# Elements of Special Functions

The theory of special functions began as an independent domain around the $15^{th}$ century with the work of Bernoulli, Euler and Bessel, and imperiously still stands unfinished until today. The $4^{th}$ unsolved problem from the last six Millennium Prize Problems is the Riemann Hypothesis on the Zeta Function (according to the Clay Mathematics Institute, May 2012). The field of special functions is vivid, is complex and it is rich in connections to almost any part of mathematics, from numbers to nonlinear pseudo-differential equations.

What is a special function, and what makes it important and good for pure and applied mathematics? First, the special functions have a small number of parameters allowing them to have simpler geometrical representations. Secondly, the special functions are extremely well connected by functional and differential relations to themselves and to other special functions. Thirdly, there is rather the exception than the rule when a special function is not involved in solving a mathematical or physical science problem.

It is this connectedness which is essential for the global image of mathematics and sciences, because this is what makes us able to generate theories that predict things. Based on the rigor of pure mathematics, the applied mathematics and theoretical physics show us that any prediction can end up into a formula. Which moves the problem of prediction into the simple problem of evaluating a formula.

In these lectures we present elements of special functions in this course based on the unifying vision and theories developed by Vilenkin [1], Gel'fand and Miller [3] which relate each special function to a representation of a famous Lie group. After a short introduction in Lie groups and algebras representations, we discuss in the first four chapters classes of geometrical symmetries and their associated special functions. We develop more applications related to orthogonal polynomials and hypergeometric functions. In the last four chapters we present modern topics like q-deformed groups, irregular singularities confluence and Bäcklund transformations. In the spirit of the conference, we devote a special chapter to the Painlevé equations as nonlinear analogues of the classical special functions. These equations possess hierarchies of rational solutions, one-parameter families of solutions expressible in terms of the classical special functions, and admit symmetries under affine Weyl groups [4,5].

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   Affine Weyl groups.
7. Zeta functions, Riemann Zeta Function.

References:

[1] N. Ja. Vilenkin and A. U. Klimyk, Representation of Lie Groups and Special Functions (Springer-Verlag, Heidelberg 1995).

[2] I.M. Gelfand and D. Kazhdan, Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971 (Halsted, New York 1975) pp. 95--118.

[3] W. Miller, Lie Theory and Special Functions (Academic Press, New York 2010)

[4] P. A. Clarkson, Painlevé Equations – Special Nonlinear Functions, *J. Comput. Appl. Math.*, **153** (2003) 127-140.

[5] C. Truesdell, An Essay toward a Unified Theory of Special Functions (Princeton University Press, Princeton, 2010).