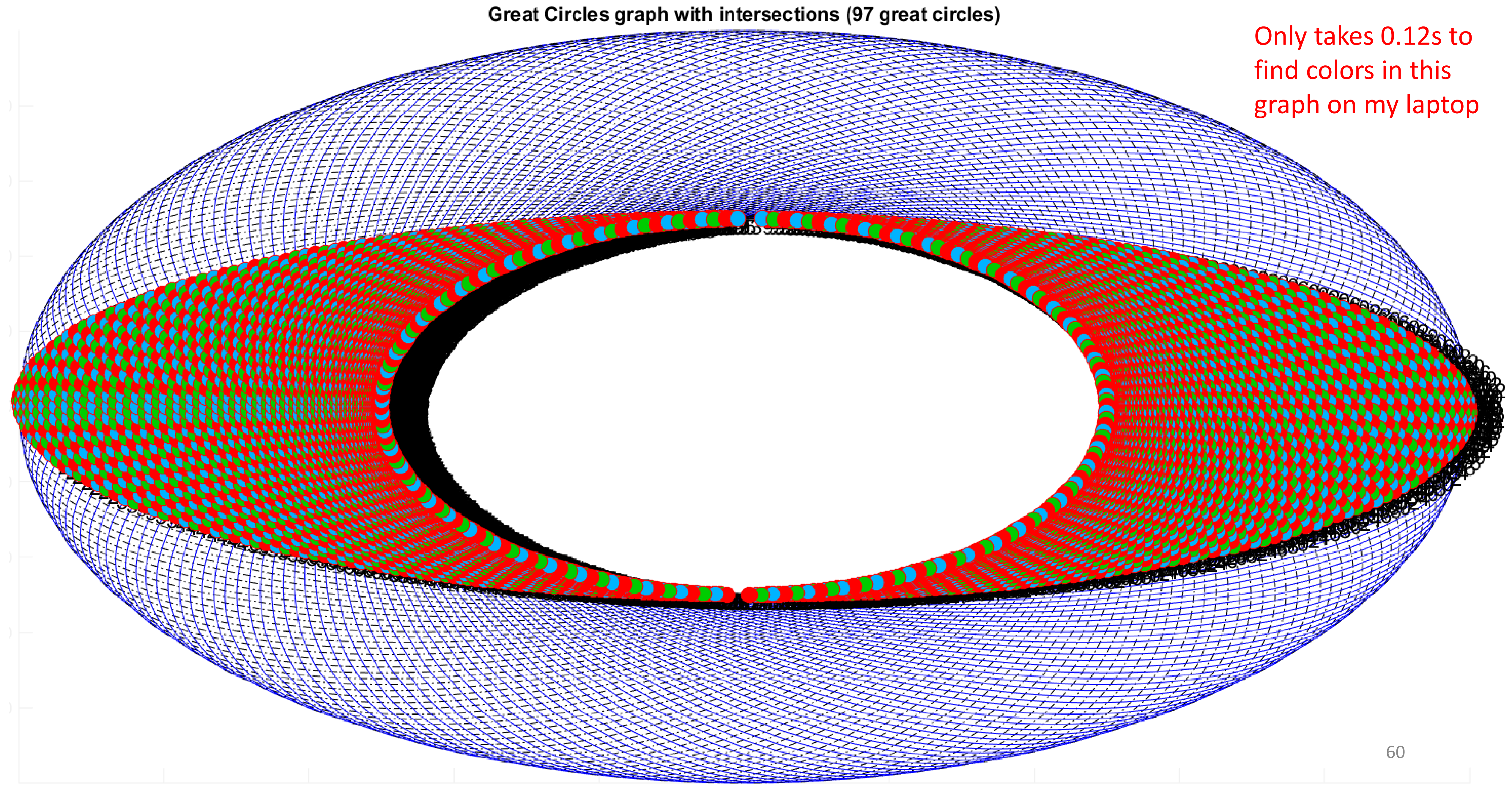
- According to the techniques to color the graphs including 3k, 2k, 6k+1 and 6k+5 great circles, there are only 3 colors used

➔ $\boldsymbol{\chi(G) = 3}$

Great Circles graph with intersections (30 great circles)



Only takes 0.12s to find colors in this graph on my laptop



5. Algorithms used in the visualization

Visualization

- The problem is visualized on MATLAB. All the great circles are generated randomly.
- I have used the library geom3d, written by David Legland, to draw the problem in 3-D view
- The outcome of the application
  - Draw the problem in 3D
  - Adjacency Matrix
  - Edges matrix
  - Matrix_Circles, Matrix_Vertices (which are defined in the next 4 slide)
  - Points what forms a single circle (2000)
  - And properties of all circles (**THETA** θ, **PHI** φ)

Creating great circles

drawCircle3d([C_x C_y C_z R THETA PHI])

- Description:
  - Spherical coordinates
    - **C_x C_y C_z** are coordinates of circle center
    - **R** is the circle radius
    - **THETA** θ between 0 and 180 degrees, corresponding to the azimuth angle
    - **PHI** φ between 0 and 360 degrees, corresponding to the zenith angle
- References: [Parametric Equation of a Circle in 3D](),

[Spherical Coordinates]()

Creating great circles

$$P(t) = rcos(t)u + rsin(t)n \times u + C$$

- $n = \begin{bmatrix} sin\theta cos\phi \\ sin\theta sin\phi \\ cos\theta \end{bmatrix}; \quad u = \begin{bmatrix} -sin\phi \\ cos\phi \\ 0 \end{bmatrix}; \quad n \times u = \begin{bmatrix} cos\theta cos\phi \\ cos\theta sin\phi \\ -sin\theta \end{bmatrix}$

➔ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -sin\phi \cos(t) + cos\theta cos\phi \sin(t) \\ cos\phi \cos(t) + cos\theta sin\phi \sin(t) \\ -sin\theta \sin(t) \end{bmatrix}$

Creating great circles - Algorithm

Input: parameters ([XC YC ZC R THETA PHI])

Output: a great circle

Pseudocode:

- Create k linearly spaced points in the interval (0, 2π)
- Calculate the points coordinates:
  - x = r(i) * cos(t);
  - y = r(i) * sin(t);
  - z = 0;
  - circle0 = [x y z];
- Compute transformation from local basis to world basis
  - trans = localToGlobal3d(xc(i), yc(i), zc(i), theta(i), phi(i), psi(i)); // psi = 0;
- Compute points of transformed circle
  - circle = transformPoint3d(circle0, trans);

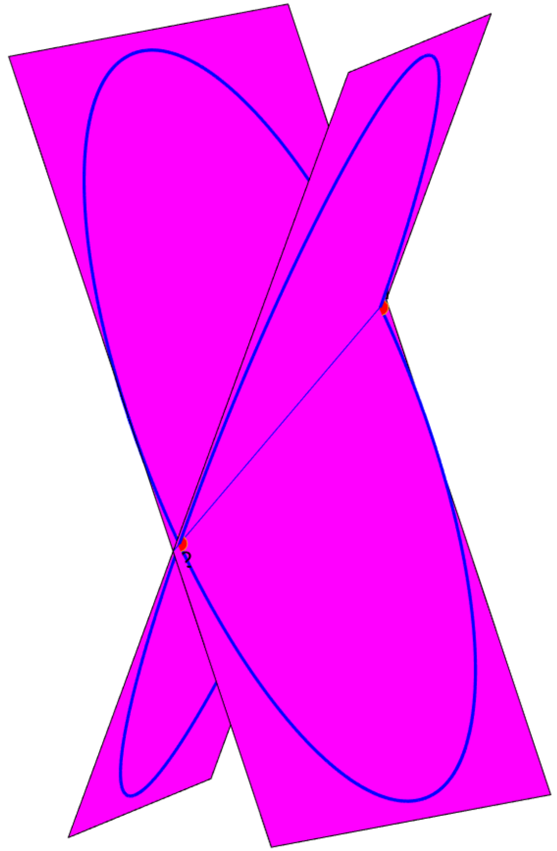
Detecting intersections

Input: a pair of great circles [XC YC ZC R THETA PHI]

Output: 2 intersection coordinates (X,Y,Z)

Pseudocode:

- Find the planes containing each great circles
- Find the line intersected by the 2 planes
- Output 2 intersection points made by the line and a circle



Matrix_Circles, Matrix_Vertices

- Matrix_Circles: a 3-dimensional-array contains lists of what vertices are in each great circle

- Matrix_Vertices: a 2-dimensional-array contains adjacency nodes of each vertex

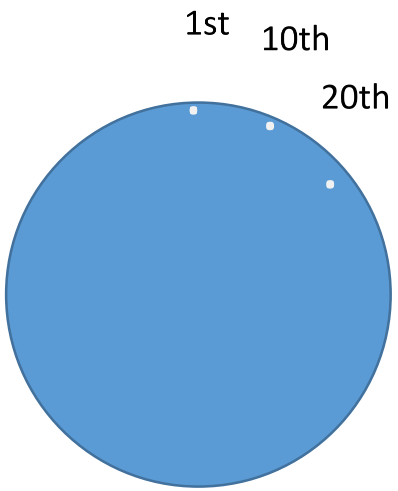
Finding edges

Input: a list in Matrix_Circles in term of vertices on a great circle

Output: Edges

Pseudocode:

- Easily figure out the order of vertices on the circle by the formula to generate them
- Each vertex will have an edge to 2 its adjacency nodes



Thank you