LINEAR TRANSFORMATIONS

ELECTRONIC VERSION OF LECTURE

HoChiMinh City University of Technology
Faculty of Applied Science, Department of Applied Mathematics

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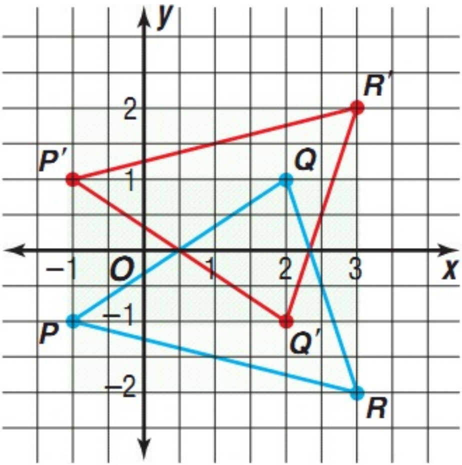
OUTLINE

Modelling Motion with Matrices

In 1995, animation took a giant step forward with the release of the first major motion picture to be created entirely on computers. Animators use computer software to create three-dimensional computer models of characters, props, and sets.

These computer models describe the shape of the object as well as the motion controls that the animators use to create movement and expressions. The animation models are actually very large matrices.



$\triangle PQR \to \triangle P'Q'R'$ is the reflection over the $x-$axis. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the **reflection matrix.**

$$P \to P' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$Q \to Q' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$R \to R' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -a-b=-1, & -c-d=1 \\ 2a+b=2, & 2c+d=-1 \\ 3a-2b=3, & 3c-2d=2 \end{cases}$$

$$\Rightarrow a=1, b=0, c=0, d=-1$$

Therefore, for every point in the plane (x_1, x_2), the matrix that results in a reflection over the $x-$axis and then we obtain a new point in the plane (y_1, y_2)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

Definition 2.1

If $f : E \mapsto F$ is a mapping from a vector space E to a vector space F, then f is called a ***linear transformation*** *from E to F if the following 2 properties hold for all vectors x and y in E and for all scalars λ:*

$$\begin{cases} f(x+y) = f(x) + f(y), \forall x, y \in E \\ f(\lambda x) = \lambda f(x), \forall \lambda \in \mathbb{R}, \forall x \in E. \end{cases}$$

We denote the set of all linear transformations from E to F by $\mathscr{L}(E, F)$.

EXAMPLE 2.1

The mapping $f : \mathbb{R}_2 \to \mathbb{R}_3$ *which is defined for* $\forall x = (x_1, x_2)$, *by* $f(x) = (3x_1 - x_2, x_1, x_1 + x_2)$ *is the linear transformation.*

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- $\forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}_2,$

$$\begin{aligned}\mathbf{f(x{+}y)} &= (3(x_1 + y_1) - (x_2 + y_2), x_1 + y_1, (x_1 + y_1) + (x_2 + y_2)) \\ &= (3x_1 - x_2, x_1, x_1 + x_2) + (3y_1 - y_2, y_1, y_1 + y_2) \\ &= \mathbf{f(x){+}f(y)}.\end{aligned}$$

- $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}_2,$

$$\begin{aligned} f(\lambda x) &= (3\lambda x_1 - \lambda x_2, \lambda x_1, \lambda x_1 + \lambda x_2) \\ &= \lambda(3x_1 - x_2, x_1, x_1 + x_2) \\ &= \lambda f(x) \end{aligned}$$

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EXAMPLE 2.2

The mapping $f : \mathbb{R}_2 \to \mathbb{R}_2$ defined by $\forall x = (x_1, x_2)$, $f(x) = (2x_1^2 - x_2, x_2)$ is NOT a linear transformation.

- $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}_2,$

$$\begin{aligned} f(\lambda x) &= (3\lambda x_1 - \lambda x_2, \lambda x_1, \lambda x_1 + \lambda x_2) \\ &= \lambda(3x_1 - x_2, x_1, x_1 + x_2) \\ &= \lambda f(x) \end{aligned}$$

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Indeed, $f(\lambda x) = (2(\lambda x_1)^2 - \lambda x_2, \lambda x_2) = (2\lambda^2 x_1^2 - \lambda x_2, \lambda x_2) \neq \lambda(2x_1^2 - x_2, x_2) = \lambda.f(x)$, if $\lambda \neq 1$

Relation between the coordinate vectors

Let $f : E \to F, y = f(x)$.

1. $B = \{e_1, e_2, \ldots, e_n\}$ is the basis for E

RELATION BETWEEN THE COORDINATE VECTORS

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3. $X = [x]_B = (x_1, x_2, \ldots, x_n)^T$ or $x = \sum_{i=1}^{n} x_i e_i$

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4. $Y = [y]_C = (y_1, y_2, \ldots, y_m)^T$ or $y = \sum_{k=1}^{m} y_k f_k$.

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4. $Y = [y]_C = (y_1, y_2, \dots, y_m)^T$ or $y = \sum_{k=1}^{m} y_k f_k$.

Find the relation between $[x]_B, [y]_C$?

We have

$$y = f(x) = \sum_{k=1}^{m} y_k f_k = f(\sum_{i=1}^{n} x_i e_i)$$

$$= \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i (\sum_{k=1}^{m} a_{ki} f_k) = \sum_{k=1}^{m} (\sum_{i=1}^{n} a_{ki} x_i) f_k$$

$$\Rightarrow y_k = \sum_{i=1}^{n} a_{ki} x_i, k = 1, 2, \ldots, m.$$

$$\begin{cases} y_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{cases}$$

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In matrix form $[y]_C = A_{BC}[x]_B$

Matrices for linear transformation relative to 2 bases

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$$\begin{cases} f(e_1) = a_{11} f_1 + a_{21} f_2 + \ldots + a_{i1} f_i + \ldots + a_{m1} f_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f(e_j) = a_{1j} f_1 + a_{2j} f_2 + \ldots + a_{ij} f_i + \ldots + a_{mj} f_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f(e_n) = a_{1n} f_1 + a_{2n} f_2 + \ldots + a_{in} f_i + \ldots + a_{mn} f_m \end{cases}$$

Then the matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \vdots & & \vdots & & \vdots \\ \left[f(e_1)\right]_C & \dots & \left[f(e_j)\right]_C & \dots & \left[f(e_n)\right]_C \\ \vdots & & \vdots & & \vdots \end{pmatrix}$$

is called the matrix for f relative to the bases B and C.

EXAMPLE 3.2

Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ *be the linear transformation and the matrix for* f *relative to the basis* $B = \{(1,1),(-1,1)\}$ *be* $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$. *Find* $f(-1,5)$.

Example 3.2

Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ *be the linear transformation and the matrix for* f *relative to the basis* $B = \{(1,1), (-1,1)\}$ *be* $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$. *Find* $f(-1,5)$.

We have $x = (-1,5) = \alpha(1,1) + \beta(-1,1) \Rightarrow \alpha = 2, \beta = 3$
$\Rightarrow [x]_B = (2,3)^T$.

Therefore $[f(-1,5)]_B = A.[x]_B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$.
So $f(-1,5) = -1(1,1) + 6(-1,1) = (-7,5)$

Example 3.3

Let $f:\mathbb{R}_2 \to \mathbb{R}_3$ *be defined by* $(f(x))^T = Ax^T$, *where* $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ 4 & 3 \end{pmatrix}$. *Find the matrix for* f *relative to the bases* $B = \{(1,1),(1,2)\}$ *and* $C = \{(1,0,1),(1,1,1),(1,0,0)\}$

We have $(f(1,1))^T = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 7 \end{pmatrix}$.

We have $(f(1,1))^T = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 7 \end{pmatrix}$. We compute the coordinate vector $(f(1,1))^T$ relative to the basis C

$$\begin{pmatrix} -2 \\ 2 \\ 7 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

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Therefore, $\alpha = 5, \beta = 2, \gamma = -9$.
So $[f(1,1)]_C = (5,2,-9)^T$.

Similarly, we compute $[f(1,2)]_C = \begin{pmatrix} 6 \\ 4 \\ -15 \end{pmatrix}$.

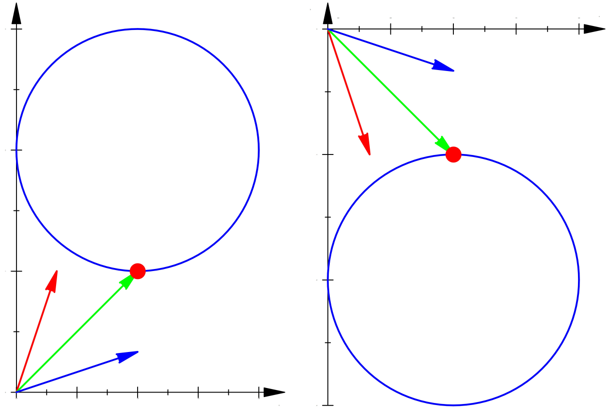
Similarly, we compute $[f(1,2)]_C = \begin{pmatrix} 6 \\ 4 \\ -15 \end{pmatrix}$.

Thus, the matrix for f relative to the bases B, C is

$$A = \begin{pmatrix} 5 & 6 \\ 2 & 4 \\ -9 & -15 \end{pmatrix}.$$

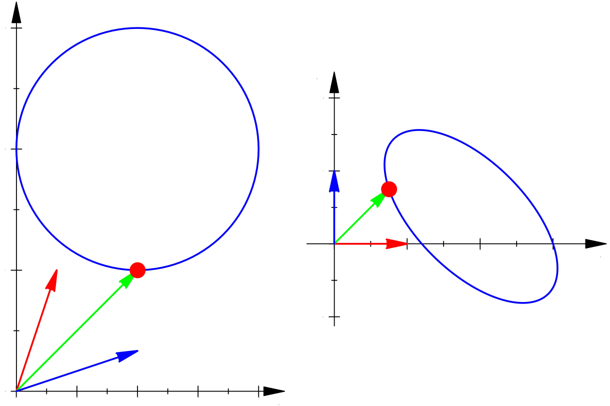
$f : \mathbb{R}_2 \longrightarrow \mathbb{R}_2$

1. The standard basis of $\mathbb{R}_2$ is $B = \{(1,0),(0,1)\}$
2. f is the reflection over the $x-$axis.



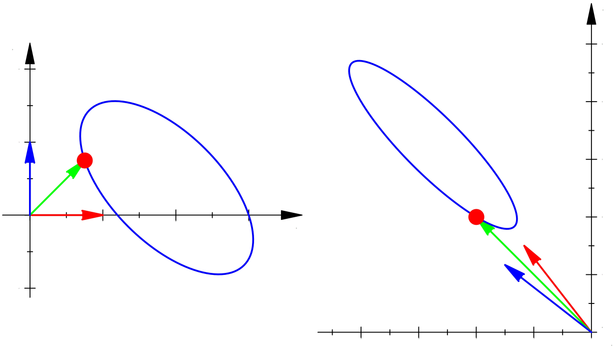
The Transition Matrix

1. The standard basis of $\mathbb{R}_2$ is $B = \{(1,0),(0,1)\}$
2. The non-standard basis is $B' = \{(\frac{1}{3},1),(1,\frac{1}{3})\}$



$f : \mathbb{R}_2 \to \mathbb{R}_2$

1. The non-standard basis of $\mathbb{R}_2$ is $B' = \{(\frac{1}{3}, 1), (1, \frac{1}{3})\}$
2. f the reflection over the $x-$ axis.



Matrices for General Linear Transformation relative to different bases

Consider the linear operator $f : E \to E$, $y = f(x)$

1. $B = \{e_1, e_2, \ldots, e_n\}, B' = \{e'_1, e'_2, \ldots, e'_n\}$ are the 2 bases for E.

MATRICES FOR GENERAL LINEAR TRANSFORMATION RELATIVE TO DIFFERENT BASES

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2. $A = Mat_B(f)$ is the matrix for f relative to basis B

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Find the matrix for f relative to basis B' ?

Let $P_{B' \to B}$ be the transition matrix from B' to B. Then

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$$[f(x)]_B = A_B.[x]_B \Leftrightarrow P_{B'\to B}[f(x)]_{B'} = A_B.P_{B'\to B}[x]_{B'}$$
$$\Leftrightarrow [f(x)]_{B'} = P^{-1}_{B'\to B}A_B P_{B'\to B}[x]_{B'}.$$

 Matrices for General Linear Transformations

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$$\Leftrightarrow [f(x)]_{B'} = P^{-1}_{B'\to B}A_B P_{B'\to B}[x]_{B'}.$$

Therefore, matrix $P^{-1}_{B'\to B}A_B P_{B'\to B}$ is the matrix for f relative to the basis B'

Example 3.4

Let $f:\mathbb{R}_2 \to \mathbb{R}_2$ *be the linear operator and matrix for* f *relative to the basis* $B = \{(1,0),(0,1)\}$ *be* $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Find the matrix for f *relative to the basis* $B' = \left\{(\frac{1}{3},1),(1,\frac{1}{3})\right\}$.

Example 3.4

Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ be the linear operator and matrix for f relative to the basis $B = \{(1,0),(0,1)\}$ be $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the matrix for f relative to the basis $B' = \left\{(\frac{1}{3},1),(1,\frac{1}{3})\right\}$.

Find the matrix for f relative to the basis B' can be found by $A' = P^{-1}_{B' \to B} A P_{B' \to B}$ where $P_{B' \to B}$ is the transition matrix from B' to B.

Find S.

$$\begin{cases} (\frac{1}{3},1) = s_{11}(1,0) + s_{21}(0,1) \\ (1,\frac{1}{3}) = s_{12}(1,0) + s_{22}(0,1) \end{cases} \Rightarrow \begin{cases} s_{11} = \frac{1}{3}; s_{21} = 1 \\ s_{12} = 1; s_{22} = \frac{1}{3} \end{cases}$$

So $P_{B'\to B} = \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & \frac{1}{3} \end{pmatrix} \Rightarrow P_{B\to B'} = P^{-1}_{B'\to B} = \begin{pmatrix} -\frac{3}{8} & \frac{9}{8} \\ \frac{9}{8} & -\frac{3}{8} \end{pmatrix}$.

Therefore,

$$\begin{aligned} A' = P^{-1}_{B'\to B} A P_{B'\to B} &= \begin{pmatrix} -\frac{3}{8} & \frac{9}{8} \\ \frac{9}{8} & -\frac{3}{8} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{5}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}. \end{aligned}$$

Matrices for General Linear Transformations relative to different bases

Let $f : E \to F$, $y = f(x)$

1. $B = \{e_1, e_2, \dots, e_n\}, B' = \{e'_1, e'_2, \dots, e'_n\}$ are the 2 bases for E.

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3. $A = Mat_{BC}(f)$ is the matrix for f relative to the bases BC.

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3. $A = Mat_{BC}(f)$ is the matrix for f relative to the bases BC.

Find the matrix for f relative to the bases B', C' ?

Let S be the transition matrix from B' to B, P be the transition matrix from C' to C. Then

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$$[f(x)]_C = A_{BC}.[x]_B \Leftrightarrow P[f(x)]_{C'} = A_{BC}.S[x]_{B'}$$
$$\Leftrightarrow [f(x)]_{C'} = P^{-1}A_{BC}S[x]_{B'}.$$

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$$\Leftrightarrow [f(x)]_{C'} = P^{-1}A_{BC}S[x]_{B'}.$$

Therefore, $P^{-1}A_{BC}S$ is the matrix for f relative to the bases B', C'.

EXAMPLE 3.5

Let $f : \mathbb{R}_3 \to \mathbb{R}_2$ *be linear transformation, and the matrix for* f *relative to 2 bases* $B = \{(1,1,1),(1,1,0),(1,0,0)\}$ *and* $C = \{(1,0),(0,1)\}$ *be* $A = \begin{pmatrix} 8 & 4 & 2 \\ -4 & -2 & -1 \end{pmatrix}$. *Find* $f(x_1, x_2, x_3)$

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}, P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Matrix for f relative to 2 standard bases is

$$A' = P^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} . \begin{pmatrix} 8 & 4 & 2 \\ -4 & -2 & -1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 2 & 4 \\ -1 & -1 & -2 \end{pmatrix} \Rightarrow [f(x)]_{CT} = A'[x]_{CT}$$

$$f(x_1, x_2, x_3) = (2x_1 + 2x_2 + 4x_3, -x_1 - x_2 - 2x_3)$$

EXAMPLE 3.6

Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ *be defined if* $f(1,1) = (-1,1)$, $f(1,0) = (1,2)$. *Find the matrix for* f *relative to standard basis.*

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Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ *be defined if* $f(1,1) = (-1,1)$, $f(1,0) = (1,2)$. *Find the matrix for* f *relative to standard basis.*

$B = \{(1,1),(1,0)\}$, $CT = \{(1,0),(0,1)\}$

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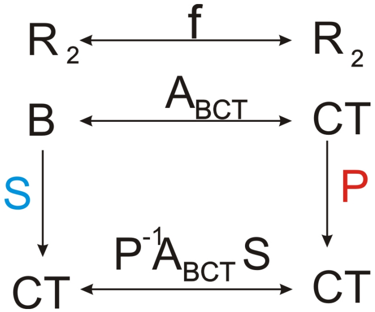
Let $f:\mathbb{R}_2 \to \mathbb{R}_2$ be defined if $f(1,1)=(-1,1)$, $f(1,0)=(1,2)$. Find the matrix for f relative to standard basis.

$B=\{(1,1),(1,0)\}$, $CT=\{(1,0),(0,1)\}$



$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}, P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_{BCT} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}.$$

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Matrix for f relative to the standard basis is

$$\begin{aligned} A' &= P^{-1}AS \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}. \\ &= \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \end{aligned}$$

DEFINITION 4.1

If $f : E \mapsto F$ is a linear transformation, then

1. *The set of all vectors in E that maps into $\mathbf{0}$ is called the kernel of f:*

$$\boxed{Ker(f) = \{x \in E : f(x) = 0\}} \tag{1}$$

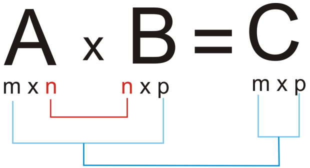
2. *The set of all vectors in F that are images under f of at least one vector in E is called the range of f:*

$$\boxed{Im(f) = \{y \in F : \exists x \in E, y = f(x)\} = f(E)} \tag{2}$$

THEOREM 4.1

If $f : E \mapsto F$ is a linear transformation, then

1. *The range $Im(f)$ of f is a subspace of F*
2. *The kernel $Ker(f)$ of f is a subspace of E*


A x B = C
m x n
n x p
m x p

Example 4.1

If $f : \mathbb{R}_3 \to \mathbb{R}_2$ *is defined by* $f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$, *then*

1. *Find* $Ker(f)$, *its basis and dimension.*
2. *Find* $Im(f)$, *its basis and dimension.*

Example 4.1

If $f : \mathbb{R}_3 \to \mathbb{R}_2$ *is defined by* $f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$, *then*

1. *Find* $Ker(f)$, *its basis and dimension.*
2. *Find* $Im(f)$, *its basis and dimension.*

$Ker(f) = \left\{(x_1, x_2, x_3) : x_1 - x_2 = 0, x_2 + x_3 = 0\right\}$. Solving the linear system, we obtain $x_1 = \alpha, x_2 = \alpha, x_3 = -\alpha, \forall \alpha \in \mathbb{R}$. Therefore, $Ker(f) = \{\alpha(1, 1, -1) : \forall \alpha \in \mathbb{R}\}$.

EXAMPLE 4.1

If $f : \mathbb{R}_3 \to \mathbb{R}_2$ *is defined by* $f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$, *then*

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FIND Imf

Step 1. Choose a basis for $E = \mathbb{R}_3 : e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

FIND Imf

Step 1. Choose a basis for $E = \mathbb{R}_3 : e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

Step 2. Find $f(e_1) = (1,0)$, $f(e_2) = (-1,1)$, $f(e_3) = (0,1)$.

FIND Imf

Step 1. Choose a basis for $E = \mathbb{R}_3$: $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

Step 2. Find $f(e_1) = (1,0)$, $f(e_2) = (-1,1)$, $f(e_3) = (0,1)$.

Step 3. We have

$$y = f(x) = f(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) =$$

$$= \lambda_1 f(e_1) + \lambda_2 f(e_2) + \lambda_3 f(e_3)$$

$$\Rightarrow Im(f) = span\left\{f(e_1), f(e_2), f(e_3)\right\}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore, the set $\left\{f(e_1), f(e_2)\right\} = \left\{(1,0), (-1,1)\right\}$ is the basis for $Im(f)$ and $dim(Im(f)) = 2$.

Example 4.2

Let $f : \mathbb{R}_3 \to \mathbb{R}_3$ *be defined by* $f(1,0,0) = (1,1,1)$, $f(-1,1,0) = (-2,-1,0)$, $f(0,-1,1) = (2,1,3)$.

- *Find* $f(x_1, x_2, x_3)$
- *Find the basis and dimension for* $Ker f$
- *Find the basis and dimension for* $Im f$

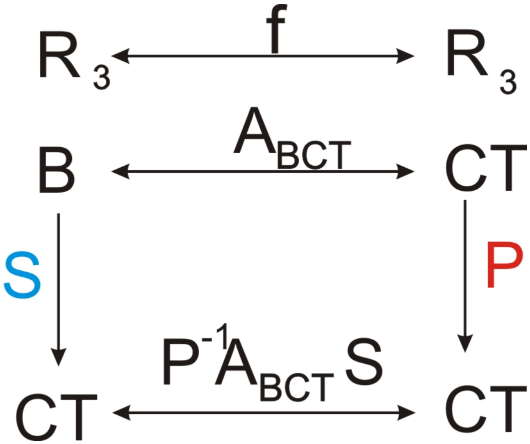
Example 4.2

Let $f:\mathbb{R}_3 \to \mathbb{R}_3$ *be defined by* $f(1,0,0)=(1,1,1)$, $f(-1,1,0)=(-2,-1,0)$, $f(0,-1,1)=(2,1,3)$.

- *Find* $f(x_1,x_2,x_3)$
- *Find the basis and dimension for* $Kerf$
- *Find the basis and dimension for* Imf

Choose the basis $B=\left\{(1,0,0),(-1,1,0),(0,-1,1)\right\}$, and the standard basis $CT=\{(1,0,0),(0,1,0),(0,0,1)\}$.

FIND $f(x_1, x_2, x_3)$



$$S = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}, P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{BCT} = \begin{pmatrix} 1 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Matrix for f relative to the standard bases

$$A' = P^{-1}AS = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix} \Rightarrow [f(x)]_{CT} = A'[x]_{CT}$$

$$f(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_1 + x_3, x_1 + x_2 + 4x_3)$$

THE BASIS AND DIMENSION FOR $Ker(f)$.

$$f(x_1,x_2,x_3)=(x_1-x_2+x_3,x_1+x_3,x_1+x_2+4x_3)$$

$\forall x\in Ker(f)\Leftrightarrow f(x)=0$

$$\Leftrightarrow\begin{cases} x_1-x_2+x_3 &= 0\\ x_1+x_3 &= 0\\ x_1+x_2+4x_3 &= 0\end{cases}\Leftrightarrow x_1=x_2=x_3=0\ Ker(f)=\left\{\mathbf{0}\right\}.$$

$Dim(Ker(f))=0.$

THE BASIS AND DIMENSION FOR $Ker(f)$.

$$f(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_1 + x_3, x_1 + x_2 + 4x_3)$$

$\forall x \in Ker(f) \Leftrightarrow f(x) = 0$

$$\Leftrightarrow \begin{cases} x_1 - x_2 + x_3 & = & 0 \\ x_1 + x_3 & = & 0 \\ x_1 + x_2 + 4x_3 & = & 0 \end{cases} \Leftrightarrow x_1 = x_2 = x_3 = 0 \; Ker(f) = \left\{ \mathbf{0} \right\}.$$

$Dim(Ker(f)) = 0.$

The basis for $Ker(f)$ does NOT exist.

THE BASIS AND DIMENSION FOR $Im(f)$.

The basis for $\mathbb{R}_3$ is $\left\{(1,0,0),(-1,1,0),(0,-1,1)\right\}$.

THE BASIS AND DIMENSION FOR $Im(f)$.

The basis for $\mathbb{R}_3$ is $\Big\{(1,0,0),(-1,1,0),(0,-1,1)\Big\}$.

$$\begin{aligned} Im(f) &= span\Big\{f(1,0,0), f(-1,1,0), f(0,-1,1)\Big\} \\ &= span\Big\{(1,1,1),(-2,-1,0),(2,1,3)\Big\} \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

The basis for $Im(f)$ is $\Big\{f(e_1), f(e_2), f(e_3)\Big\} = \Big\{(1,1,1),(-2,-1,0),(2,1,3)\Big\}$ and $Dim(Im(f)) = 3$.

Theorem 4.2

Let $f : E \to F$ *be a linear transformation and* $A \in M_{m\times n}(K)$ *be the matrix for* f *relative to 2 bases* $B = \{e_1, e_2, \ldots, e_n\} \subset E$ *and* $C = \{f_1, f_2, \ldots, f_m\} \subset F$ *or* $A = Mat_{BC}(f)$. *Then the coordinate vector of* $x \in Ker(f)$ *relative to the basis* B *satisfies the linear system* $A[x]_B = 0$.

Theorem 4.2

Let $f: E \to F$ be a linear transformation and $A \in M_{m\times n}(K)$ be the matrix for f relative to 2 bases $B = \{e_1, e_2, \ldots, e_n\} \subset E$ and $C = \{f_1, f_2, \ldots, f_m\} \subset F$ or $A = Mat_{BC}(f)$. Then the coordinate vector of $x \in Ker(f)$ relative to the basis B satisfies the linear system $A[x]_B = 0$.

$x \in E, x \in Ker(f) \Leftrightarrow f(x) = 0 \Leftrightarrow [f(x)]_C = 0 \Leftrightarrow A[x]_B = 0.$
Therefore, the coordinate vector of $x \in Ker(f)$ relative to the basis B satisfies the linear system $A[x]_B = 0$.

EXAMPLE 4.3

Let $f : \mathbb{R}_3 \to \mathbb{R}_2$ *be the linear transformation and the matrix for* f *relative to 2 bases*
$B = \{e_1 = (1,0,0), e_2 = (1,0,1), e_3 = (1,1,1)\}$ *and*
$C = \{f_1 = (1,0), f_2 = (1,1)\}$ *be*

$$A_{BC} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

Find a basis and dimension of $Ker(f)$.

$x \in Ker(f) \Leftrightarrow f(x) = 0 \Leftrightarrow [f(x)]_C = 0 \Leftrightarrow A_{BC}[x]_B = 0.$
Suppose $[x]_B = (x_1, x_2, x_3)^T$. Then

$x \in Ker(f) \Leftrightarrow f(x) = 0 \Leftrightarrow [f(x)]_C = 0 \Leftrightarrow A_{BC}[x]_B = 0.$
Suppose $[x]_B = (x_1, x_2, x_3)^T$. Then

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix}$$

$$\begin{aligned} \Rightarrow x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \\ &= \alpha(1,0,0) - 2\alpha(1,0,1) + \alpha(1,1,1) \\ &= (0, \alpha, -\alpha) = \alpha(0,1,-1) \end{aligned}$$

Therefore $\{(0,1,-1)\}$ is the basis for $Ker(f)$
$\Rightarrow dim(Ker(f)) = 1$

Relation between the dimensions of Kernel and Range

Theorem 4.3

Let $f : E \to F$ be the linear transformation. Then

$$rank(f) + dim(ker(f)) = dim(E)$$

or

$$\boxed{dim(Im(f)) + dim(ker(f)) = dim(E)}$$

Even though large matrices are used for computer animation, you can use a simple matrix to decribe many of the motions called transformations.

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Even though large matrices are used for computer animation, you can use a simple matrix to decribe many of the motions called transformations.
Some of the transformations we will examine in this lesson are

1. translations (slides)
2. reflections (flips)
3. rotations (turns)
4. dilatations (enlargements or reductions).

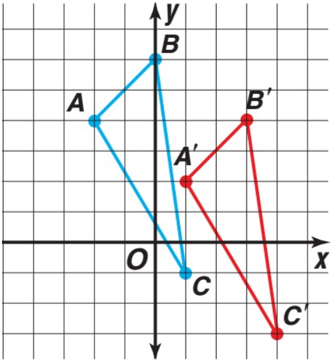
A $2 \times n$ matrix can be used to express the vertices of an n-gon (a polygon with n sides) with the first row of elements representing the $x-$coordinates and the second row the $y-$coordinates of the vertices.

EXAMPLE 5.1

Triangle ABC can be represented by the following vertex matrix

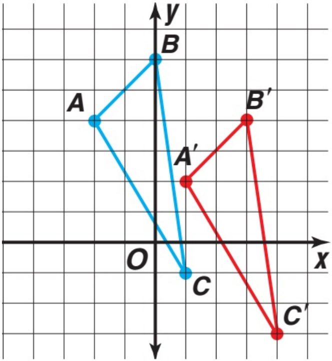
$$\begin{array}{lccc} & A & B & C \\ x-coordinate & -2 & 0 & 1 \\ y-coordinate & 4 & 6 & -1 \end{array}$$

TRANSLATION MATRIX



Find the vertex matrix of $\Delta A'B'C'$?

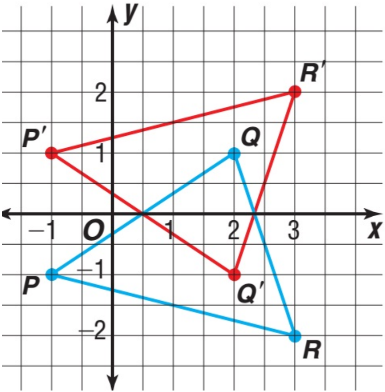
Translation matrix



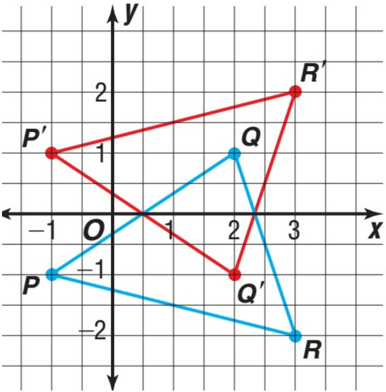
Find the vertex matrix of $\Delta A'B'C'$?

$$\begin{pmatrix} -2 & 0 & 1 \\ 4 & 6 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 \\ -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & -3 \end{pmatrix}$$

Reflection matrix



Reflection matrix

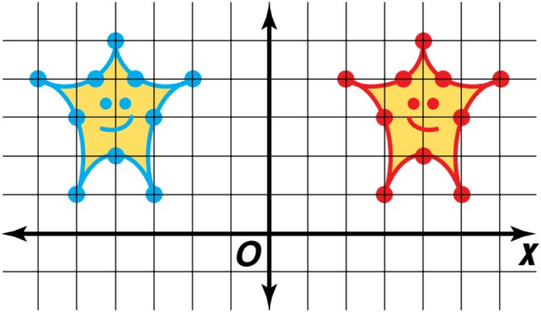


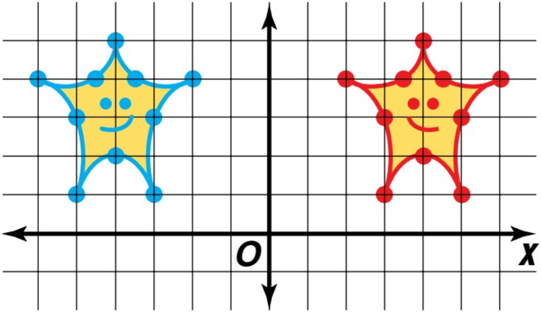
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 3 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{pmatrix}$$

Reflection Matrices		
For a reflection over the:	**Symbolized by:**	**Multiply the vertex matrix by:**
x-axis	$R_{x\text{-axis}}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
y-axis	$R_{y\text{-axis}}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
line $y = x$	$R_{y=x}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

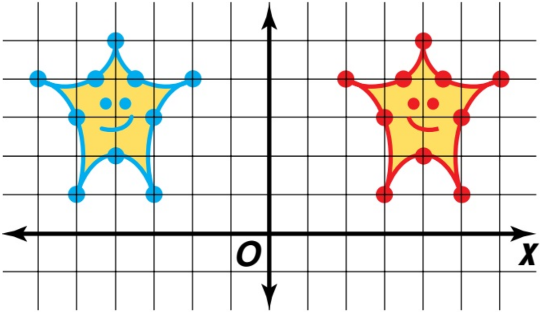
EXAMPLE 5.2

To create an image that appears to be reflected in a mirror, an animator will use a matrix to reflect this image over the $y-$*axis. Use a reflection matrix to find the coordinates of the vertices of a star reflected in a mirror (the* $y-$*axis) if the coordinates of the points connected to create the star are* $(-2,4),(-3.5,4),(-4,5),(-4.5,4),(-6,4),(-5,3),$ $(-5,1),(-4,2),(-3,1),$ *and* $(-3,3).$


O
x



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{pmatrix}$$



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{pmatrix}$$

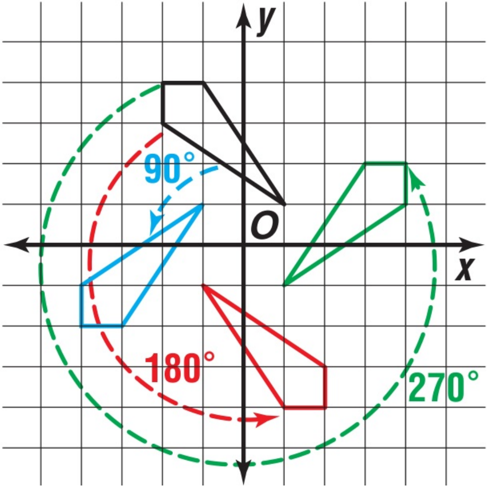
$$= \begin{pmatrix} 2 & 3.5 & 4 & 4.5 & 6 & 5 & 5 & 4 & 3 & 3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{pmatrix}$$

Rotation matrix

Rotation Matrices		
For a counterclockwise rotation about the origin of:	Symbolized by:	Multiply the vertex matrix by:
90°	Rot_{90}	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
180°	Rot_{180}	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
270°	Rot_{270}	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example 5.3

Suppose a figure is animated to spin around a certain point. Numerous rotation images would be necessary to make a smooth movement image. If the image has key points at $(1,1)$, $(-1,4)$, $(-2,4)$, *and* $(-2,3)$ *and the rotation is about the origin, find the location of these points at the* 90^o, 180^o, *and* 270^o *counterclockwise rotations.*


y
O
x
90°
180°
270°

$Rot_{90}:$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -4 & -3 \\ 1 & -1 & -2 & -2 \end{pmatrix}$$

Rot_{90} :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -4 & -3 \\ 1 & -1 & -2 & -2 \end{pmatrix}$$

Rot_{180} :

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 & 2 \\ -1 & -4 & -4 & -3 \end{pmatrix}$$

$Rot_{90}:$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -4 & -3 \\ 1 & -1 & -2 & -2 \end{pmatrix}$$

$Rot_{180}:$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 & 2 \\ -1 & -4 & -4 & -3 \end{pmatrix}$$

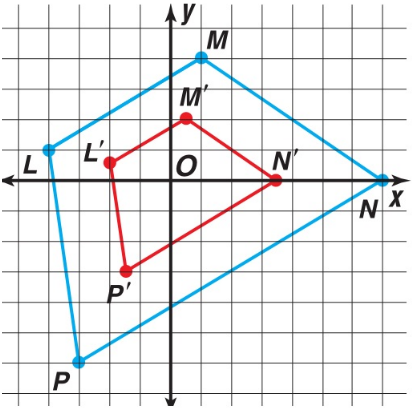
$Rot_{270}:$

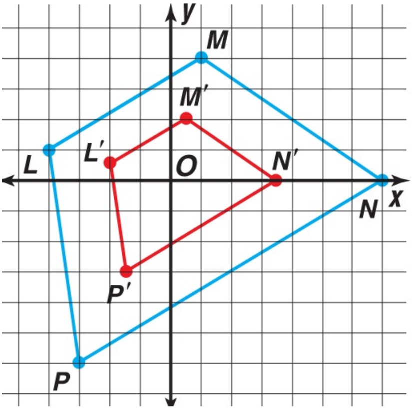
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 4 & 3 \\ -1 & 1 & 2 & 2 \end{pmatrix}$$

DILATATION

EXAMPLE 5.4

A trapezoid has vertices at $L(-4,1)$, $M(1,4)$, $N(7,0)$, *and* $P(-3,-6)$. *Find the coordinates of the dilated trapezoid* $L'M'N'P'$ *for a scale factor of* 0.5. *Describe the dilatation.*


y
M
M′
L
L′
O
N′
N
x
P′
P



$$0.5\begin{pmatrix} -4 & 1 & 7 & -3 \\ 1 & 4 & 0 & -6 \end{pmatrix} = \begin{pmatrix} -2 & 0.5 & 3.5 & -1.5 \\ 0.5 & 2 & 0 & -3 \end{pmatrix}$$

LET $f:\mathbb{R}_n \to \mathbb{R}_m$ BE THE LINEAR TRANSFORMATION

$$(f(x))^T = A_{m\times n}.x^T$$

1. The basis for $Ker f : null(A,'r')$
2. Choose the standard basis $\{e_1, e_2, \dots e_n\}$.

$$M = \begin{pmatrix} f(e_1) \\ f(e_2) \\ \dots \\ f(e_n) \end{pmatrix} \Rightarrow rref(M) \Rightarrow \text{The basis for } Imf$$

THE MATRIX FOR f RELATIVE TO 2 BASES B, C

Let $f : E \mapsto F$ be the linear transformation and $dimE = n, dimF = m$.

THE MATRIX FOR f RELATIVE TO 2 BASES B, C

Let $f : E \mapsto F$ be the linear transformation and $dimE = n, dimF = m$. Suppose that $B = \{e_1, e_2, \ldots, e_n\}$ is the basis for E,

THE MATRIX FOR f RELATIVE TO 2 BASES B, C

Let $f : E \mapsto F$ be the linear transformation and $dimE = n, dimF = m$. Suppose that $B = \{e_1, e_2, \ldots, e_n\}$ is the basis for E, $C = \{f_1, f_2, \ldots, f_m\}$ is the basis for F. The matrix for f relative to 2 bases B, C is A, which is defined by

$$B = \begin{pmatrix} f_1.' & f_2.' & \ldots & f_m.' \end{pmatrix};$$

$$C = \begin{pmatrix} f(e_1).' & f(e_2).' & \ldots & f(e_n).' \end{pmatrix}$$

$$B * A = C \Rightarrow A = inv(B) * C$$

THANK YOU FOR YOUR ATTENTION