SYSTEMS OF LINEAR EQUATIONS

ELECTRONIC VERSION OF LECTURE

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OUTLINE

1. DEFINITION
2. NON-HOMOGENEOUS LINEAR SYSTEM
3. HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS
4. MATLAB

SPORTS

In 1998, Cynthia Cooper of the WNBA Houston Comets basketball team was named Team Sportswoman of the Year by the Women's Sports Foundation. Cooper scored 680 points in the 1998 season by hitting 413 of her 1−point, 2−point, and 3−point attempts. She made 40% of her 160 3−point field goal attempts. How many 1−, 2−, and 3−point baskets did Ms. Cooper complete?

Let x, y, z be the number of $1-$ point free throws, $2-$ and $3-$point field goals respectively. Then we can write a system of equations

$$\begin{cases} x+2y+3z &= 680 \\ x+y+z &= 413 \\ \dfrac{z}{160} &= 0.4 \end{cases}$$

$$\Leftrightarrow \begin{cases} x &= 210 \\ y &= 139 \\ z &= 64 \end{cases}$$

Linear systems in two unknowns arise in connection with intersections of lines.

EXAMPLE 1.1

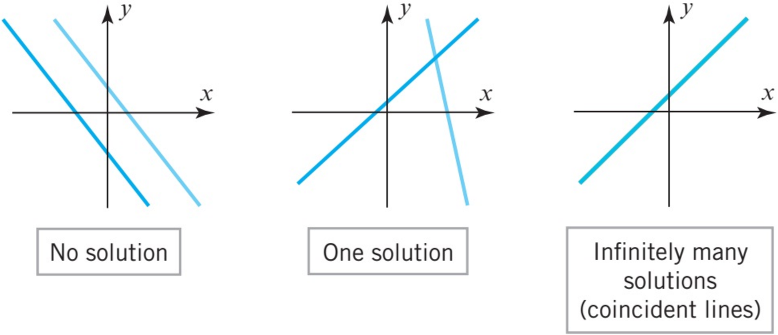
Consider the linear system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

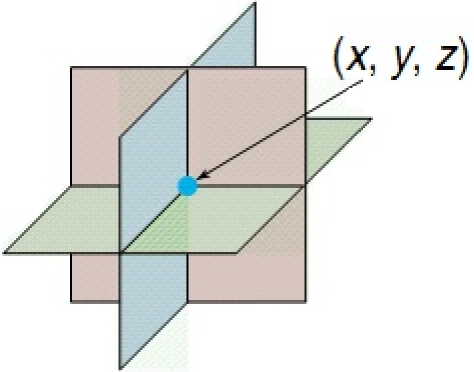
in which the graphs of the equations are lines in the $xy-$*plane. Each solution* (x, y) *of this system corresponds to a point of intersection of the lines.*

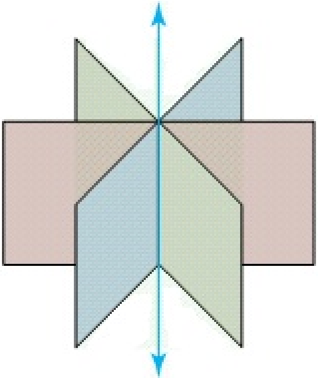
There are 3 possibilities:

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions.


y
x
No solution
y
x
One solution
y
x
Infinitely many solutions (coincident lines)

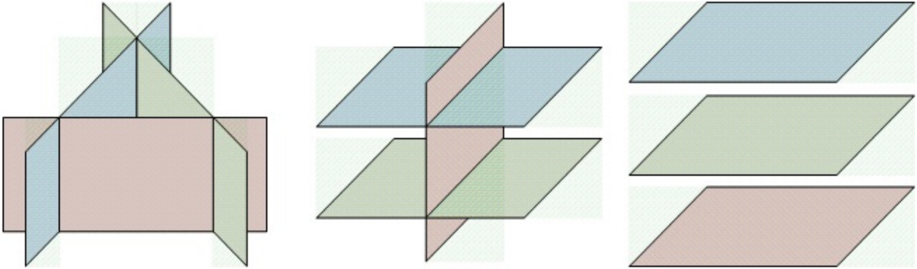
LINEAR SYSTEMS IN THREE UNKNOWNS





Unique solution

Infinitely many solutions



No solution

DEFINITION 1.1

A *general linear system* *of m equations in the n unknowns can be written as:*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n & = & b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n & = & b_i \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n & = & b_m \end{cases} \tag{1}$$

where a_{ij} are the ***coefficients*** *of the system, b_i are* ***constants*** *of the system, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $x_1, x_2, \ldots, x_n$ are the* ***unknowns****.*

The double subscripting on the coefficients of the unknowns is a useful device that is used to specify the location of the coefficient in the system.

1. The first subscript on the coefficient a_{ij} indicates the equation in which the coefficient occurs,
2. and the second subscript indicates which unknown it multiplies.

DEFINITION 1.2

A ***solution*** *of the system (1) is a sequence of* n *numbers* $(s_1, s_2, \ldots, s_n)$ *such that the equations of the system (1) are satisfied when we substitute* $x_1 = s_1$, $x_2 = s_2, \ldots,$ $x_n = s_n$.

Definition 1.3

Matrix $A=(a_{ij})_{m\times n}$ is called the coefficient matrix of the system (1).

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}_{m\times n}$$

Definition 1.4

Matrix

$$A_B = \left(\begin{array}{cccccc|c} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array}\right)_{m\times(n+1)}$$

is called the augmented matrix for the system (1), which is obtained by adjoining column B to matrix A as the last column.

If we let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ then the system (1) can be written in the matrix form

$$\begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n \\ \vdots & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A_{m\times n}X_{n\times 1} = B_{m\times 1}.$$

DEFINITION 1.5

The system (1) is called a homogeneous if $B = 0_{m\times 1}$ and a nonhomogeneous if $B \neq 0_{m\times 1}$.

EVERY NON-HOMOGENEOUS LINEAR SYSTEM HAS:

- no solution
- unique solution
- infinitely many solutions

DEFINITION 2.1

A linear system is consistent if it has at least one solution (unique solution or infinitely many solutions) and inconsistent if it has no solutions.

Solving system of linear equations

- In this section we shall develop a systematic procedure for solving systems of linear equations.
- The procedure is based on the idea of reducing the augmented matrix of a system to another augmented matrix that is simple enough that the solution of the system can be found by inspection.

Consider the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n & = & b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n & = & b_i \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n & = & b_m \end{cases}$$

If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in reduced row-echelon form, then the solution set of the system will be evident by inspection or after a few simple steps.

Operations that lead to equivalent systems of Equations

If we perform the following elementary row operations on the system (1):

1. Interchange two equations $(r_i \leftrightarrow r_j)$
2. Multiply an equation through by a nonzero constant $\lambda \neq 0 (r_i \rightarrow \lambda r_i)$.
3. Add a constant times one equation to another $(r_i \rightarrow r_i + \lambda r_j)$

then we obtain a new system that has the same solution set but is easier to solve.

EXAMPLE 2.1

Solve the system by Gaussian elimination

$$\begin{cases} x_1 + 2x_2 + 3x_3 &= 7 \\ 2x_1 + x_2 + 2x_3 &= 6 \\ 3x_1 + 2x_2 + x_3 &= 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 2 & 1 & 2 & 6 \\ 3 & 2 & 1 & 7 \end{array}\right) \xrightarrow[]{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -3 & -4 & -8 \\ 0 & -4 & -8 & -14 \end{array}\right)$$

$$\xrightarrow{r_2 \to r_2 - r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & 4 & 6 \\ 0 & -4 & -8 & -14 \end{array}\right) \xrightarrow{r_3 \to r_3 + 4r_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 8 & 10 \end{array}\right)$$

The system corresponding to this matrix is

$$\begin{cases} x_1 + 2x_2 + 3x_3 &= 7 \\ x_2 + 4x_3 &= 6 \\ 8x_3 &= 10 \end{cases} \Leftrightarrow \begin{cases} x_1 &= \dfrac{5}{4} \\ x_2 &= 1 \\ x_3 &= \dfrac{5}{4} \end{cases}$$

Thus the system has unique solution

$$(x_1, x_2, x_3)^T = \left(\frac{5}{4}, 1, \frac{5}{4}\right)^T$$

EXAMPLE 2.2

Solve the system by Gaussian elimination

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 1 \\ x_1 + 3x_2 - 13x_3 = -1 \\ 3x_1 + 5x_2 + x_3 = 5 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 1 & 3 & -13 & -1 \\ 3 & 5 & 1 & 5 \end{array}\right) \xrightarrow[r_3 \to r_3 - 3r_1]{r_2 \to r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -10 & -2 \\ 0 & -1 & 10 & 2 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -10 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to this matrix is

$$\begin{cases} x_1 + 2x_2 - 3x_3 & = & 1 \\ \quad\quad\;\; x_2 - 10x_3 & = & -2 \end{cases}$$

Solving for the leading variables, we obtain

$$\begin{cases} x_1 & = & 1 - 2x_2 + 3x_3 \\ x_2 & = & -2 + 10x_3 \end{cases}$$

Finally, we express the general solution of the system parametrically by assigning the free variables x_3 arbitrary value α. This means that $x_3 = \alpha$, where $\alpha \in \mathbb{R}$, we can find

$$\begin{cases} x_2 &= -2 + 10x_3 = -2 + 10\alpha \\ x_1 &= 1 - 2x_2 + 3x_3 = 5 - 17\alpha \end{cases}$$

So the system has infinitely many solutions $(x_1, x_2, x_3)^T = (5 - 17\alpha, -2 + 10\alpha, \alpha)^T$, where $\alpha \in \mathbb{R}$ is arbitrary number.

Example 2.3

Solve the system by Gaussian elimination

$$\begin{cases} x_1 & -2x_2 & +3x_3 & = & 2 \\ 3x_1 & +3x_2 & & = & -3 \\ 3x_1 & & +3x_3 & = & 8 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 3 & 3 & 0 & -3 \\ 3 & 0 & 3 & 8 \end{array}\right) \xrightarrow[]{\substack{r_2 \to r_2 - 3r_1 \\ r_3 \to r_3 - 3r_1}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 9 & -9 & -9 \\ 0 & 6 & -6 & 2 \end{array}\right)$$

$$\xrightarrow{r_2 \leftrightarrow r_2/9} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -6 & 2 \end{array}\right) \xrightarrow{r_3 \to r_3 - 6r_2}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

The system corresponding to this matrix is

$$\begin{cases} x_1 - 2x_2 + 3x_3 &= 2 \\ x_2 - x_3 &= -1 \\ 0 &= 8 \end{cases}$$

This system has no solution.

Definition 3.1

A system of linear equations is said to be homogeneous if the constant terms are all zero.

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n & = & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n & = & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n & = & 0 \end{array}\right. \qquad (2)$$

1. The trivial solution is $X = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix}^T$.
2. The nontrivial solution is $X \neq \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix}^T$.

HOMOGENEOUS LINEAR SYSTEM ALWAYS HAS:

1. only the trivial solution.
2. or infinitely many solutions in addition to the trivial solution, i.e. nontrivial solutions

THEOREM 3.1

A homogeneous linear system (2) has *non-trivial solutions* *if and only if*

$$r(A) < n,$$

where n is the number of unknowns.

Indeed, if $r(A) = n$, then the system (2) has only trivial solution $X = 0$.
If $r(A) < n$, then the system (2) has infinitely many solutions or non-trivial solutions.

If $r(A) = r < n$, then the system (2) has general solution:

$$\left\{\begin{array}{rcl} x_1 & = & \varphi_1(t_1, t_2, \ldots, t_{n-r}) \\ x_2 & = & \varphi_2(t_1, t_2, \ldots, t_{n-r}) \\ \ldots & & \\ x_r & = & \varphi_r(t_1, t_2, \ldots, t_{n-r}) \\ x_{r+1} & = & t_1 \\ \ldots & & \\ x_n & = & t_{n-r} \end{array}\right. \tag{3}$$

where $t_1, \ldots, t_{n-r}$ are arbitrary numbers, which are called free variables.

Example 3.1

Solve the system by Gaussian elimination

$$\begin{cases} x_1+3x_2+3x_3+2x_4+4x_5 &= 0 \\ x_1+4x_2+5x_3+3x_4+7x_5 &= 0 \\ 2x_1+5x_2+4x_3+x_4+5x_5 &= 0 \\ x_1+5x_2+7x_3+6x_4+10x_5 &= 0 \end{cases}$$

Solution.

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 4 \\ 1 & 4 & 5 & 3 & 7 \\ 2 & 5 & 4 & 1 & 5 \\ 1 & 5 & 7 & 6 & 10 \end{pmatrix} \xrightarrow{\substack{r_2\to r_2-r_1 \\ r_3\to r_3-2r_1 \\ r_4\to r_4-r_1}}$$

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & -1 & -2 & -3 & -3 \\ 0 & 2 & 4 & 4 & 6 \end{pmatrix} \xrightarrow[r_4 \to r_4 - 2r_2]{r_3 \to r_3 + r_2}$$

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \xrightarrow{r_4 \to r_4 + r_3} \begin{pmatrix} 1 & 3 & 3 & 2 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Free variables: x_3, x_5

The system corresponding to this matrix is

$$\begin{cases} x_1 + 3x_2 + 3x_3 + 2x_4 + 4x_5 &= 0 \\ x_2 + 2x_3 + x_4 + 3x_5 &= 0 \\ -2x_4 &= 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_2 &= -2x_3 - 3x_5 \\ x_1 &= 3x_3 + 5x_5 \\ x_4 &= 0 \end{cases}$$

Let $x_3 = t_1, x_5 = t_2$. The general solution of this system is

$$X(t_1, t_2) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3t_1 + 5t_2 \\ -2t_1 - 3t_2 \\ t_1 \\ 0 \\ t_2 \end{pmatrix}$$

where t_1, t_2 are arbitrary numbers.

MATLAB

- Gauss-Jordan Elimination: $rref([A\ B])$
- General solution of homogeneous system $AX = 0$: $null(A, 'r')$

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 & 4 \\ 1 & 4 & 5 & 3 & 7 \\ 2 & 5 & 4 & 1 & 5 \\ 1 & 5 & 7 & 6 & 10 \end{pmatrix}$$

$$>> null(A, 'r')$$

$$ans = \begin{pmatrix} 3 & 5 \\ -2 & -3 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

THANK YOU FOR YOUR ATTENTION