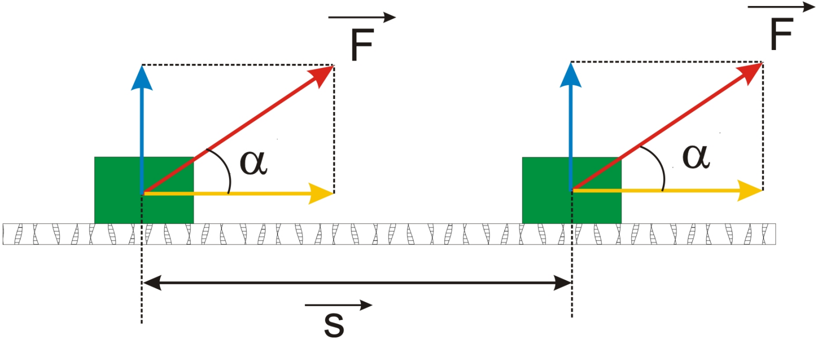
INNER PRODUCT SPACES

ELECTRONIC VERSION OF LECTURE

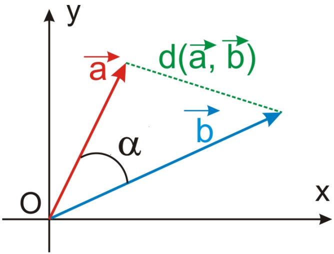
HoChiMinh City University of Technology
Faculty of Applied Science, Department of Applied Mathematics

Outline

WORK DONE BY A FORCE $\overrightarrow{F}$



$$W = \overrightarrow{F} . \overrightarrow{s} = F.s.\cos\alpha$$



$$\overrightarrow{a} = (a_1, a_2), \overrightarrow{b} = (b_1, b_2).$$

$$< \overrightarrow{a}, \overrightarrow{b} >= a_1.b_1 + a_2.b_2; ||\overrightarrow{a}|| = \sqrt{a_1^2 + a_2^2}$$

$$\cos\alpha = \frac{< \overrightarrow{a}, \overrightarrow{b} >}{||\overrightarrow{a}||.||\overrightarrow{b}||}; d(\overrightarrow{a}, \overrightarrow{b}) = ||\overrightarrow{a} - \overrightarrow{b}||$$

A real vector space V is called a real Euclidean inner product space if

- $< \cdot, \cdot >: V \times V \to \mathbb{R}$
  $(x, y) \longmapsto < x, y >$ – which is called inner product of 2 vectors.
- The following axioms are satisfied

1. $< x, y > = < y, x >, \ \forall x, y \in V$
2. $< x + y, z > = < x, z > + < y, z >, \forall x, y, z \in V$
3. $< \alpha x, y > = \alpha < x, y >, \forall x, y \in V, \forall \alpha \in \mathbb{R}.$
4. $< x, x > > 0, x \neq 0$ and $< x, x > = 0 \Leftrightarrow x = 0$

Example 2.1

On $\mathbb{R}_3$ we define the standard inner product

$$(x, y) \longmapsto < x, y >= x_1.y_1 + x_2.y_2 + x_3.y_3 = x.y^T$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$.

Example 2.2

On $\mathbb{R}_n$ we define the standard inner product

$$(x, y) \longmapsto < x, y >= \sum_{i=1}^{n} x_i y_i = x.y^T$$

where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$.

Example 2.3

On $\mathbb{R}_2$ we define the ***weighted Euclidean inner product*** *of 2 vectors*

$$(x, y) \longmapsto < x, y >= x_1.y_1 + 2x_2.y_2$$

where $x = (x_1, x_2), y = (y_1, y_2)$.

- $< x, y >= x_1.y_1 + 2x_2.y_2 = y_1.x_1 + 2y_2.x_2 =< y, x >$
- $< x + y, z >= (x_1 + y_1)z_1 + 2(x_2 + y_2)z_2 =$ $(x_1z_1 + 2x_2z_2) + (y_1z_1 + 2y_2z_2) =< x, z > + < y, z >$
- $< \alpha x, y >= \alpha.x_1.y_1 + 2\alpha.x_2.y_2 = \alpha(x_1y_1 + 2x_2y_2) =$ $\alpha. < x, y >$
- $< x, x >= x_1.x_1 + 2x_2.x_2 = x_1^2 + 2x_2^2 \geqslant 0.$ $< x, x >= 0 \Leftrightarrow x_1 = x_2 = 0$

EXAMPLE 2.4

On $\mathbb{R}_2$ the following function is not a inner product

$$(x, y) \longmapsto < x, y >= x_1.y_1 - 3x_2.y_2$$

where $x = (x_1, x_2), y = (y_1, y_2)$.

Let $x = (1,2)$. Then $< x, x >= 1 \times 1 - 3 \times 2 \times 2 = -11 < 0$.
Axiom 4 is not satisfied.

Definition 2.1

If V is a real inner product space, then the ***norm*** *(or* ***length****) of a vector $x \in V$ is denoted by $||x||$ and is defined by*

$$\boxed{||x|| = \sqrt{< x, x >}} \tag{1}$$

Example 2.5

On $\mathbb{R}_2$ the inner product is given

$$< x, y >= 3x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

where $x = (x_1, x_2), y = (y_1, y_2)$. Find the length of vector $u = (1, 2)$.

The length of vector u is $||u|| = \sqrt{< u, u >}$.

$$< u, u >= 3 \times 1 \times 1 + 1 \times 2 + 2 \times 1 + 2 \times 2 = 11$$

$$\Rightarrow ||u|| = \sqrt{11}$$

Definition 2.2

*If V is a real inner product space, then the **distance** between two vectors $u, v \in V$ is denoted by $d(u, v)$ and is defined by*

$$\boxed{d(u,v) = ||u - v|| = \sqrt{< u - v, u - v >}} \tag{2}$$

EXAMPLE 2.6

On $\mathbb{R}_2$ the inner product is given

$$< x, y >= x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 5x_2 y_2$$

where $x = (x_1, x_2), y = (y_1, y_2)$. *Find the distance between* $u = (1, -1), v = (0, 2)$.

$u - v = (1, -3)$. The distance between u, v is

$$d(u, v) = ||u - v|| = \sqrt{< u - v, u - v >} =$$

$$\sqrt{1 \times 1 - 2 \times 1 \times (-3) - 2 \times (-3) \times 1 + 5 \times (-3) \times (-3)}$$

$$= \sqrt{58}.$$

Definition 2.3

The angle α between two vectors $x, y \in V$ is defined by

$$\cos\alpha = \frac{<x, y>}{||x||.||y||}, (0 \leqslant \alpha \leqslant \pi)$$

$$<x, y> = ||x||.||y||.\cos\alpha.$$

Example 2.7

On $\mathbb{R}_2$ the inner product is given

$$< x, y >= x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 5x_2 y_2$$

where $x = (x_1, x_2), y = (y_1, y_2)$. Find the angle between 2 vectors $u = (1,1), v = (1,0)$.

We have

$$\cos\alpha = \frac{< u, v >}{||u||.||v||}$$

$$< u, v > = 1 \times 1 + 2 \times 1 \times 0 + 2 \times 1 \times 1 + 5 \times 1 \times 0 = 3$$

$$||u|| = \sqrt{< u, u >} =$$

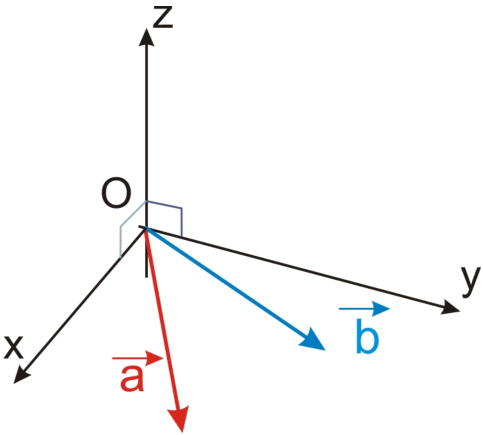
$$= \sqrt{1 \times 1 + 2 \times 1 \times 1 + 2 \times 1 \times 1 + 5 \times 1 \times 1} = \sqrt{10}$$

$$||v|| = \sqrt{< v, v >} =$$

$$= \sqrt{1 \times 1 + 2 \times 1 \times 0 + 2 \times 0 \times 1 + 5 \times 0 \times 0} = 1$$

Therefore $\cos\alpha = \dfrac{3}{\sqrt{10}} \Rightarrow \alpha = \arccos\dfrac{3}{\sqrt{10}}$

ORTHOGONALITY



ORTHOGONALITY

DEFINITION 3.1

1. *Two vectors* $x, y \in V$ *in an inner product space* V *is called* ***orthogonal*** $\Leftrightarrow < x, y >= 0$. *We denote it by* $x \perp y$.
2. *Vector* x *is* ***orthogonal to the set*** $M \subset V$ *if* x *is orthogonal to every vector in* M. *We denote it by* $x \perp M$.

Example 3.1

On $\mathbb{R}_2$ the inner product is given

$$< x, y >= 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$$

where $x = (x_1, x_2), y = (y_1, y_2)$, *and let* $u = (1, -1)$, $v = (2, m)$. *Find* m *such that* $u \perp v$.

In order to $u \perp v$, then $< u, v >= 0$

$\Leftrightarrow 2 \times 1 \times 2 - 1 \times m - (-1) \times 2 + (-1) \times m = 6 - 2m = 0$

$\Leftrightarrow m = 3$

EXAMPLE 3.2

On $\mathbb{R}_3$ the standard inner product is given and let $M = span\Big\{(1,1,1),(2,1,3)\Big\}$. Show that $u = (-2,1,1) \perp M$.

For every $v \in M$, we have

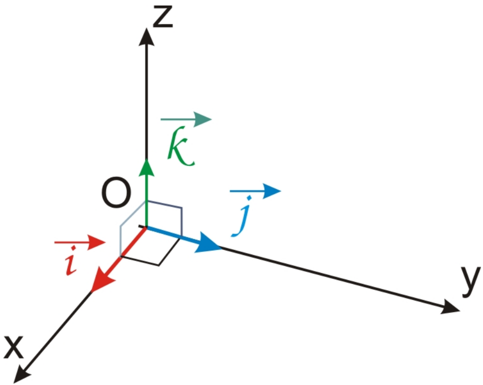
$$v = \alpha(1,1,1) + \beta(2,1,3) = (\alpha + 2\beta, \alpha + \beta, \alpha + 3\beta),$$

$\forall \alpha, \beta \in \mathbb{R}$. We have

$$< u, v >= -2.(\alpha + 2\beta) + 1.(\alpha + \beta) + 1.(\alpha + 3\beta) = 0$$

Therefore, $u \perp M$.

ORTHOGONAL AND ORTHONORMAL SETS



DEFINITION 3.2

1. *A set of two or more vectors in a real inner product space* $\left\{x_1, x_2, \ldots, x_n\right\}$ *is called orthogonal* $\Leftrightarrow$ *all pairs of distinct vectors in the set are* ***orthogonal***.
2. *An orthogonal set in which each vector has norm* 1 *is said to be orthonormal*

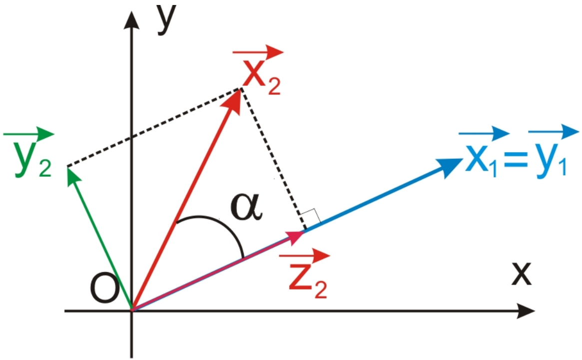
$$||x_k|| = 1, (k = 1, 2, \ldots, n)$$

Example 3.3

On $\mathbb{R}_2$ *the standard inner product is given. Then the set* $M = \left\{(1,-2),(2,1)\right\}$ *is orthogonal set.*
$N = \left\{\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\right\}$ *is the orthonormal set.*

- $<(1,-2),(2,1)>= 1\times 2+(-2)\times 1 = 0 \Rightarrow M$ is orthogonal set.
- N is the orthonormal set because
$$\left\langle\left(\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\right),\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)\right\rangle = \frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}}+\frac{(-2)}{\sqrt{5}}\times\frac{1}{\sqrt{5}} = 0$$
$$\left\|\left(\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\right)\right\| = \sqrt{\frac{1}{\sqrt{5}^2}+\frac{4}{\sqrt{5}^2}} = 1$$
$$\left\|\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)\right\| = \sqrt{\frac{4}{\sqrt{5}^2}+\frac{1}{\sqrt{5}^2}} = 1$$

THE GRAM-SCHMIDT PROCESS

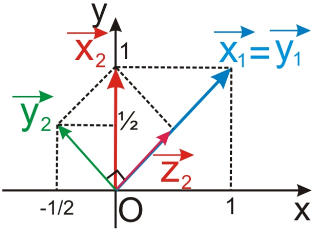


$$\overrightarrow{z_2} = ||\overrightarrow{x_2}||.||\overrightarrow{y_1}||.\cos\alpha.\frac{\overrightarrow{y_1}}{||y_1||^2} = \frac{<\overrightarrow{x_2}, \overrightarrow{y_1}>}{||\overrightarrow{y_1}||^2}.\overrightarrow{y_1}$$

$$\overrightarrow{y_2} = \overrightarrow{x_2} - \overrightarrow{z_2}$$

Example 3.4

On $\mathbb{R}_2$, construct an orthogonal Set from 2 vectors $x_1 = (1,1), x_2 = (0,1)$.



$$y_1 = x_1 = (1,1),$$

$$y_2 = x_2 - \frac{< x_2, y_1 >}{||y_1||^2}.y_1 = (0,1) - \frac{1}{2}.(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

THE GRAM-SCHMIDT PROCESS

To convert a basis $\{x_1, x_2, x_3\}$ into an orthogonal basis $\{y_1, y_2, y_3\}$, perform the following computations

$$\begin{cases} y_1 &= x_1 \\ y_2 &= \lambda_{21} y_1 + x_2 \\ y_3 &= \lambda_{31} y_1 + \lambda_{32} y_2 + x_3 \end{cases}$$

Since $y_1 \perp y_2$ then

$$< y_1, y_2 > =< y_1, \lambda_{21} y_1 + x_2 >= \lambda_{21} < y_1, y_1 > + < x_2, y_1 >= 0$$

$$\Rightarrow \lambda_{21} = -\frac{< x_2, y_1 >}{< y_1, y_1 >}$$

Similarly, $y_3 \perp y_1, y_2$ so

$$
\begin{aligned}
< y_3, y_1 > &=< \lambda_{31} y_1 + \lambda_{32} y_2 + x_3, y_1 > \\
&= \lambda_{31} < y_1, y_1 > + \lambda_{32} < y_2, y_1 > + < x_3, y_1 > \\
&= \lambda_{31} < y_1, y_1 > + < x_3, y_1 > = 0 \\
\Rightarrow \lambda_{31} &= -\frac{< x_3, y_1 >}{< y_1, y_1 >}
\end{aligned}
$$

$$
\begin{aligned}
< y_3, y_2 > &=< \lambda_{31} y_1 + \lambda_{32} y_2 + x_3, y_2 > \\
&= \lambda_{31} < y_1, y_2 > + \lambda_{32} < y_2, y_2 > + < x_3, y_2 > \\
&= \lambda_{32} < y_2, y_2 > + < x_3, y_2 > = 0 \\
\Rightarrow \lambda_{32} &= -\frac{< x_3, y_2 >}{< y_2, y_2 >}
\end{aligned}
$$

Example 3.5

In $\mathbb{R}_3$, construct an orthogonal Set from 3 vectors $(1,1,1),(0,1,1),(0,0,1)$

$$y_1 = x_1 = (1,1,1),$$
$$y_2 = -\frac{<x_2,y_1>}{<y_1,y_1>}y_1 + x_2 = -\frac{2}{3}(1,1,1) + (0,1,1) = \left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$$
$$y_3 = -\frac{<x_3,y_1>}{<y_1,y_1>}y_1 - \frac{<x_3,y_2>}{<y_2,y_2>}y_2 + x_3$$
$$= -\frac{1}{3}(1,1,1) - \frac{1}{2}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) + (0,0,1) = \left(0,-\frac{1}{2},\frac{1}{2}\right)$$

$(1,1,1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, -\frac{1}{2}, \frac{1}{2}\right)$ is the orthogonal set.
The orthonormal set can be obtained by setting
$e_1 = \frac{y_1}{||y_1||}$, $e_2 = \frac{y_2}{||y_2||}$, $e_3 = \frac{y_3}{||y_3||}$.

ORTHOGONAL COMPLEMENTS

THEOREM 3.1

If W is a subspace of a real inner product space V, then

1. $\forall x \in V, x \perp W \Leftrightarrow x$ *is orthogonal to a basis of* W
2. *The set* $W^{\perp}$ *of all vectors in* V *that are orthogonal to* W *is called the* ***orthogonal complement*** *of* W.

Example 3.6

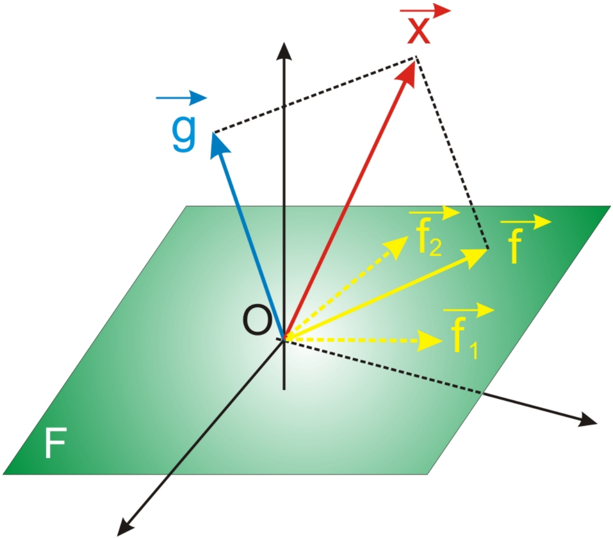
Let $W = \left\{(x_1, x_2, x_3) \in \mathbb{R}_3 : x_1 + x_2 + x_3 = 0\right\}$ *be the subspace of* $\mathbb{R}_3$. *Find a basis for the orthogonal complement* $W^{\perp}$ *of* W.

Step 1. The basis of W is $\left\{(-1,1,0),(-1,0,1)\right\}$.

Step 2. $x = (x_1, x_2, x_3) \in W^{\perp}$ so $x \perp (-1,1,0)$ and $x \perp (-1,0,1)$. Therefore, $\begin{cases} -x_1 + x_2 = 0 \\ -x_1 + x_3 = 0 \end{cases}$

$\Rightarrow x_1 = x_3, x_2 = x_3 \Rightarrow (x_1, x_2, x_3) = x_3(1,1,1)$. So $dim(W^{\perp}) = 1$ and the basis of $W^{\perp}$ is $\left\{(1,1,1)\right\}$.

ORTHOGONAL PROJECTIONS



Theorem 3.2

If F is a finite-dimensional subspace of an inner product space V, then every vector x in V can be expressed in exactly one way as

$$x = f + g,$$

where $f \in F, g \in F^{\perp}$.

Definition 3.3

Vector f is called the orthogonal projection of x on F. We denote it by $f = proj_F x$.

DEFINITION 3.4

The distance between a vector x and the subspace F is defined by

$$\boxed{d(x,F) = ||g|| = ||x - f||} \tag{3}$$

Example 3.7

On $\mathbb{R}_3$ the standard inner product, the subspace $F = span\left\{(1,1,1),(0,1,1)\right\}$ and the vector $x = (1,1,2)$ are given. Find the orthogonal projection $pr_F x$ of x on F and the distance between x and F.

Step 1. The basis of F : $f_1 = (1,1,1)$, $f_2 = (0,1,1)$

Step 2.

$$\begin{aligned} x = f + g &= \lambda_1.f_1 + \lambda_2.f_2 + g \\ &= \lambda_1(1,1,1) + \lambda_2(0,1,1) + g, \end{aligned}$$

where $f \in F$, $g \in F^{\perp}$

Step 3.

$$
\begin{aligned}
<x, f_1> &= <\lambda_1.f_1 + \lambda_2.f_2 + g, f_1> \\
&= \lambda_1. <f_1, f_1> + \lambda_2. <f_1, f_2> + <f_1, g> \\
&= \lambda_1.3 + \lambda_2.2 = <(1,1,2),(1,1,1)> = 4
\end{aligned}
$$

$$
\begin{aligned}
<x, f_2> &= <\lambda_1.f_1 + \lambda_2.f_2 + g, f_2> \\
&= \lambda_1. <f_2, f_1> + \lambda_2. <f_2, f_2> + <f_2, g> \\
&= \lambda_1.2 + \lambda_2.2 = <(1,1,2),(0,1,1)> = 3
\end{aligned}
$$

$$
\Rightarrow \begin{cases} 3\lambda_1 + 2\lambda_2 = 4 \\ 2\lambda_1 + 2\lambda_2 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \dfrac{1}{2} \end{cases}
$$

Step 4. Conclusion

- The orthogonal projection $pr_F x$ of x on F is
$$f = \lambda_1.f_1 + \lambda_2.f_2 = 1.(1,1,1) + \frac{1}{2}(0,1,1) = \left(1, \frac{3}{2}, \frac{3}{2}\right)$$
- The distance between x and F is

$$d(x,F) = ||g|| = ||x - f|| = \left\| (1,1,2) - \left(1, \frac{3}{2}, \frac{3}{2}\right) \right\|$$
$$= \left\| \left(0, -\frac{1}{2}, \frac{1}{2}\right) \right\| = \sqrt{0^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}$$

THE STANDARD INNER PRODUCT ON $\mathbb{R}_n$

1. $< x, y >= dot(x, y)$
2. $||x|| = norm(x)$
3. $d(x, y) = norm(x - y)$
4. $\cos\alpha = dot(x, y)/(norm(x) * norm(y))$

Orthogonal Complement

1. $f_1, f_2, \ldots, f_m$: basis of F. $A = [f_1; f_2; \ldots; f_m]$

$$A = \begin{pmatrix} f_1 \\ f_2 \\ \ldots \\ f_m \end{pmatrix} \Rightarrow \text{Basis of } F^{\perp} : null(A,'r')$$

2. If F is the solution subspace of homogeneous system $AX = 0$ then the basis of $F^{\perp}$ consists of all row vectors of matrix B

$$B = rref(A)$$

THE STANDARD INNER PRODUCT ON $\mathbb{R}_n$

Suppose the set of $f_1, f_2, \ldots, f_m$ is a basis of F.

$$A = \begin{pmatrix} dot(f_1, f_1) & dot(f_1, f_2) & \ldots & dot(f_1, f_m) \\ dot(f_2, f_1) & dot(f_2, f_2) & \ldots & dot(f_2, f_m) \\ \ldots & \ldots & \ldots & \ldots \\ dot(f_m, f_1) & dot(f_m, f_2) & \ldots & dot(f_m, f_m) \end{pmatrix},$$

$$B = \begin{pmatrix} dot(x, f_1) \\ dot(x, f_2) \\ \ldots \\ dot(x, f_m) \end{pmatrix}, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T = inv(A) * B$$

1. Projection $f = \lambda(1) * f_1 + \lambda(2) * f_2 + \ldots + \lambda(m) * f_m$
2. Distance $||g|| = ||x - f|| = norm(x - f)$

THANK YOU FOR YOUR ATTENTION