H CORRECTED PRIVACY BUDGET BOUNDS IN LATENT [5]

In this section, we aim at providing corrected privacy budget bounds for LATENT [5]. LATENT first encodes embedded features e into an dl-bit bit-string B. Then, each bit $j \in [0, dl-1]$ is randomized by a RR mechanism (i.e., the MOUE algorithm for high sensitivities in Theorem 3.3 [5]), denoted f-LT, as follows:

$$\forall j \in [0, dl-1] : P(\tilde{B}_j = 1) = \begin{cases} p_j = \dfrac{1}{1+\alpha}, & \text{if } B_j = 1 \\ q_j = \dfrac{1}{1+\alpha\exp(\frac{\varepsilon_f}{dl})}, & \text{if } B_j = 0 \end{cases} \tag{30}$$

From Eq. 30, we also have that $P(\tilde{B}_j = 0) = 1 - p_j = \frac{\alpha}{1+\alpha}$ if $B_j = 1$, and $P(\tilde{B}_j = 0) = 1 - q_j = \frac{\alpha\exp(\frac{\varepsilon_f}{dl})}{1+\alpha\exp(\frac{\varepsilon_f}{dl})}$ if $B_j = 0$.

Theorem 8. *LATENT with the randomization probabilities as in Eq. 30 preserves $\epsilon_{corrected}$-LDP, where $\epsilon_{corrected} = \frac{(1+\alpha)(1+\alpha\exp(\frac{\varepsilon_f}{dl}))}{\alpha(1+\exp(\frac{\varepsilon_f}{dl}))}\varepsilon_f$.*

Proof. Similar to the analysis in **Appx.F**, we obtain the following inequality:

$$\frac{P(f\text{-LT}(B) = \tilde{B})}{P(f\text{-LT}(B') = \tilde{B})} \leq \prod_{j=0}^{dl-1}\Big(\frac{P(f\text{-LT}(B_j) = \tilde{B}_j)}{P(f\text{-LT}(B'_j) = \tilde{B}_j)}\Big)^{\frac{\Delta_i}{\mathbb{E}|\mathcal{R}(f\text{-LT}(v,i)) - \mathcal{R}(f\text{-LT}(v^{\neq i},i))|}} \leq \exp(\varepsilon_f) \tag{31}$$

and the expectation $\mathbb{E}|\mathcal{R}(f\text{-LT}(B,j)) - \mathcal{R}(f\text{-LT}(B',j))|$ is computed as follows:

$$\begin{aligned}
&\mathbb{E}|\mathcal{R}(f\text{-LT}(B,j)) - \mathcal{R}(f\text{-LT}(B',j))| \\
&= \Big(P(f\text{-LT}(B_j) = 1|B_j = 1)P(f\text{-LT}(B'_j) = 0|B'_j = 0)P(B'_j = 0) \\
&\qquad + P(f\text{-LT}(B_j) = 1|B_j = 0)P(f\text{-LT}(B'_j) = 0|B'_j = 1)P(B'_j = 1)\Big)\Delta_j \\
&\quad + \Big(P(f\text{-LT}(B_j) = 0|B_j = 1)P(f\text{-LT}(B'_j) = 1|B'_j = 0)P(B'_j = 0) \\
&\qquad + P(f\text{-LT}(B,j) = 0|B_j = 0)P(f\text{-LT}(B'_j) = 1|B'_j = 1)P(B'_j = 1)\Big)\Delta_j \\
&= \Big(p_j(1-q_j)P(B'_j = 0) + q_j(1-p_j)P(B'_j = 1) + (1-p_j)q_jP(B'_j = 0) + (1-q_j)p_jP(B'_j = 1)\Big)\Delta_j \\
&= \Big(p_j(1-q_j) + q_j(1-p_j)\Big)\Delta_j
\end{aligned} \tag{32}$$

Furthermore, we have:

$$p_j(1-q_j) + q_j(1-p_j) = \frac{\alpha(1+\exp(\frac{\varepsilon_f}{dl}))}{(1+\alpha)(1+\alpha\exp(\frac{\varepsilon_f}{dl}))} \tag{33}$$

From Eqs. 31-33, we have that

$$\begin{aligned}
&\frac{P(f\text{-LT}(B) = \tilde{B})}{P(f\text{-LT}(B') = \tilde{B})} \leq \prod_{j=0}^{dl-1}\Big(\frac{P(f\text{-LT}(B_j) = \tilde{B}_j)}{P(f\text{-LT}(B'_j) = \tilde{B}_j)}\Big)^{\frac{\Delta_j}{\mathbb{E}|\mathcal{R}(f\text{-LT}(B,j)) - \mathcal{R}(f\text{-LT}(B',j))|}} \\
&= \prod_{j=0}^{dl-1}\Big(\frac{P(f\text{-LT}(B_j) = 1|B_j = 1)}{P(f\text{-LT}(B_j) = 0|B_j = 1)}\Big)^{\frac{\Delta_j}{(p_j(1-q_j)+q_j(1-p_j))\Delta_j}} \times \prod_{j=0}^{dl-1}\Big(\frac{P(f\text{-LT}(B_j) = 0|B_j = 0)}{P(f\text{-LT}(B_j) = 1|B_j = 0)}\Big)^{\frac{\Delta_j}{(p_j(1-q_j)+q_j(1-p_j))\Delta_j}} \\
&= \prod_{j=0}^{dl-1}\Big(\exp(\frac{\varepsilon_f}{dl})\Big)^{\frac{1}{p_j(1-q_j)+q_j(1-p_j)}}
\end{aligned} \tag{34}$$

Then, from Eq. 34, we have:

$$\epsilon_{corrected} = \ln\Big(\Pi_{j=0}^{dl-1}\big(\exp(\frac{\varepsilon_f}{dl})\big)^{\frac{1}{p_j(1-q_j)+q_j(1-p_j)}}\Big) = \frac{(1+\alpha)(1+\alpha\exp(\frac{\varepsilon_f}{dl}))}{\alpha(1+\exp(\frac{\varepsilon_f}{dl}))}\varepsilon_f \tag{35}$$

Consequently, Theorem 8 holds. □

I CORRECTED PRIVACY BUDGET BOUNDS IN OME [29]

In this section, we aim at providing corrected privacy budget bounds for OME. OME first encodes the embedding features z into an dl-bit binary vector B. Then, each bit $j \in [0, dl-1]$ is randomized by the following f-OME mechanism:

$$\forall j \in [0, dl-1] : P(\tilde{B}_j = 1) = \begin{cases} p_{1j} = \dfrac{\alpha}{1+\alpha}, & \text{if } j \in 2k, B_j = 1 \\ p_{2j} = \dfrac{1}{1+\alpha^3}, & \text{if } j \in 2k+1, B_j = 1 \\ q_j = \dfrac{1}{1+\alpha \exp(\frac{\varepsilon_f}{dl})}, & \text{if } B_j = 0 \end{cases} \tag{36}$$

From Eq. 36, we also have that $P(\tilde{B}_j = 0) = 1 - p_{1j} = \frac{1}{1+\alpha}$ if $B_j = 1$ and $j \in 2k$, $P(\tilde{B}_j = 0) = 1 - p_{2j} = \frac{\alpha^3}{1+\alpha^3}$ if $B_j = 1$ and $j \in 2k+1$, and $P(\tilde{B}_j = 0) = 1 - q_j = \frac{\alpha \exp(\frac{\varepsilon_f}{dl})}{1+\alpha \exp(\frac{\varepsilon_f}{dl})}$ if $B_j = 0$.

Theorem 9. *OME with the randomization probabilities as in Eq. 36 preserves $\epsilon_{corrected}$-LDP, where $\epsilon_{corrected} = (\frac{dl}{Q_1} - \frac{dl}{Q_2})\ln(\alpha) + \frac{\varepsilon_f}{2Q_1} + \frac{\varepsilon_f}{2Q_2}$ in which $Q_1 = \frac{\alpha}{1+\alpha}\frac{\alpha \exp(\frac{\varepsilon_f}{dl})}{1+\alpha \exp(\frac{\varepsilon_f}{dl})} + \frac{1}{1+\alpha \exp(\frac{\varepsilon_f}{dl})}\frac{1}{1+\alpha}$ and $Q_2 = \frac{1}{1+\alpha^3}\frac{\alpha \exp(\frac{\varepsilon_f}{dl})}{1+\alpha \exp(\frac{\varepsilon_f}{dl})} + \frac{1}{1+\alpha \exp(\frac{\varepsilon_f}{dl})}\frac{\alpha^3}{1+\alpha^3}$.*

Proof. Similar to the analysis in **Appx. F** and **Appx. H**, we obtain:

$$\frac{P(f\text{-OME}(B) = \tilde{B})}{P(f\text{-OME}(B') = \tilde{B})} \leq \prod_{j=0}^{dl-1} \Big(\frac{P(f\text{-OME}(B_j) = \tilde{B}_j)}{P(f\text{-OME}(B'_j) = \tilde{B}_j)}\Big)^{\frac{\Delta_j}{\mathbb{E}|\mathcal{R}(f\text{-OME}(B,j)) - \mathcal{R}(f\text{-OME}(B',j))|}} \leq \exp(\varepsilon_f) \tag{37}$$

and the expectation $\mathbb{E}|\mathcal{R}(f\text{-OME}(B,j)) - \mathcal{R}(f\text{-OME}(B',j))|$ is computed as follows:

$$\mathbb{E}|\mathcal{R}(f\text{-OME}(B,j)) - \mathcal{R}(f\text{-OME}(B',j))| = \begin{cases} \big(p_{1j}(1-q_j) + q_j(1-p_{1j})\big)\Delta_j = Q_1\Delta_j, \text{if } j \in 2k \\ \big(p_{2j}(1-q_j) + q_j(1-p_{2j})\big)\Delta_j = Q_2\Delta_j, \text{if } j \in 2k+1 \end{cases} \tag{38}$$

where $Q_1 = p_{1j}(1-q_j) + q_j(1-p_{1j}) = \frac{\alpha}{1+\alpha}\frac{\alpha \exp(\frac{\varepsilon_f}{dl})}{1+\alpha \exp(\frac{\varepsilon_f}{dl})} + \frac{1}{1+\alpha \exp(\frac{\varepsilon_f}{dl})}\frac{1}{1+\alpha}$, and $Q_2 = p_{2j}(1-q_j) + q_j(1-p_{2j}) = \frac{1}{1+\alpha^3}\frac{\alpha \exp(\frac{\varepsilon_f}{dl})}{1+\alpha \exp(\frac{\varepsilon_f}{dl})} + \frac{1}{1+\alpha \exp(\frac{\varepsilon_f}{dl})}\frac{\alpha^3}{1+\alpha^3}$.

From Eqs. 37 and 38, we have:

$$\begin{aligned} \frac{P(f\text{-OME}(B) = \tilde{B})}{P(f\text{-OME}(B') = \tilde{B})} &\leq \prod_{j=0}^{dl-1} \Big(\frac{P(f\text{-OME}(B_j) = \tilde{B}_j)}{P(f\text{-OME}(B'_j) = \tilde{B}_j)}\Big)^{\frac{\Delta_j}{\mathbb{E}|\mathcal{R}(f\text{-OME}(B,j)) - \mathcal{R}(f\text{-OME}(B',j))|}} \\ &= \prod_{j\in 2k} \Big(\frac{P(f\text{-OME}(B_j) = 1|B_j = 1)P(f\text{-OME}(B_j) = 0|B_j = 0)}{P(f\text{-OME}(B_j) = 1|B_j = 0)P(f\text{-OME}(B_j) = 0|B_j = 1)}\Big)^{\frac{\Delta_j}{Q_1\Delta_j}} \\ &\times \prod_{j\in 2k+1} \Big(\frac{P(f\text{-OME}(B_j) = 1|B_j = 1)P(f\text{-OME}(B_j) = 0|B_j = 0)}{P(f\text{-OME}(B_j) = 1|B_j = 0)P(f\text{-OME}(B_j) = 0|B_j = 1)}\Big)^{\frac{\Delta_i}{Q_2\Delta_j}} \\ &= \alpha^{\frac{dl}{Q_1} - \frac{dl}{Q_2}} \exp(\frac{\varepsilon_f}{2Q_1} + \frac{\varepsilon_f}{2Q_2}) \end{aligned} \tag{39}$$

Then, from Eq. 39, we have:

$$\begin{aligned} \epsilon_{corrected} &= \ln\Big(\alpha^{\frac{dl}{Q_1} - \frac{dl}{Q_2}} \exp(\frac{\varepsilon_f}{2Q_1} + \frac{\varepsilon_f}{2Q_2})\Big) &(40) \\ &= (\frac{dl}{Q_1} - \frac{dl}{Q_2})\ln(\alpha) + \frac{\varepsilon_f}{2Q_1} + \frac{\varepsilon_f}{2Q_2} &(41) \end{aligned}$$

Consequently, Theorem 9 does hold. □