(1)

ELECTRIC FIELD PERT.

Finite sys: $H = H_0 + V \quad , \; V = e\mathcal{E}x \qquad (q_e = -e)$

$$Q_n |\partial_{\mathcal{E}} n\rangle = T_n (\partial_{\mathcal{E}} H)|n\rangle$$
$$= e\, T_n\, x\, |n\rangle$$

Crystal:

$$Q_{nk} |\partial_{\mathcal{E}} u_{nk}\rangle = e\, T_n\, x\, |u_{nk}\rangle \qquad ?$$

We haven't yet discussed (4.16)

1D: $\boxed{Q_{nk}\, x\, |u_{nk}\rangle = i\hbar\, T_{nk}\, v_k\, |u_{nk}\rangle}$ — $\boxed{x \text{ formula}}$

Then $\boxed{Q_{nk} |\partial_{\mathcal{E}} u_{nk}\rangle = ie\hbar\, T_{nk}^2\, v_k\, |u_{nk}\rangle}$

$\mathcal{E}$-field pert. of $|u_{nk}\rangle$

<u>x-formula</u>:

$$v_k = -\frac{i}{\hbar}[x, H_k]$$

$$\therefore \langle u_{mk} | v_k | u_{nk} \rangle = -\frac{i}{\hbar}(E_{nk} - E_{mk}) \langle u_{mk} | x | u_{nk} \rangle$$

Turn around:

$$\sum_m |u_{mk}\rangle \left(\langle u_{mk} | x | u_{nk} \rangle \right) = \sum_m |u_{mk}\rangle \left(i\hbar \frac{1}{E_{nk} - E_{mk}} \langle u_{mk} | v_k | u_{nk} \rangle \right)$$

$$Q_{nk}\, x\, |u_{nk}\rangle = i\hbar\, T_{nk}\, v_k\, |u_{nk}\rangle$$

Really allowed?

Two arguments yes:

- Express $\mathcal{E} = -\frac{1}{c}\frac{\partial A}{\partial t}$, A = vector pot of E+M HW
- $\partial_\lambda P$ turned around

$$H = H_0 + \lambda_A A + \lambda_B B$$

Let $\boxed{\Sigma} = \frac{V}{(2\pi)^3} \sum_n^{occ} \int d^3k$

$$\partial_A \langle B \rangle = \partial_B \langle A \rangle = \boxed{\Sigma}\, 2\,\mathrm{Re} \langle u_{n\mathbf{k}} | A\, T_{n\mathbf{k}}\, B\, | u_{n\mathbf{k}} \rangle$$

$$\lambda_B = e\mathcal{E}_\nu \quad B = r_\nu$$

Naive: $\partial_A \langle r_\nu \rangle = \boxed{\Sigma}\, 2\,\mathrm{Re} \langle u_{n\mathbf{k}} | A\, T_{n\mathbf{k}}\, \underbrace{r_\nu | u_{n\mathbf{k}} \rangle}_{\text{TRICKS} \Rightarrow}$

$$= - \boxed{\Sigma}\, 2\,\mathrm{Im} \langle u_{n\mathbf{k}} | A\, T_{n\mathbf{k}} \left(\hbar\, T_{n\mathbf{k}}\, v_{\mathbf{k}} \right) | u_{n\mathbf{k}} \rangle$$

$\partial_{(e\mathcal{E}_\nu)} \langle A \rangle = \text{same}$

$$\boxed{\partial_\nu \langle A \rangle = -e\hbar \boxed{\Sigma}\, 2\,\mathrm{Im} \langle u_{n\mathbf{k}} | A\, T_{n\mathbf{k}}^2\, v_{\mathbf{k}\nu} | u_{n\mathbf{k}} \rangle}$$

4.22

(4.22) $\partial_\nu \langle A \rangle = -e\hbar \, \blacksquare \, 2\,\mathrm{Im} \langle u_{nk} | A \, T_{nk}^2 \, v_{k,\nu} | u_{nk} \rangle$

$A = J_\mu = -\frac{e}{V} v_\mu$

$\partial_\nu \langle J_\mu \rangle = \sigma_{\mu\nu} = e^2 \hbar \, \blacksquare \, 2\,\mathrm{Im} \langle u_{nk} | v_{k,\mu} \, T_{nk}^2 \, v_{k,\nu} | u_{nk} \rangle$

But remember $\quad Q_{nk} | \partial_k u_{nk} \rangle = \hbar T_{nk} v_k | u_{nk} \rangle$

$\left(= -i \, Q_{nk} \, x \, | u_{nk} \rangle \right)$

so $\quad \sigma_{\mu\nu} = \frac{e^2}{\hbar} \, \blacksquare \, 2\,\mathrm{Im} \langle \partial_\mu u_{nk} | \cancel{Q_{nk}} | \partial_\nu u_{nk} \rangle$

OUT

$= -\frac{e^2}{\hbar} \, \blacksquare \, \Omega_{n,\mu\nu}$

$$\boxed{\sigma_{\nu\mu} = \frac{e^2}{\hbar} \frac{1}{(2\pi)^3} \sum_n \int d^3k \; \Omega_{n,\mu\nu}}$$

Intrinsic AHC

Karplus + Luttinger 1954

$$\sigma_{\mu\nu} = \frac{e^2\hbar}{(2\pi)^3} \int d^3k \sum_n^{occ} \sum_m^{empty} 2\,\mathrm{Im} \frac{\langle u_{nk} | v_\mu | u_{mk}\rangle \langle u_{mk} | v_\nu | u_{nk}\rangle}{(E_{nk} - E_{mk})^2}$$

Modern view:

$$\sigma_{\nu\mu} = \frac{e^2}{\hbar} \frac{1}{(2\pi)^3} \sum_n \int d^3k\, f_{nk}\, \Omega_{n,\mu\nu}$$

occupation, 0 or 1

"Intrinsic" only.

Effect on single Bloch state (or wave packet)

In absence of $\mathcal{E}$:

$$\langle v_\mu \rangle_{nk} = \frac{1}{\hbar} \partial_\mu E_{nk}$$

also $\partial_{\mathcal{E}_\nu} \langle v_\mu \rangle_{nk} = \frac{e}{\hbar} \Omega_{nk,\mu\nu}$

$$\therefore \quad \boxed{\langle v_\mu \rangle_{nk} = \frac{1}{\hbar} \partial_\mu E_{nk} + \frac{e}{\hbar} \Omega_{nk,\mu\nu} \mathcal{E}_\nu}$$

↑ "anomalous velocity" term

See sec. 5.1.7 in book.

Revised Boltzmann transport equation:

$$\boxed{\begin{aligned} \dot{\underline{r}} &= \underline{v}_g - \dot{\underline{k}} \times \underline{\Omega} \\ \dot{\underline{k}} &= -\frac{e}{\hbar} \underline{\mathcal{E}} - \frac{e}{\hbar c} \dot{\underline{r}} \times \underline{B} \end{aligned}} \quad (5.11)$$

Make it look more symmetric:

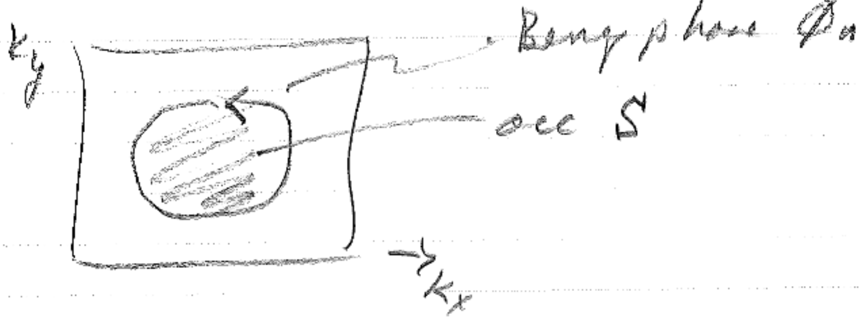
$$\boxed{\begin{aligned} \dot{\underline{r}} &= \frac{1}{\hbar}\underline{\nabla}_{\underline{k}} E_{n\underline{k}} - \dot{\underline{k}} \times \underline{\Omega}_{\underline{k}} \\ \dot{\underline{k}} &= -\frac{1}{\hbar}\underline{\nabla}_{\underline{r}} U(\underline{r}) - \frac{e}{\hbar c}\dot{\underline{r}} \times \underline{B}(\underline{r}) \end{aligned}}$$

$$\boxed{\underline{B} = \underline{\nabla}_{\underline{r}} \times \underline{A}^{EM}} \qquad \boxed{\underline{\Omega} = \underline{\nabla}_{\underline{k}} \times \underline{A}^{Berry}_{n\underline{k}}}$$

See Chiang + Niu, JPCM 20, 193202 (2008)

Xiao, Chang & Niu, RMP 82, 1959 (2010)

__Metal__



$$\sigma_{AHC}^{(n)} = \frac{e^2}{h}\frac{1}{2\pi}\int d^2k\, f_n(\underline{k})\, \Omega_{n\underline{k}}$$

occ, $T=0$, $\Theta(E_f - E_{nk})$

$$= \frac{e^2}{h}\frac{1}{2\pi}\int_S d^2k\, \Omega_{nk} \quad (*)$$

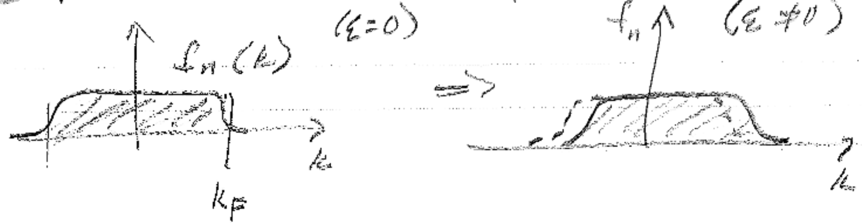
Stokes

$$= \frac{e}{h}\frac{\Phi_n}{2\pi}$$

But why is $\sigma_{xx} = \sigma_{yy} = 0$? Shouldn't it be zero

Big picture: $\mathcal{E}$ does two things:

① $\underline{\mathcal{E}}$ perturbs occupations of states



Described by "Boltzmann Eq" (BE)

S_F is displaced by $\mathcal{E}$ by amount $\propto$ $1/\tau$ ← relax. time

② $\underline{\mathcal{E}}$ perturbs the states themselves.

For metals we were neglecting (1) (which only affects metals)

Contrib (2) is present for both metals + insulators.

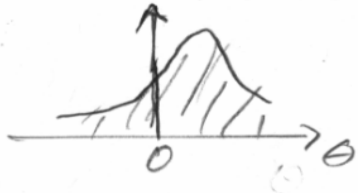
Combine

$\sigma_{AHC} = \sigma_{yx}$ in real FM metal comes from two sources:

(I) Ω via Karplus-Luttinger term

(II) Magnetic impurity scattering

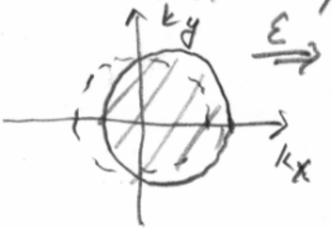
Wavepacket 



"skew scatt."

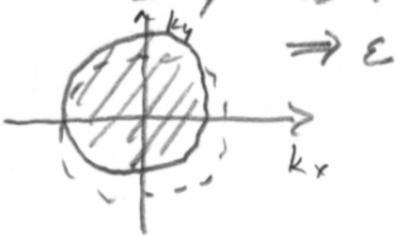
also "side-jump": 

Effect on Fermi surface in $\mathcal{E}$-field:



Normal scatt

$\Rightarrow \sigma_{xx}$



Skew scatt

$\Rightarrow \sigma_{yx}$

σ_{xx}, σ_{yx}

PROP. TO $\frac{1}{n_i}$

DOMINATES IN CLEAN SAMPLE, LOW T