# DETERMINANTS

## ELECTRONIC VERSION OF LECTURE

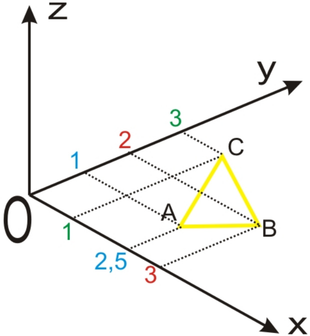
**HoChiMinh City University of Technology**
**Faculty of Applied Science, Department of Applied Mathematics**

# OUTLINE

1. REAL-WORLD PROBLEMS
2. DETERMINANTS
3. INVERSE OF AN MATRIX
4. MATLAB

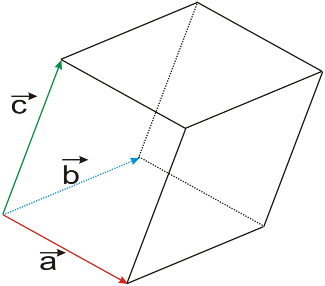
# EVALUATING THE AREA OF THE TRIANGLE


$\Longrightarrow$




$$S = \frac{1}{2} abs|[\overrightarrow{AB}, \overrightarrow{AC}]| = \frac{1}{2} abs \begin{vmatrix} 2,5 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \frac{5}{4}$$

# EVALUATING THE VOLUME OF THE PARALLELEPIPED




$$\overrightarrow{a} = (a_1, a_2, a_3);$$

$$\overrightarrow{b} = (b_1, b_2, b_3); \overrightarrow{c} = (c_1, c_2, c_3)$$

$$\Rightarrow V = abs([\overrightarrow{a} \times \overrightarrow{b}], \overrightarrow{c}) = abs \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Definition 2.1**

*If $A = (a_{ij})$ is a square matrix, then the determinant of $A$ is a number. We denote it by $det(A)$ or $|A|$.*

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*If $A = (a_{ij})$ is a square matrix, then the* *determinant* *of $A$ is a* *number*. *We denote it by $det(A)$ or $|A|$.*

So

$$det : M_n(K) \to K$$

$$A \to det A.$$

### DEFINITION 2.2

*If* $A = (a_{ij})_{n\times n}$ *is a square matrix, then the minor of entry* $a_{ij}$ *is denoted by* $M_{ij}$ *and is defined to be the determinant of the submatrix of order* $(n-1)$ *that remains after the* $i-th$ *row and* $j-column$ *are deleted from* $A$.

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n)(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{n\times n}$$

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$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{(n-1)\times(n-1)}$$

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## Definition 2.3

*If $A = (a_{ij})_{n\times n}$ is a square matrix, then the number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of entry $a_{ij}$.*

# DEFINITION 2.4

*If $A$ is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of $A$ by the corresponding cofactors and adding the resulting products is called the determinant of $A$, and the sums themselves are called cofactor expansion of $A$. That is,*

$$det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

$$det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

# COFACTOR EXPANSION ALONG THE FIRST ROW

$$det(A) = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{1j} C_{1j} =$$

$$= \sum_{j=1}^{n} a_{1j}.(-1)^{1+j} M_{1j}.$$

1. $n = 1$, $A = (a_{11}) \Rightarrow |A| = a_{11}$.

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2. $n = 2, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow |A| =$
$(-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} = a_{11}a_{22} - a_{12}a_{21}.$

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$(-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} = a_{11}a_{22} - a_{12}a_{21}$.
3. $n = 3, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow |A| =$
$(-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$
$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +$
$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

## Example 2.1

*Find the determinant* $\det A$ *of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

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*Find the determinant* $\det A$ *of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

**Solution.** Cofactor expansion along the first row: $|A| = 1.C_{11} + 2.C_{12} + 3.C_{13}$.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 2 \times 5 - 1 \times 1 = 9,$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix} = -(4 \times 5 - 1 \times 3) = -17,$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4 \times 1 - 2 \times 3 = -2.$$

Therefore,

$$|A| = 1 \times 9 + 2 \times (-17) + 3 \times (-2) = -31.$$

# SMART CHOICE OF ROW OR COLUMN

# Smart Choice of Row or Column

We can find determinant using cofactor expansion along any row.

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Determinant also can be found using cofactor expansion along any column.

$$det A = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Determinant also can be found using cofactor expansion along any column.

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It will be easiest to use cofactor expansion along the row or column which has the most zeros.

# Example 2.2

$$Evaluate\ det A\ where\ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$$

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*Evaluate* $det A$ *where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$

**Solution.** Cofactor expansion along the second row:

$$|A| = 0.C_{21} + 2.C_{22} + 0.C_{23} =$$

$$= 2.(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 2(1 \times 5 - 3 \times 3) = -8.$$

# Example 2.3

$$\text{Evaluate } \det A \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

# EXAMPLE 2.3

*Evaluate* $det A$ *where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$

**Solution.** Cofactor expansion along the third column

$$|A| = 3.C_{13} + 0.C_{23} + 0.C_{33} =$$

$$= 3.(-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 3(2 \times 1 - 1 \times 3) = -3.$$

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2. If $A \xrightarrow{r_i \to \lambda r_i (c_i \to \lambda c_i)} B$ then $detB = \lambda detA$ where $\lambda \neq 0$.

# HOW AN ELEMENTARY ROW (OR COLUMN) OPERATION ON A SQUARE MATRIX AFFECTS THE VALUE OF ITS DETERMINANT

1. If $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} B$ then $detB = -detA$ .
2. If $A \xrightarrow{r_i \to \lambda r_i (c_i \to \lambda c_i)} B$ then $detB = \lambda detA$ where $\lambda \neq 0$.
3. If $A \xrightarrow{r_i \to r_i + \lambda . r_j (c_i \to c_i + \lambda c_j)} B$ then
$$detB = detA, \forall \lambda \in K$$

## COROLLARY 2.1

1. *If $A$ is a square matrix with 2 equal rows or 2 equal columns then $det(A) = 0$.*

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$det A = -det A \Rightarrow det A = 0.$

# Corollary 2.1

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   $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} A$ *where $i, j$ are 2 equal rows or 2 equal columns*
   $det A = -det A \Rightarrow det A = 0.$
2. *If $A$ is a square matrix with 2 proportional rows or 2 proportional columns then $det(A) = 0$.*

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   $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} A$ *where $i, j$ are 2 equal rows or 2 equal columns*
   $det A = -det A \Rightarrow det A = 0.$
2. *If $A$ is a square matrix with 2 proportional rows or 2 proportional columns then $det(A) = 0$. Since*
   $A \xrightarrow{r_i \to \lambda r_i (c_i \to \lambda c_i)} B$ *where $\lambda \neq 0$ is the ratio of 2 rows or 2 columns, $det B = \lambda det A$, where*

# EXAMPLE 2.4

*Use Row Reduction to evaluate the determinant* $\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix}$

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$$\begin{vmatrix} 0 & 19 & -16 \\ 1 & -8 & 6 \\ 0 & 44 & -27 \end{vmatrix} \overset{\text{Cofactor expansion along the first column}}{=\!=\!=\!=}$$

$$= 1.(-1)^{2+1}.\begin{vmatrix} 19 & -16 \\ 44 & -27 \end{vmatrix} = -191.$$

# DETERMINANT OF A MATRIX PRODUCT

**THEOREM 2.1**

*If $A, B$ are square matrices of the same size, then*

$$\boxed{det(AB) = det(A).det(B)} \tag{1}$$

## Example 2.5

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 6 \\ 2 & 8 & 9 \end{pmatrix}, B = \begin{pmatrix} 7 & 8 & 9 \\ 4 & -3 & 6 \\ -1 & 2 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 12 & 8 & 30 \\ 14 & 50 & 42 \\ 37 & 10 & 93 \end{pmatrix}$$

*Verify that*

$$det(A).det(B) = (-6).(-246) = det(AB) = 1476$$

# Corollary 2.2

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1. $det(A^k) = (det A)^k$.

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$$det(A^k) = det(\underbrace{A.A\ldots A}_{k\ times}) = \underbrace{det A.det A\ldots det A}_{k\ times} = (det A)^k.$$

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2. $det(\alpha AB) = \alpha^n.det A.det B.$
*Indeed,* $det(\alpha AB) = det(\alpha A).det B = \underbrace{\alpha.\alpha\dots\alpha}_{n \text{ times}} det A.det B$

# Example 2.6

*Evaluate* $det(X)$ *if* $X$ *satisfies*

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{pmatrix}$$

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*Evaluate* $det(X)$ *if* $X$ *satisfies*

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We have $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} .det(X) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{vmatrix}$

$\Rightarrow$

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$$\Rightarrow 1.det(X) = 3 \Rightarrow det(X) = 3.$$

## Example 2.7

*If* $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$, *then evaluate* $det(A^{2011})$.

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*If* $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$, *then evaluate* $det(A^{2011})$.

We have

$$det(A^{2011}) = (det A)^{2011} = (-1)^{2011} = -1.$$

## Example 2.8

$$\textit{If } A = \begin{pmatrix} 3 & -2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 5 \\ 1 & -2 & 7 \end{pmatrix}, \textit{ then}$$

*evaluate* $det(2AB)$.

## Example 2.8

*If* $A = \begin{pmatrix} 3 & -2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 5 \\ 1 & -2 & 7 \end{pmatrix}$, *then evaluate* $det(2AB)$.

We have

$$det(2AB) = 2^3.det A.det B = 8 \times 3 \times 2 = 48.$$

# Adjoint of a Matrix

**Definition 3.1**

*If $A = (a_{ij})$ is any $n \times n$ matrix and $C_{ij}$ is cofactor of entry $a_{ij}$, then*

$$P_A = \begin{pmatrix} C_{11} & \dots & C_{1j} & \dots & C_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{i1} & \dots & C_{ij} & \dots & C_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nj} & \dots & C_{nn} \end{pmatrix}^T =$$

# ADJOINT OF A MATRIX

## DEFINITION 3.1

*If $A = (a_{ij})$ is any $n \times n$ matrix and $C_{ij}$ is cofactor of entry $a_{ij}$, then*

$$P_A = \begin{pmatrix} C_{11} & \dots & C_{1j} & \dots & C_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{i1} & \dots & C_{ij} & \dots & C_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nj} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & \dots & C_{i1} & \dots & C_{n1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1j} & \dots & C_{ij} & \dots & C_{nj} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{in} & \dots & C_{nn} \end{pmatrix}$$

*is called the adjoint of $A$ and is denoted by $adj(A)$ or $P_A$.*

## Example 3.1

*If* $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 5 & 3 & -1 \end{pmatrix}$, *then find the adjoint of* $A$.

## Example 3.1

*If* $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 5 & 3 & -1 \end{pmatrix}$, *then find the adjoint of A.*

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix}; \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix};$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix}; \quad C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix};$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 5 & -1 \end{vmatrix}; \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix},$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}; \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix};$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}.$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}; \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix};$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}.$$

Therefore, the adjoint of $A$ is

$$P_A = \begin{pmatrix} -10 & 13 & -11 \\ 6 & -7 & 9 \\ 2 & -1 & -1 \end{pmatrix}^T = \begin{pmatrix} -10 & 6 & 2 \\ 13 & -7 & -1 \\ -11 & 9 & -1 \end{pmatrix}.$$

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$\frac{1}{det A}.P_A.A =$

$$\frac{1}{det A}.\begin{pmatrix} C_{11} & \dots & C_{i1} & \dots & C_{n1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1j} & \dots & C_{ij} & \dots & C_{nj} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{in} & \dots & C_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

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Since $\begin{vmatrix} a_{12} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{22} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_{i2} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_{n2} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nn} \end{vmatrix} = 0$

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$$= \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

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where $C_{ij} = (-1)^{i+j} M_{ij}$.

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3. **Step 3.** $A^{-1} = \dfrac{1}{det A} P_A$

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*Find the inverse of the matrix*

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## Example 3.3

*Find matrix $X$ which satisfies*

$$\begin{pmatrix} 0 & -8 & 3 \\ 1 & -5 & 9 \\ 2 & 3 & 8 \end{pmatrix} X = \begin{pmatrix} -25 & 23 & -30 \\ -36 & -2 & -26 \\ -16 & -26 & 7 \end{pmatrix}$$

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# MATLAB

- **Evaluating determinant of matrix $A$:**
  $det(A)$
- **Finding inverse of Matrix $A$:**
  $A\hat{}(-1)$ or $inv(A)$

# THANK YOU FOR YOUR ATTENTION