SUBSPACES

Electronic version of lecture

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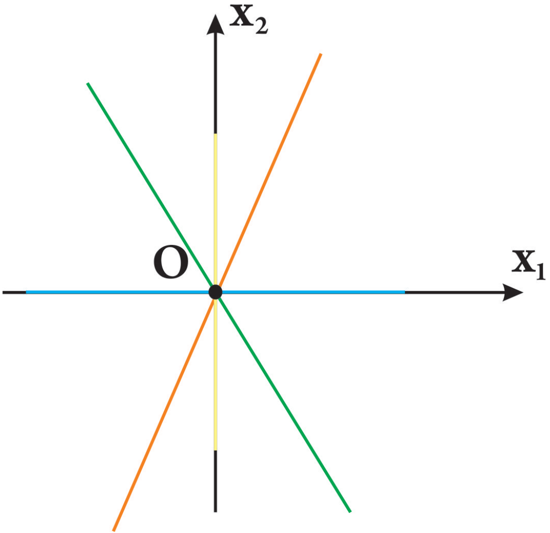
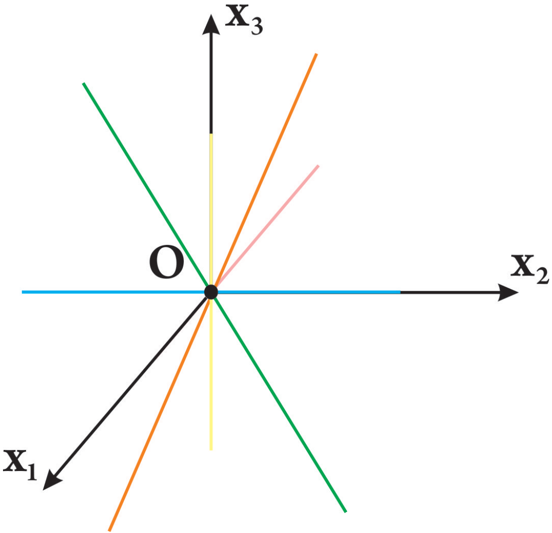
OUTLINE

1. SUBSPACES
2. OPERATIONS WITH SUBSPACES

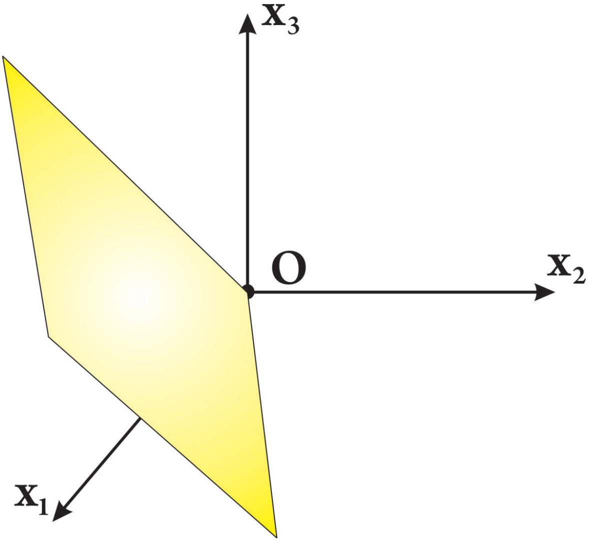
SUBSPACES OF $\mathbb{R}_2$: LINES THROUGH THE ORIGIN



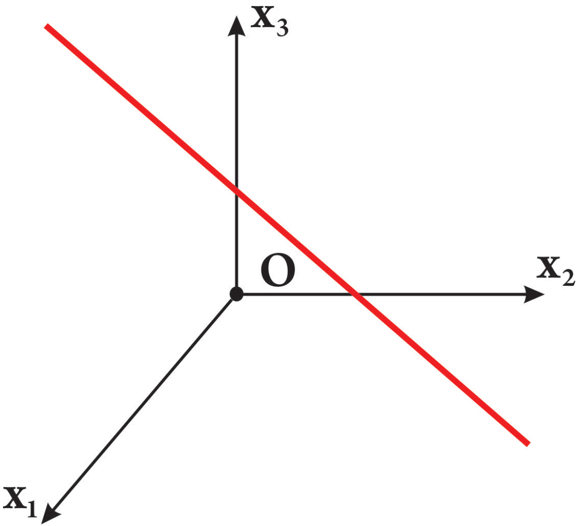
SUBSPACES OF $\mathbb{R}_3$: LINES THROUGH THE ORIGIN



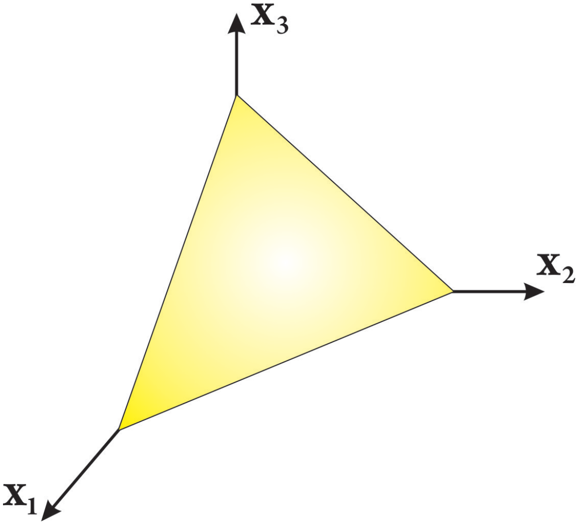
SUBSPACES OF $\mathbb{R}_3$: PLANES THROUGH THE ORIGIN



A subset of $\mathbb{R}_3$ that is Not a subspace



A SUBSET OF $\mathbb{R}_3$ THAT IS NOT A SUBSPACE



DEFINITION 1.1

A subset W of a vector space V is called a subspace of V if and only if the following conditions are satisfied

1. $W \neq \varnothing$
2. $\forall x, y \in W, x + y \in W$
3. $\forall \lambda \in \mathbb{R}, \forall x \in W, \lambda x \in W.$

EXAMPLE 1.1

$W = \mathbb{R} \times \{0\} = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 = 0\}$ *is the subspace of* $\mathbb{R}_2$.

We have $W \subset \mathbb{R}_2$, $(0,0) \in W \Rightarrow W \neq \varnothing$. For all $x = (x_1, 0), y = (y_1, 0) \in W$ we have

$$x + y = (x_1 + y_1, 0) \in W,$$

$$\forall \lambda \in \mathbb{R}, \lambda x = (\lambda x_1, 0) \in W.$$

Therefore, W is the subspace of $\mathbb{R}_2$.

EXAMPLE 1.2

$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - 2x_2 + x_3 = 0\}$ *is the subspace of* $\mathbb{R}_3$.

We have $W \subset \mathbb{R}_3$, $(0,0,0) \in W \Rightarrow W \neq \varnothing$.

$$\forall x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in W$$

$$\Rightarrow 2x_1 - 2x_2 + x_3 = 0, \quad 2y_1 - 2y_2 + y_3 = 0.$$

Therefore, $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, and
$2(x_1 + y_1) - 2(x_2 + y_2) + (x_3 + y_3) =$
$(2x_1 - 2x_2 + x_3) + (2y_1 - 2y_2 + y_3) = 0 \Rightarrow x + y \in W$.

$\forall \lambda \in \mathbb{R}, \lambda x = (\lambda x_1, \lambda x_2, \lambda x_3)$, then

$$2\lambda x_1 - 2\lambda x_2 + \lambda x_3 = \lambda(2x_1 - 2x_2 + x_3) = 0$$

$$\Rightarrow \lambda x \in W.$$

Thus, W is the subspace of $\mathbb{R}_3$.

EXAMPLE 1.3

$W = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}, x_1 + 2x_2 + x_3 = 1\}$ *is NOT a subspace of* $\mathbb{R}_3$.

Indeed, if $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in W$, then

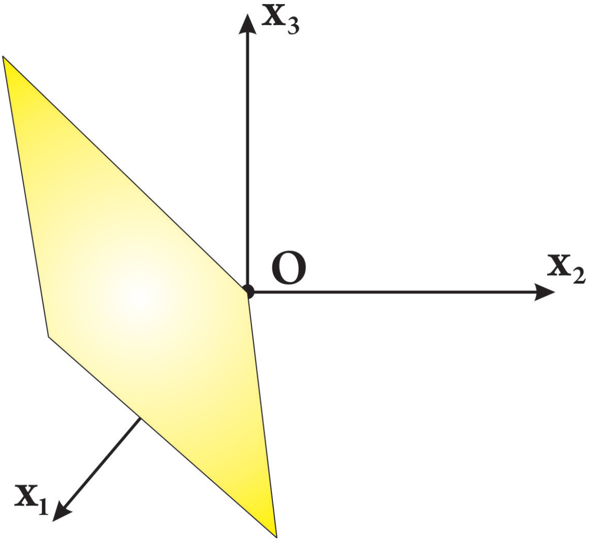
$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

and

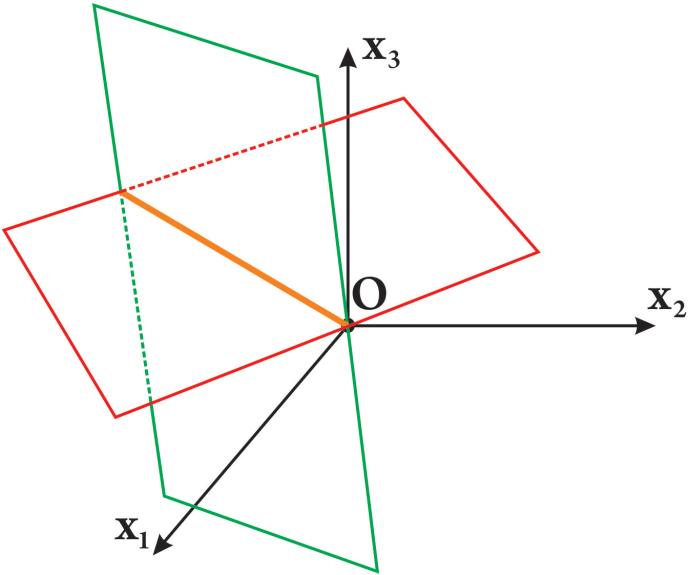
$$\begin{aligned}(x_1 + y_1) &+ 2(x_2 + y_2) + (x_3 + y_3) \\ &= (x_1 + 2x_2 + x_3) + (y_1 + 2y_2 + y_3) = 1 + 1 = 2. \quad (1)\end{aligned}$$

Thus, $x + y \notin W$.

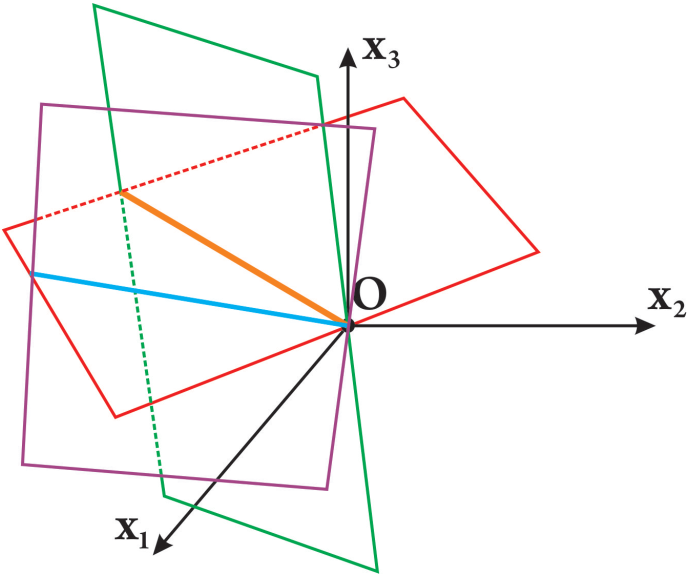
SOLUTION SPACES OF HOMOGENEOUS SYSTEMS



SOLUTION SPACES OF HOMOGENEOUS SYSTEMS



SOLUTION SPACES OF HOMOGENEOUS SYSTEMS



THEOREM 1.1

The solution set of a homogeneous linear system $A_{m\times n}X_{n\times 1} = 0_{m\times 1}$ of m equations in n unknowns is a subspace of $\mathbb{R}^n$, which is called the null space of matrix $A_{m\times n}$.

THEOREM 1.2

The dimension of the null space is $n - r$ where $r = rank(A)$ and n is the number of unknowns.

EXAMPLE 1.4

Find a basis and dimension for the subspace W of $\mathbb{R}_3$ which is defined by

$$W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$$

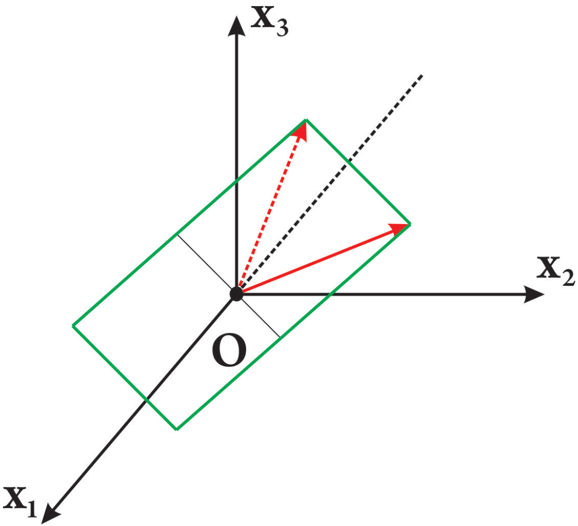
In order to find a basis of W, we solve the homogeneous system of linear equations

$$x_1 + x_2 + x_3 = 0 \Leftrightarrow x_1 = -x_2 - x_3.$$

The general solution

$$(x_1, x_2, x_3) = (-t_1 - t_2, t_1, t_2) = t_1(-1, 1, 0) + t_2(-1, 0, 1).$$

SOLUTION SPACE OF HOMOGENEOUS SYSTEM



- The 2 vectors $(-1,1,0)$ and $(-1,0,1)$ are linearly independent.

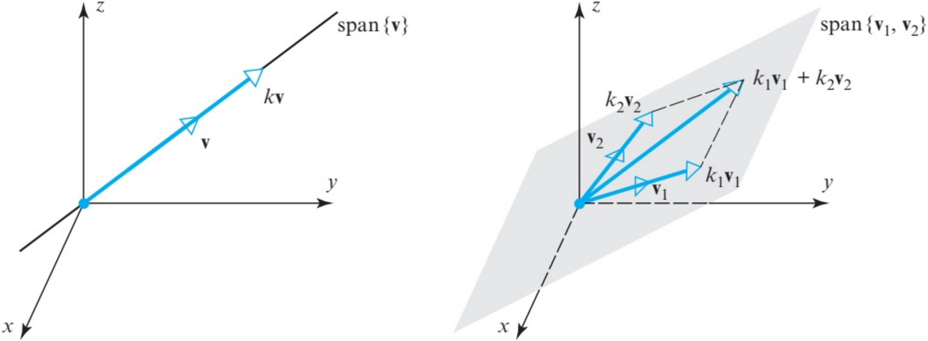
$$\alpha(-1,1,0)+\beta(-1,0,1)=\mathbf{0}$$
$$\Rightarrow (-\alpha-\beta,\alpha,\beta)=(0,0,0)\Rightarrow \alpha=\beta=0.$$

- We show that the 2 vectors $(-1,1,0)$ and $(-1,0,1)$ span W. Indeed, $\forall\, w=(x_1,x_2,x_3)\in W$ we have

$$w=x_2(-1,1,0)+x_3(-1,0,1)=(-x_2-x_3,x_2,x_3).$$

Therefore, the set of the 2 vectors $(-1,1,0)$ and $(-1,0,1)$ is the basis of W and the dimension of W is $\dim(W)=2$.

SPACE SPANNED BY A SET OF VECTORS



(a) Span $\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

(b) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

SPACE SPANNED BY A SET OF VECTORS

THEOREM 1.3

If $S = \{w_1, w_2, \ldots, w_n\}$ is a non-empty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V generated by S or subspace spanned by S. We denote this subspace as

$$W = span\left\{w_1, w_2, \ldots, w_n\right\} \text{ or } W = span(S)$$

EXAMPLE 1.5

The following vectors span a subspace of $\mathbb{R}_3$. *Find the subset of these vectors that forms a basis of this subspace.*

$$w_1 = (1, 2, 1), \quad w_2 = (2, 1, -1), \quad w_3 = (0, 4, 4).$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 0 & 4 & 4 \end{pmatrix}$$

$$A \xrightarrow{r_2 \to r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 4 & 4 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 4/3 r_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will start by expressing w_3 as linear combinations of w_1 and w_2.

$$w_3 + \frac{4}{3}\left(w_2 - 2w_1\right) = 0 \Rightarrow w_3 = -\frac{4}{3}w_2 + \frac{8}{3}w_1.$$

Now we show that the set of w_1 and w_2 is the basis of the subspace spanned by w_1, w_2, w_3.

1. The set $\{w_1, w_2\}$ is linear independent

$$\lambda_1 w_1 + \lambda_2 w_2 = 0 \Rightarrow \begin{cases} \lambda_1 + 2\lambda_2 = 0 \\ 2\lambda_1 + \lambda_2 = 0 \\ \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

2. w_1, w_2 span $W = spans\{w_1, w_2, w_3\}$. For all $w \in W$, we have

$$w = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = \lambda_1 w_1 + \lambda_2 w_2 +$$

$$+\lambda_3 \left(-\frac{4}{3} w_2 + \frac{8}{3} w_1\right) = \left(\lambda_1 + \frac{8\lambda_3}{3}\right) w_1 + \left(\lambda_2 - \frac{4\lambda_3}{3}\right) w_2$$

THEOREM 2.1

If U, W are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Proof.

1. $U \bigcap W \neq \varnothing$ because $\mathbf{0} \in U, W \Rightarrow \mathbf{0} \in U \bigcap W$.
2. $\forall x, y \in U \bigcap W \Rightarrow x, y \in U, W \Rightarrow x + y \in U, W$
   $\Rightarrow x + y \in U \bigcap W$
3. $x \in U \bigcap W \Rightarrow \forall \lambda \in \mathbb{R}, \lambda x \in U, W \Rightarrow \lambda x \in U \bigcap W$.

Definition 2.1

If U and W are subspaces of a vector space V. The set of all vectors of the form $u+w$, where u belongs to U and w to W, is called the sum of U and W.

$$U+W=\left\{x\in V,\exists u\in U,\exists w\in W: x=u+w\right\}$$

Theorem 2.2

$U+W$ is the subspace of V.

FORMULA CONNECTING THE DIMENSIONS OF THE SUM AND THE INTERSECTION OF 2 SUBSPACES

THEOREM 2.3

Let U and W be subspaces of a finitely generated vector space V. Then

$$dim(U+W) = dim(U) + dim(W) - dim(U \cap W)$$

EXAMPLE 2.1

In $\mathbb{R}_3$ *the following vectors* $u_1 = (1,2,1)$, $u_2 = (3,6,5)$, $u_3 = (4,8,6)$, $u_4 = (8,16,12)$ *and* $w_1 = (1,3,3)$, $w_2 = (2,5,5)$, $w_3 = (3,8,8)$, $w_4 = (6,16,16)$ *are given. Let*

$$U = span\left\{u_1, u_2, u_3, u_4\right\},$$

$$W = span\left\{w_1, w_2, w_3, w_4\right\}.$$

Find a basis and dimension of the sum $U + W$ *and the intersection* $U \cap W$.

- The basis and dimension of U

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 5 \\ 4 & 8 & 6 \\ 8 & 16 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $dim(U) = 2$ and the basis of U is

$$\left\{(1,2,1),(3,6,5)\right\}$$

- The basis and dimension of W

$$\begin{pmatrix} 1 & 3 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 8 \\ 6 & 16 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $dim(W) = 2$ and the basis of W is

$$\left\{(1,3,3),(2,5,5)\right\}$$

THE BASIS AND DIMENSION OF THE SUM $U+W$

The sum $U+W$ is spanned by the vectors $\Big\{(1,2,1),(3,6,5),(1,3,3),(2,5,5)\Big\}$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 5 \\ 1 & 3 & 3 \\ 2 & 5 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $dim(U+W)=3$ and the basis of $U+W$ is $\Big\{(1,2,1),(3,6,5),(1,3,3)\Big\}$.

THE BASIS AND DIMENSION OF $U \cap W$.

$$x \in U \cap W \Leftrightarrow \begin{cases} x = \alpha_1(1,2,1) + \alpha_2(3,6,5) \\ x = \alpha_3(1,3,3) + \alpha_4(2,5,5) \end{cases}$$

$$\Rightarrow \alpha_1(1,2,1) + \alpha_2(3,6,5) =$$

$$= \alpha_3(1,3,3) + \alpha_4(2,5,5)$$

$$\Leftrightarrow \alpha_3 = -\alpha_4 = -2\alpha_2 = 2\alpha_1$$

$$\Rightarrow x = \alpha_1(1,2,1) + \alpha_2(3,6,5) = \alpha_2(2,4,4).$$

Therefore, $dim(U \cap W) = 1$ and the basis of $U \cap W$ is $\left\{(2,4,4)\right\}$

EXAMPLE 2.2

For $\mathbb{R}_3$ the 2 subspaces

$$U = \left\{ (x_1, x_2, x_3) \;\middle|\; \begin{array}{l} x_1 + x_2 - 2x_3 = 0 \\ x_1 - x_2 - 2x_3 = 0 \end{array} \right\}$$

and

$$W = \left\{ (x_1, x_2, x_3) : x_1 = x_2 \right\}$$

are given. Find a basis and dimension of $U + W$ and $U \cap W$.

- The basis and dimension of U

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

Therefore, $dim(U) = 1$ and the basis of U is $\left\{(2,0,1)\right\}$

- The basis and dimension of W.

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$$

Therefore, $dim(W) = 2$ and the basis of W is $\left\{(1,1,0),(0,0,1)\right\}$

THE BASIS AND DIMENSION OF THE SUM $U+W$

The sum $U+W$ is spanned by the vectors $\left\{(2,0,1),(1,1,0),(0,0,1)\right\}$.

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $dim(U+W)=3$ and the basis of $U+W$ is $\left\{(2,0,1),(1,1,0),(0,0,1)\right\}$.

THE BASIS AND DIMENSION OF THE INTERSECTION $U \cap W$.

$$x \in U \cap W \Leftrightarrow \begin{cases} x_1 + x_2 - 2x_3 = 0 \\ x_1 - x_2 - 2x_3 = 0 \\ x_1 = x_2 \end{cases}$$

$$\Leftrightarrow x_1 = x_2 = x_3 = 0$$

Therefore, $dim(U \cap W) = 0$ and there is no basis of $U \cap W$.

EXAMPLE 2.3

For $\mathbb{R}_3$, the 2 subspaces

$$U = \left\{(x_1, x_2, x_3) \in \mathbb{R}_3 : x_1 + x_2 + x_3 = 0\right\},$$

and

$$W = span\left\{(1;0;1), (2;3;1)\right\}$$

are given. Find a basis and dimension of $U \cap W$ and $U + W$.

- The basis and dimension of U

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$

Therefore, $dim(U) = 2$ and the basis of U is $\left\{(-1,1,0),(-1,0,1)\right\}$

- The basis and dimension of W

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

Therefore, $dim(W) = 2$ and the basis of W is $\left\{(1,0,1),(2,3,1)\right\}$

THE BASIS AND DIMENSION OF THE SUM $U+W$

The sum $U+W$ is spanned by the vectors $\left\{(-1,1,0),(-1,0,1),(1,0,1),(2,3,1)\right\}$.

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $dim(U+W)=3$ and the basis of $U+W$ is $\left\{(-1,1,0),(-1,0,1),(1,0,1)\right\}$.

THE BASIS AND DIMENSION OF $U \cap W$

$\forall x \in U \cap W \iff x \in U \wedge x \in W$.
$x \in W \iff x = \alpha(1,0,1) + \beta(2,3,1)$
$\iff x = (\alpha + 2\beta, 3\beta, \alpha + \beta)$.
$x \in U \iff x$ satisfies the condition of U:

$$\alpha + 2\beta + 3\beta + \alpha + \beta = 0 \iff \alpha = -3\beta.$$

$$\begin{aligned} x &= (\alpha + 2\beta, 3\beta, \alpha + \beta) \\ &= (-\beta, 3\beta, -2\beta) = \beta(-1,3,-2). \end{aligned}$$

Therefore, $\left\{(-1,3,-2)\right\}$ is the basis of $U \cap W$ and $\dim(U \cap W) = 1$.

THANK YOU FOR YOUR ATTENTION