

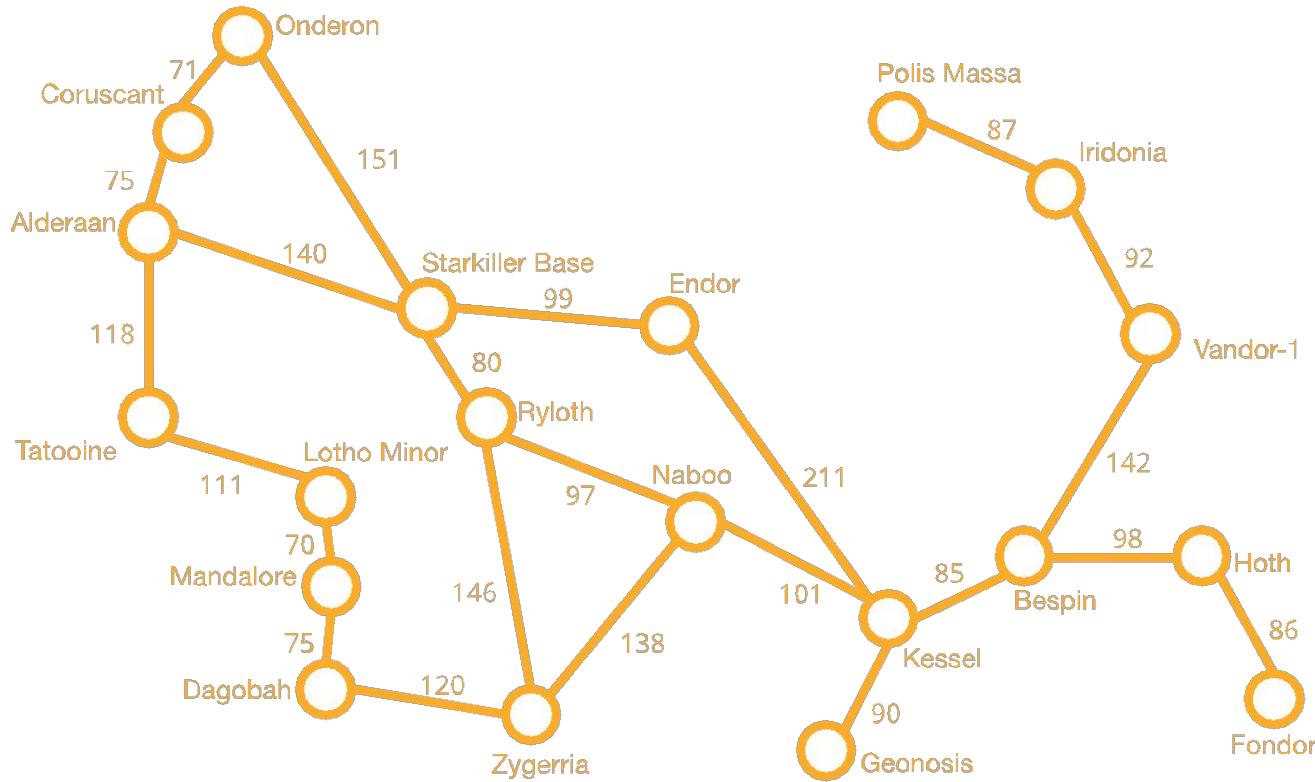
“Uniform Cost” Search

“In computer science, *uniform-cost* search (UCS) is a tree search algorithm used for traversing or searching a *weighted* tree, tree structure, or graph.” - Wikipedia

Motivation: Map Navigation Problems

All our search methods so far
assume *step-cost* = 1

This is only true for some problems



$g(N)$: the path cost function

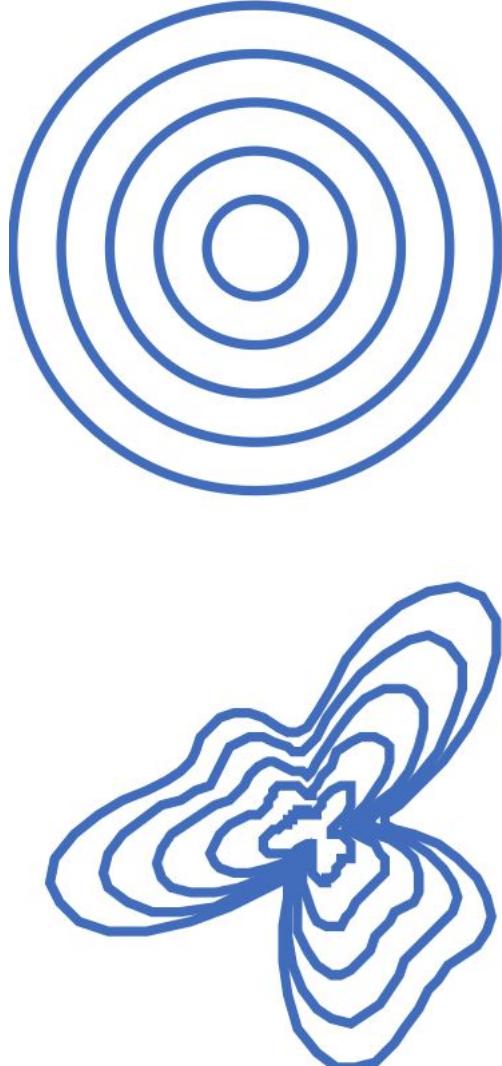
- Our assumption so far: All moves equal in cost
 - Cost = # of nodes in path-1
 - $g(N) = \text{depth}(N)$ in the search tree
- More general: Assigning a (potentially) unique cost to each step
 - N_0, N_1, N_2, N_3 = nodes visited on path p from N_0 to N_3
 - $C(i,j)$: Cost of going from N_i to N_j
 - If N_0 the root of the search tree,
$$g(N_3) = C(0,1) + C(1,2) + C(2,3)$$

Uniform-cost search (UCS)

- Extension of BF-search:
 - **Expand node with *lowest path cost***
- Implementation:
 - *frontier* = priority queue ordered by $g(n)$
- Subtle but significant difference from BFS:
 - Tests if a node is a goal state when it is selected for expansion, **not when it is added to the frontier.**
 - Updates a node on the frontier if a better path to the same state is found.
 - So always enqueues a node *before checking whether it is a goal.*

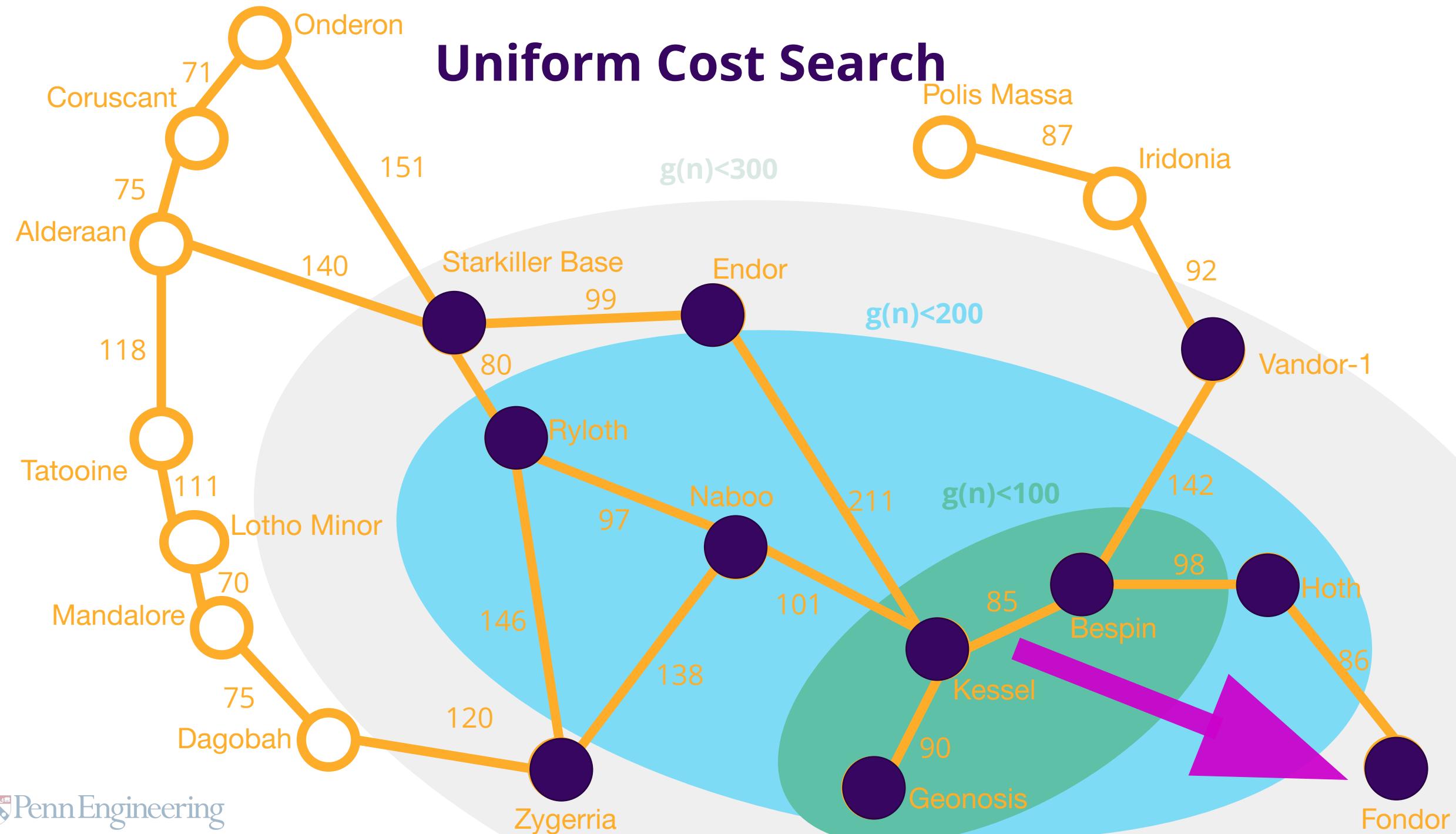
WHY???

Shape of Search

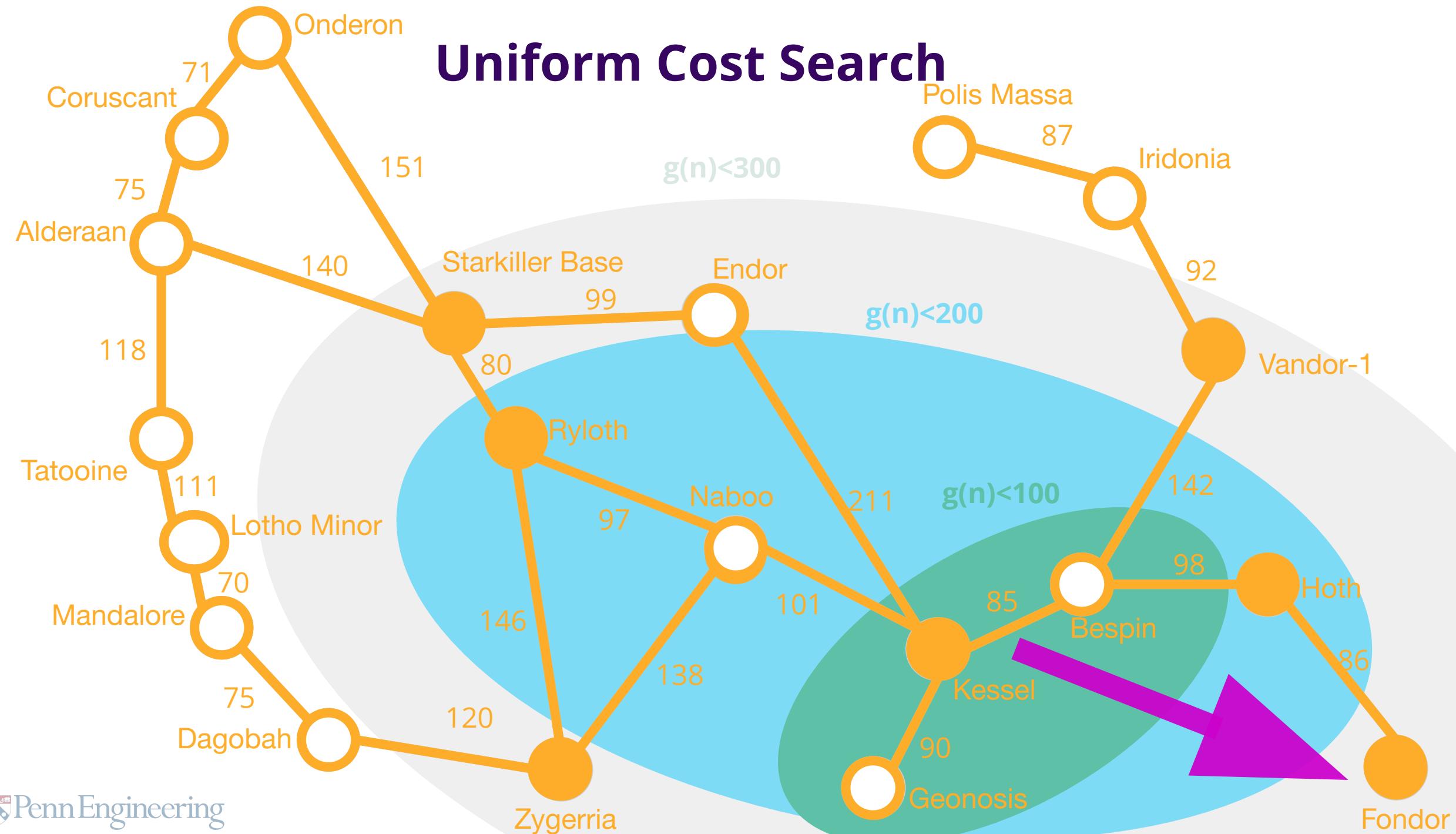


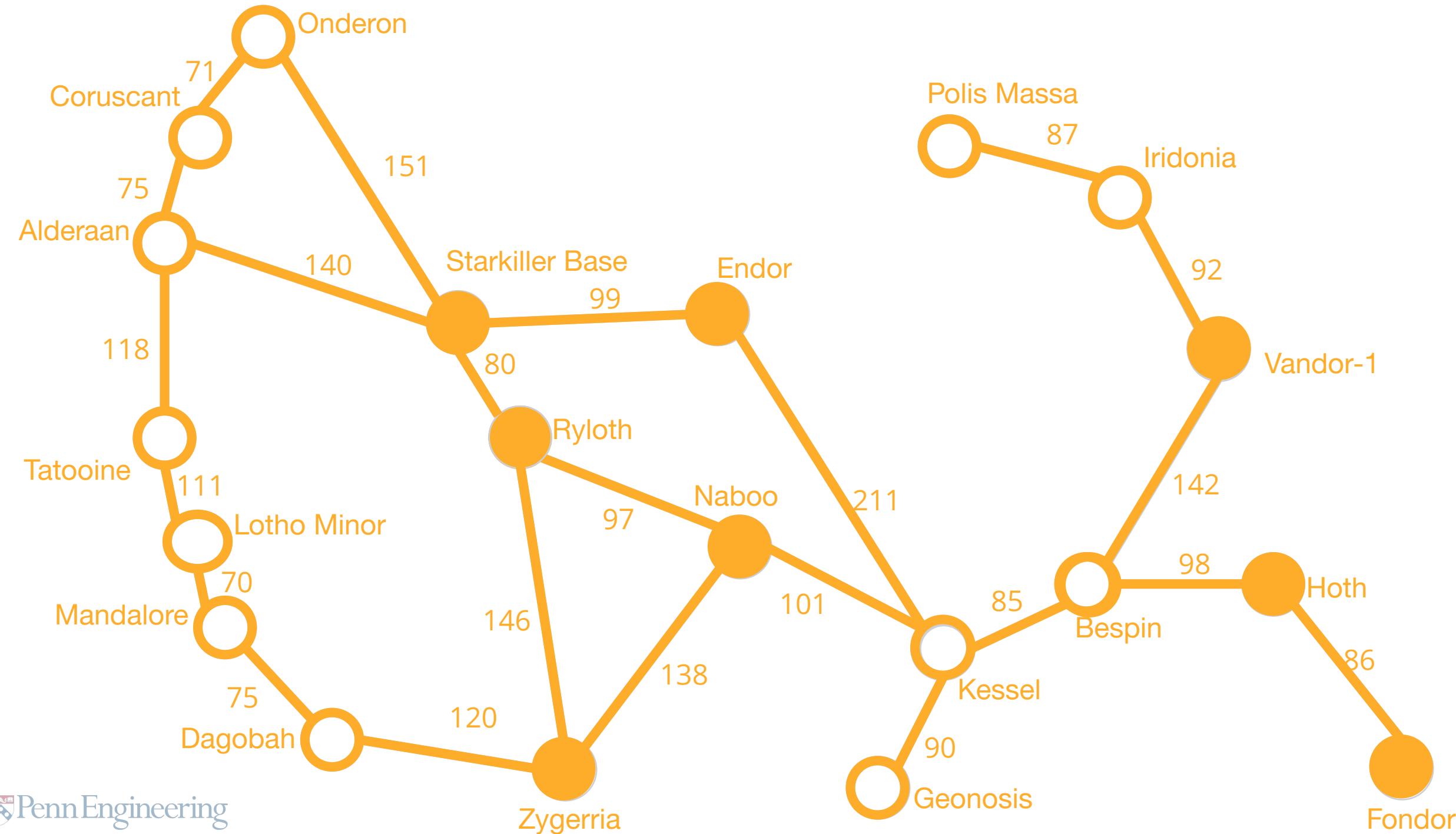
- **Breadth First Search** explores equally in all directions. Its frontier is implemented as a FIFO queue. This results in smooth contours or “plys”.
- **Uniform Cost Search** lets us prioritize which paths to explore. Instead of exploring all possible paths equally, it favors lower cost paths. Its frontier is a priority queue. This results in “cost contours”.

Uniform Cost Search

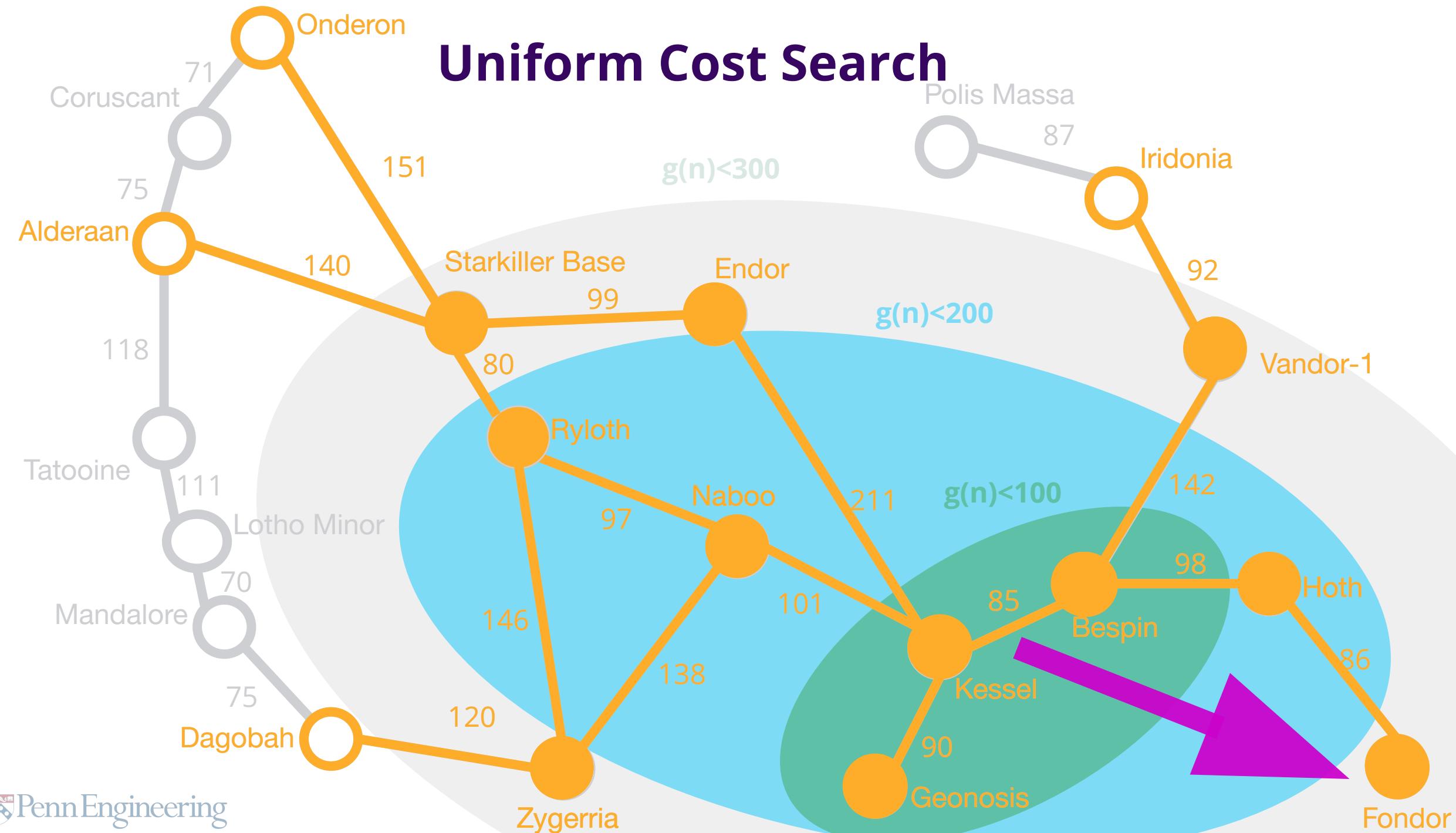


Uniform Cost Search





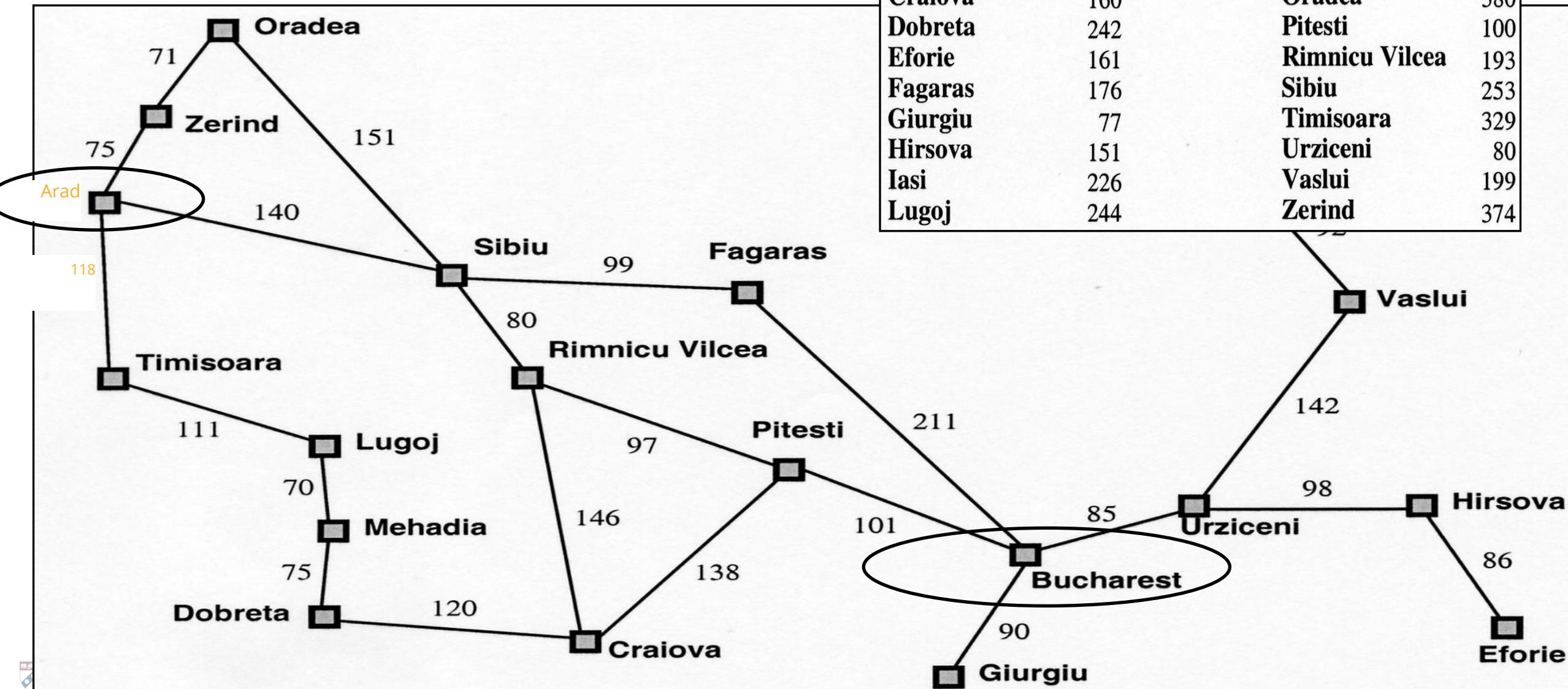
Uniform Cost Search



A Better Idea...

- Node expansion based on *an estimate* which *includes distance to the goal*
- General approach of informed search:
 - *Best-first search*: node selected for expansion based on an *evaluation function $f(n)$*
✓ $f(n)$ includes *estimate* of distance to goal (*new idea!*)
- Implementation: Sort frontier queue by this new $f(n)$.
 - Special cases: **greedy search**, and *A* search*

Simple, useful estimate heuristic: straight-line distances



Heuristic (estimate) functions



Heureka! ---Archimedes

[dictionary] “**A rule of thumb, simplification, or educated guess that reduces or limits the search for solutions in domains that are difficult and poorly understood.**”

Heuristic knowledge is useful, but not necessarily correct.

Heuristic algorithms use heuristic knowledge to solve a problem.

A heuristic function $h(n)$ takes a state n and returns an *estimate* of the distance from n to the goal.

Greedy Best-First Search

First attempt at integrating heuristic knowledge

Review: Best-first search

Basic idea:

select node for expansion with minimal *evaluation function $f(n)$*

- where $f(n)$ is some function that includes *estimate heuristic $h(n)$* of the remaining distance to goal

Implement using priority queue

Exactly UCS with $f(n)$ replacing $g(n)$

Greedy best-first search: $f(n) = h(n)$

Expands the node that *is estimated* to be closest to goal

Completely ignores $g(n)$: the cost to get to n

In our Romanian map, $h(n) = h_{SLD}(n)$ = straight-line distance from n to Bucharest

In a grid, the heuristic distance can be calculated using the “Manhattan distance”:

```
def heuristic(a, b):
    # Manhattan distance on a square grid
    return abs(a.x - b.x) + abs(a.y - b.y)
```

Greedy best-first search

```
frontier = PriorityQueue()
frontier.put(start, 0)
came_from = {}
came_from[start] = None

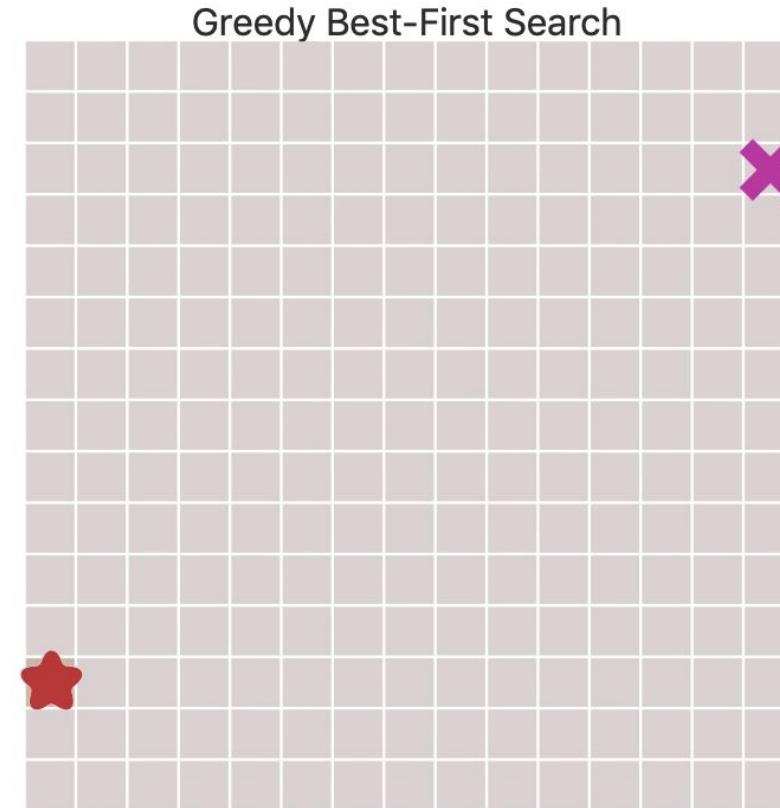
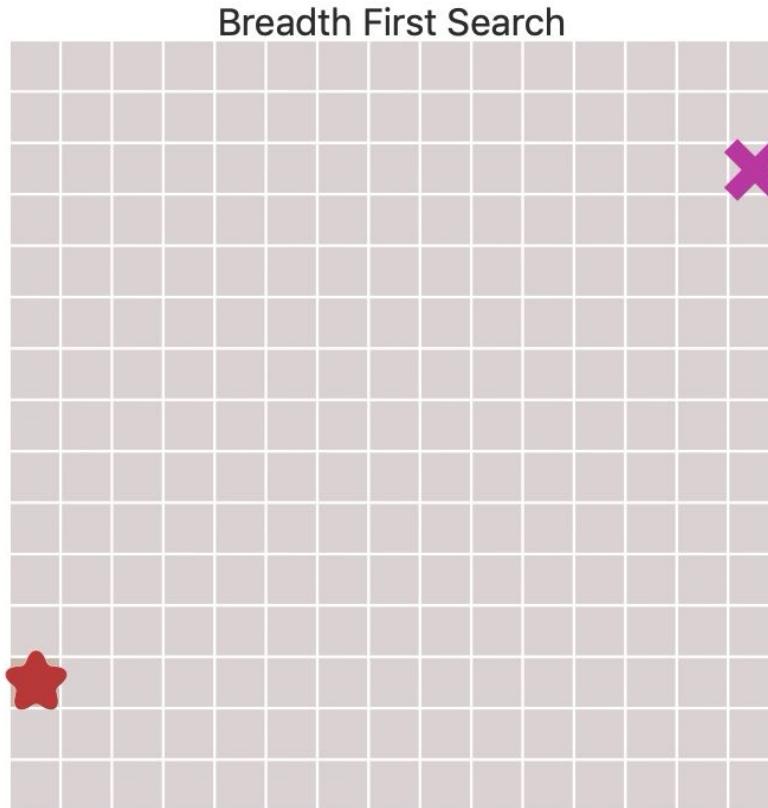
while not frontier.empty():
    current = frontier.get()

    if current == goal:
        break

    for next in graph.neighbors(current):
        if next not in came_from:
            priority = heuristic(goal, next)
            frontier.put(next, priority)
            came_from[next] = current
```

Code from Amit Patel
of Red Blob Games

BFS v. Greedy Best-First Search



<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Greedy best-first search example

Frontier queue:

Arad 366



- **Initial State = Arad**
- **Goal State = Bucharest**

Arad	366	Mehadia	241
Bucharest	0	Neamt	234
Craiova	160	Oradea	380
Dobreta	242	Pitesti	100
Eforie	161	Rimnicu Vilcea	193
Fagaras	176	Sibiu	253
Giurgiu	77	Timisoara	329
Hirsova	151	Urziceni	80
Iasi	226	Vaslui	199
Lugoj	244	Zerind	374

Greedy best-first search example

Frontier queue

Arad 366

	Planet	SLD	Planet	SLD
	Alderaan	366	Lotho Minor	244
	Bespin	80	Mandalore	241
	Coruscant	374	Naboo	100
	Dagobah	242	Onderon	380
	Endor	176	Polis Massa	234
	Fondor	161	Ryloth	193
	Geonosis	77	Starkiller Base	253
	Hoth	151	Tatooine	329
• Initial	Iridonia	226	Vandor-1	199
• Goal	Kessel	0	Zygerria	160

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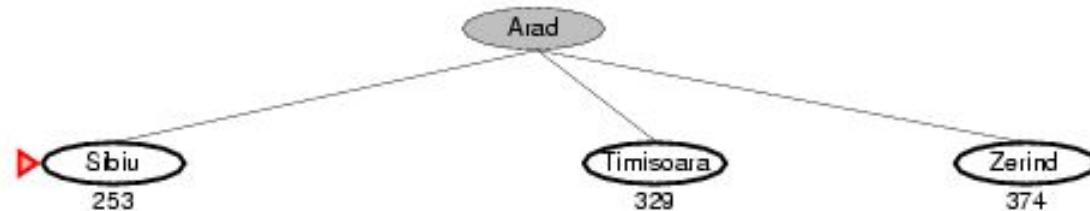
Greedy best-first search example

Frontier queue:

Sibiu 253

Timisoara 329

Zerind 374



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Greedy best-first search example

Frontier queue:

Fagaras 176

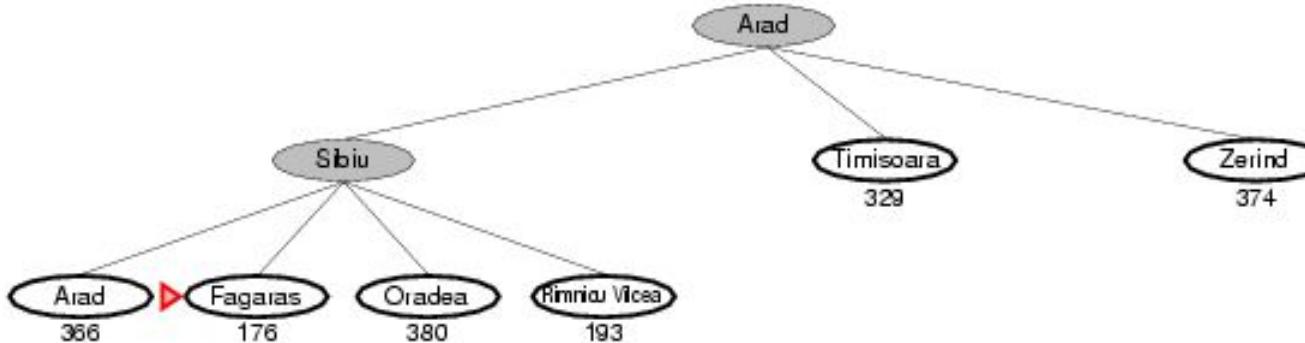
Rimnicu Vilcea
193

Timisoara 329

Arad 366

Zerind 374

Oradea 380



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Greedy best-first search example

Frontier queue:

Bucharest 0

Rimnicu Vilcea
193

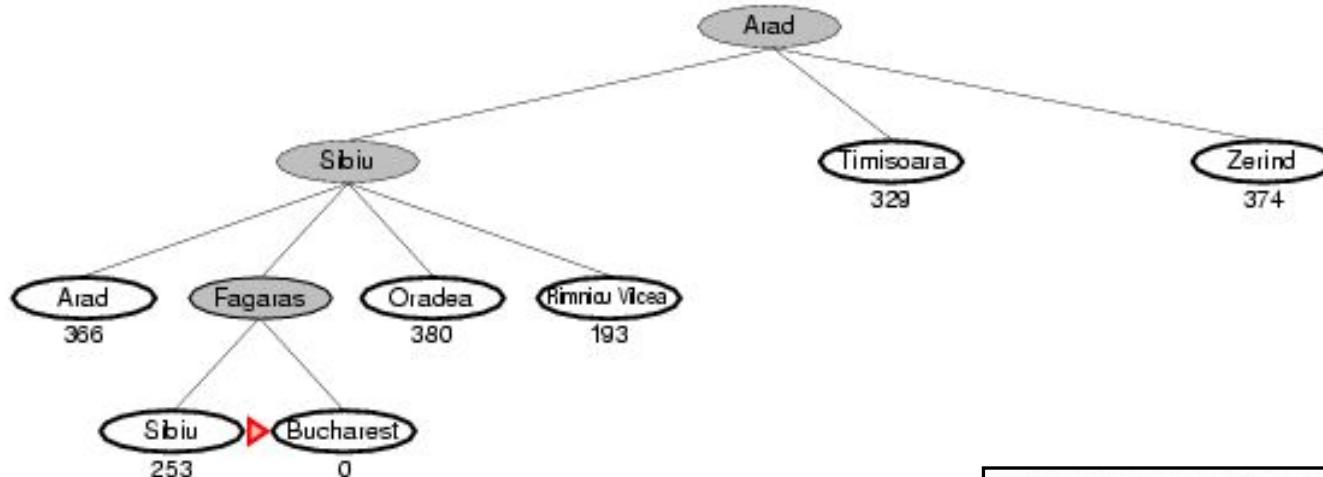
Sibiu 253

Timisoara 329

Arad 366

Zerind 374

Oradea 380



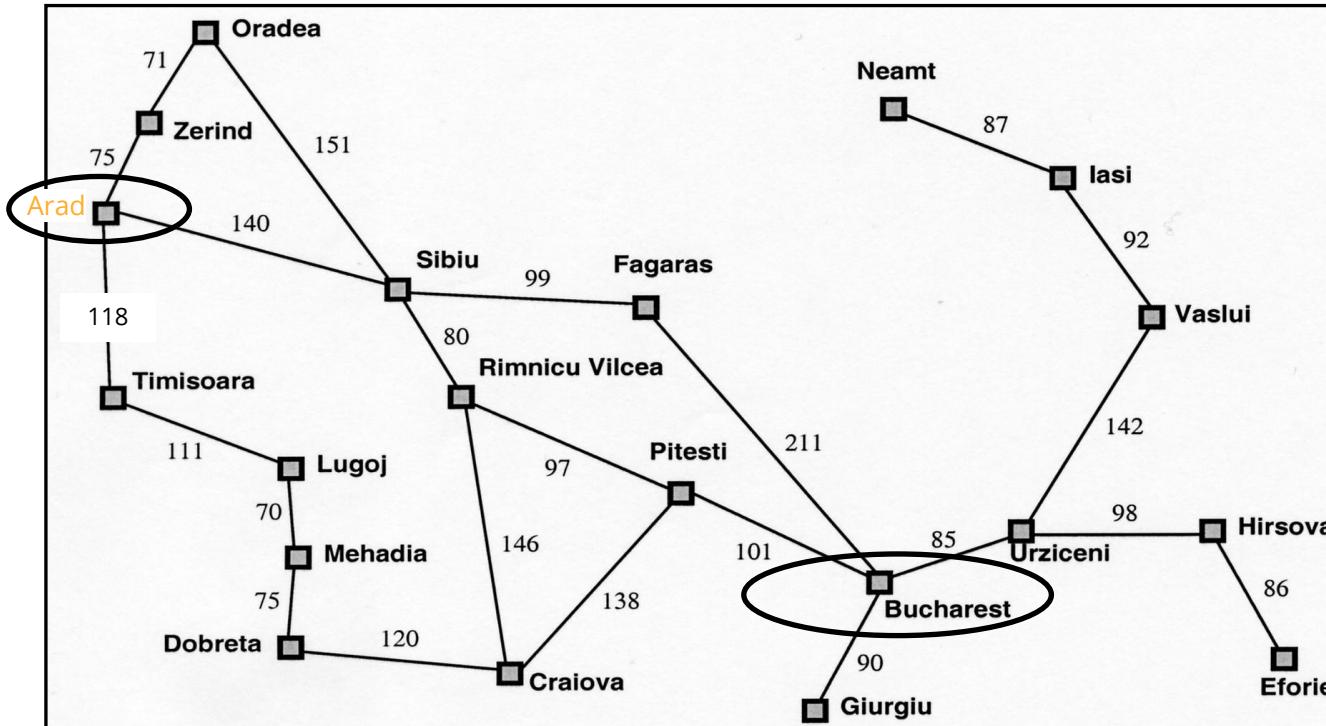
Goal reached !!

Arad	366	Mehadia	241
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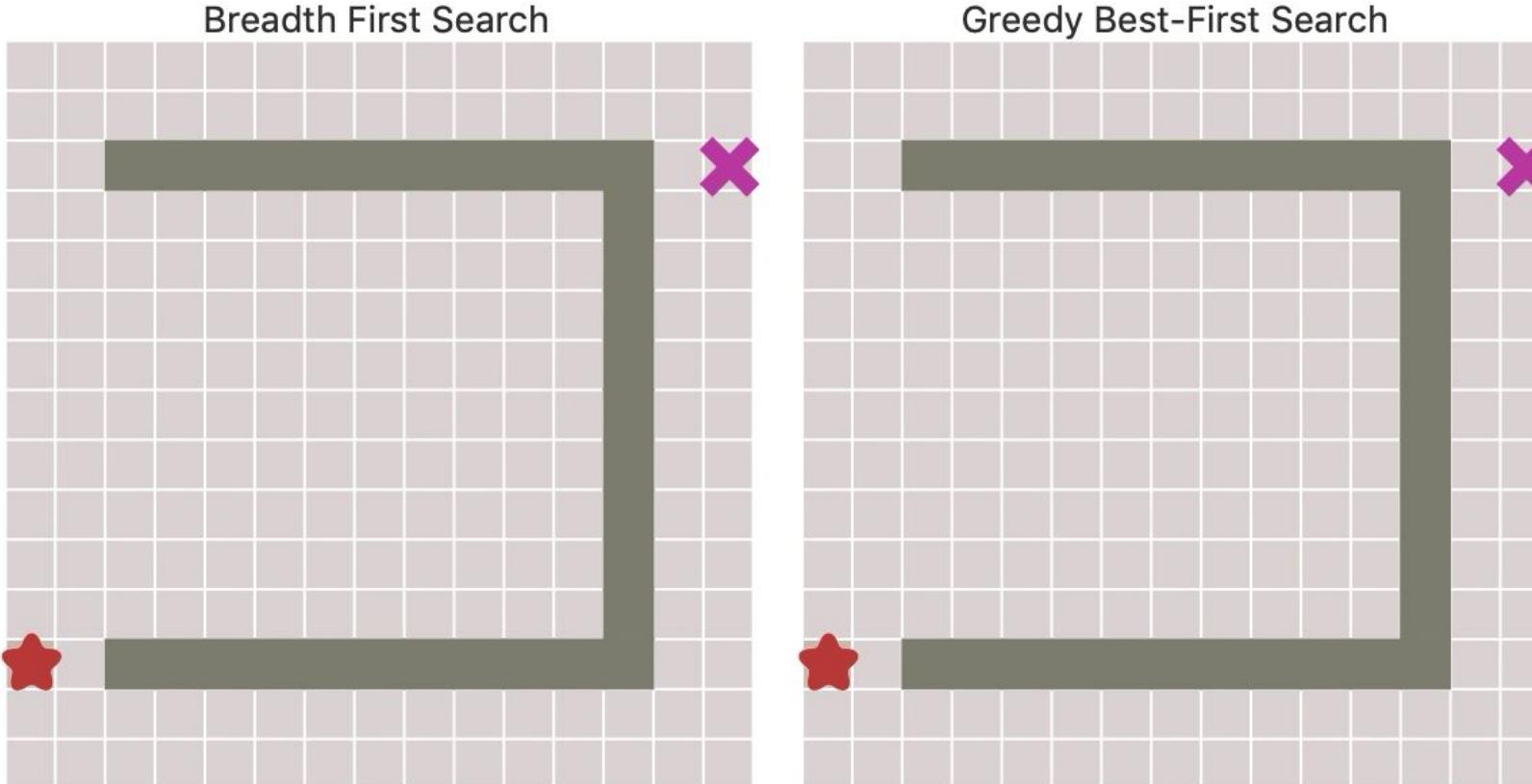
Properties of greedy best-first search

Optimal?

- No!
 - Found: Arad □ Sibiu □ Fagaras □ Bucharest (450km)
 - Shorter: Arad □ Sibiu □ Rimnicu Vilcea □ Pitesti □ Bucharest (418km)



BFS v. Greedy Best-First Search

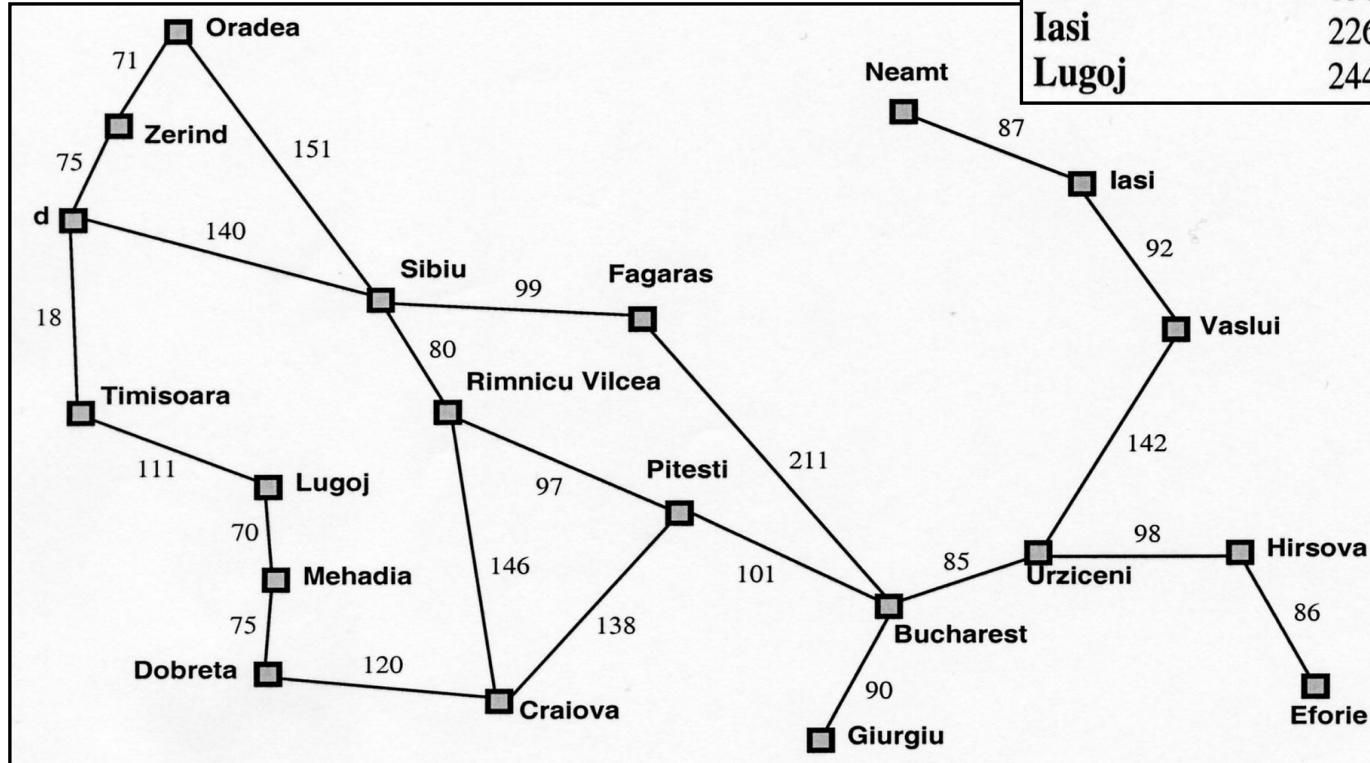


<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Properties of greedy best-first search

Complete?

- No – can get stuck in loops,
- e.g., Iasi \square Neamt \square Iasi \square Neamt \square ...



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A* search

AIMA 3.5



A* search

Best-known form of best-first search.

Key Idea: avoid expanding paths that are already expensive, but expand most promising first.

Simple idea: $f(n) = g(n) + h(n)$

- $g(n)$ the actual cost (so far) to *reach* the node
- $h(n)$ estimated cost to *get from the node to the goal*
- $f(n)$ estimated *total cost* of path through n to goal

Implementation: Frontier queue as priority queue by increasing $f(n)$ (*as expected...*)

Key concept: Admissible heuristics

A heuristic $h(n)$ is *admissible* if it *never overestimates* the cost to reach the goal; i.e. it is *optimistic*

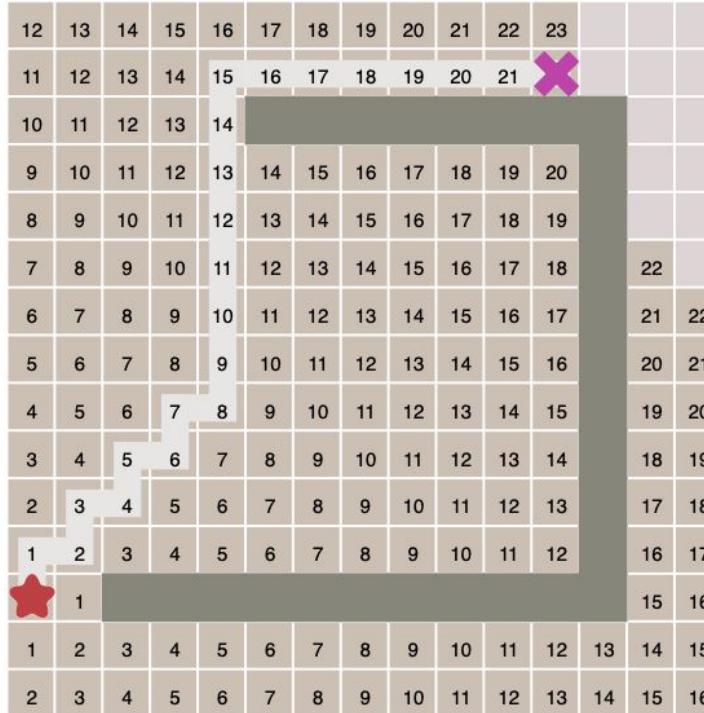
- Formally: $\forall n$, n a node:
 - $h(n) \leq h^*(n)$ where $h^*(n)$ is the true cost from n
 - $h(n) \geq 0$ so $h(G)=0$ for any goal G .

Example: $h_{SLD}(n)$ never overestimates the actual road distance

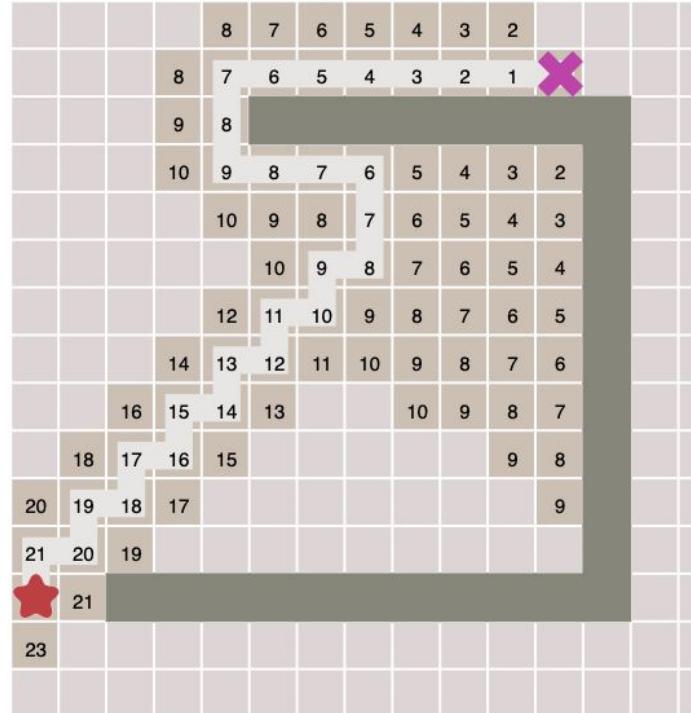
Theorem: If $h(n)$ is *admissible*, A* using Tree Search is *optimal*

A* is optimal with admissible heuristic

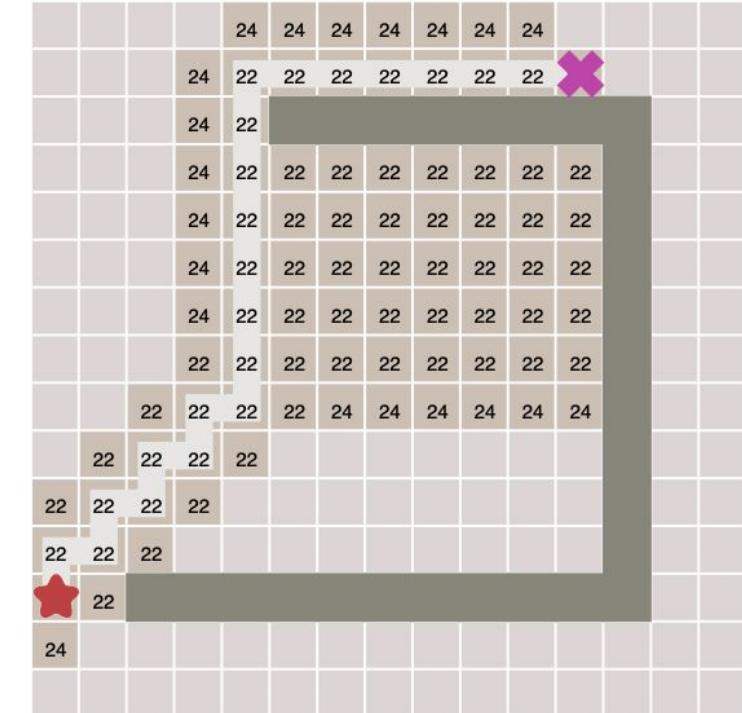
Dijkstra's



Greedy Best-First

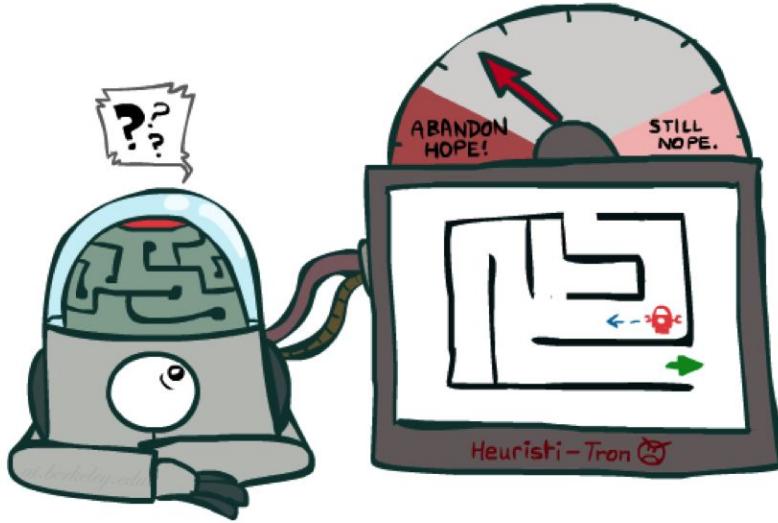


A* Search

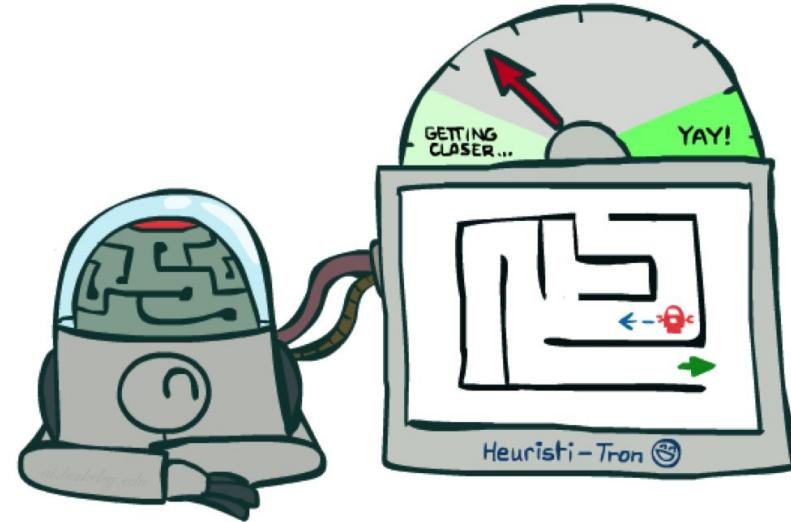


<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Idea: Admissibility



Inadmissible
(pessimistic) heuristics
break optimality by
trapping good plans on
the frontier

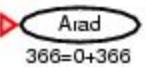


Admissible (optimistic)
heuristics slow down
bad plans but never
outweigh true costs

A* search example

Frontier queue:

Arad 366



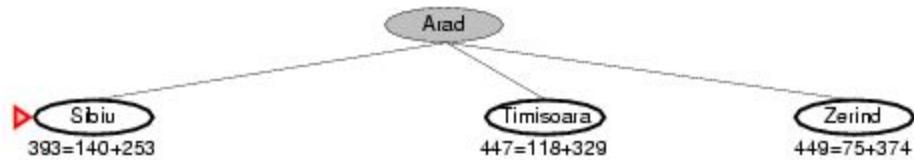
A* search example

Frontier queue:

Sibiu 393

Timisoara 447

Zerind 449



We add the three nodes we found to the Frontier queue.

We sort them according to the **$g() + h()$** calculation.

A* search example

Frontier queue:

Rimnicu Vilcea

413

Fagaras 415

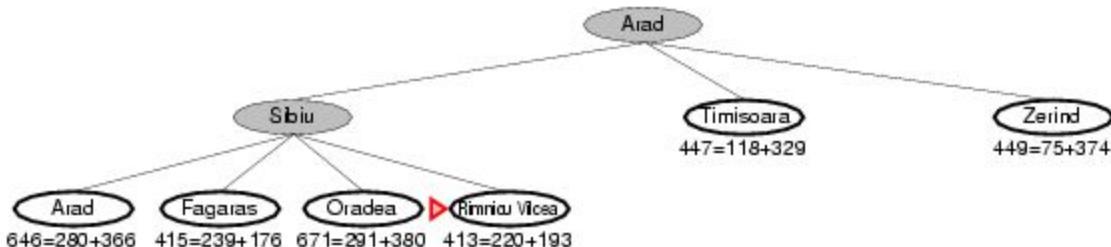
Timisoara 447

Zerind 449

Arad 646



Oradea 671



When we expand Sibiu, we run into Arad again. Note that we've already expanded this node once; but we still add it to the Frontier queue again.

A* search example

Frontier queue:

Fagaras 415

Pitesti 417

Timisoara 447

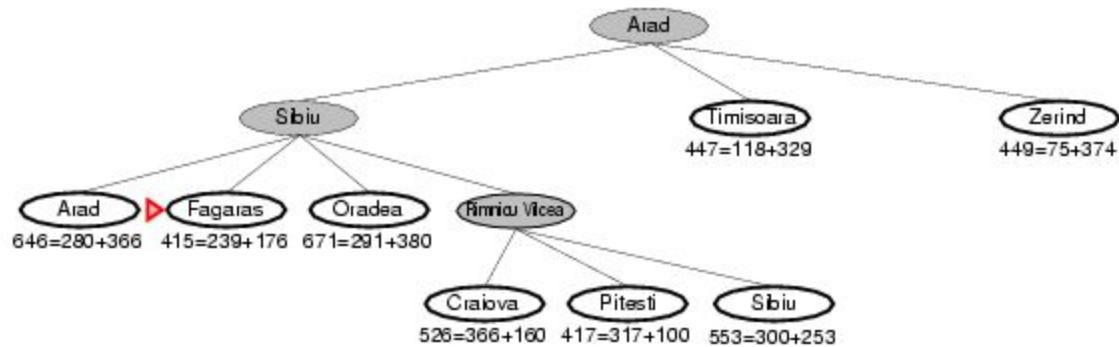
Zerind 449

Craiova 526

Sibiu 553

Arad 646

Oradea 671



We expand Rimnicu Vicea.

A* search example

Frontier queue:

Pitesti 417

Timisoara 447

Zerind 449

Bucharest 450

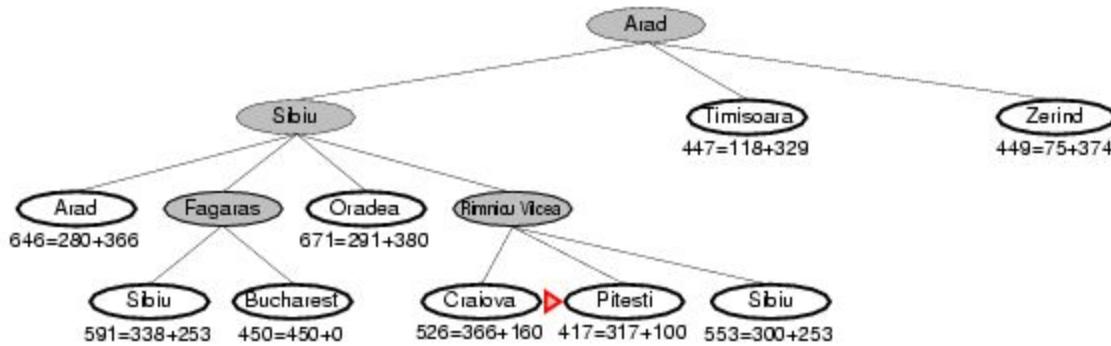
Craiova 526

Sibiu 553

Sibiu 591

Arad 646

Oradea 671



When we expand Fagaras, we find Bucharest, but we're not done. The algorithm doesn't end until we "expand" the goal node – it has to be at the top of the Frontier queue.

A* search example

Frontier queue:

Bucharest 418

Timisoara 447

Zerind 449

Bucharest 450

Craiova 526

Sibiu 553

Sibiu 591

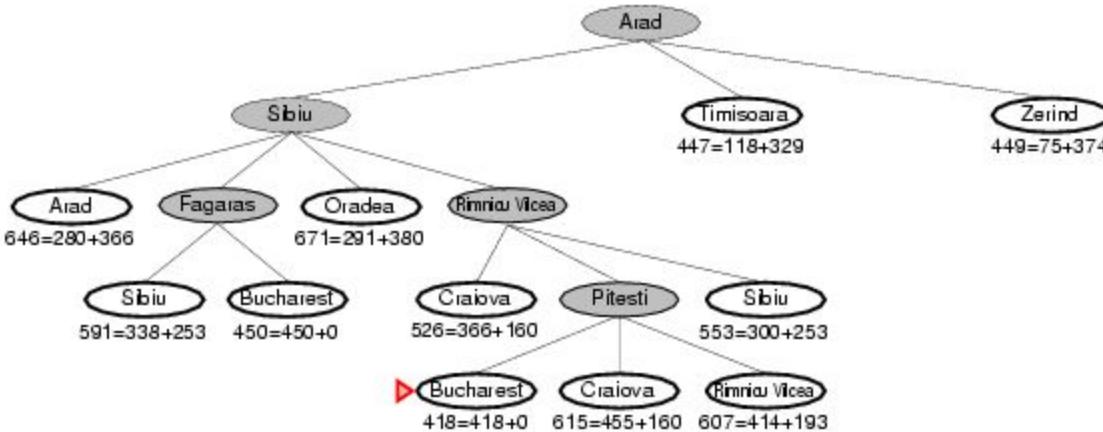
Rimnicu Vicea

607

Craiova 615

Arad 646

Oradea 671



Note that we just found a better value for Bucharest!

Now we expand this better value for Bucharest since it's at the top of the queue.

We're done and we know the value found is optimal!

Heuristic functions

For the 8-puzzle

- Avg. solution cost is about 22 steps
 - (branching factor ≤ 3)
 - (branching factor ≤ 3)
 - A good heuristic function can reduce the search process



Start State



Goal State

Example Admissible heuristics

For the 8-puzzle:

$h_{oop}(n)$ = number of out of place tiles

$h_{md}(n)$ = total Manhattan distance (i.e., # of moves from desired location of each tile)

$$h_{oop}(S) = 8$$

$$h_{md}(S) = 3+1+2+2+2+3+3+2 = 18$$



Start State



Goal State

Relaxed problems

A problem with fewer restrictions on the actions than the original is called a *relaxed problem*

The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem

If the rules of the 8-puzzle are relaxed so that a tile can move *anywhere*, then $h_{oop}(n)$ gives the shortest solution

If the rules are relaxed so that a tile can move to *any adjacent square*, then $h_{md}(n)$ gives the shortest solution

Defining Heuristics: $h(n)$

Cost of an exact solution to a *relaxed* problem (fewer restrictions on operator)

Constraints on *Full* Problem:

A tile can move from square A to square B *if A is adjacent to B and B is blank.*

- Constraints on *relaxed* problems:
 - A tile can move from square A to square B *if A is adjacent to B.* (h_{md})
 - A tile can move from square A to square B *if B is blank.*
 - A tile can move from square A to square B. (h_{oop})

Dominance: A metric on better heuristics

If $h_2(n) \geq h_1(n)$ for all n (both admissible)

- then h_2 *dominates* h_1

So h_2 is optimistic, but more accurate than h_1 ,

- h_2 is therefore better for search
- Notice: h_{md} dominates h_{oop}

Typical search costs (average number of nodes expanded):

$d=12$ Iterative Deepening Search = 3,644,035 nodes

$A^*(h_{oop}) = 227$ nodes, $A^*(h_{md}) = 73$ nodes

$d=24$ IDS = too many nodes

$A^*(h_{oop}) = 39,135$ nodes, $A^*(h_{md}) = 1,641$ nodes

The best and worst admissible heuristics

$h^*(n)$ - the (unachievable) Oracle heuristic

- $h^*(n)$ = the true distance from the n to goal

$$h_{\text{we're here already}}(n) = h_{\text{teleportation}}(n) = 0$$

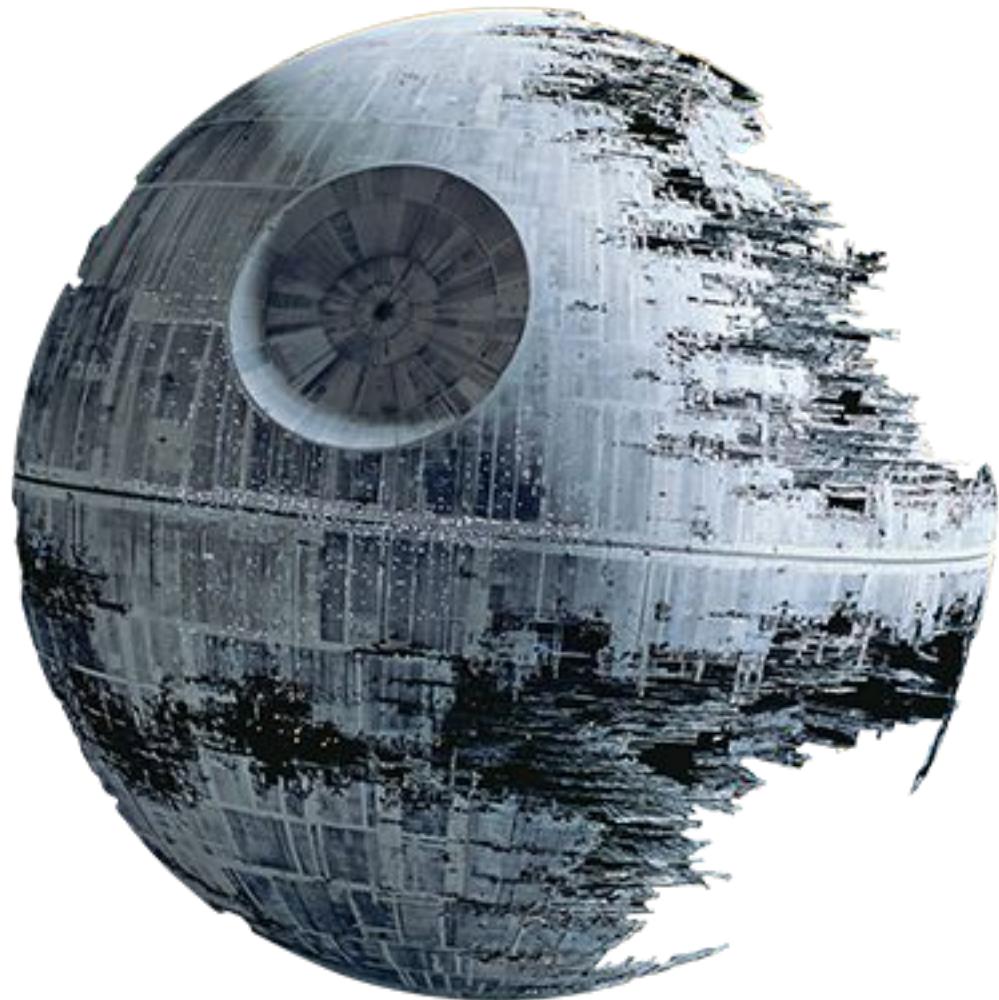
Admissible: both yes!!!

$h^*(n)$ *dominates all other heuristics*

$h_{\text{teleportation}}(n)$ *is dominated by all heuristics*

A* search is Optimal

AIMA 3.5



A* search

Best-known form of best-first search.

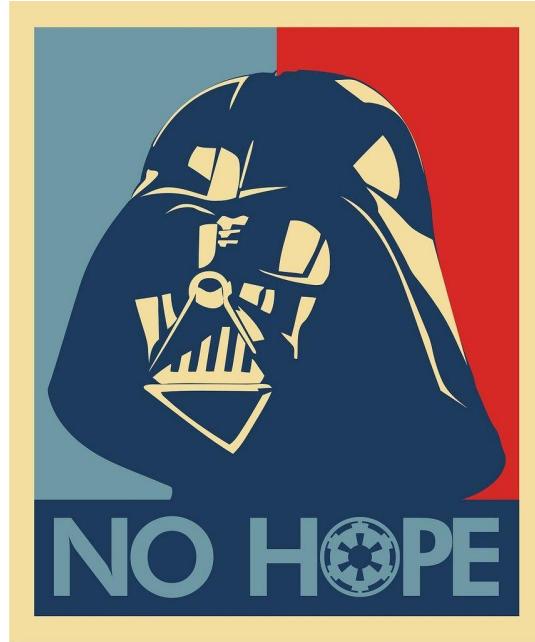
Key Idea: avoid expanding paths that are already expensive, but expand most promising first.

Simple idea: $f(n) = g(n) + h(n)$

- $g(n)$ the actual cost (so far) to *reach* the node
- $h(n)$ estimated cost to *get from the node to the goal*
- $f(n)$ estimated *total cost* of path through n to goal

Implementation: Frontier queue as priority queue by increasing $f(n)$ (*as expected...*)

Key: Admissibility



Inadmissible (pessimistic) heuristics break optimality by pushing good plans too far back on the frontier, which means they may never get expanded.



Admissible (optimistic) heuristics slow down bad plans but never outweigh true costs. That means that the true best plan will always be expanded.

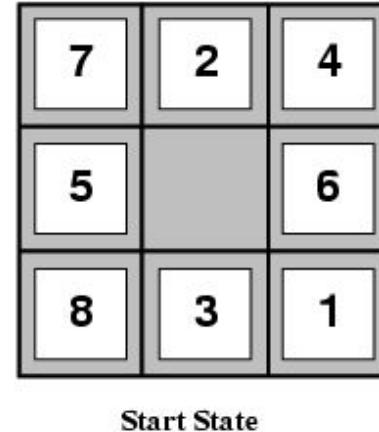
Admissible Heuristics

A heuristic h is *admissible* (optimistic) if:

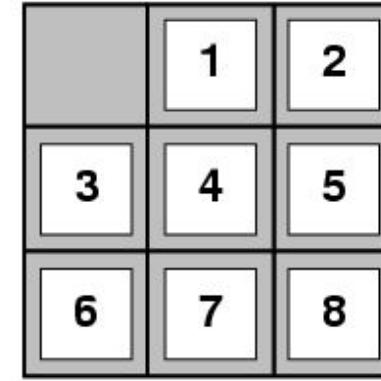
$$0 \leq h(n) \leq h^*(n)$$

where $h^*(n)$ is the true cost to a nearest goal

Is Manhattan Distance admissible?



Start State



Goal State

Coming up with admissible heuristics is most of what's involved in using A* in practice.

Optimality of A* Tree Search

Assume:

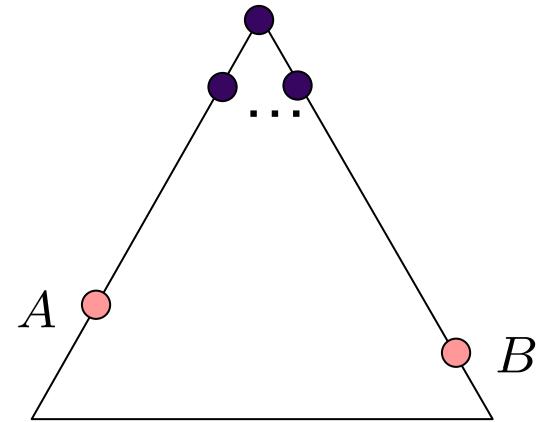
A is an optimal goal node

B is a suboptimal goal node

h is admissible

Claim:

A will exit the frontier before B



Slide credit: Dan Klein and Pieter
Abbeel
<http://ai.berkeley.edu>

Optimality of A* Tree Search

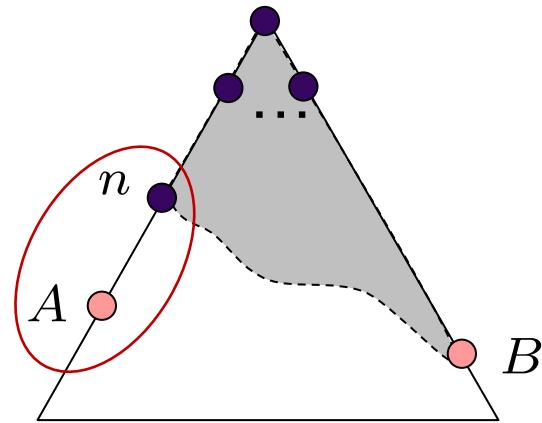
Proof:

Imagine B is on the frontier

Some ancestor n of A is on the frontier, too (maybe A!)

Claim: n will be expanded before B

- $f(n)$ is less or equal to $f(A)$



$$\begin{aligned}f(n) &= g(n) + h(n) \\f(n) &\leq g(A) \\g(A) &= f(A)\end{aligned}$$

Definition of f-cost
Admissibility of h
 $h = 0$ at a goal

Slide credit: Dan Klein and Pieter
Abbeel
<http://ai.berkeley.edu>

Optimality of A* Tree Search

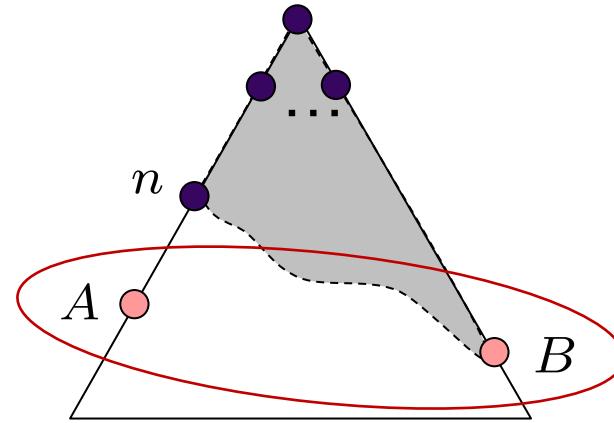
Proof:

Imagine B is on the frontier

Some ancestor n of A is on the frontier, too (maybe A!)

Claim: n will be expanded before B

- $f(n)$ is less or equal to $f(A)$
- $f(A)$ is less than $f(B)$



$$\begin{aligned} g(A) &< g(B) \\ f(A) &< f(B) \end{aligned}$$

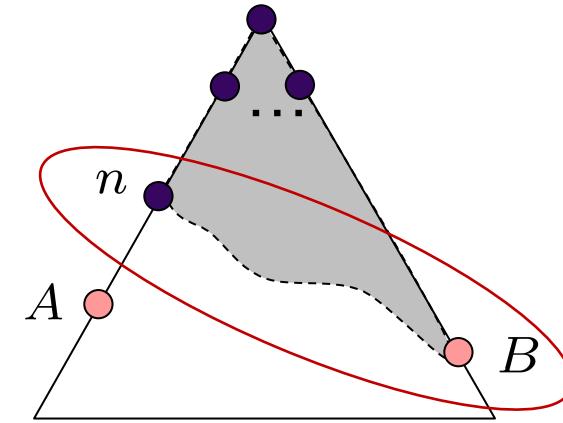
B is suboptimal
 $h = 0$ at a goal

Slide credit: Dan Klein and Pieter
Abbeel
<http://ai.berkeley.edu>

Optimality of A* Tree Search

Proof:

- Imagine B is on the frontier
- Some ancestor n of A is on the frontier, too (maybe A!)
- Claim: n will be expanded before B
 - $f(n)$ is less or equal to $f(A)$
 - $f(A)$ is less than $f(B)$
 - n expands before B
- All ancestors of A expand before B
- A expands before B
- A* search is optimal



$$f(n) \leq f(A) < f(B)$$

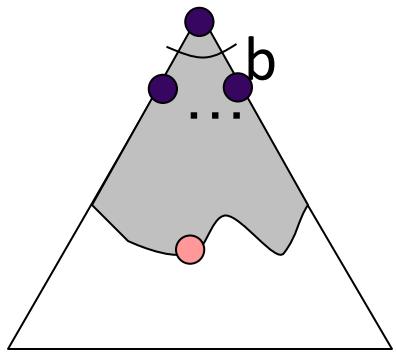
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Properties of A*

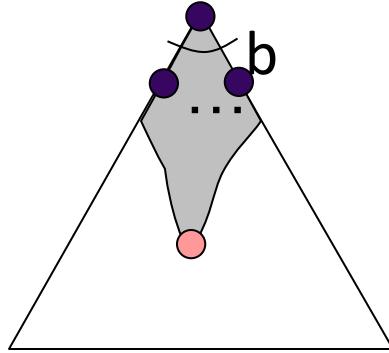
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Properties of A*

Uniform-Cost



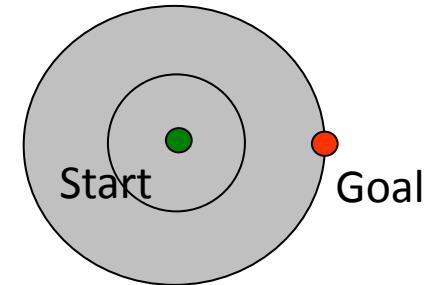
A*



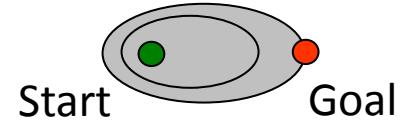
Slide credit: Dan Klein and Pieter
Abbeel
<http://ai.berkeley.edu>

UCS vs A* Contours

Uniform-cost expands equally in all “directions”



A* expands mainly toward the goal, but does hedge its bets to ensure optimality



Slide credit: Dan Klein and Pieter
Abbeel
<http://ai.berkeley.edu>

A* Applications

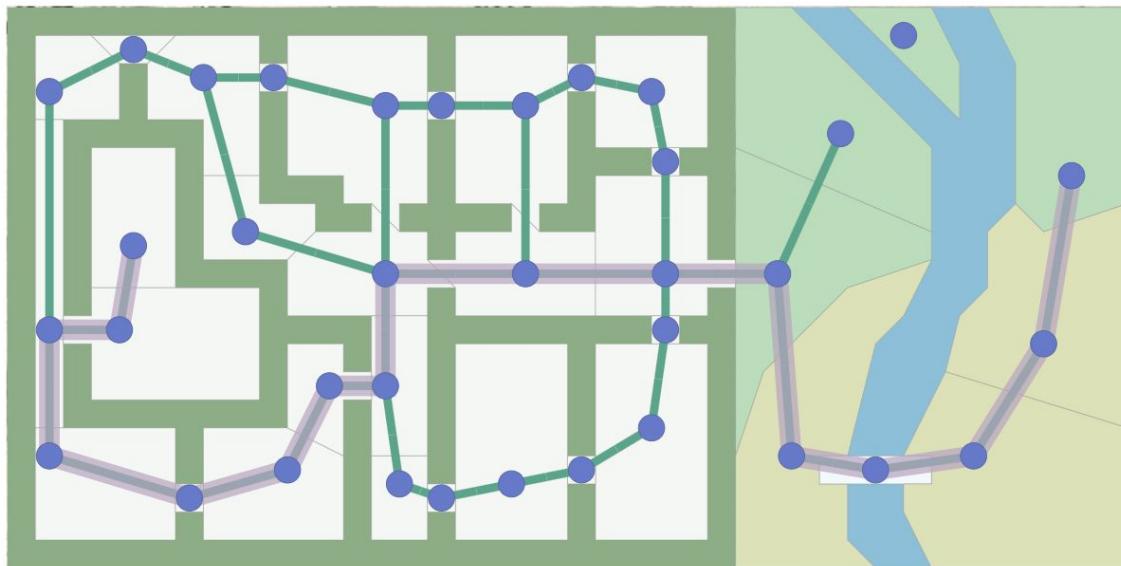
Pathing / routing problems (A* is in your GPS!)

Video games

Robot motion planning

Resource planning problems

...



Supplemental Reading

I recommend this A* tutorial by Amit Patel of Red Blob Games

<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Introduction to the A* Algorithm
from Red Blob Games

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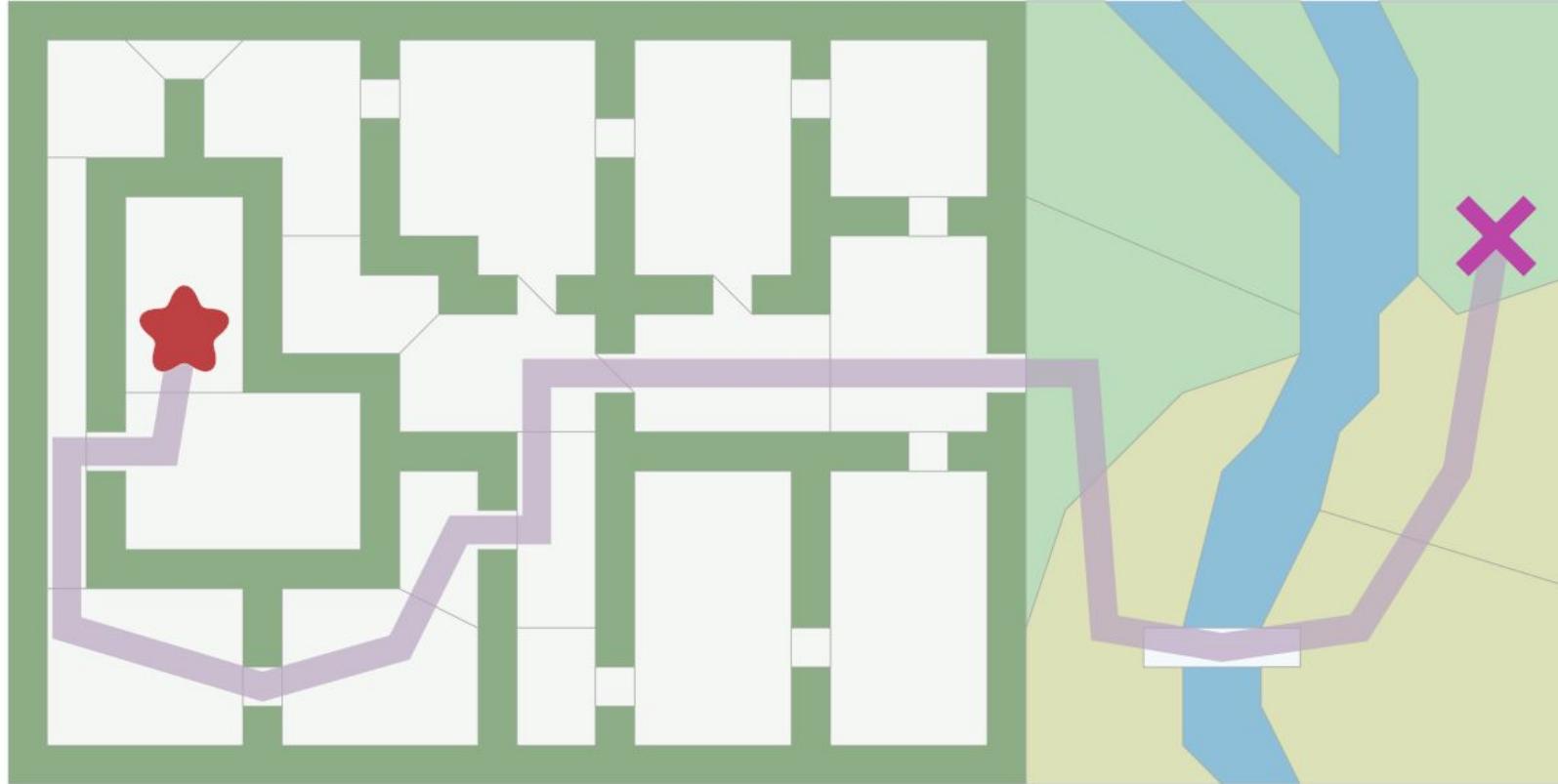
Created 26 May 2014, updated Aug 2014, Feb 2016, Jun 2016

In games we often want to find paths from one location to another. We're not only trying to find the shortest distance; we also want to take into account travel time. Move the blob  (start point) and cross  (end point) to see the shortest path.



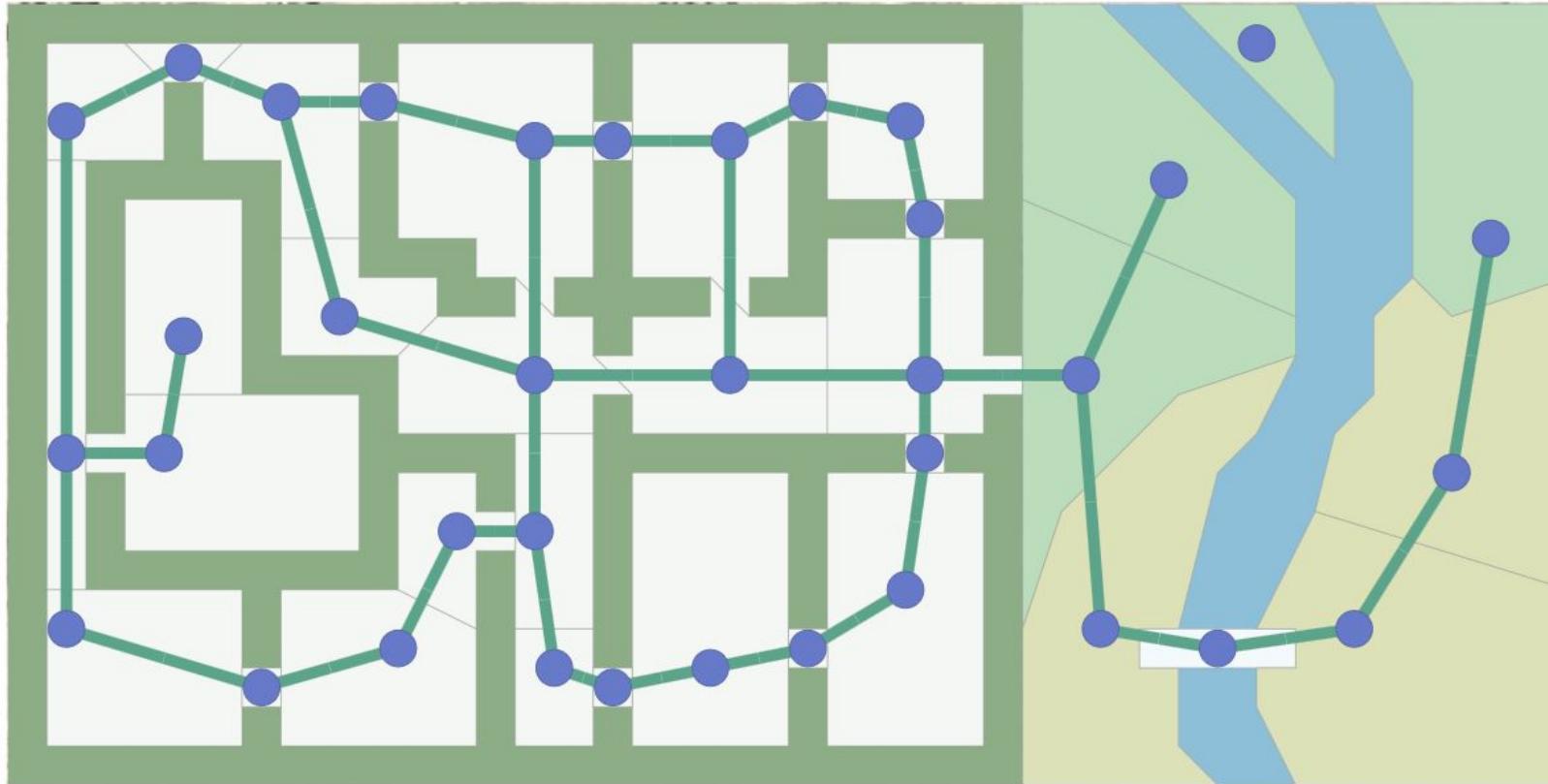
To find this path we can use a *graph search* algorithm, which works when the map is represented as a graph. **A*** is a popular choice for graph search. **Breadth First Search** is the simplest of the graph search algorithms, so let's start there, and we'll work our way up to A*.

Pathfinding in Games



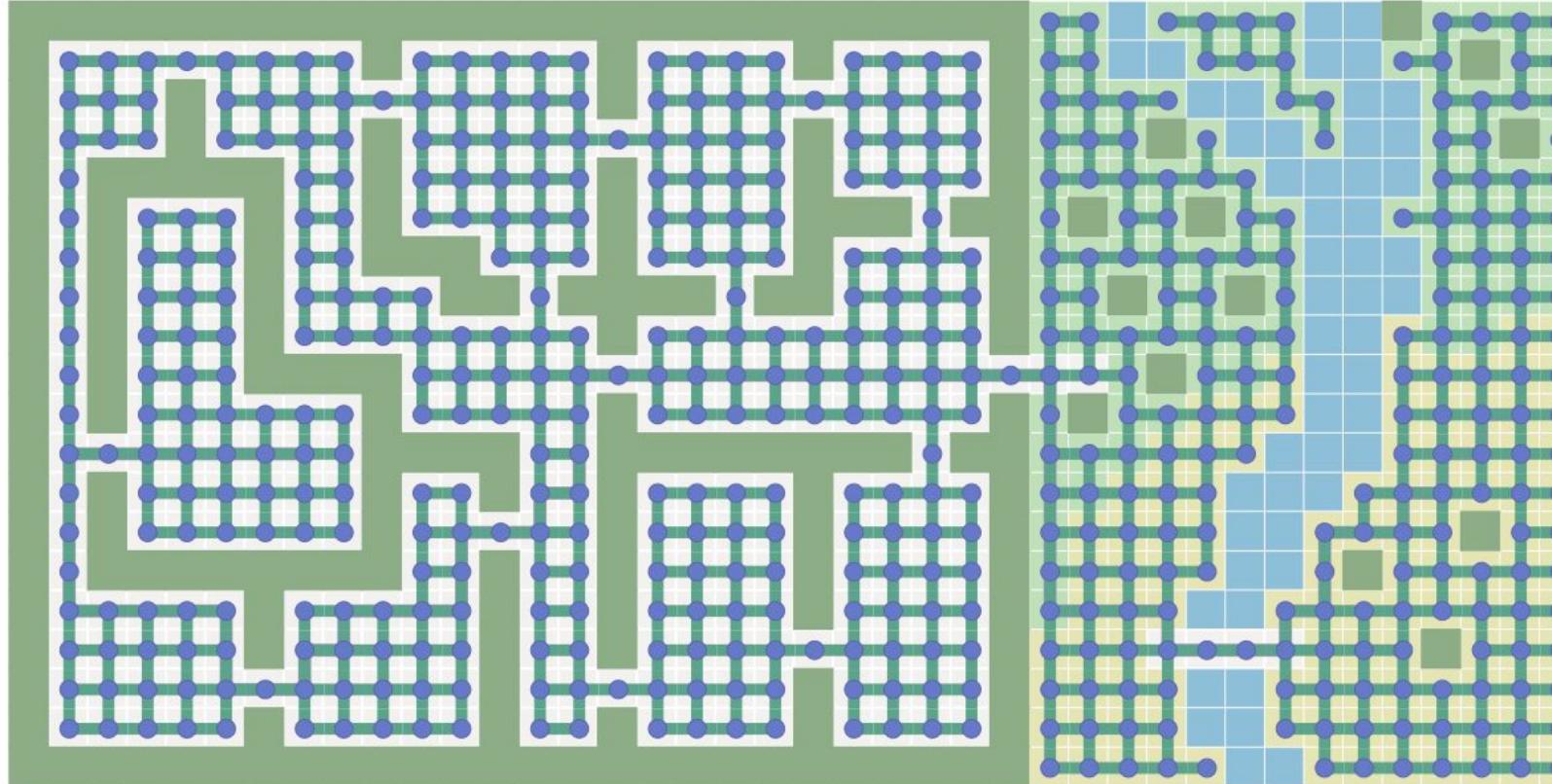
<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Pathfinding in Games



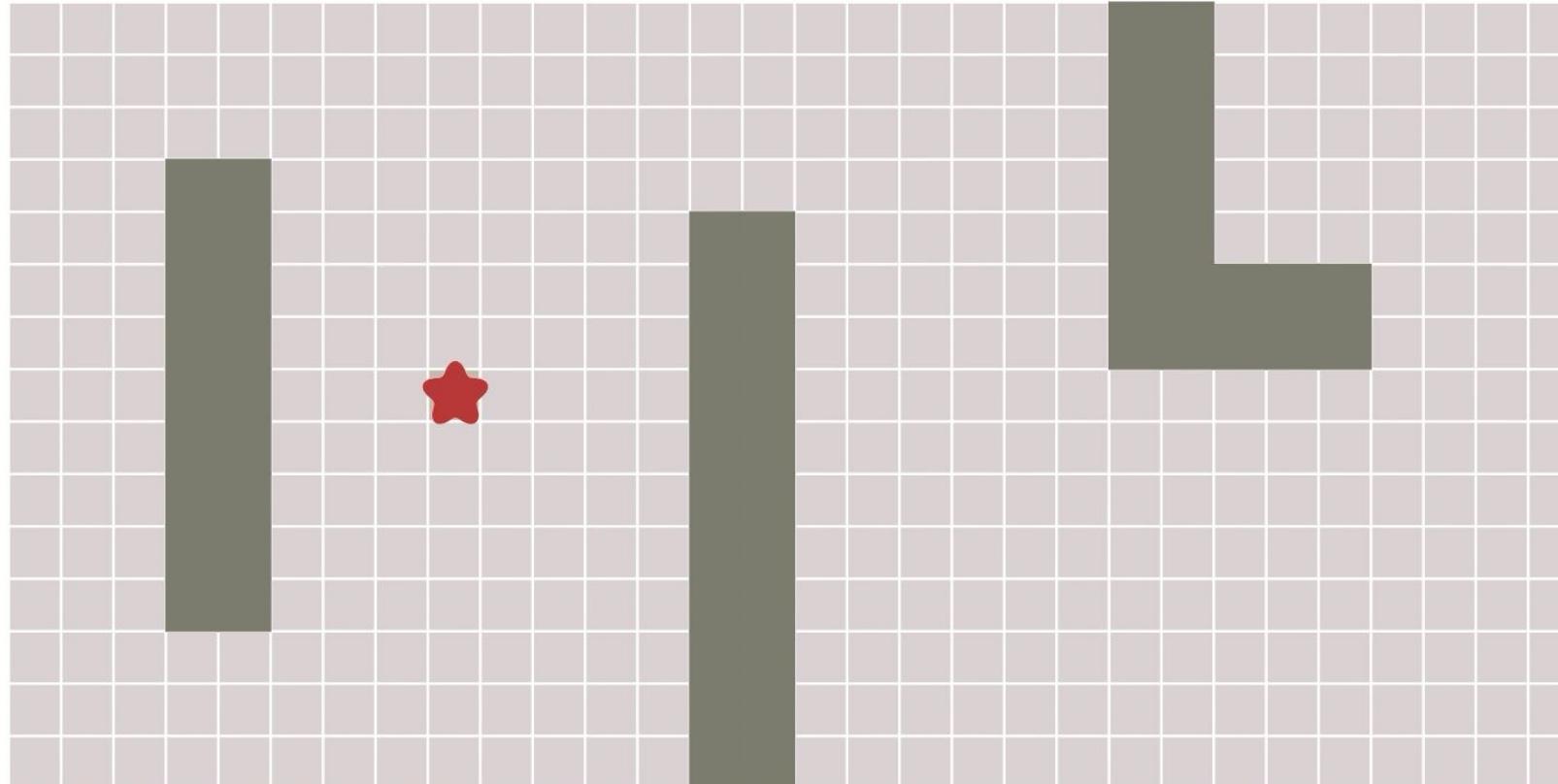
<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Pathfinding in Games



<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Breadth First Search



<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

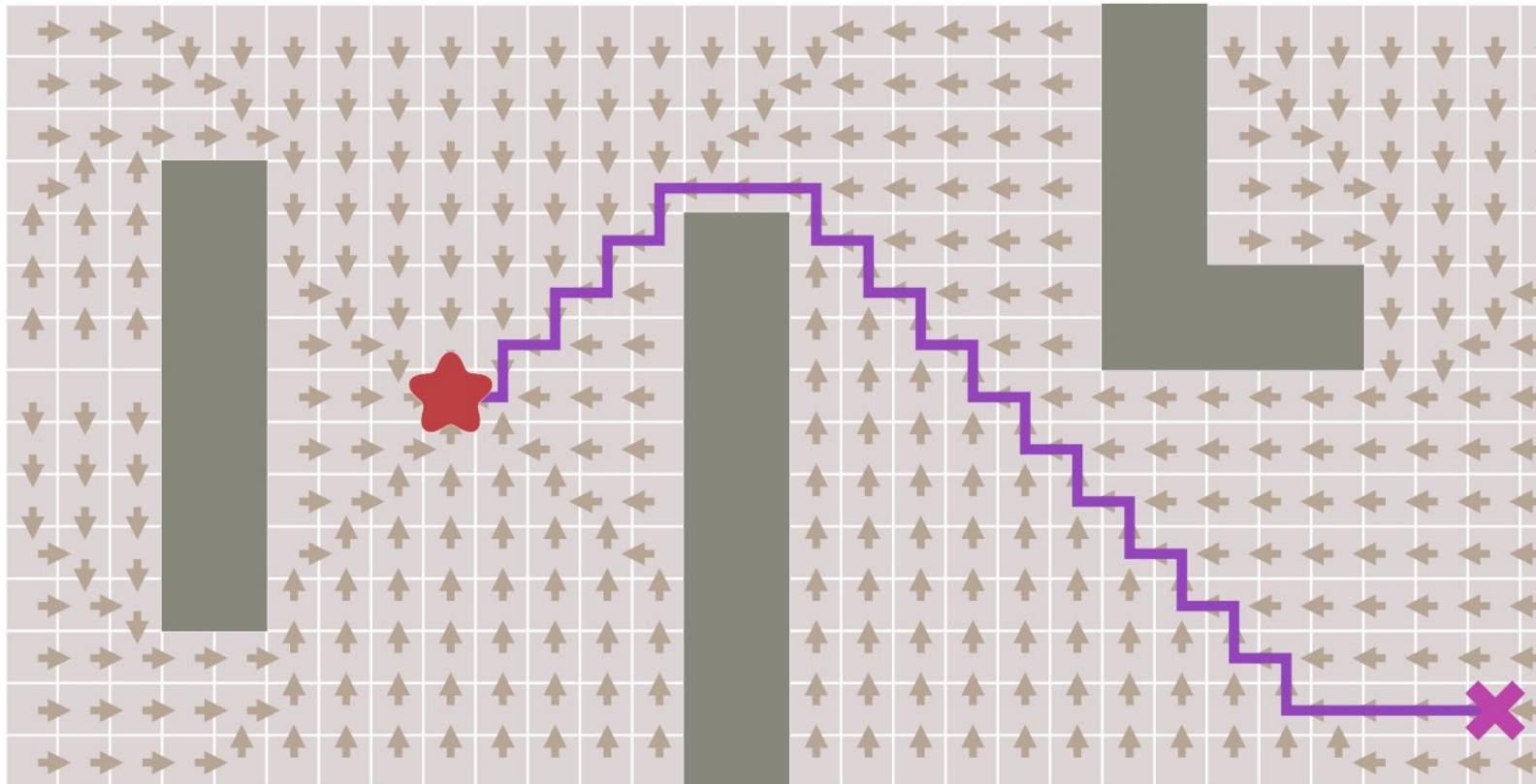
BFS in 10 lines of Python

```
frontier = Queue()
frontier.put(start ★)
visited = {}
visited[start] = True

while not frontier.empty():
    current = frontier.get()
    for next in graph.neighbors(current):
        if next not in visited:
            frontier.put(next)
            visited[next] = True
```

<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Finding the shortest path



<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

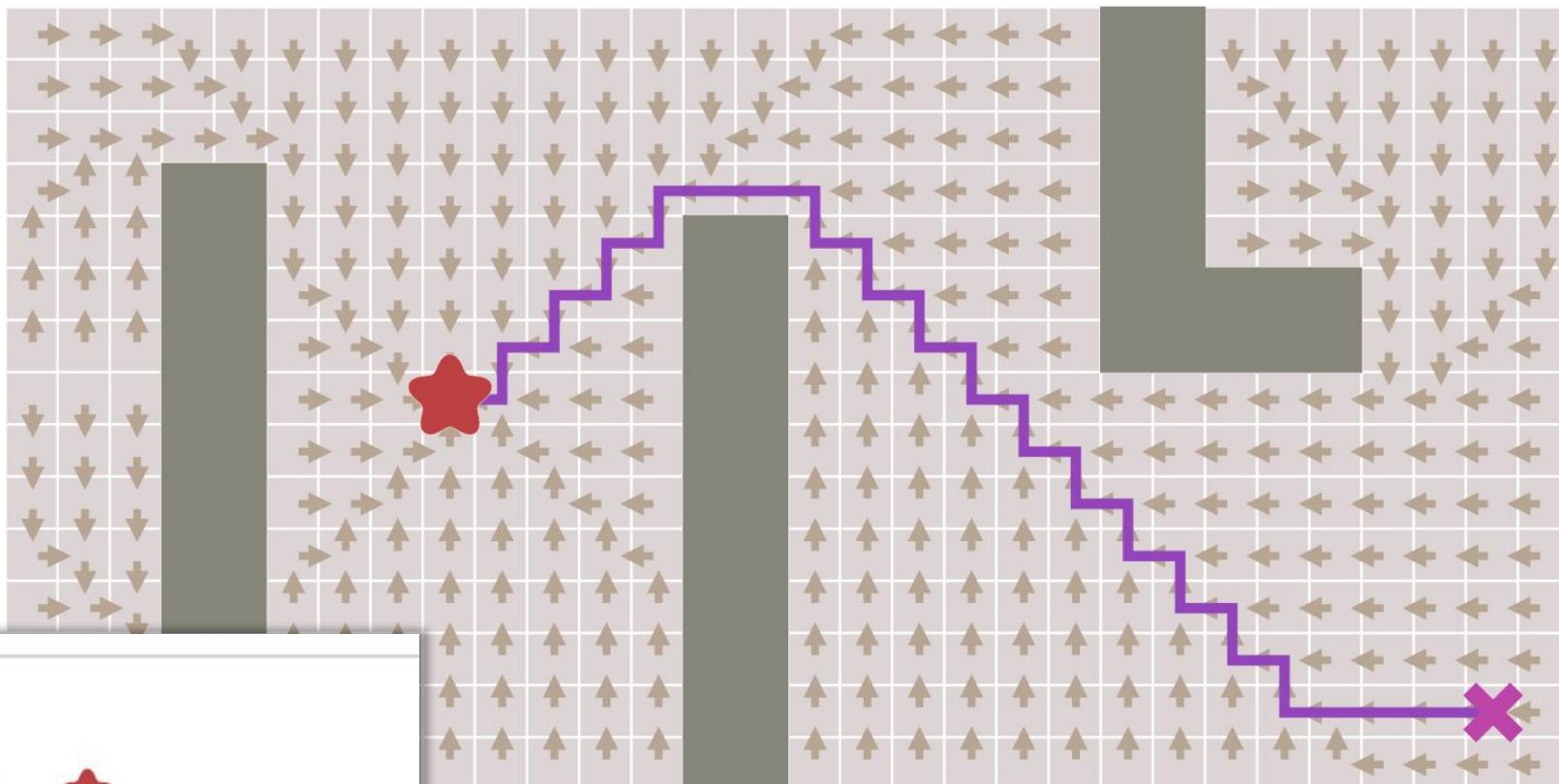
Finding the shortest path

```
frontier = Queue()
frontier.put(start ★)
came_from = {}
came_from[start] = None

while not frontier.empty():
    current = frontier.get()
    for next in graph.neighbors(current):
        if next not in came_from:
            frontier.put(next)
            came_from[next] = current
```

<https://www.redblobgames.com/pathfinding/a-star/introduction.html>

Finding the shortest path



```
current = goal ✎  
path = []  
while current != start: ★  
    path.append(current)  
    current = came_from[current]  
path.append(start) # optional  
path.reverse() # optional
```