Assignment 4
Question 6

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(a) Given $P = A^T A$ (nxn matrix) and $Q = AA^T$ (mxm matrix), and y is a $nx1$ real-valued vector, we have (given the correct dimension of real-valued y):

$$y^T P y = y^T A^T A y$$
$$= (Ay)^T (Ay)$$

Now let $Ay = w$ is a nx1 vector. Hence, we have

$$(Ay)^T (Ay) = w^T w = ||w||^2 \geq 0$$
$$y^T P y \geq 0 \quad \forall y$$

Similarly, for a real-valued mx1 vector z, we have (given the correct dimension of real-valued z):

$$z^T Q y = z^T A A^T z$$
$$= (A^T z)^T (A^T z)$$

Now let $A^T z = u$ is a mx1 vector. Hence, we have

$$(A^T z)^T (A^T z) = u^T u = ||u||^2 \geq 0$$
$$z^T Q z \geq 0 \quad \forall z$$

Proof of non-negative eigenvalues of P and Q: For P, let the non-zero eigen vector be v with corresponding eigenvalue λ:

$$Pv = \lambda v$$

Pre-multiplying both sides by v^T, we get,

$$v^T P v = \lambda v^T v = \lambda ||v||^2$$

$$\implies \lambda = \frac{v^T P v}{||v||^2}$$

Now, $||v||^2$ is always positive (by-definition). Also, $v^T P v \geq 0$ (proved earlier). Hence, value of λ is always non-negative. Since we made no assumptions for v or λ, the condition holds for all eigenvalues of P.

Similarly, for Q, let:

$$Qv = \lambda v$$

Pre-multiplying both sides by v^T, we get,

$$v^T Q v = \lambda v^T v = \lambda ||v||^2$$

$$\implies \lambda = \frac{v^T Q v}{||v||^2}$$

Now, $||v||^2$ is always positive (by-definition). Also, $v^T Q v \geq 0$ (proved earlier). Hence, value of λ is always non-negative. Since we made no assumptions for v or λ, the condition holds for all eigenvalues of Q.

(b) u is an eigenvector of P with eigenvalue λ. Now,

$$A \in R^{m*n} \implies A^T A = P \in R^{n*n} \implies u \in R^n \quad \text{and}$$

$$Pu = \lambda u$$

Pre-multiplying both sides with A, we get

$$APu = \lambda Au$$

Using $P = A^T A$, we simplify this to,

$$AA^T Au = (AA^T)Au = \lambda Au$$

Simplify $AA^T = Q$, and $Au = v \in R^m$ to get,

$$Q(Au) = Qv = \lambda(Au) = \lambda v$$

Hence, Au is an eigenvector of Q with eigenvalue λ.

Similarly to prove for Q, we assume an eigenvector v with eigenvalue μ. Now,

$$A \in R^{m*n} \implies AA^T = Q \in R^{m*m} \implies v \in R^m \quad \text{and}$$

$$Qv = \mu v$$

Pre-multiplying both sides with A^T, we get

$$A^T Qv = \mu A^T v$$

Using $Q = AA^T$, we simplify this to,

$$A^T AA^T v = (A^T A)A^T v = \mu A^T v$$

Simplify $A^T A = P$, and $A^T v = w \in R^n$ to get,

$$P(A^T v) = Pw = \mu(A^T v) = \mu w$$

Hence, $A^T v$ is an eigenvector of P with eigenvalue μ.

(c) We are given that v_i is an eigenvector of Q (with eigenvalue α_i). Which gives us the first equation

$$Qv_i = AA^Tv_i = \alpha_i v_i \qquad \text{(i)}$$

Now, given

$$u_i = \frac{A^Tv_i}{||A^Tv_i||_2}$$

we have

$$Au_i = \frac{AA^Tv_i}{||A^Tv_i||_2}$$

$$\implies Au_i = \frac{Qv_i}{||A^Tv_i||_2} = \frac{\alpha_i v_i}{||A^Tv_i||_2} = \left(\frac{\alpha_i}{||A^Tv_i||_2}\right)v_i$$

Substitute $\left(\frac{\alpha_i}{||A^Tv_i||_2}\right) = \gamma_i$ to get the required equation.

$$Au_i = \gamma_i v_i$$

Now, α_i is non-negative since it's an eigenvalue of Q (proved earlier), and $||A^Tv_i||_2$ is positive by definition, the value of γ_i is non-negative.

There exists a non-negative γ_i for which the given equation holds.

(d) We know as a result that

$$u_i^Tu_j = 0 \qquad \text{for} \quad i \neq j$$

and for

$$u_i^Tu_i = \frac{(A^Tv_i)^T(A^Tv_i)}{||A^Tv_i||^2} = 1$$

So, the matrix $V = [u_1|u_2|...|u_n]$ is orthonormal because $VV^T = V^TV = I_n$

Similarly, for $U = [v_1|v_2|...|v_m], UU^T = U^TU = I_m$

because $v_i^Tv_j = 0 \quad$ for $\quad i \neq j$ and $v_i^Tv_i = 1$ (because v_i and v_j are eigenvectors of P, and which can be assumed to be of unit length for consistency.

Given these results, the value of U^TAV

$$= U^TAV = U^TA[u_1|u_2|...|u_n]$$

$$= \begin{bmatrix} v_1^T \\ v_2^T \\ .. \\ v_m^T \end{bmatrix} A[u_1|u_2|...|u_n]$$

From the previous result, we have $Au_i = \gamma_i v_i$

$$= \begin{bmatrix} v_1^T \\ v_2^T \\ .. \\ v_m^T \end{bmatrix} [\gamma_1 v_1|\gamma_2 v_2|...|\gamma_n v_n]$$

$$= [\Gamma_{ij}^T]_{mxn}$$

Where

$$\Gamma_{ij} = v_i^T \gamma_j v_j = \begin{cases} 0 & \text{if} \quad i \neq j \\ \gamma_j & \text{otherwise} \end{cases}$$

Hence, $U^T AV = \Gamma$ where T is a diagonal matrix, with i^{th} diagonal element $= \gamma_i$

$$\implies (UU^T)A(VV^T) = U\Gamma V^T$$

$$\implies A = U\Gamma V^T$$

This holds since U and V are orthonormal.
Hence, proved.