triangle inequality

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Given an inner product space V over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). The inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ satisfies these properties:

1. **Linearity in the first argument**
   - **Additivity** $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
   - **Homogeneity** $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$

   for all $u, v, w \in V$ and $\lambda \in \mathbb{F}$
2. **Conjugate symmetry**
   - $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$
3. **Positivity**
   - $\langle v, v \rangle \geq 0$ for all $v \in V$
4. **Definiteness**
   - $\langle v, v \rangle = 0$ if and only if $v = 0$

In the definition, **Linearity** and **Conjugate symmetry** implicitly imply that $\langle v, v \rangle \in \mathbb{R}$ for all $v \in V$. Hence, the inequality and equality in **Positivity** and **Definiteness** are justifiable. Furthermore, **Linearity** and **Conjugate symmetry** also imply **Conjugate Linearity**.

- **Conjugate Additivity** $\langle w, u + v \rangle = \langle w, u \rangle + \langle w, v \rangle$
- **Conjugate Homogeneity** $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$

Theorem 1 (Pythagorean Theorem) *Suppose u and v are orthogonal in V namely $\langle u, v \rangle = 0$ and $\langle v, u \rangle = 0$, then*

$$||u + v||^2 = ||u||^2 + ||v||^2$$

where $||v|| = \sqrt{\langle v, v \rangle}$ denotes the norm of v.

Proof

$$\begin{aligned}
||u+v||^2 &= \langle u+v, u+v\rangle && \text{(rewrite)}\\
&= \langle u, u+v\rangle + \langle v, u+v\rangle && \text{(additivity)}\\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle && \text{(conjugate additivity)}\\
&= \langle u,u\rangle + \langle v,v\rangle && (\langle u,v\rangle = 0 \text{ and } \langle v,u\rangle = 0)\\
&= ||u||^2 + ||v||^2 && \text{(rewrite)}
\end{aligned}$$

Theorem 2 (Cauchy-Schwarz Inequality)

$$|\langle u,v\rangle| \leq ||u||||v||$$

for any $u, v \in V$ where $|z| = \sqrt{z\bar{z}}$ denotes the complex modulus of z

Proof

If both $||u|| = 0$ and $||v|| = 0$, trivial. Without loss of generality, assume that $||v|| > 0$, we then project u into 1-dimension subspace of v. The projection can be obtained by

$$P_v u = \frac{\langle u,v\rangle}{||v||^2} v$$

Let $w = u - P_v u$, hence w and v are orthogonal or w and $P_v u$ are orthogonal. We have

$$\begin{aligned}
||u||^2 &= ||w + P_v u||^2 && \text{(rewrite)}\\
&= ||w||^2 + ||P_v u||^2 && \text{(pythagorean theorem)}\\
&= ||w||^2 + \langle \frac{\langle u,v\rangle}{||v||^2} v, \frac{\langle u,v\rangle}{||v||^2} v\rangle && \text{(rewrite)}\\
&= ||w||^2 + \frac{\langle u,v\rangle}{||v||^2} \langle v, \frac{\langle u,v\rangle}{||v||^2} v\rangle && \text{(homogeneity)}\\
&= ||w||^2 + \frac{\langle u,v\rangle}{||v||^2} (\overline{\frac{\langle u,v\rangle}{||v||^2}}) \langle v,v\rangle && \text{(conjugate homogeneity)}\\
&= ||w||^2 + \frac{\langle u,v\rangle \overline{\langle u,v\rangle}}{||v||^2 \overline{||v||^2}} \langle v,v\rangle && \text{(property of complex conjugate)}\\
&= ||w||^2 + \frac{|\langle u,v\rangle|^2}{||v||^2} && (\text{rewrite, } ||v|| \in \mathbb{R})
\end{aligned} \tag{1}$$

Multiply both sides by $||v||^2$

$$||u||^2||v||^2 = ||w||^2||v||^2 + |\langle u,v\rangle|^2$$

Since $||w||^2||v||^2 \leq 0$,

$$|\langle u,v\rangle|^2 \leq ||u||^2||v||^2$$

Or,

$$|\langle u, v\rangle| \leq ||u||||v||$$

with equality if $||w|| = 0$. In other words, u and v are linearly dependent.

Theorem 3 (Triangle Inequality)

$$||u + v|| \leq ||u|| + ||v||$$

for any $u, v \in V$

Proof

$$\begin{aligned}
||u+v||^2 &= \langle u+v, u+v\rangle && \text{(rewrite)}\\
&= \langle u, u+v\rangle + \langle v, u+v\rangle && \text{(additivity)}\\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle && \text{(conjugate additivity)}\\
&= ||u||^2 + ||v||^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} && \text{(conjugate symmetry, rewrite)}\\
&\leq ||u||^2 + ||v||^2 + 2|\langle u, v\rangle| && \text{(property of complex number)}\\
&\leq ||u||^2 + ||v||^2 + 2||u||||v|| && \text{(cauchy-schwarz inequality)}\\
&= (||u|| + ||v||)^2 && \text{(rewrite)}
\end{aligned}$$

Or,

$$||u + v|| \leq ||u|| + ||v||$$

with equality if and only two conditions are satisfied

- cauchy-schwarz equality for u and v
- $\langle u, v\rangle + \overline{\langle u, v\rangle} = 2|\langle u, v\rangle|$

If both $||u|| = 0$ and $||v|| = 0$, trivial. Without loss of generality, assume that $||v|| > 0$, from the cauchy-schwarz equality, $u = \lambda v$ for any $\lambda \in \mathbb{F}$

$$\begin{aligned}
LHS &= \langle u, v\rangle + \overline{\langle u, v\rangle} && (2)\\
&= \langle u, v\rangle + \langle v, u\rangle && \text{(conjugate symmetry)}\\
&= \langle \lambda v, v\rangle + \langle v, \lambda v\rangle && \text{(rewrite)}\\
&= \lambda\langle v, v\rangle + \overline{\lambda}\langle v, v\rangle && \text{(homogeneity and conjugate homogeneity)}\\
&= (\lambda + \overline{\lambda})||v||^2 && \text{((rewrite))}\\
RHS &= 2|\langle \lambda v, v\rangle| && \text{(rewrite)}\\
&= 2|\lambda\langle v, v\rangle| && \text{(homogeneity)}\\
&= 2|\lambda||v||^2| && \text{(rewrite)}\\
&= 2|\lambda|||v||^2 && (||v||^2 \in \mathbb{R})
\end{aligned} \tag{3}$$

The condition in which $LHS = RHS$ is when $\lambda \geq 0$