APPLICATIONS OF LINEAR SYSTEMS

ELECTRONIC VERSION OF LECTURE

HoChiMinh City University of Technology
Faculty of Applied Science, Department of Applied Mathematics

OUTLINE

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DEFINITION 1.1

A network is a set of branches through which something "flows".

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EXAMPLE 1.1

The branches might be electrical wires through which electricity flows, pipes through which water or oil flows, traffic lanes through which vehicular traffic flows,...

Definition 1.2

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There if generally some numerical measure of the rate *at which the medium flows through a branch.* ***For example,*** *the flow rate of electricity is measured in amperes, the flow rate of water or oil in gallons per minute, the flow rate of traffic in vehicles per hour,...*

Flow conservation at each node.

The rate of flow into any node is equal to the rate of flow out of that node.

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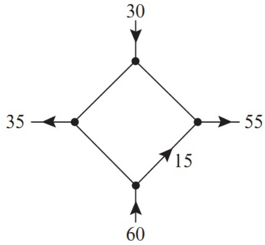
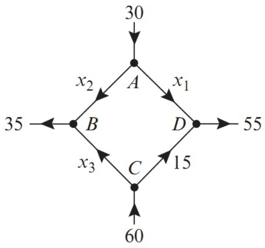
Example 1.2

Find the flow rates and directions of flow in the branches



Solution. We will assign arbitrary directions to the unknown flow rates x_1, x_2, and x_3. We need not be concerned if some of the directions are incorrect, since an incorrect direction will be signalled by a negative value for the flow rate when we solve for the unknowns.



Node	**Flow In**		**Flow Out**
A	30	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	35
C	60	$=$	$x_3 + 15$
D	$x_1 + 15$	$=$	55

These conditions produce the linear system

$$\begin{cases} x_1 + x_2 = 30 \\ x_2 + x_3 = 35 \\ x_3 + 15 = 60 \\ x_1 + 15 = 55 \end{cases} \Rightarrow \begin{cases} x_1 = 40 \\ x_2 = -10 \\ x_3 = 45 \end{cases}$$

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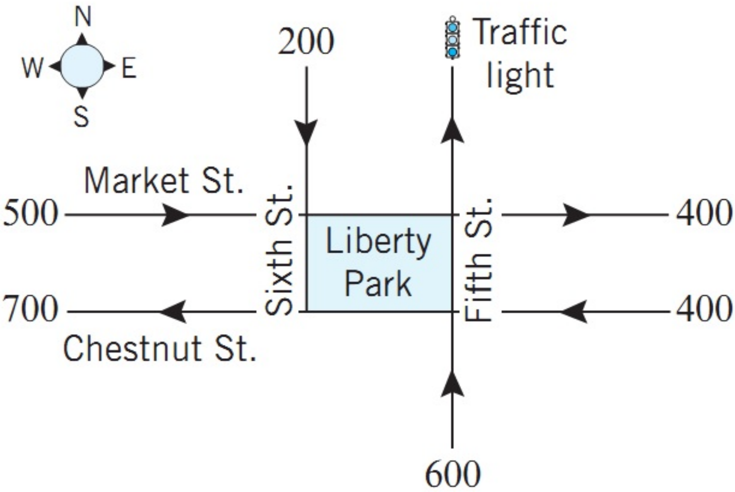
The flow in branch x_2 is into node A.

Example 1.3 (Design of Traffic Patterns.)

The following network shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.


N
W
E
S
200
Traffic light
Market St.
500
400
Sixth St.
Liberty Park
Fifth St.
700
400
Chestnut St.
600

1. How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

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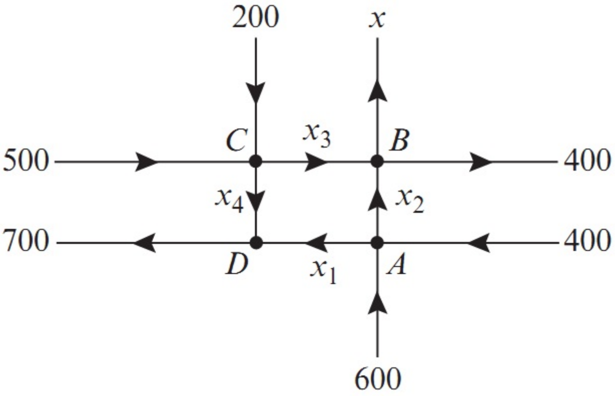
2. Assuming that the traffic light has been set to balance the total flow in and out of the complex, what is the average number of vehicles per hour that will flow along the streets that border the complex?

Solution. Let x denote the number of vehicles per hour that the traffic light must be let through, then the total number of vehicles per hour that flow in and out of the complex will be

Total	**Flowing In**		**Flowing Out**
	$500+400+600+200$	$=$	$x+700+400$

$$x+1100=1700 \Rightarrow x=600.$$

To avoid traffic congestion, the flow in must equal the flow out at each intersection.



Intersection	Flow In		Flow Out
A	$400+600$	$=$	x_1+x_2
B	x_2+x_3	$=$	$400+x$
C	$500+200$	$=$	x_3+x_4
D	x_1+x_4	$=$	700

Because the total flow in and out of the complex are equal, so $x=600$.

We obtain

$$\begin{cases} x_1 + x_2 = 1000 \\ x_2 + x_3 = 1000 \\ x_3 + x_4 = 700 \\ x_1 + x_4 = 700 \end{cases} \Rightarrow \begin{cases} x_1 = 700 - t \\ x_2 = 300 + t \\ x_3 = 700 - t \\ x_4 = t \end{cases}$$

The average flow rates must be non-negative since we have assumed the streets to be one-way. Therefore, $0 \leqslant t \leqslant 700 \Rightarrow 0 \leqslant x_1 \leqslant 700$; $300 \leqslant x_2 \leqslant 1000$; $0 \leqslant x_3 \leqslant 700$; $0 \leqslant x_4 \leqslant 700$.

Electrical Circuits

Definition 2.1

A battery is a source of electric energy, and a resistor is an element that dissipates electric energy.

Theorem 2.1 (Ohm's Law)

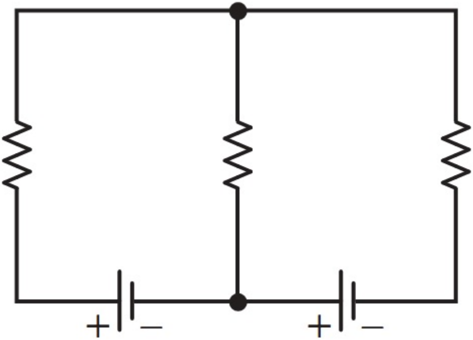
If a current of I amperes passes through a resistor with a resistance of R ohms, then there is a resulting drop of E volts in electrical potential

$$\boxed{E = IR} \tag{1}$$

DEFINITION 2.2

A typical electrical network will have multiple batteries and resistors joined by some configuration of wires. A point at which three or more wires in a network are joined is called a node (or junction point). A branch is a wire connecting 2 nodes, and a closed loop is a succession of connected branches that begin and end at the same node.

EXAMPLE 2.1



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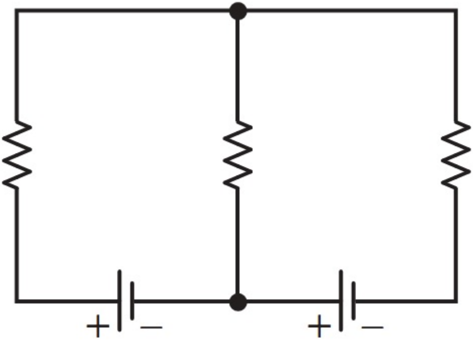


This electrical network has 2 nodes and three closed loops - 2 inner loops and 1 outer loop.

KIRCHHOFF'S LAWS

DEFINITION 2.3

As current flows through an electrical network, it undergoes increases and decreases in electrical potential, called voltage rises and voltage drops, respectively.

The behavior of the current at the nodes and around closed loops is governed by two fundamental laws:

Theorem 2.2 (Kirchhoff's Current Law)

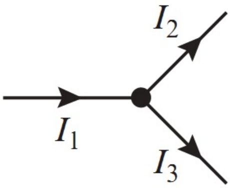
The sum of the currents flowing into any node is equal to the sum of the currents flowing out.

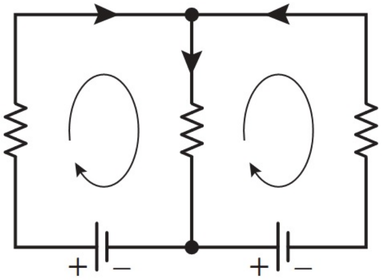
Theorem 2.3 (Kirchhoff's Voltage Law)

In one traversal of any closed loop, the sum of the voltage rises equals the sum of the voltage drops.

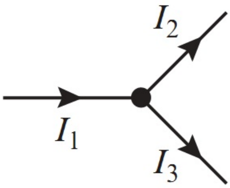
In circuits with multiple loops and batteries there is usually no way to tell in advance which way the currents are flowing, so for consistency we will always take the direction of travel for each closed loop to be clockwise.

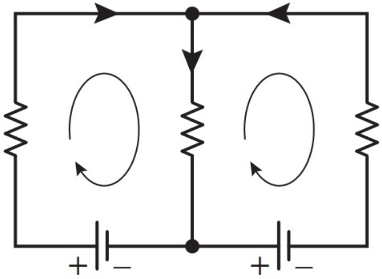
EXAMPLE 2.2





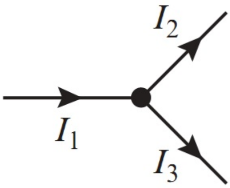
EXAMPLE 2.2

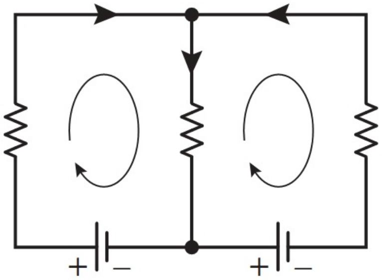




$$I_1 = I_2 + I_3.$$

EXAMPLE 2.2





$I_1 = I_2 + I_3$. *Clockwise closed-loop convention with arbitrary direction assignments to currents in the branches.*

Theorem 2.4 (Convention for resistor)

1. *A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned to the loop.*

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2. *A voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that assigned to the loop.*

Theorem 2.5 (Convention for battery)

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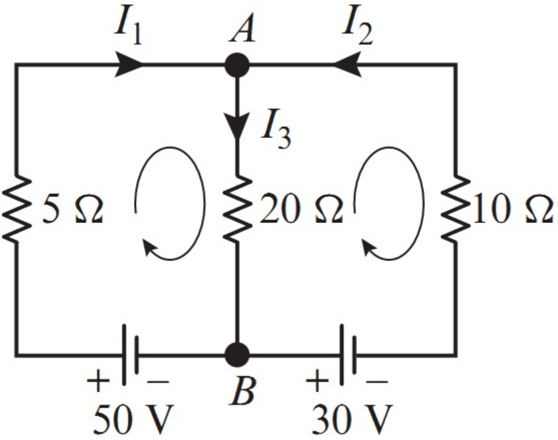
Those currents whose directions were assigned correctly will have positive values and those whose directions were assigned incorrectly will have negative values.

EXAMPLE 2.3

Determine the currents I_1, I_2, and I_3 in the following circuit.



KIRCHHOFF'S CURRENT LAW

Using the assigned directions for the currents, Kirchhoff's current law provides one equation for each node.

Node	Current In		Current Out
A	$I_1 + I_2$	$=$	I_3
B	I_3	$=$	$I_1 + I_2$

Kirchhoff's Voltage Law

	Voltage Rises		Voltage Drops
Left Inside Loop	50	$=$	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	$=$	0
Outside Loop	$30 + 50 + 10I_2$	$=$	$5I_1$

KIRCHHOFF'S VOLTAGE LAW

	Voltage Rises		Voltage Drops
Left Inside Loop	50	$=$	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	$=$	0
Outside Loop	$30 + 50 + 10I_2$	$=$	$5I_1$

Solving the following linear system

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 5I_1 + 20I_3 = 50 \\ 10I_2 + 20I_3 = -30 \\ 5I_1 - 10I_2 = 80 \end{cases} \Rightarrow \begin{cases} I_1 = 6 \\ I_2 = -5 \\ I_3 = 1. \end{cases}$$

THANK YOU FOR YOUR ATTENTION