Assignment 5
Question 2

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PART 1 (For 1D)

Question:
Given g and h derive a formula to determine f where g is the gradient image (in 1D), h is the convolution kernel to represent the gradient operation, and f is the original 1D image.

Answer

Given the equation

$$g(x, y) = (h * f)(x, y)$$

Taking Fourier transform of this equation and using convolution theorem (Fourier transform of a convolution is the pointwise product of the Fourier transforms) we get-

$$G(u, v) = H(u)F(u, v)$$

where $G(u, v) = \mathcal{F}(g(x, y)), H(u) = \mathcal{F}(h(x)), F(u, v) = \mathcal{F}(f(x, y))$

$$\hat{F}(u, v) = \frac{G(u, v)}{H(u)} = F(u, v)$$

Taking Inverse Fourier Transform we get

$$f(x, y) = \mathcal{F}^{-1}(\hat{F}(u, v))$$

The fundamental difficulties he will face will be that as $h(x)$ is a convolution kernel to represent the gradient operation hence $H(u)$ is a high pass filter and so at low frequencies it will tend to 0. If the value of $H(u)$ becomes 0, then he cannot use this approach to to extract low frequency components of $\hat{F}(u, v)$. Now if $H(u)$ does not become 0, then he can calculate $f(x, y)$ accurately provided there is no noise. If we consider noise then if $H(u)$ tends to 0, the value of $\hat{F}(u, v)$ will blow up amplifying noise in $f(x, y)$

This is problematic as images generally have large magnitude of these components.

PART 2 (For 2D)

Question:
Given the gradients of a 2D image in the X and Y directions, determine the original image.

Answer

Lets say g_{x} is the gradient of the 2D image in the X direction,g_{y} is the gradient of the 2D image in the Y direction, h_{x} is the convolution kernel for gradient operation in the X direction, h_{y} is the convolution kernel for gradient operation in the Y direction and f is the original 2D image

Then we have the equations

$$g_{\mathrm{x}}(x,y) = (h_{\mathrm{x}} * f)(x,y)$$
$$g_{\mathrm{y}}(x,y) = (h_{\mathrm{y}} * f)(x,y)$$

Taking Fourier transform of these equations and using convolution theorem (Fourier transform of a convolution is the pointwise product of the Fourier transforms) we get-

$$G_{\mathrm{x}}(u,v) = H_{\mathrm{x}}(u,v)F(u,v)$$
$$G_{\mathrm{y}}(u,v) = H_{\mathrm{y}}(u,v)F(u,v)$$

where $G_{\mathrm{x}}(u,v) = \mathcal{F}(g_{\mathrm{x}}(x,y)), G_{\mathrm{y}}(u,v) = \mathcal{F}(g_{\mathrm{y}}(x,y)), H_{\mathrm{x}}(u,v) = \mathcal{F}(h_{\mathrm{x}}(x,y)), H_{\mathrm{y}}(u,v) = \mathcal{F}(h_{\mathrm{y}}(x,y)), F(u,v) = \mathcal{F}(f(x,y))$

Using these two equations we get -

$$\hat{F}(u,v) = \frac{G_{\mathrm{x}}(u,v)}{H_{\mathrm{x}}(u,v)} = F(u,v) \tag{1}$$

and also

$$\hat{F}(u,v) = \frac{G_{\mathrm{y}}(u,v)}{H_{\mathrm{y}}(u,v)} = F(u,v) \tag{2}$$

Now as $h_{\mathrm{x}}(x,y)$ is a gradient kernel in x direction thus $H_{\mathrm{x}}(u,v)$ will be a high pass filter in u and so when u is small $H_{\mathrm{x}}(u,v)$ will be small and calculating $\hat{F}(u,v)$ by (1) would not be appropriate as its value would blow up though we can use (1) in case v is small(or large) and u is large. Similarly as $h_{\mathrm{y}}(x,y)$ is a gradient kernel in y direction thus $H_{\mathrm{y}}(u,v)$ will be a high pass filter in v and so when v is small $H_{\mathrm{y}}(u,v)$ will be small and so we should not calculate $\hat{F}(u,v)$ by (2) though we can use (2) in case u is small(or large) and v is large.

Thus when both u and v are large we can choose any one of the two equations to get the value of $\hat{F}(u,v)$, when u is large,v is small then we will choose (1), when u is small,v is large then we will choose (2) but when u and v both are small then we again have the problem that the value will blow up by both the equations.

Taking Inverse Fourier Transform we get

$$f(x,y) = \mathcal{F}^{-1}(\hat{F}(u,v))$$

The fundamental difficulties he will face will be in case of low frequencies in both u and v directions, if the value of both $H_{\mathrm{x}}(u, v)$ and $H_{\mathrm{y}}(u, v)$ become 0, then he cannot use this approach to extract low frequency components of $\hat{F}(u, v)$. Now if any one does not become 0, then he can calculate $f(x, y)$ accurately provided there is no noise. If we consider noise then if $H_{\mathrm{x}}(u, v)$ or $H_{\mathrm{y}}(u, v)$ tend to 0 then the corresponding value of $\hat{F}(u, v)$ will blow up amplifying noise in $f(x, y)$.

This is problematic as images generally have large magnitude of these components.