Assignment 4
Question 5

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October 9, 2017

To prove:
The direction $\boldsymbol{f}$ perpendicular to $\boldsymbol{e}$ for which $\boldsymbol{f^t C f}$ is maximized, is the eigenvector of $\boldsymbol{C}$ with the second highest eigenvalue assuming that all non-zero eigenvalues of $\boldsymbol{C}$ are distinct and that $rank(\boldsymbol{C}) > 2$

Proof
We have to maximise $\boldsymbol{f^t C f}$ given the constraints that $\boldsymbol{f}$ is perpendicular to $\boldsymbol{e}$ and $\boldsymbol{f}$ is a unit vector and $\boldsymbol{e^t C e} = \lambda_1$
We use the method of Lagrange multipliers to do so, as in the proof in class, with two constraints - $\boldsymbol{f^t f} - 1 = 0$ and $\boldsymbol{f^t e} = 0$

$$J(\boldsymbol{f}) = \boldsymbol{f^t C f} - \lambda(\boldsymbol{f^t f} - 1) - \delta(\boldsymbol{f^t e})$$

Taking derivative of $J(\boldsymbol{f})$ with respect to $\boldsymbol{f^t}$ and setting it to zero, we get

$$2\boldsymbol{C f} - 2\lambda \boldsymbol{f} - \delta \boldsymbol{e} = 0$$

Left multiplying both sides by $\boldsymbol{e}$, we get

$$2\boldsymbol{e^t C f} - 2\lambda \boldsymbol{e^t f} - \delta \boldsymbol{e^t e} = 0$$

We have $\boldsymbol{C e} = \lambda_1 \boldsymbol{e}$ or taking transpose both sides, $\boldsymbol{e^t C^t} = \lambda_1 \boldsymbol{e^t}$

Now, as $\boldsymbol{C}$ is a covariance matrix, hence it is symmetric as covariance of $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$ is the same as covariance of $\boldsymbol{x_j}$ and $\boldsymbol{x_i}$. Thus we have $\boldsymbol{e^t C} = \boldsymbol{e^t C^t} = \lambda_1 \boldsymbol{e^t}$.

Right Multiplying with $\boldsymbol{f}$ we get $\boldsymbol{e^t C f} = \lambda_1 \boldsymbol{e^t f} = 0$

Thus as $\boldsymbol{e^t f} = 0$ ($\boldsymbol{f}$ and $\boldsymbol{e}$ are perpendicular) and $\boldsymbol{e^t C f} = 0$, hence $\delta = 0$ and so

$$\boldsymbol{C f} - \lambda \boldsymbol{f} = 0$$

$$\boldsymbol{f^t C f} = \lambda$$

. To maximise $\boldsymbol{f^t C f}$ we have to choose the largest λ. As we have assumed that $rank(\boldsymbol{C}) > 2$ and as $\boldsymbol{e}$ is the eigenvector corresponding to λ_1 hence the nest largest value of λ can be λ_2 (the second largest eigenvalue).

Thus $\boldsymbol{f}$ is the eigenvector corresponding to λ_2