Major Exam

Indian Institute of Technology Jodhpur
Mathematics-II: MAL1020
(May 6, 2024)

Duration: 120 minutes **Maximum Marks: 30**

Answer All Six Questions (Check both pages): $(6 \times 5 = 30)$

1. (*a*) Show that $n \times n$ real symmetric matrices form a subspace of a $n \times n$ real matrix space. Find the dimension of this subspace. What is the dimension of the subspace formed by the skew-symmetric matrices? $[1 + 1 + 1]$

   (*b*) Define the linear transformation $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by
$$T(f(x)) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}, \; f \in P_2(\mathbb{R})$$
   Let $\{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$. Find a basis for the range space of T and what is the dimension of the null space of T? $[1 + 1]$

2. (*a*) Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f'(x)$. Find the matrix representation of T with respect to standard basis of $P_3(\mathbb{R})$ and standard basis of $P_2(\mathbb{R})$. [2]

   (*b*) Let $V = P_2(\mathbb{R})$ with inner product
$$\langle f(x),\, g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$
   Use the Gram-Schmidt orthogonalization process to orthonormalize the standard basis $\{1, x, x^2\}$. [3]

3. (*a*) Give an example of a 2×2 real matrix which doesn't have any eigen vector. Justify your answer. [1]

   (*b*) Prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. [1]

   (*c*) Let T be the linear transformation on $M_{2\times 2}(\mathbb{R})$ defined by
$$T(A) = A^t.$$
   Find the characteristic equation, eigenvalues and eigenvectors of T. $[1 + 1 + 1]$

4. (*a*) Find the solution of
$$y'' - y' - 2y = 4x^2$$
   that satisfies $y(0) = 0$ and $y'(0) = 1$. Here ' $'$ ' denotes differentiation with respect to x. [3]

   (*b*) Consider the initial value problem (IVP)
$$y' = \sqrt{|y|},\; y(0) = 0.$$
   Find one solution of the IVP. Does this IVP have a unique solution (Justify your answer)? [2]

5. (*a*) Consider the differential equation

$$2x^2y'' + x(2x+1)y' - y = 0.$$

Check the behaviour of the point $x = 0$ and find the indicial equation associated with $x = 0$. How many linearly independent solutions can it have? [1 + 1]

(*b*) For $\lambda \in \mathbb{R}$, solve the SLP

$$y'' + \lambda y = 0 \quad \text{with} \quad y(0) - y(\pi) = 0,\ y'(0) - y'(\pi) = 0.$$

Also, show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal. [2 + 1]

6. (*a*) Use the method of power series to find one solution of

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

[2]

(*b*) Solve $X' = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} X$. What can you say about the stability of this linear system? [2 + 1]

* * *

Solution-1 ⇒ To show that $n \times n$ real symmetric matrices forms a subspace of a $n \times n$ real Matrix space $M_{2\times2}(\mathbb{R})$

Let V represent the $n \times n$ real symmetric matrices
V is a non empty subset of $M_{2\times2}(\mathbb{R})$

$I \in V$
↳ Identity matrix

let $A, B \in V$ & $\alpha \in \mathbb{R}$

then $A+B = C$
$[a_{ij}]_{n\times n} + [b_{ij}]_{n\times n} = [a_{ij} + b_{ij}] = [c_{ij}]_{n\times n} \in V$
$= [a_{ji} + b_{ji}] = [c_{ji}]_{n\times n}$

(a)

(1)

$\alpha A + B = \alpha[a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}$
$= [\alpha a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}$
$= [\alpha a_{ij} + b_{ij}]_{n\times n} \in V$

So V is a subspace of $M_{2\times2}(\mathbb{R})$

(1)

dimension of symmetric matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & & & | \\ | & | & & & \\ a_{1n} & a_{2n} & & & a_{nn} \end{bmatrix} \begin{array}{l} \to n \text{ element} \\ \to n-1 \text{ element} \\ \\ \to 1 \text{ element} \end{array}$$

total $n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$

So. dim of $\boxed{V = \frac{n(n+1)}{2}}$

. dimension of the skew-symmetric matrix

(1) let $A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \cdots & | \\ | & | & | & \ddots & | \\ -a_{1n} & -a_{2n} & | & \cdots & 0 \end{bmatrix}$ $\begin{array}{l} \rightarrow (n-1) \text{ Independent element} \\ \rightarrow (n-2) \text{ element} \\ \\ \\ \rightarrow 0 \text{ Independent element} \end{array}$

So total of $(n-1) + (n-2) + \cdots + 1 + 0$

$= \frac{(n-1)n}{2}$ Independent element

So $\boxed{\dim = \frac{(n-1)n}{2}}$

(b) $T : P_2(\mathbb{R}) \rightarrow M_{2\times2}(\mathbb{R})$

$T(f(x)) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$, $f \in P_2(\mathbb{R})$

(1) $\{1, x, x^2\}$ are the standard basis of $P_2(\mathbb{R})$

$T(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$

$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1), T(x), T(x^2)\})$

$= \text{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right)$

$= \text{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right)$

$\dim(R(T)) = 2$

Rank Nullity Theorem $\rightarrow$

Rank + Nullity $= \dim(P_2(R))$

$2 + \text{Nullity}(T) = 3$

$\boxed{\text{Nullity}(T) = 3-2 = 1}$

1

2. $\boxed{(a)}$ $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ where,

$$T(f(x)) = f'(x)$$

- Find the matrix representation of T?

- Standard basis of $P_3(\mathbb{R})$ are $1, x, x^2, x^3$.

- Standard basis of $P_2(\mathbb{R})$ are $1, x, x^2$.

$T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ where,

$$T(f(x)) = f'(x) = \frac{d}{dx}[f(x)]$$

1

$f(x) = 1$, $T(1) = \frac{d}{dx}[1] = 0 = 0.1 + 0.x + 0.x^2$

$f(x) = x$, $T(x) = \frac{d}{dx}[x] = 1 = 1.1 + 0.x + 0.x^2$

$f(x) = x^2$, $T(x^2) = \frac{d}{dx}[x^2] = 2x = 0.1 + 2.x + 0.x^2$

$f(x) = x^3$, $T(x^3) = \frac{d}{dx}[x^3] = 3x^2 = 0.1 + 0.x + 3.x^2$

If $\alpha = \{1, x, x^2, x^3\}$ & $\beta = \{1, x, x^2\}$, then matrix representation of T with respect to standard basis α of $P_3(\mathbb{R})$ and standard basis β of $P_2(\mathbb{R})$ is given by

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}. \quad \text{(ANS)} \quad \text{— } 1$$

$\boxed{(b)}$ Let, $V = P_2(\mathbb{R})$.

- Standard basis of $P_2(\mathbb{R})$:

$$\beta = \{1, x, x^2\}$$

Note: We use the Gram-Schmidt process to replace β by an orthogonal basis $\{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$. Then, use this orthogonal basis $\{v_1, v_2, v_3\}$ to obtain an orthonormal basis for $P_2(\mathbb{R})$.

- **Gram-Schmidt process**

Let, $S = \{w_1, w_2, \ldots, w_n\}$.

orthogonal set $\leftarrow S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$

and, $$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, \quad 2 \leq k \leq n.$$

- For $k=3$,

$$v_1 = w_1.$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1.$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

Case-1

Here, $w_1 = 1$, $w_2 = x$, $w_3 = x^2$ and

inner product, $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t) g(t)\, dt$

Take, $v_1 = w_1 = 1$, $\|v_1\|^2 = \langle v_1, v_1 \rangle = \langle 1, 1 \rangle$

$$= \int_{-1}^{1} 1^2\, dt$$

$$= 2$$

Now, $v_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$.

$$= x - \frac{\langle x, 1 \rangle}{2} \times (1).$$

$$= x - \frac{1}{2}\left[\int_{-1}^{1} t\, dt\right] \quad \left[\because f(x) = x,\ g(x) = 1 \quad \langle f(x), g(x) \rangle = \int_{-1}^{1} f(t) g(t)\, dt\right]$$

$$= x - \frac{1}{2} \times 0 = x$$

$$\|v_2\|^2 = \langle v_2, v_2\rangle = \langle x, x\rangle = \int_{-1}^{1} t^2\,dt$$
$$= \frac{2}{3}$$

1.5

Now, $v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$

$$= x^2 - \frac{\langle x^2, 1\rangle}{2} \times (1) - \frac{\langle x^2, x\rangle}{2/3} \times (x).$$

$$= x^2 - \frac{1}{2}\left[\int_{-1}^{1} t^2\,dt\right] - \frac{3x}{2}\left[\int_{-1}^{1} t^3\,dt\right]$$

$$= x^2 - \frac{1}{2} \times \frac{2}{3} - \frac{3x}{2} \times 0$$

$$= x^2 - \frac{1}{3}$$

So, we conclude that $\{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal basis for $P_2(\mathbb{R})$.

- To obtain an orthonormal basis (say) $\{u_1, u_2, u_3\}$, we normalize v_1, v_2 and v_3.

Now, $u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\langle v_1, v_1\rangle}} v_1$

$$= \frac{1}{\sqrt{\int_{-1}^{1} 1^2\,dt}} \times (1).$$

$$= \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\langle v_2, v_2 \rangle}} \, v_2$$

$$= \frac{1}{\sqrt{\int_{-1}^{1} t^2 \, dt}} \times (x)$$

$$= \sqrt{\frac{3}{2}} \, x$$

1.5

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{\langle v_3, v_3 \rangle}} \, v_3 .$$

$$= \frac{1}{\sqrt{\int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt}} \times \left(x^2 - \frac{1}{3}\right)$$

$$= \sqrt{\frac{5}{8}} \left(3x^2 - 1\right).$$

So, the required orthonormal basis for $P_2(\mathbb{R})$ is $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{5}{8}} \left(3x^2 - 1\right) \right\}$

(ANS)

This problem can be solved by considering the following cases.

- Case-2 : $w_1 = 1$, $w_2 = x^2$, $w_3 = x$
- Case-3 : $w_1 = x$, $w_2 = 1$, $w_3 = x^2$.
- Case-4 : $w_1 = x$, $w_2 = x^2$, $w_3 = 1$.
- Case-5 : $w_1 = x^2$, $w_2 = 1$, $w_3 = x$.
- Case-6 : $w_1 = x^2$, $w_2 = x$, $w_3 = 1$.

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(3) (a) Take $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, $a > 0$

char. eq^n of A —

$$\lambda^2 + a^2 = 0$$

$$\boxed{\lambda = \pm ia}$$

since eigen value of the real matrix is not exist because the roots of the char. eq^n is img. So eigen vector also does not exist.

(b) we have to prove that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diag.

If the matrix is diag then it is similar to some diagonal matrix i.e. $\exists$ an invertible matrix P s.t.

$$PAP^{-1} = D$$

& since eigen value of A is 1

So, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow P\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1}P\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}P^{-1} = P^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}P^{-1}P = P^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = P^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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contradiction.

Hence, A is not a diagonalizable matrix.

(c) Let T be the L.T. on $M_{2\times2}(\mathbb{R})$ defined by, $T(A) = A^t$

<u>Method - 1</u>

we have to find the matrix rep. of T w.r.to standard basis & then find the char. eq^n, eigen value & eigen vector of the matrix.

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, matrix of T is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

char. eq^n $\rightarrow$ $|A - \lambda I| = 0$

$$\boxed{(\lambda-1)^3(\lambda+1) = 0}$$

e. value, $\boxed{\lambda = 1, 1, 1, -1}$

eigen vector w.r.to. eigen value
$\lambda = 1$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0-1 & 1 & 0 \\ 0 & 1 & 0-1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$\Rightarrow \boxed{x_2 = x_3}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

& eigen vector w.r.to. eigen value
$\lambda = -1$

$(A + \lambda I)x = 0$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 = 0$, $x_2 + x_3 = 0$, $x_2 + x_3 = 0$

$x_4 = 0$

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$x_1 = 0$, $x_2 = -x_3$, $x_4 = 0$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

So, eigen vectors -

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

This is the set of one of eigen vectors.

we can find others also.

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Method-2 $\quad T(A) = A^T$

If α is an eigen value, then.

$$T(A) = A^T = \alpha A$$

Take transpose on both side

$$A = \alpha A^T$$

$$A = \alpha^2 A \qquad \because A^T = \alpha A$$

$$(\alpha^2 - 1)A = 0 \qquad \because A \neq 0$$

$$\Rightarrow \alpha^2 = 1$$

$$\Rightarrow \boxed{\alpha = \pm 1}$$

If $\alpha = 1$, then $A = A^T$

$\Rightarrow$ A is symm. matrix.

If $\alpha = -1$, then $A = -A^T$

$\Rightarrow$ A is skew symm. matrix.

dim of eigen space corr. to e. value $\alpha = 1$ is $\frac{n(n+1)}{2} = 3$

& dim of eigen space corr. to e.value $\alpha = -1$ is $\frac{n(n-1)}{2} = 1$

So, char. eq^n $\rightarrow$ $\boxed{(x-1)^3(x+1) = 0}$

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is eigen vectors corr. to. e.value $\alpha = 1$

& $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is eigen vectors corr to $\alpha = -1$

Solution – 4

(a) Given diff. eqn. is –

$$y'' - y' - 2y = 4x^2$$

with $y(0) = 0$, $y'(0) = 1$.

→ The Auxilary equation is –

$$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$$

So the roots of auxilary eqn. are $2, -1$.

So the $C.F = c_1 e^{2x} + c_2 e^{-x}$. 1 marks upto here

And ~~For~~ $P.I = \frac{1}{D^2 - D - 2}(4x^2)$

$$= -\frac{1}{2}\left[1 + \left(\frac{D}{2} - \frac{D^2}{2}\right)\right]^{-1} 4x^2$$

$$= -\frac{1}{2}\left[1 - \left(\frac{D}{2} - \frac{D^2}{2}\right) + \left(\frac{D}{2} - \frac{D^2}{2}\right)^2 - \dots\right] 4x^2$$

$$= -\frac{1}{2}\left[4x^2 - \frac{8x}{2} + \frac{8}{2} + \frac{8}{4} + 0\right]$$

$$= -2x^2 + 2x - 3$$

So the general solution is – 2 marks upto here

$$y(x) = c_1 e^{2x} + c_2 e^{-x} - 2x^2 + 2x - 3$$

and $y'(x) = 2c_1 e^{2x} - c_2 e^{-x} - 4x + 2$

As $y(0) = 0 \Rightarrow y(0) = c_1 + c_2 - 3 = 0 \Rightarrow c_1 + c_2 = 3$

As $y'(0) = 1 \Rightarrow y'(0) = 2c_1 - c_2 + 2 = 1 \Rightarrow 2c_1 - c_2 = -1$

After solving $c_1 = \frac{2}{3}$ and $c_2 = \frac{7}{3}$. Hence the final solution is-

$$\boxed{y(x) = \frac{2}{3}e^{2x} + \frac{7}{3}e^{-x} - 2x^2 + 2x - 3}$$ 3 marks

(b) Given I.V.P. is -

$$y' = \sqrt{|y|}, \quad y(0) = 0.$$

Clearly, $y(x) = 0$ is a solution of given I.V.P.

1 marks

Suppose $x > 0$

Then $y' = \sqrt{y}$

or $\frac{dy}{dx} = y^{1/2}$

$\Rightarrow y^{-1/2} dy = dx$

$\Rightarrow 2y^{1/2} = x + c$

$\Rightarrow y^{1/2} = \frac{x+c}{2} \Rightarrow y(x) = \left(\frac{x+c}{2}\right)^2$

As $y(0) = 0 \Rightarrow y(0) = \left(\frac{c}{2}\right)^c = 0 \Rightarrow c = 0$.

Hence

$$y(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & x \geq 0. \end{cases}$$

$\Rightarrow$ Given I.V.P. doesn't have unique solution

2 marks

Q-5 (a) $$2x^2y'' + x(2x+1)y' - y = 0 \quad \text{——— (1)}$$

$$\Rightarrow \quad y'' + \frac{x(2x+1)}{2x^2}y' - \frac{1}{2x^2}y = 0$$

$$\Rightarrow \quad P(x) = \frac{(2x+1)}{2x} \quad , \quad Q(x) = -\frac{1}{2x^2}$$

For $x=0$

$$x P(x) = x\left(\frac{2x+1}{2x}\right) = 2x \quad \text{analytic}$$

$$\& \quad x^2 Q(x) = x^2\left(-\frac{1}{2x^2}\right) = -\frac{1}{2} \quad \text{analytic}$$

So, $x=0$ is a regular singular Point.

~~Sol~~ let $$y = \sum_{j=0}^{\infty} a_j x^{j+m}$$

$$y' = \sum_{j=0}^{\infty} (m+j)a_j x^{m+j-1} \quad \& \quad y'' = \sum_{j=0}^{\infty} (m+j)(m+j-1)a_j x^{m+j-2}$$

Put this values in Equation (1), we get

$$\Rightarrow \sum_{j=0}^{\infty} 2(m+j)(m+j-1)a_j x^{m+j} + \sum_{j=0}^{\infty} 2(m+j)a_j x^{m+j+1} + \sum_{j=0}^{\infty}(m+j)a_j x^{m+j}$$

$$- \sum_{j=0}^{\infty} a_j x^{j+m} = 0$$

$$\Rightarrow \sum_{j=0}^{\infty}\left[2(m+j)(m+j-1)+(m+j)-1\right]a_j x^{m+j}+\sum_{j=0}^{\infty}2(m+j)a_j x^{m+j+1}=0$$

Now for constant term to compare both side we get

$$\cancel{2(m+j)}\quad 2(m)(m-1)+m-1=0$$

1 marks

$$\Rightarrow 2\left[m^2-m\right]+m-1=0$$

$$\Rightarrow 2m^2-m-1=0$$

$$\Rightarrow 2m^2-2m+m-1=0$$

$$\Rightarrow 2m(m-1)+1(m-1)=0$$

$$\Rightarrow m=-\frac{1}{2},\ 1$$

2 marks for complete justification

Q-5 (b) $y'' + \lambda y = 0$, $y(0) - y(\pi) = 0$

$y'(0) - y'(\pi) = 0$

$\Rightarrow \quad D^2 = -\lambda$

For $\lambda = 0$, $y = Ax + B$

By using the condition , we get

$A = 0$, B is an arbitrary constant.

$\Rightarrow \boxed{\lambda_0 = 0 , \quad \phi_0 = 1}$

For $\lambda < 0$

i.e $\lambda = -\mu^2$ $\quad y = Ae^{-\mu x} + Be^{\mu x}$

By using BC , we obtain

$$A(1 - e^{-\mu\pi}) + B(1 - e^{\mu\pi}) = 0$$

$$A(1 + e^{-\mu\pi}) + B(1 - e^{\mu\pi}) = 0$$

we get $\boxed{A = B = 0}$

i.e. only trivial sol^n.

$\cancel{\sum_{j=1} (m\pi j)}$

For $\underline{\lambda = \mu^2 > 0}$

$$y = A\cos(\mu x) + B\sin(\mu x)$$

By BC, we get

$$\left\{\begin{array}{l} A(1-\cos(\mu\pi)) - B\sin(\mu\pi) = 0 \\ A\sin(\mu\pi) + B(1-\cos(\mu\pi)) = 0 \end{array}\right\}$$

After solving.

$$\underline{\mu = 2n}, \quad \underline{\lambda_n = 4n^2}$$

By taking $A = B = 1$

Eigen function $\longrightarrow$ $\varphi_n = \cos(\sqrt{\lambda_n}\, x)$,

$\psi_n = \sin(\sqrt{\lambda_n}\, x)$

2 marks

NOTE $\rightarrow$ ~~All Eigenvalues λ_n are positive and~~

To show Eigenfunction are orthogonal for distinct Eigenvalue

$\cancel{\int_0^{\pi} \cos\sqrt{\lambda_n}}$ $\int_0^{\pi}$ (Eigenfunction of λ_1)$\cdot$(Eigenfunction of λ_2) $dx = 0$

3 marks for complete

6 (a) $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ — ①

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)y}{1-x^2} = 0$$

Here $x=0$ is an ordinary point due to the fact that coefficients of y' & y are real analytic.

Let the solution of ① is of the form,

$$\left.\begin{aligned} y &= \sum_{i=0}^{\infty} a_i x^i \\ y' &= \sum_{i=1}^{\infty} i a_i x^{i-1} \\ y'' &= \sum_{i=2}^{\infty} i(i-1) a_i x^{i-2} \end{aligned}\right]$$

1/2 marks

Substitute y, y', y'' in ① and get -

$$\sum_{i=2}^{\infty} i(i-1) a_i x^{i-2} - \sum_{i=2}^{\infty} i(i-1) a_i x^i - \sum_{i=1}^{\infty} 2 i a_i x^i + p(p+1)\sum_{i=0}^{\infty} a_i x^i = 0$$

adjusting the index of summation in the first sum & get -

$$\sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} x^i - \sum_{i=2}^{\infty} i(i-1) a_i x^i - \sum_{i=1}^{\infty} 2 i a_i x^i + p(p+1)\sum_{i=0}^{\infty} a_i x^i = 0$$

1 marks

$\Rightarrow 2a_2 + 6a_3x - 2a_1x + p(p+1)[a_0 + a_1x] +$

$$\sum_{i=2}^{\infty} [(i+2)(i+1)a_{i+2} - i(i-1)a_i - 2ia_i + p(p+1)a_i]x^i$$
$$= 0$$

Comparing the coefficients of both sides I get.

$$\begin{cases} 2a_2 + p(p+1)a_0 = 0 \\ 6a_3 - 2a_1 + p(p+1)a_1 = 0 \\ (i+2)(i+1)a_{i+2} - i(i-1)a_i - 2ia_i + p(p+1)a_i = 0 \\ \qquad i = 2, 3, \ldots \end{cases}$$

recurrence relation

$$\begin{cases} a_2 = \dfrac{-p(p+1)a_0}{1.2} \\ a_3 = -\dfrac{(p+2)(p-1)}{2.3}a_1 \\ a_{i+2} = -\dfrac{(p+i+1)(p-i)}{(i+2)(i+1)}a_i \; ; \; i = 2, 3, \ldots \end{cases}$$

1.5 marks

Set a_0 & a_0 as A and B respectively and treated as constants.

$$a_2 = \frac{-p(p+1)}{1\cdot 2}A$$

$$a_3 = -\frac{(p-1)(p+2)}{2\cdot 3}B$$

$$a_4 = \frac{-(p-2)(p+3)}{3\cdot 4}a_2 = \frac{p(p-2)(p+1)(p+3)}{4!}A$$

$$a_5 = \frac{-(p-3)(p+4)}{4\cdot 5} a_3 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!} B$$

$$a_6 = \frac{-p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} A$$

$$a_7 = -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} B$$

$\vdots \qquad \vdots \qquad \vdots$

putting these coefficients into the assumed solution, I get -

$$y = A\left(1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6 + \cdots\right)$$

$$+ B\left(x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}x^7 + \cdots\right)$$

2 marks

6) (b) $X' = \underbrace{\begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}}_{A} X$

Characteristic equation:-

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)^3 = 0 \quad \Rightarrow \lambda = 2, 2, 2$

$\lambda = 2$ is an eigenvalue of A with multiplicity 3.

1/2 marks

By solving $(A - 2I)K = 0$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get only eigenvector $K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

To get the another two L.I eigenvector, we solve $(A - 2I)P = K$ and then

solve $(A - 2I)Q = P$ and get

$$P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ \& \ Q = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}$$

Hence the solution $\rightarrow$ $X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$

2 marks

$$+ c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} \cdot \frac{t^2}{2} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} e^{2t} \right]$$

Since all $\lambda > 0$ Hence system is unstable

3 marks for complete