VECTOR SPACES

ELECTRONIC VERSION OF LECTURE

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Outline

1. Vector Space Axioms
2. Linear Independence and Dependence
3. Spanning set and Basis
4. Coordinates Relative to a Basis

Real Vector Spaces

Let $V \neq \varnothing$ on which 2 operations are defined:

1. $$\begin{aligned} + : V \times V &\to V \\ (x, y) &\longmapsto x + y \end{aligned}$$
2. $$\begin{aligned} \bullet : \mathbb{R} \times V &\to V \\ (\lambda, x) &\longmapsto \lambda.x \end{aligned}$$

VECTOR SPACE AXIOMS

If the following 8 axioms are satisfied by: $\forall x, y, z \in V, \forall \lambda, \mu \in \mathbb{R}$

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. $\exists \mathbf{0} \in V : x + \mathbf{0} = x$
4. $\exists (-x) \in V : x + (-x) = \mathbf{0}$
5. $(\lambda + \mu)x = \lambda x + \mu x$
6. $\lambda(x + y) = \lambda x + \lambda y$
7. $\lambda(\mu x) = (\lambda.\mu)x$
8. $\mathbf{1}.x = x$

then V is called real vector space.

Linear combination of vectors

Definition 2.1

If w is a vector in a vector space V, then w is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_n \in V$, if w can be expressed in the form

$$w = \sum_{i=1}^{n} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are scalars. These scalars are called the coefficients of the linear combination.

Show that w is a linear combination of $v_1, v_2, \ldots, v_n$

In order for w to be a linear combination of $v_1, v_2, \ldots, v_n$, there must be scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$w = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$

- If this system is consistent then w is a linear combination of $v_1, v_2, \ldots, v_n$.
- If this system is inconsistent then w is NOT a linear combination of $v_1, v_2, \ldots, v_n$.

Example 2.1

Show that $w = (1, 4, -3)$ *is a linear combination of* $v_1 = (2, 1, 1), v_2 = (-1, 1, -1), v_3 = (1, 1, -2)$.

In order for w to be a linear combination of v_1, v_2, v_3, there must be scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = w$$

$\Leftrightarrow (2\lambda_1, \lambda_1, \lambda_1) + (-\lambda_2, \lambda_2, -\lambda_2) + (\lambda_3, \lambda_3, -2\lambda_3) = (1, 4, -3)$

$$\Leftrightarrow \begin{cases} 2\lambda_1 - \lambda_2 + \lambda_3 &= 1 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 4 \\ \lambda_1 - \lambda_2 - 2\lambda_3 &= -3 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 1 \end{cases}$$

Therefore, $w = (1, 4, -3)$ is a linear combination of

$v_1 = (2, 1, 1)$, $v_2 = (-1, 1, -1)$, $v_3 = (1, 1, -2)$ and

$$w = v_1 + 2v_2 + v_3.$$

Example 2.2

Determine whether $w = (4, 3, 5)$ is a linear combination of $v_1 = (1, 2, 5), v_2 = (1, 3, 7), v_3 = (-2, 3, 4)$ or not?

In order for w to be a linear combination of v_1, v_2, v_3, there must be scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = w$$

$$\Leftrightarrow (\lambda_1, 2\lambda_1, 5\lambda_1) + (\lambda_2, 3\lambda_2, 7\lambda_2) + (-2\lambda_3, 3\lambda_3, 4\lambda_3) = (4, 3, 5) \quad (1)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 5 \end{array}\right) \xrightarrow[\;]{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 5r_1}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -15 \end{array}\right) \xrightarrow{r_3 \to r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

This system is inconsistent, so no such scalars $\lambda_1, \lambda_2, \lambda_3$ exist. Consequently, $w = (4, 3, 5)$ is NOT a linear combination of $v_1 = (1, 2, 5)$, $v_2 = (1, 3, 7)$, $v_3 = (-2, 3, 4)$

Example 2.3

Determine whether $w=(4,3,10)$ *is a linear combination of* $v_1=(1,2,5), v_2=(1,3,7), v_3=(-2,3,4)$ *or not?*

In order for w to be a linear combination of v_1, v_2, v_3, there must be scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = w$$

$$\Leftrightarrow (\lambda_1, 2\lambda_1, 5\lambda_1) + (\lambda_2, 3\lambda_2, 7\lambda_2) + (-2\lambda_3, 3\lambda_3, 4\lambda_3) = (4,3,10) \quad (2)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 10 \end{array}\right) \xrightarrow[]{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 5r_1}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -10 \end{array}\right) \xrightarrow[]{\substack{r_3 \to r_3 - 2r_2 \\ r_1 \to r_1 - r_2}} \left(\begin{array}{ccc|c} 1 & 0 & -9 & 9 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This system has infinitely many solutions

$$(\lambda_1, \lambda_2, \lambda_3) = (9+9t, -5-7t, t), \qquad t \in \mathbb{R}.$$

Therefore, $w = (4, 3, 10)$ is a linear combination of

$$v_1 = (1,2,5), v_2 = (1,3,7), v_3 = (-2,3,4)$$

and

$$w = (9+9t)v_1 + (-5-7t)v_2 + tv_3, \; t \in \mathbb{R}.$$

 Linear Independence and Dependence

$\left\{v_1, v_2, \ldots, v_m\right\}$ is a linear dependent set

$\longleftrightarrow$

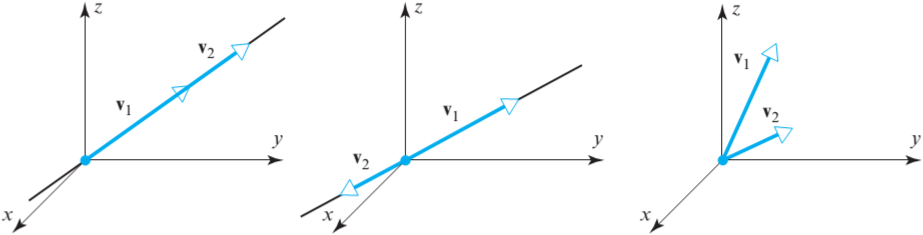
$\exists \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$: $\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_m^2 \neq 0$ such that $\sum_{i=1}^{m} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_m v_m = 0$

$\sum_{i=1}^{m} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_m v_m = 0$ $\Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$

$\longleftrightarrow$

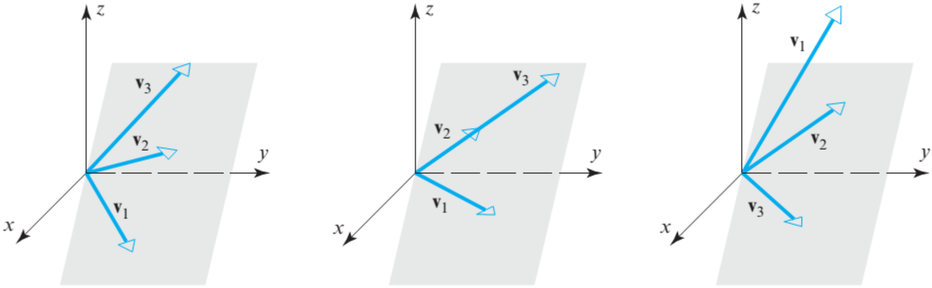
$\left\{v_1, v_2, \ldots, v_m\right\}$ is a linear independent set

A Geometric Interpretation of Linear Independence



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

A Geometric Interpretation of Linear Independence



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

The linear independence or linear dependence of $v_1, v_2, \ldots, v_m$ is determined by whether there exist non-trivial solutions of the system $\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_m v_m = 0$, where $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ are the unknowns.

- If this system has trivial solution $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$ then $v_1, v_2, \ldots, v_m$ are linearly independent.
- If this system has non-trivial solutions then $v_1, v_2, \ldots, v_m$ are linearly dependent.

When $v_1, v_2, \ldots, v_m \in \mathbb{R}_n$

Let $A = \begin{pmatrix} v_1^T & v_2^T & \ldots & v_m^T \end{pmatrix}$ and determine $r(A)$.

- If $r(A) = m$ then $v_1, v_2, \ldots, v_m$ are linearly independent.
- If $r(A) < m$ then $v_1, v_2, \ldots, v_m$ are linearly dependent.

SPECIAL CASE $m = n$

- If $det(A) \neq 0$ then $v_1, v_2, \ldots, v_m$ are linearly independent.
- If $det(A) = 0$ then $v_1, v_2, \ldots, v_m$ are linearly dependent.

Example 2.4

Determine whether $v_1 = (2,1,2), v_2 = (3,2,1), v_3 = (1,1,4)$ *are linearly dependent or linearly independent?*

Let

$$A = \begin{pmatrix} v_1^T & v_2^T & v_3^T \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

We have $det(A) = 5 \neq 0$, thus v_1, v_2, v_3 are linearly independent.

EXAMPLE 2.5

Determine whether $v_1 = (1,2,3), v_2 = (4,5,6), v_3 = (7,8,9)$ *are linearly independent or linearly dependent?*

$$A = \begin{pmatrix} v_1^T & v_2^T & v_3^T \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

We have $det(A) = 0$, therefore, v_1, v_2, v_3 are linearly dependent.

EXAMPLE 2.6

Determine whether $v_1 = (1,1,2,3), v_2 = (2,3,3,1), v_3 = (1,2,1,-2)$ *are linearly independent or linearly dependent?*

$$A = \begin{pmatrix} v_1^T & v_2^T & v_3^T \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & -2 \end{pmatrix} \xrightarrow{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \\ r_4 \to r_4 - 3r_1}}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -5 & -5 \end{pmatrix} \xrightarrow[]{\substack{r_3 \to r_3 + r_2 \\ r_4 \to r_4 + 5r_2}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow r(A) = 2 < 3 = m.$$

Therefore, v_1, v_2, v_3 are linearly dependent.

Spanning Set

Definition 3.1

The set $S = \left\{v_1, v_2, \ldots, v_m\right\}$ *of the vector space* V ***spans*** V *if* $\forall w \in V, \exists \lambda_i \in \mathbb{R}, i = 1, 2, \ldots, m:$

$$w = \sum_{i=1}^{m} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_m v_m.$$

We denote it by

$$V = Span(S) = Span\left\{v_1, v_2, \ldots, v_m\right\}.$$

Example 3.1

In $\mathbb{R}_2$ *consider* $S = \left\{(1,0);(0,1)\right\}$. *For all* $w = (x_1, x_2) \in \mathbb{R}_2$ *we have*

$$w = (x_1, x_2) = x_1(1,0) + x_2(0,1)$$

thus, S is the spanning set of $\mathbb{R}_2$.

EXAMPLE 3.2

In $\mathbb{R}_2$ *consider* $S = \{(1,2);(1,1)\}$. *For all* $w = (x_1, x_2) \in \mathbb{R}_2$, *we find* $a, b \in \mathbb{R}$ *such that*

$$w = (x_1, x_2) = a(1,2) + b(1,1) = (a+b, 2a+b)$$

$$\Leftrightarrow \begin{cases} a+b = x_1 \\ 2a+b = x_2 \end{cases}$$

This system is consistent because $\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0$. *Therefore, S spans* $\mathbb{R}_2$.

Example 3.3

The set $S = \left\{(1,1,1);(1,0,2)\right\}$ *does not span* $\mathbb{R}_3$.

S spans $\mathbb{R}_3$ if the system

$$\alpha(1,1,1)+\beta(1,0,2)=(x_1,x_2,x_3)$$

$$\left\{\begin{array}{rcl} \alpha+\beta & = & x_1 \\ \alpha & = & x_2 \\ \alpha+2\beta & = & x_3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \alpha+\beta & = & x_1 \\ 0\alpha-\beta & = & x_2-x_1 \\ 0\alpha+0\beta & = & x_3+x_2-2x_1 \end{array}\right.$$

is consistent for all x_1, x_2, x_3.

This system may have no solution or may have solutions depending on x_1, x_2, x_3. Choosing $(x_1, x_2, x_3) = (1, 1, 2)$, this system is inconsistent. Therefore, $(1, 1, 2)$ is not a linear combination of vectors in S.

Note. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \Rightarrow rank(A) = 2 < 3$

S does not span $\mathbb{R}_3$

WHEN $v_1, v_2, \ldots, v_m \in \mathbb{R}_n$

Let $A = \begin{pmatrix} v_1^T & v_2^T & \ldots & v_m^T \end{pmatrix}$ and determine $r(A)$.

- If $r(A) = n$ then $v_1, v_2, \ldots, v_m$ span $\mathbb{R}_n$.
- If $r(A) < n$ then $v_1, v_2, \ldots, v_m$ does not span $\mathbb{R}_n$.

Note. n is the number of coordinates of vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}_n$.

Special case $m = n$

- If $det(A) \neq 0$ then $v_1, v_2, \ldots, v_m$ span $\mathbb{R}_n$.
- If $det(A) = 0$ then $v_1, v_2, \ldots, v_m$ does not span $\mathbb{R}_n$.

Basis for a Vector Space

Definition 3.2

If $S = \{v_1, v_2, \ldots, v_n\}$ *is a set of vectors in vector space* V*, then* S *is called a* ***basis*** *for* V *if*

1. S *spans* V
2. S *is linearly independent*

The number of vectors in a basis S *for* V *is called the* *dimension* *of vector space* V*. We denote it by* $dim(V)$.

Example 3.4

The set $S = \left\{i, j, k\right\} \subset \mathbb{R}_3$, *where* $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$, *is the standard basis for* $\mathbb{R}_3$.

- Indeed, $\forall x = (x_1, x_2, x_3) \in \mathbb{R}_3$ we have $x = x_1.i + x_2.j + x_3.k \Rightarrow S$ spans $\mathbb{R}_3$.
- Consider $\alpha.i + \beta.j + \gamma.k = \mathbf{0}$ $\Leftrightarrow (\alpha, \beta, \gamma) = (0,0,0) \Leftrightarrow \alpha = \beta = \gamma = 0$ $\Rightarrow S$ is linear independent.

Therefore, S is the basis for $\mathbb{R}_3 \Rightarrow dim(\mathbb{R}_3) = 3$.

DEFINITION 4.1

If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V, then every vector $w \in V$ can be expressed in the form

$$w = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$$

in exactly one way. The scalars $x_1, x_2, \dots, x_n$ are called the coordinates of w relative to the basis S. We denote $\left[w\right]_S = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$.

Example 4.1

Find the coordinate vector of $w = (6,5,4)$ relative to the basis S: $v_1 = (1,1,0)$, $v_2 = (2,1,3)$, $v_3 = (1,0,2)$.

We must find x_1, x_2, x_3 such that

$$w = (6,5,4) = x_1(1,1,0) + x_2(2,1,3) + x_3(1,0,2)$$

$$\Leftrightarrow \begin{cases} x_1 + 2x_2 + x_3 = 6 \\ x_1 + x_2 = 5 \\ 3x_2 + 2x_3 = 4 \end{cases} \Leftrightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \\ x_3 = -1 \end{cases}$$

Therefore, $\left[w\right]_S = (3,2,-1)^T$.

In matrix form

$$\begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ e_1 & \cdots & e_i & \cdots & e_n \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{pmatrix} . \begin{pmatrix} \vdots \\ [x]_B \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ x \\ \vdots \end{pmatrix}$$

We have

$$B[x]_B = x^T \Rightarrow [x]_B = B^{-1}.x^T$$

The Change-of-Basis Problem

If $B = \{e_1, e_2, \ldots, e_n\}$ and $B' = \{e'_1, e'_2, \ldots, e'_n\}$ are 2 bases for a vector space V.

Suppose that $w \in V$, then

$w = \sum_{k=1}^{n} x_k e_k$ or $\left[w\right]_B = (x_1, x_2, \ldots, x_n)^T$ and

$w = \sum_{i=1}^{n} x'_i e'_i$ or $\left[w\right]_{B'} = (x'_1, x'_2, \ldots, x'_n)^T$

How are the coordinate vectors $\left[w\right]_B$ and $\left[w\right]_{B'}$ related?

Suppose that there is a relation between B and B':

$$e'_i = \sum_{k=1}^{n} s_{ki} e_k = s_{1i} e_1 + s_{2i} e_2 + \ldots + s_{ni} e_n,\ i = 1, 2, \ldots n.$$

$$\Leftrightarrow \begin{cases} e'_1 & = & s_{11}e_1 + s_{21}e_2 + \ldots + s_{n1}e_n \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e'_n & = & s_{1n}e_1 + s_{2n}e_2 + \ldots + s_{nn}e_n \end{cases}$$

Definition 4.2

The matrix $S = \begin{pmatrix} s_{11} & \dots & s_{1i} & \dots & s_{1n} \\ s_{21} & \dots & s_{2i} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n1} & \dots & s_{ni} & \dots & s_{nn} \end{pmatrix}$ *is called the transition matrix from B' to B. We denote it by* $S = P_{B' \to B}$. *And*

$$[w]_B = P_{B' \to B}[w]_{B'}$$

$$
\begin{aligned}
w &= \sum_{i=1}^{n} x'_i e'_i \\
&= x'_1 e'_1 + x'_2 e'_2 + \ldots + x'_n e'_n \\
&= x'_1 (s_{11} e_1 + s_{21} e_2 + \ldots + s_{n1} e_n) + x'_2 (s_{12} e_1 + s_{22} e_2 + \\
&\ldots + s_{n2} e_n) + \ldots + x'_n (s_{1n} e_1 + s_{2n} e_2 + \ldots + s_{nn} e_n) \\
&= (s_{11} x'_1 + s_{12} x'_2 + \ldots + s_{1n} x'_n) e_1 + (s_{21} x'_1 + s_{22} x'_2 + \\
&\ldots + s_{2n} x'_n) e_2 + \ldots + (s_{n1} x'_1 + s_{n2} x'_2 + \ldots + s_{nn} x'_n) e_n \\
&= \sum_{k=1}^{n} x_k e_k = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n
\end{aligned}
$$

$$\begin{cases} x_1 &= & s_{11}x'_1 + s_{12}x'_2 + \ldots + s_{1n}x'_n \\ x_2 &= & s_{21}x'_1 + s_{22}x'_2 + \ldots + s_{2n}x'_n \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots \\ x_n &= & s_{n1}x'_1 + s_{n2}x'_2 + \ldots + s_{nn}x'_n \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ s_{n1} & s_{n2} & \ldots & s_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

$$\Rightarrow \left[w\right]_B = P_{B'\to B}\left[w\right]_{B'}, \left[w\right]_{B'} = P^{-1}_{B'\to B}\left[w\right]_B = P_{B\to B'}[w]_B.$$

EXAMPLE 4.2

Consider the bases
$B = \left\{(2,1,0),(1,0,3),(0,0,1)\right\}$,
$B' = \left\{(1,0,1),(0,1,-2),(0,1,3)\right\}$ *for* $\mathbb{R}_3$ *and*
$w = (8,-4,6)$.

1. *Find the transition matrix* S *from* B' *to* B.
2. *Find the coordinate vector of* w *relative to 2 bases* B, B'.

The old basis vectors are $e_1 = (2,1,0), e_2 = (1,0,3), e_3 = (0,0,1)$ and the new basis vectors are $e'_1 = (1,0,1), e'_2 = (0,1,-2), e'_3 = (0,1,3)$. We want to find the coordinate vectors of e'_1, e'_2, e'_3 relative to basis B :

$$\Leftrightarrow \left\{ \begin{array}{lcl} e'_1 & = & s_{11}e_1 + s_{21}e_2 + s_{31}e_3 \\ e'_2 & = & s_{12}e_1 + s_{22}e_2 + s_{32}e_3 \\ e'_3 & = & s_{13}e_1 + s_{23}e_2 + s_{33}e_3 \end{array} \right.$$

$$e'_1 = s_{11}e_1 + s_{21}e_2 + s_{31}e_3$$

$$\Leftrightarrow s_{11}(2,1,0) + s_{21}(1,0,3) + s_{31}(0,0,1) = (1,0,1)$$

$$\Leftrightarrow \left\{ \begin{array}{rcrcrcl} 2s_{11} & + & s_{21} & & & = & 1 \\ s_{11} & & & & & = & 0 \\ & & 3s_{21} & + & s_{31} & = & 1 \end{array} \right.$$

$$\Leftrightarrow s_{11} = 0, s_{21} = 1, s_{31} = -2.$$

In matrix form: $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} s_{11} \\ s_{21} \\ s_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$e_2' = s_{12}e_1 + s_{22}e_2 + s_{32}e_3$$

$$\Leftrightarrow s_{12}(2,1,0) + s_{22}(1,0,3) + s_{32}(0,0,1) = (0,1,-2)$$

$$\Leftrightarrow \begin{cases} 2s_{12} + s_{22} = 0 \\ s_{12} = 1 \\ 3s_{22} + s_{32} = -2 \end{cases}$$

$$\Leftrightarrow s_{12} = 1, s_{22} = -2, s_{32} = 4.$$

In matrix form: $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} s_{12} \\ s_{22} \\ s_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$$e'_3 = s_{13}e_1 + s_{23}e_2 + s_{33}e_3$$

$$\Leftrightarrow s_{13}(2,1,0) + s_{23}(1,0,3) + s_{33}(0,0,1) = (0,1,3)$$

$$\Leftrightarrow \begin{cases} 2s_{13} + s_{23} = 0 \\ s_{13} = 1 \\ 3s_{23} + s_{33} = 3 \end{cases}$$

$$\Leftrightarrow s_{13} = 1, s_{23} = -2, s_{33} = 9.$$

In matrix form: $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} s_{13} \\ s_{23} \\ s_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

Therefore, the transition matrix S from B' to B is

$$S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -2 \\ -2 & 4 & 9 \end{pmatrix}$$

Note. $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} . S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix}$

$$\Rightarrow S = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}^{-1} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix}$$

In matrix form

$$\begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ e_1 & \cdots & e_i & \cdots & e_n \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{pmatrix} . \begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ [e'_1]_B & \cdots & [e'_i]_B & \cdots & [e'_n]_B \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ e'_1 & \cdots & e'_i & \cdots & e'_n \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{pmatrix}$$

We have

$$BS = B' \Rightarrow S = B^{-1}.B'$$

$$B[x]_B = x^T = B'[x]_{B'} \Rightarrow [x]_B = B^{-1}.B'[x]_{B'} = S[x]_{B'}.$$

2. The Coordinate vectors of w relative to 2 bases B, B'.

The coordinates of w relative to basis B are $\lambda_1, \lambda_2, \lambda_3$ which satisfies $w = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$

$\Leftrightarrow \lambda_1(2,1,0) + \lambda_2(1,0,3) + \lambda_3(0,0,1) = (8,-4,6)$

$$\Leftrightarrow \begin{cases} 2\lambda_1 + \lambda_2 = 8 \\ \lambda_1 = -4 \\ 3\lambda_2 + \lambda_3 = 6 \end{cases}$$

$\Leftrightarrow \lambda_1 = -4, \lambda_2 = 16, \lambda_3 = -42$

$\Rightarrow \left[w\right]_B = (-4,16,-42)^T \Rightarrow \left[w\right]_{B'} = S^{-1}.\left[w\right]_B = (8,-2,-2)^T.$

THANK YOU FOR YOUR ATTENTION