

From Mass-Spring Systems to Spectral Graph Neural Networks

Nguyen Ngoc Khanh

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1 Mass-Spring System

1.1 The two particles system

Consider a spring follows Hook's law. Let two particles i and j connected by a spring located at x_i and x_j respectively and $e_{ij} = \frac{x_j - x_i}{\|x_j - x_i\|_2}$ be the direction from x_i to x_j then the force that i affects j can be represented as:

$$F_{ij} = -k(\|x_j - x_i\|_2 - L)e_{ij} = -k(x_j - x_i) + kLe_{ij} \quad (1)$$

where k is a positive real number, the characteristic of the spring and L is the initial length the of spring. The magnitude of the force is proportional to the displacement from the initial distance between two particles.

Let two particles connected by a spring sit in an Euclidean space such that the particles can freely move on a particular z axis. At the initial condition, the two particles are located at x_i and x_j and the spring is at its length ($\|x_j - x_i\|_2 = L$) (no force).

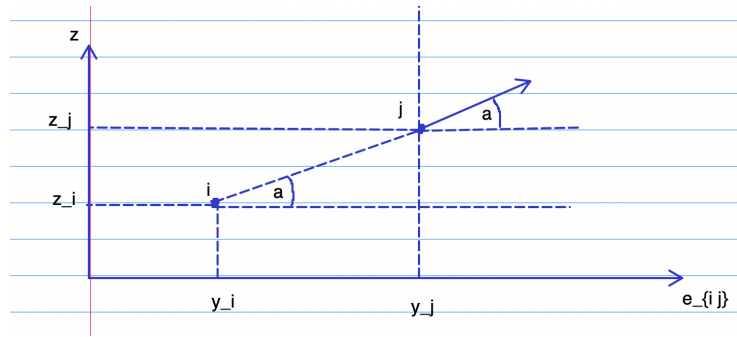


Figure 1: subspace of e_{ij} and z

Since, i and j can only move in the z axis, we can rewrite

$$x_i = x_i^{(0)} + z_i z$$

$$x_j = x_j^{(0)} + z_j z$$

Where $x_i^{(0)}$ and $x_j^{(0)}$ are the initial positions of i and j , z_i and z_j are the displacements on the z axis. Hence, the projected force on the z axis can be written as

$$F_{ij} \cdot z = (-k(x_j - x_i) + kLe_{ij}) \cdot z = (-k(x_j^{(0)} - x_i^{(0)}) + kLe_{ij}) \cdot z + (-k(z_j - z_i)z) \cdot z \quad (2)$$

The first term is the dot product of the initial force with the z direction which is essential zero since there is no force at the beginning. Hence, the projected force on the z axis can be written as

$$F_{ij} \cdot z = -k(z_j - z_i) \quad (3)$$

The projected force on the z axis linearly depends on the corresponding displacement.

1.2 The n particles system

Let n particles with the same weight m on an an Euclidean space that can freely move on a particular z axis. Some of them are connected by springs of the same characteristic k which is denoted by a undirected unweighted graph $G = (V, E)$. A particular node i is affected by all of its neighbours where the projected force on i is

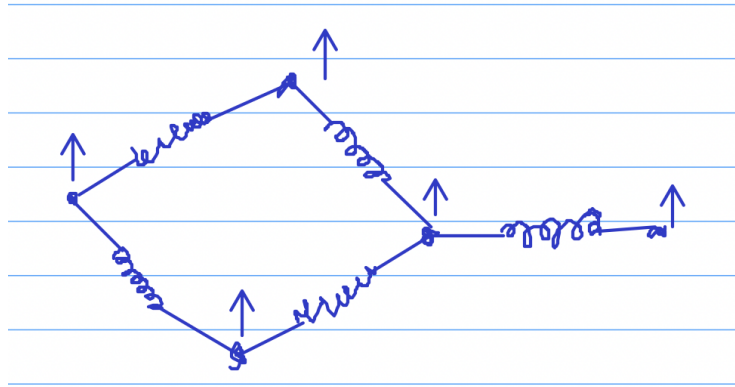


Figure 2: 4 particles system

$$F_i \cdot z = -k \sum_{e_{ji} \in E} z_i - z_j = -k(d_i z_i - \sum_{e_{ji} \in E} z_j) \quad (4)$$

Where d_i denotes the degree of the node i . Newton's Second Law of Motion:

$$\begin{aligned} a_i \cdot z &= \frac{F_i}{m} \cdot z \\ \ddot{z}_i &= -\frac{k}{m}(d_i z_i - \sum_{e_{ji} \in E} z_j) \end{aligned}$$

We can rewrite in the matrix form

$$\ddot{z} = -\frac{k}{m}(D - A)z = -\frac{k}{m}Lz \quad (5)$$

Where $z = (z_1, z_2, \dots, z_n)^T$, A is the adjacency matrix of G and D is the degree matrix of A (diagonal matrix where each entry equals to the corresponding degree of the node). $L = D - A$ is called Laplacian matrix of A .

We are seeking the mode of oscillation of the system. A mode of oscillation is a particular frequency where all particles oscillate at the same frequency. At that frequency, the differential equation for each particle must be in the form:

$$\ddot{z}_i = -\omega^2 z_i \quad (6)$$

Where ω is the frequency. In the matrix form:

$$\ddot{z} = -\omega^2 z \quad (7)$$

From 5 and 7, we have

$$Lz = \frac{m}{k}\omega^2 z \quad (8)$$

From 8, the oscillation mode frequencies are equivalent to the eigenvalues of the Laplacian matrix, and the initial condition to achieve each of the frequencies is the corresponding eigenvector.

Since L is real symmetric, by the Spectral Theorem, it has an eigenbasis. Furthermore, L is positive semi definite, then all of its eigenvalues are positive, hence the frequencies make sense.

For an arbitrary initial condition, since the system is linear, we can decompose the displacement z into the eigenbasis of the Laplacian matrix then solve each of the component individually.

2 Graph Laplacian Basis

Recall that, the eigen decomposition of L is as follow:

$$L = U\Lambda U^T = U\Lambda U^{-1} \quad (9)$$

Where each column vector in U is a normalized eigenvector.

Analogous to Fourier Transform, the eigenvalues of Laplacian matrix can serve as the frequency and the eigenbasis is corresponding to the Fourier basis.

Let $x \in \mathbb{R}^n$ be a graph signal on $G = (V, E)$ where each component of x is a real number corresponding to a node in G .

The convolution in spatial domain is equivalent to multiplication in spectral domain. Define the convolution operation as:

$$y(x) = U(U^T w \odot U^T x) \quad (10)$$

Where $w \in \mathbb{R}^n$ is called filter or kernel. Define $W = \text{diag}(U^T w)$ be the diagonal matrix whose entries are the entries of $U^T w$, we can rewrite 10 as

$$y(x) = (UWU^T)x \quad (11)$$

2.1 ChebNet

Let \mathcal{L} be the normalized laplacian matrix.

$$\mathcal{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \quad (12)$$

The decomposition of \mathcal{L}

$$\mathcal{L} = U \Lambda U^T = U \Lambda U^{-1} \quad (13)$$

Theorem 1 (Chung [2]) *All eigenvalues of \mathcal{L} are in the interval $[0, 2]$.*

ChebNet [3] approximate the diagonal matrix W using Chebyshev polynomials as the orthogonal basis in the polynomial subspace of the vector space of all functions $f : [-1, +1] \rightarrow \mathbb{R}$ with respect to the inner product.

$$\int_{-1}^{+1} f(x)g(x) \frac{dx}{\sqrt{1-x^2}} \quad (14)$$

Chebyshev polynomials of the first kind:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (15)$$

Where $x \in [-1, +1]$, $T_0(x) = 1$ and $T_1(x) = x$. Let $f_i : [-1, +1] \rightarrow \mathbb{R}$ is an arbitrary function such that $f_i(\tilde{\lambda}_i) = w_i$ where $\tilde{\lambda}_i = \frac{2\lambda_i}{\lambda_{\max}} - 1 \in [-1, +1]$, λ_{\max} is the largest eigenvalue. We want to project f_i into the subspace with the orthogonal basis of the first K terms of Chebyshev polynomials of the first kind. We can write f_i as

$$\hat{f}_i(t) = \sum_{k=0}^{K-1} \theta_{ki} T_k(t) \quad (16)$$

Hence, w_i is approximated as

$$\hat{w}_i = \hat{f}_i(\tilde{\lambda}_i) = \sum_{k=0}^{K-1} \theta_k T_k(\tilde{\lambda}_i) \quad (17)$$

Matrix form of the approximation on W :

$$\hat{W} = \sum_{k=0}^{K-1} \theta_k T_k(\tilde{\Lambda}) \quad (18)$$

Where $\tilde{\Lambda}$ is the diagonal matrix of $\tilde{\lambda}_i$. Moreover,

$$U\hat{W}U^T = \sum_{k=0}^{K-1} \theta_k U T_k(\tilde{\Lambda}) U^T = \sum_{k=0}^{K-1} \theta_k T_k(\tilde{\mathcal{L}}) \quad (19)$$

Where $\tilde{\mathcal{L}} = \frac{2\mathcal{L}}{\lambda_{\max}} - I$. The convolution operation is

$$y(x) = \sum_{k=0}^{K-1} \theta_k T_k(\tilde{\mathcal{L}}) x \quad (20)$$

The construction of ChebNet avoids decomposing the matrix L as compare to 10.

2.2 Graph Convolutional Network

Similar to ChebNet, GCN [4] limits $K = 2$ and sets $\lambda_{\max} = 2$ hence $\tilde{\mathcal{L}} = \mathcal{L} - I$.

$$\begin{aligned} U\hat{W}U^T &= \theta_0 T_0(\tilde{\mathcal{L}}) + \theta_1 T_1(\tilde{\mathcal{L}}) \\ &= \theta_0 I + \theta_1 \tilde{\mathcal{L}} \\ &= \theta_0 I - \theta_1 D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \end{aligned}$$

The convolution operation is

$$y(x) = \theta_0 x - \theta_1 D^{-\frac{1}{2}} A D^{-\frac{1}{2}} x \quad (21)$$

The notes here is greatly inspired by [1].

References

- [1] Xinye Chen. A note on spectral graph neural network. *arXiv preprint arXiv:2012.06660*, 2020.
- [2] Fan RK Chung and Fan Chung Graham. *Spectral graph theory*. Number 92. American Mathematical Soc., 1997.

- [3] Michaël Defferrard, Xavier Bresson, and Pierre Vandergheynst. Convolutional neural networks on graphs with fast localized spectral filtering. *arXiv preprint arXiv:1606.09375*, 2016.
- [4] Thomas N Kipf and Max Welling. Semi-supervised classification with graph convolutional networks. *arXiv preprint arXiv:1609.02907*, 2016.