18.01 Single Variable Calculus
Fall 2006

Lecture 18: Definite Integrals

Integrals are used to calculate cumulative totals, averages, areas.

Area under a curve: (See Figure 1.)

1. Divide region into rectangles

2. Add up area of rectangles
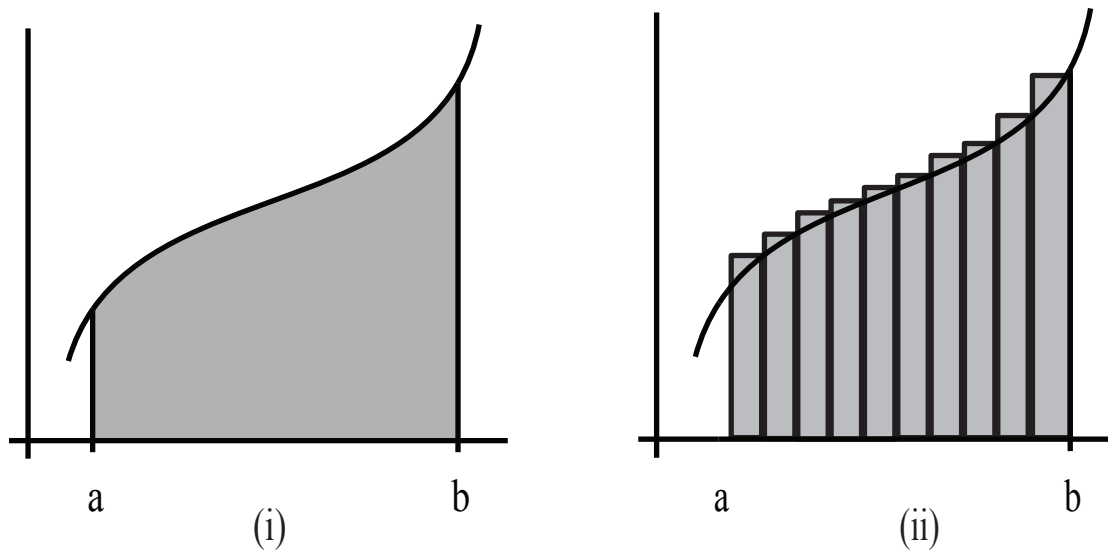
3. Take limit as rectangles become thin

Figure 1: (i) Area under a curve; (ii) sum of areas under rectangles
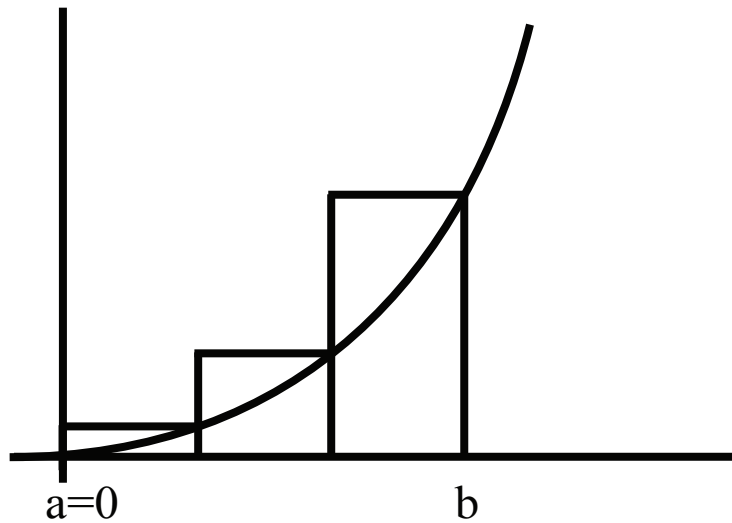
Example 1. $f(x) = x^2$, $a = 0$, b arbitrary

1. Divide into n intervals
   Length $b/n =$ base of rectangle

2. Heights:

   - 1st: $x = \dfrac{b}{n}$,   height $= \left(\dfrac{b}{n}\right)^2$

   - 2nd: $x = \dfrac{2b}{n}$,   height $= \left(\dfrac{2b}{n}\right)^2$

Sum of areas of rectangles:

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2 = \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

1

Figure 2: Area under $f(x) = x^2$ above $[0, b]$.

We will now estimate the sum using some 3-dimensional geometry.

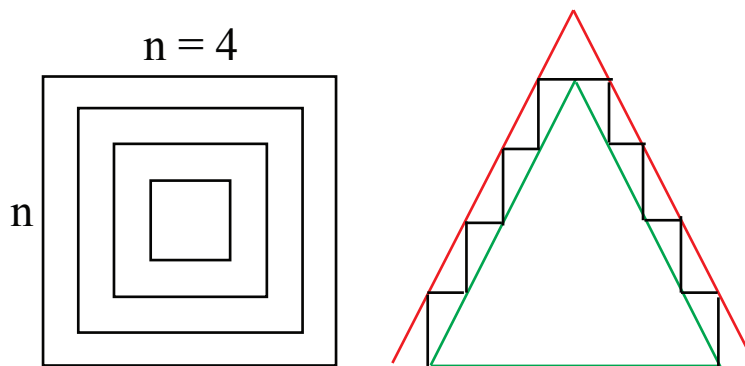Consider the staircase pyramid as pictured in Figure 3.

Figure 3: Staircase pyramid: left(top view) and right (side view)

1st level: $n \times n$ bottom, represents volume n^2.
2nd level: $(n-1) \times (n-1)$, represents volumne $(n-1)^2$), etc.
Hence, the total volume of the staircase pyramid is $n^2 + (n-1)^2 + \cdots + 1$.

Next, the volume of the pyramid is greater than the volume of the inner prism:

$$1^2 + 2^2 + \cdots + n^2 > \frac{1}{3}(\text{base})(\text{height}) = \frac{1}{3}n^2 \cdot n = \frac{1}{3}n^3$$

and less than the volume of the outer prism:

$$1^2 + 2^2 + \cdots + n^2 < \frac{1}{3}(n+1)^2(n+1) = \frac{1}{3}(n+1)^3$$

In all,

$$\frac{1}{3} = \frac{\frac{1}{3}n^3}{n^3} < \frac{1^2 + 2^2 + \cdots + n^2}{n^3} < \frac{1}{3}\frac{(n+1)^3}{n^3}$$

Therefore,

$$\lim_{n\to\infty} \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{1}{3}b^3,$$

and the area under x^2 from 0 to b is $\dfrac{b^3}{3}$.

Example 2. $f(x) = x$; area under x above $[0, b]$. Reasoning similar to Example 1, but easier, gives a sum of areas:

$$\frac{b^2}{n^2}(1 + 2 + 3 + \cdots + n) \to \frac{1}{2}b^2 \quad (\text{as } n \to \infty)$$

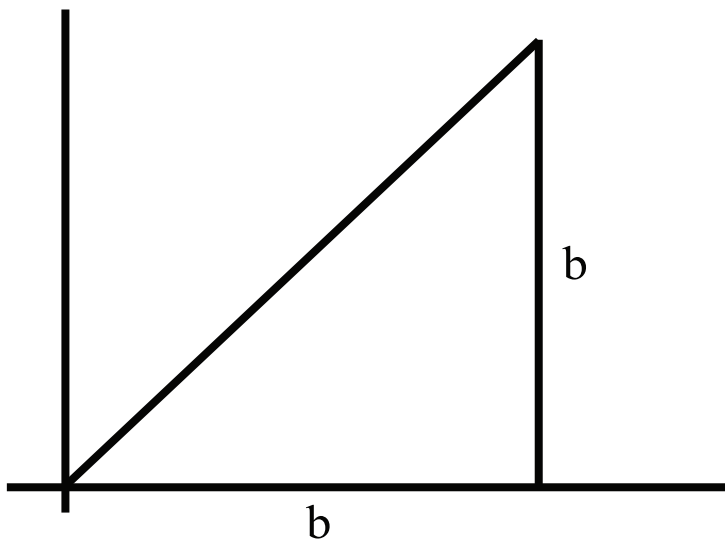This is the area of the triangle in Figure 4.

Figure 4: Area under $f(x) = x$ above $[0, b]$.

Pattern:

$$\frac{d}{db}\left(\frac{b^3}{3}\right) = b^2$$

$$\frac{d}{db}\left(\frac{b^2}{2}\right) = b$$

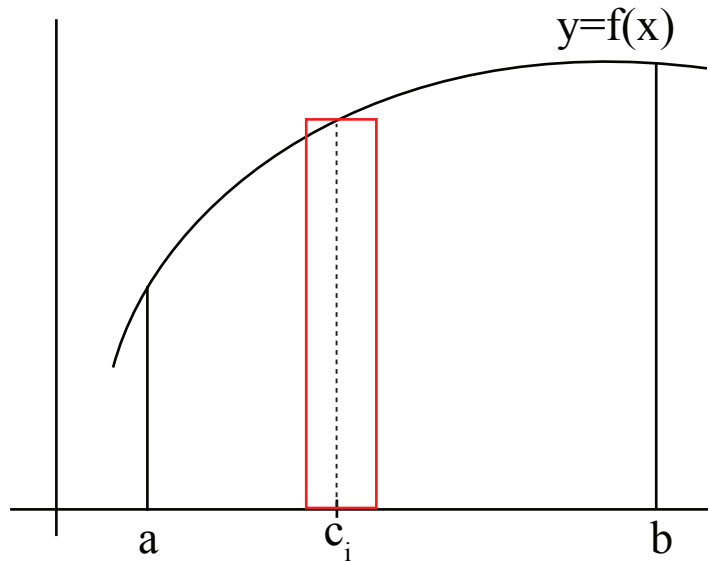The area $A(b)$ under $f(x)$ should satisfy $A'(b) = f(b)$.

General Picture

Figure 5: One rectangle from a Riemann Sum

- Divide into n equal pieces of length $= \Delta x = \dfrac{b-a}{n}$

- Pick <u>any</u> c_i in the interval; use $f(c_i)$ as the height of the rectangle

- Sum of areas: $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$

In summation notation: $\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x$ $\leftarrow$ called a *Riemann sum*.

Definition:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx \;\; \leftarrow \;\; \text{called a } \textit{definite integral}$$

This definite integral represents the area under the curve $y = f(x)$ above $[a, b]$.

Example 3. (Integrals applied to quantity besides area.) Student borrows from parents.
$P =$ principal in dollars, $t =$ time in years, $r =$ interest rate (e.g., 6 % is $r = 0.06$/year).
After time t, you owe $P(1 + rt) = P + Prt$

The integral can be used to represent the total amount borrowed as follows. Consider a function $f(t)$, the "borrowing function" in dollars per year. For instance, if you borrow $ 1000 /month, then $f(t) = 12,000$/year. Allow f to vary over time.

Say $\Delta t = 1/12$ year $= 1$ month.

$$t_i = i/12 \quad i = 1, \cdots, 12.$$

$f(t_i)$ is the borrowing rate during the ith month so the amount borrowed is $f(t_i)\Delta t$. The total is

$$\sum_{i=1}^{12} f(t_i)\Delta t.$$

In the limit as $\Delta t \to 0$, we have

$$\int_0^1 f(t)dt$$

which represents the total borrowed in one year in dollars per year.

The integral can also be used to represent the total amount owed. The amount owed depends on the interest rate. You owe

$$f(t_i)(1 + r(1 - t_i))\Delta t$$

for the amount borrowed at time t_i. The total owed for borrowing at the end of the year is

$$\int_0^1 f(t)(1 + r(1 - t))dt$$

Lecture 19: First Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (FTC 1)

If $f(x)$ is continuous and $F'(x) = f(x)$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$

Notation: $F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$

Example 1. $F(x) = \dfrac{x^3}{3}, \quad F'(x) = x^2; \quad \displaystyle\int_a^b x^2 dx = \dfrac{x^3}{3}\Big|_a^b = \dfrac{b^3}{3} - \dfrac{a^3}{3}$

Example 2. Area under one hump of $\sin x$ (See Figure 1.)

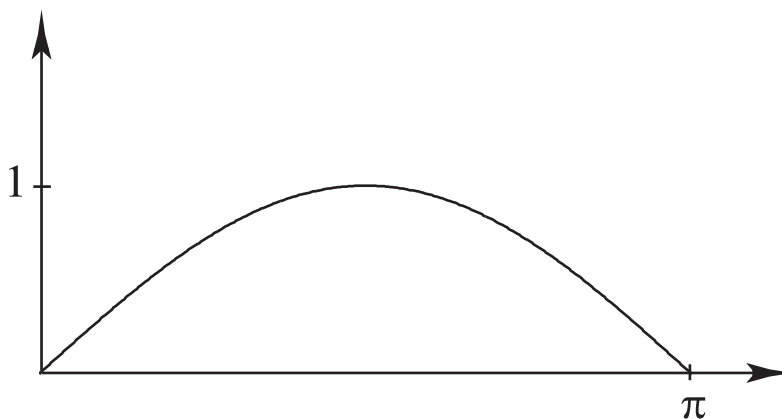$$\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$$

Figure 1: Graph of $f(x) = \sin x$ for $0 \le x \le \pi$.

Example 3. $\displaystyle\int_0^1 x^5 dx = \dfrac{x^6}{6}\Big|_0^1 = \dfrac{1}{6} - 0 = \dfrac{1}{6}$

Intuitive Interpretation of FTC:

$x(t)$ is a position; $v(t) = x'(t) = \dfrac{dx}{dt}$ is the speed or rate of change of x.

$$\int_a^b v(t)dt = x(b) - x(a) \qquad \text{(FTC 1)}$$

R.H.S. is how far $x(t)$ went from time $t = a$ to time $t = b$ (difference between two odometer readings). L.H.S. represents speedometer readings.

$$\sum_{i=1}^n v(t_i)\Delta t \quad \text{approximates the sum of distances traveled over times } \Delta t$$

The approximation above is accurate if $v(t)$ is close to $v(t_i)$ on the ith interval. The interpretation of $x(t)$ as an odometer reading is no longer valid if v changes sign. Imagine a round trip so that $x(b) - x(a) = 0$. Then the positive and negative velocities $v(t)$ cancel each other, whereas an odometer would measure the total distance not the net distance traveled.

Example 4. $\displaystyle\int_0^{2\pi} \sin x\, dx = -\cos x \Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = 0.$

The integral represents the sum of areas under the curve, above the x-axis minus the areas below the x-axis. (See Figure 2.)
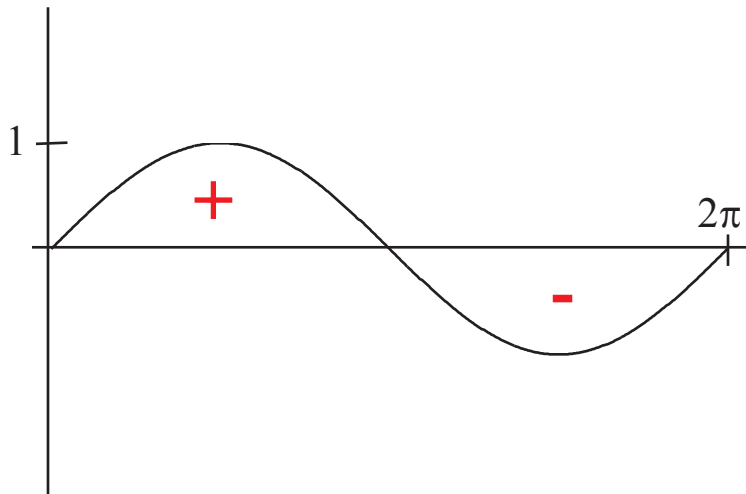

Figure 2: Graph of $f(x) = \sin x$ for $0 \le x \le 2\pi$.

Integrals have an important additive property (See Figure 3.)

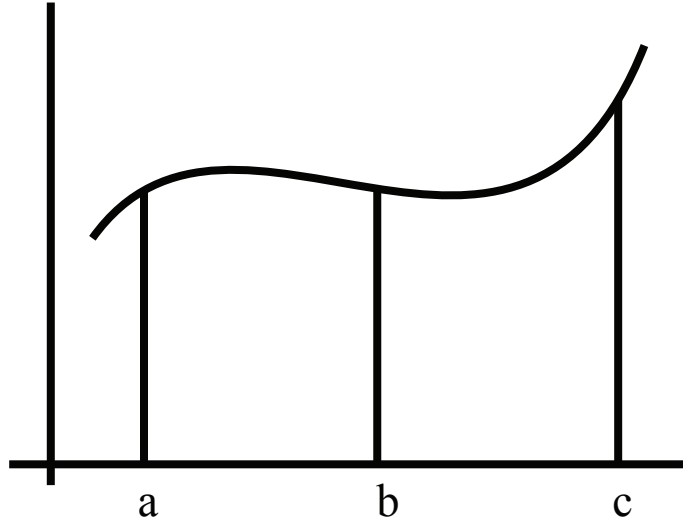$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Figure 3: Illustration of the additive property of integrals

<u>New Definition</u>:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

This definition is used so that the fundamental theorem is valid no matter if $a < b$ or $b < a$. It also makes it so that the additive property works for a, b, c in <u>any</u> order, not just the one pictured in Figure 3.

Estimation:

If $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ (only if $a < b$)

Example 5. Estimation of e^x
Since $1 \leq e^x$ for $x \geq 0$,

$$\int_0^1 1\,dx \leq \int_0^1 e^x\,dx$$

$$\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1$$

Thus $1 \leq e - 1$, or $e \geq 2$.

Example 6. We showed earlier that $1 + x \leq e^x$. It follows that

$$\int_0^1 (1+x)dx \leq \int_0^1 e^x\,dx = e - 1$$

$$\int_0^1 (1+x)dx = \left(x + \frac{x^2}{2}\right)\Big|_0^1 = \frac{3}{2}$$

Hence, $\dfrac{3}{2} \leq e - 1$, or, $e \geq \dfrac{5}{2}$.

Change of Variable:

If $f(x) = g(u(x))$, then we write $du = u'(x)dx$ and

$$\int g(u)du = \int g(u(x))u'(x)dx = \int f(x)u'(x)dx \qquad \text{(indefinite integrals)}$$

For definite integrals:

$$\int_{x_1}^{x_2} f(x)u'(x)dx = \int_{u_1}^{u_2} g(u)du \quad \text{where } u_1 = u(x_1),\ u_2 = u(x_2)$$

Example 7. $\displaystyle\int_1^2 \left(x^3 + 2\right)^4 x^2 dx$

$$\text{Let } u = x^3 + 2. \quad \text{Then } du = 3x^2 dx \implies x^2 dx = \frac{du}{3};$$

$$x_1 = 1, x_2 = 2 \implies u_1 = 1^3 + 2 = 3,\ u_2 = 2^3 + 2 = 10,\ \text{and}$$

$$\int_1^2 \left(x^3 + 2\right)^4 x^2 dx = \int_3^{10} u^4 \frac{du}{3} = \frac{u^5}{15}\Big|_3^{10} = \frac{10^5 - 3^5}{15}$$

Lecture 20: Second Fundamental Theorem

Recall: First Fundamental Theorem of Calculus (FTC 1)

> If f is continuous and $F' = f$, then
> $$\int_a^b f(x)dx = F(b) - F(a)$$

We can also write that as
$$\int_a^b f(x)dx = \int f(x)dx \Big|_{x=a}^{x=b}$$

Do all continuous functions have antiderivatives? Yes. However...
What about a function like this?
$$\int e^{-x^2}dx =??$$

Yes, this antiderivative exists. No, it's not a function we've met before: it's a new function.

The new function is defined as an integral:
$$F(x) = \int_0^x e^{-t^2}dt$$

It will have the property that $F'(x) = e^{-x^2}$.

Other new functions include antiderivatives of $e^{-x^2}, x^{1/2}e^{-x^2}, \dfrac{\sin x}{x}, \sin(x^2), \cos(x^2), \ldots$

Second Fundamental Theorem of Calculus (FTC 2)

> If $F(x) = \displaystyle\int_a^x f(t)dt$ and f is continuous, then
> $$F'(x) = f(x)$$

Geometric Proof of FTC 2: Use the area interpretation: $F(x)$ equals the area under the curve between a and x.

$$
\begin{aligned}
\Delta F &= F(x + \Delta x) - F(x) \\
\Delta F &\approx (\text{base})(\text{height}) \approx (\Delta x)f(x) \qquad (\text{See Figure } 1.) \\
\frac{\Delta F}{\Delta x} &\approx f(x) \\
\text{Hence } \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} &= f(x)
\end{aligned}
$$

But, by the definition of the derivative:
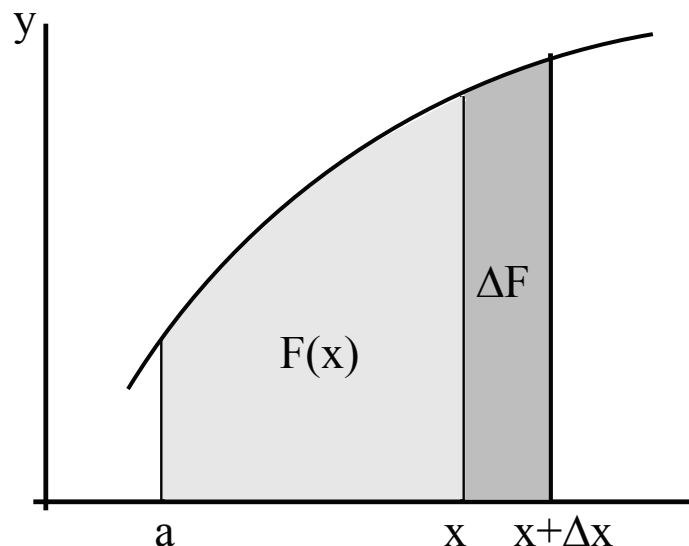$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = F'(x)$$

Figure 1: Geometric Proof of FTC 2.

Therefore,
$$F'(x) = f(x)$$

Another way to prove FTC 2 is as follows:

$$
\begin{aligned}
\frac{\Delta F}{\Delta x} &= \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt\right] \\
&= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt \quad \text{(which is the "average value" of } f \text{ on the interval } x \le t \le x+\Delta x.)
\end{aligned}
$$

As the length Δx of the interval tends to 0, this average tends to $f(x)$.

Proof of FTC 1 (using FTC 2)

Start with $F' = f$ (we assume that f is continuous). Next, define $G(x) = \int_a^x f(t)dt$. By FTC2, $G'(x) = f(x)$. Therefore, $(F - G)' = F' - G' = f - f = 0$. Thus, $F - G = $ constant. (Recall we used the Mean Value Theorem to show this).

Hence, $F(x) = G(x) + c$. Finally since $G(a) = 0$,

$$\int_a^b f(t)dt = G(b) = G(b) - G(a) = [F(b) - c] - [F(a) - c] = F(b) - F(a)$$

which is FTC 1.

Remark. In the preceding proof G was a definite integral and F could be any antiderivative. Let us illustrate with the example $f(x) = \sin x$. Taking $a = 0$ in the proof of FTC 1,

$$G(x) = \int_0^x \cos t\, dt = \sin t\Big|_0^x = \sin x \quad \text{and } G(0) = 0.$$

If, for example, $F(x) = \sin x + 21$. Then $F'(x) = \cos x$ and

$$\int_a^b \sin x \, dx = F(b) - F(a) = (\sin b + 21) - (\sin a + 21) = \sin b - \sin a$$

Every function of the form $F(x) = G(x) + c$ works in FTC 1.

Examples of "new" functions

The *error function*, which is often used in statistics and probability, is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$
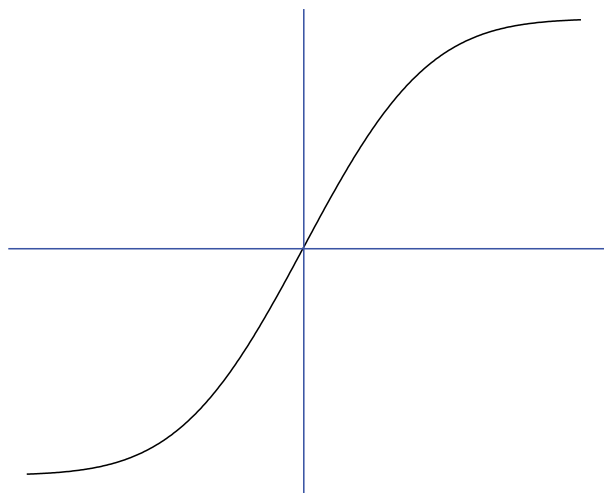$$\text{and } \lim_{x \to \infty} \mathrm{erf}(x) = 1 \quad \text{(See Figure 2)}$$

Figure 2: Graph of the error function.

Another "new" function of this type, called the *logarithmic integral*, is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

This function gives the approximate number of prime numbers less than x. A common encryption technique involves encoding sensitive information like your bank account number so that it can be sent over an insecure communication channel. The message can only be decoded using a secret prime number. To know how safe the secret is, a cryptographer needs to know roughly how many 200-digit primes there are. You can find out by estimating the following integral:

$$\int_{10^{200}}^{10^{201}} \frac{dt}{\ln t}$$

We know that

$$\ln 10^{200} = 200 \ln(10) \approx 200(2.3) = 460 \quad \text{and} \quad \ln 10^{201} = 201 \ln(10) \approx 462$$

We will approximate to one significant figure: $\ln t \approx 500$ for $200 \leq t \leq 10^{201}$.

With all of that in mind, the number of 200-digit primes is roughly [1]

$$\int_{10^{200}}^{10^{201}} \frac{dt}{\ln t} \approx \int_{10^{200}}^{10^{201}} \frac{dt}{500} = \frac{1}{500} \left(10^{201} - 10^{200} \right) \approx \frac{9 \cdot 10^{200}}{500} \approx 10^{198}$$

There are LOTS of 200-digit primes. The odds of some hacker finding the 200-digit prime required to break into your bank account number are very very slim.

Another set of "new" functions are the Fresnel functions, which arise in optics:

$$\begin{aligned} C(x) &= \int_0^x \cos(t^2) dt \\ S(x) &= \int_0^x \sin(t^2) dt \end{aligned}$$

Bessel functions often arise in problems with circular symmetry:

$$J_0(x) = \frac{1}{2\pi} \int_0^\pi \cos(x \sin \theta) d\theta$$

On the homework, you are asked to find $C'(x)$. That's easy!

$$C'(x) = \cos(x^2)$$

We will use FTC 2 to discuss the function $L(x) = \int_1^x \frac{dt}{t}$ from first principles next lecture.

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[1] The middle equality in this approximation is a very basic and useful fact

$$\int_a^b c\, dx = c(b - a)$$

Think of this as finding the area of a rectangle with base $(b - a)$ and height c. In the computation above, $a = 10^{200}, b = 10^{201}, c = \frac{1}{500}$

Lecture 21: Applications to Logarithms and Geometry

Application of FTC 2 to Logarithms

The integral definition of functions like $C(x)$, $S(x)$ of Fresnel makes them nearly as easy to use as elementary functions. It is possible to draw their graphs and tabulate values. You are asked to carry out an example or two of this on your problem set. To get used to using definite integrals and FTC2, we will discuss in detail the simplest integral that gives rise to a relatively new function, namely the logarithm.

Recall that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

except when $n = -1$. It follows that the antiderivative of $1/x$ is not a power, but something else. So let us define a function $L(x)$ by

$$L(x) = \int_1^x \frac{dt}{t}$$

(This function turns out to be the logarithm. But recall that our approach to the logarithm was fairly involved. We first analyzed a^x, and then defined the number e, and finally defined the logarithm as the inverse function to e^x. The direct approach using this integral formula will be easier.)

All the basic properties of $L(x)$ follow directly from its definition. Note that $L(x)$ is defined for $0 < x < \infty$. (We will not extend the definition past $x = 0$ because $1/t$ is infinite at $t = 0$.) Next, the fundamental theorem of calculus (FTC2) implies

$$L'(x) = \frac{1}{x}$$

Also, because we have started the integration with lower limit 1, we see that

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

Thus L is increasing and crosses the x-axis at $x = 1$: $L(x) < 0$ for $0 < x < 1$ and $L(x) > 0$ for $x > 1$. Differentiating a second time,

$$L''(x) = -1/x^2$$

It follows that L is concave down.

The key property of $L(x)$ (showing that it is, indeed, a logarithm) is that it converts multiplication into addition:

Claim 1. $L(ab) = L(a) + L(b)$

Proof: By definition of $L(ab)$ and $L(a)$,

$$L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = L(a) + \int_a^{ab} \frac{dt}{t}$$

To handle $\displaystyle\int_a^{ab} \frac{dt}{t}$, make the substitution $t = au$. Then

$$dt = adu; \quad a < t < ab \implies 1 < u < b$$

Therefore,

$$\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{adu}{au} = \int_1^b \frac{du}{u} = L(b)$$

This confirms $L(ab) = L(a) + L(b)$.

Two more properties, the end values, complete the general picture of the graph.

Claim 2. $L(x) \to \infty$ as $x \to \infty$.

Proof: It suffices to show that $L(2^n) \to \infty$ as $n \to \infty$, because the fact that L is increasing fills in all the values in between the powers of 2.

$$
\begin{aligned}
L(2^n) &= L(2 \cdot 2^{n-1}) = L(2) + L(2^{n-1}) \\
&= L(2) + L(2) + L(2^{n-2}) = L(2) + L(2) + \cdots + L(2) \quad (n \text{ times})
\end{aligned}
$$

Consequently, $L(2^n) = nL(2) \to \infty$ as $n \to \infty$. (In more familiar notation, $\ln 2^n = n \ln 2$.)

Claim 3. $L(x) \to -\infty$ as $x \to 0^+$.

Proof: $0 = L(1) = L\left(x \cdot \dfrac{1}{x}\right) = L(x) + L(1/x) \implies L(x) = -L(1/x)$. As $x \to 0^+$, $1/x \to +\infty$, so Claim 2 implies $L(1/x) \to \infty$. Hence

$$L(x) = -L(1/x) \to -\infty, \quad \text{as} \ \ x \to 0^+$$

Thus $L(x)$, defined on $0 < x < \infty$ increases from $-\infty$ to ∞, crossing the x-axis at $x = 1$. It is concave down and its graph can be drawn as in Fig. 1.

This provides an alternative to our previous approach to the exponential and log functions. Starting from $L(x)$, we can *define* the log function by $\ln x = L(x)$, *define* e as the number such that $L(e) = 1$, *define* e^x as the inverse function of $L(x)$, and *define* $a^x = e^{xL(a)}$.
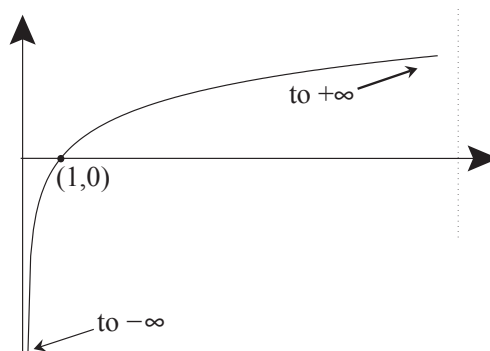

Figure 1: Graph of $y = \ln(x)$.

Application of FTCs to Geometry (Volumes and Areas)
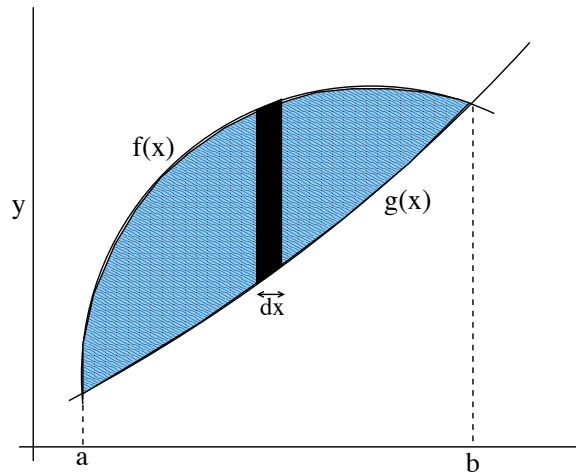
1. Areas between two curves

Figure 2: Finding the area between two curves.

Refer to Figure 2. Find the crossing points a and b. The area, A, between the curves is

$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

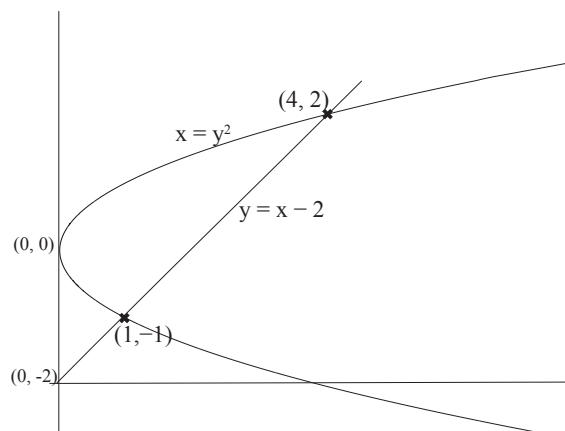**Example 1.** Find the area in the region between $x = y^2$ and $y = x - 2$.

Figure 3: The intersection of $x = y^2$ and $y = x - 2$.

First, graph these functions and find the crossing points (see Figure 3).

$$
\begin{aligned}
y + 2 &= x = y^2 \\
y^2 - y - 2 &= 0 \\
(y - 2)(y + 1) &= 0
\end{aligned}
$$

Crossing points at $y = -1, 2$. Plug these back in to find the associated x values, $x = 1$ and $x = 4$. Thus the curves meet at $(1, -1)$ and $(4, 2)$ (see Figure 3).

There are two ways of finding the area between these two curves, a hard way and an easy way.

Hard Way: Vertical Slices
 If we slice the region between the two curves vertically, we need to consider two different regions.

Figure 4: The intersection of $x = y^2$ and $y = x - 2$.

Where $x > 1$, the region's lower bound is the straight line. For $x < 1$, however, the region's lower bound is the lower half of the sideways parabola. We find the area, A, between the two curves by integrating the difference between the top curve and the bottom curve in each region:

$$
A = \int_0^1 \left\{ \sqrt{x} - (-\sqrt{x}) \right\} dx + \int_1^4 \left\{ \sqrt{x} - (x - 2) \right\} dx = \int (y_{top} - y_{bottom}) \, dx
$$

Easy Way: Horizontal Slices
Here, instead of subtracting the bottom curve from the top curve, we subtract the right curve from the left one.

$$
A = \int (x_{\text{left}} - x_{\text{right}}) \, dy = \int_{y=-1}^{y=2} \left[(y + 2) - y^2 \right] dx = \left(\frac{y^2}{2} + 2y + \frac{-y^3}{3} \right) \Big|_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3} - (\frac{1}{2} - 2 + \frac{1}{3}) = \frac{9}{2}
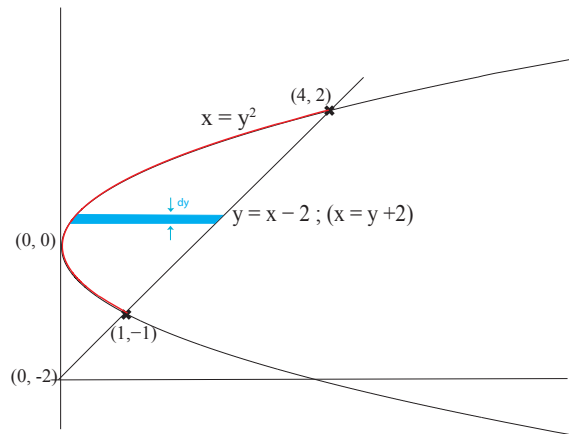$$

Figure 5: The intersection of $x = y^2$ and $y = x - 2$.

2. Volumes of solids of revolution

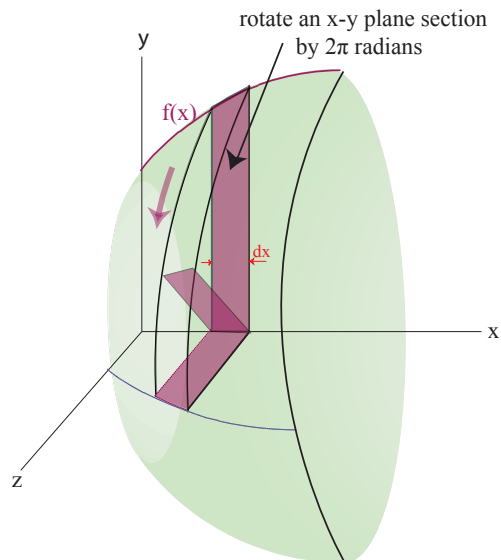Rotate $f(x)$ about the x-axis, coming out of the page, to get:

Figure 6: A solid of revolution: the purple slice is rotated by $\pi/4$ and $\pi/2$.

We want to figure out the volume of a "slice" of that solid. We can approximate each slice as a *disk* with width dx, radius y, and a cross-sectional area of πy^2. The volume of one slice is then:

$$dV = \pi y^2 dx \quad \text{(for a solid of revolution around the } x\text{-axis)}$$

Integrate with respect to x to find the total volume of the solid of revolution.

Example 2. Find the volume of a ball of radius a.

Figure 7: A ball of radius a

The equation for the upper half of the circle is

$$y = \sqrt{a^2 - x^2}.$$

If we spin the upper part of the curve about the x-axis, we get a ball of radius a. Notice that x ranges from $-a$ to $+a$. Putting all this together, we find

$$V = \int \pi y^2 dx = \int_{x=-a}^{x=a} \pi(a^2 - x^2)dx = \left(\pi a^2 x - \frac{\pi x^3}{3}\right)\Big|_{-a}^{a} = \frac{2}{3}\pi a^3 - \left(-\frac{2}{3}\pi a^3\right) = \frac{4}{3}\pi a^3$$

One can often exploit symmetry to further simplify these types of problems. In the problem above, for example, notice that the curve is symmetric about the y-axis. Therefore,

$$V = \int_{-a}^{a} \pi(a^2 - x^2)dx = 2\int_{0}^{a} \pi(a^2 - x^2)dx = 2\left(\pi a^2 x - \frac{x^3}{3}\right)\Big|_{0}^{a}$$

(The savings is that zero is an easier lower limit to work with than $-a$.) We get the same answer:

$$V = 2\left(\pi a^2 x - \frac{x^3}{3}\right)\Big|_{0}^{a} = 2\left(\pi a^3 - \frac{\pi}{3}a^3\right) = \frac{4}{3}\pi a^3$$

Lecture 22: Volumes by Disks and Shells

Disks and Shells

We will illustrate the 2 methods of finding volume through an example.
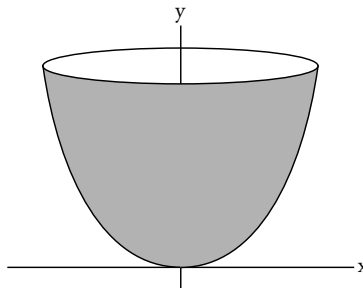
Example 1. A witch's cauldron

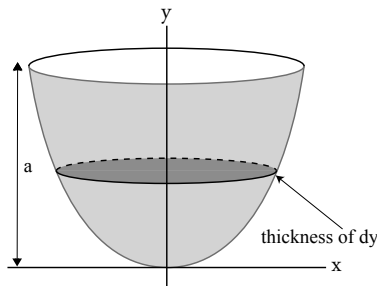Figure 1: $y = x^2$ rotated around the y-axis.

Method 1: Disks

Figure 2: Volume by Disks for the Witch's Cauldron problem.

The area of the disk in Figure 2 is πx^2. The disk has thickness dy and volume $dV = \pi x^2 dy$. The volume V of the cauldron is

$$
\begin{aligned}
V &= \int_0^a \pi x^2 \, dy \quad (\text{substitute} \quad y = x^2) \\
V &= \int_0^a \pi y \, dy = \pi \frac{y^2}{2}\Big|_0^a = \frac{\pi a^2}{2}
\end{aligned}
$$

If $a = 1$ meter, then $V = \dfrac{\pi}{2}a^2$ gives

$$V = \frac{\pi}{2}\ m^3 = \frac{\pi}{2}(100 \text{ cm})^3 = \frac{\pi}{2}10^6 \text{ cm}^3 \approx 1600 \text{ liters} \quad \text{(a huge cauldron)}$$

Warning about units.

If $a = 100$ cm, then
$$V = \frac{\pi}{2}(100)^2 = \frac{\pi}{2}10^4 \text{ cm}^3 = \frac{\pi}{2}10 \sim 16 \text{ liters}$$

But 100cm = 1m. Why is this answer different? The resolution of this paradox is hiding in the equation.
$$y = x^2$$

At the top, $100 = x^2 \implies x = 10$ cm. So the second cauldron looks like Figure 3. By contrast, when

Figure 3: The skinny cauldron.

$a = 1$ m, the top is ten times wider: $1 = x^2$ or $x = 1$ m. Our equation, $y = x^2$, is not scale-invariant. The shape described depends on the units used.

Method 2: Shells

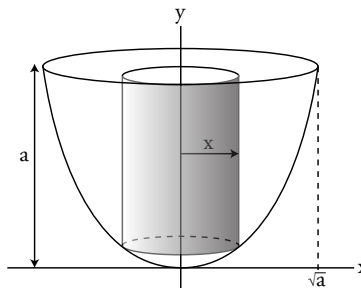This really should be called the cylinder method.

Figure 4: $x =$ radius of cylinder. Thickness of cylinder $= dx$. Height of cylinder $= a - y = a - x^2$.

The thin shell/cylinder has height $a - x^2$, circumference $2\pi x$, and thickness dx.

$$
\begin{aligned}
dV &= (a - x^2)(2\pi x)dx \\
V &= \int_{x=0}^{x=\sqrt{a}} (a - x^2)(2\pi x)dx = 2\pi \int_0^{\sqrt{a}} (ax - x^3)dx \\
&= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^{\sqrt{a}} = 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = 2\pi \left(\frac{a^2}{4} \right) = \frac{\pi a^2}{2} \quad \text{(same as before)}
\end{aligned}
$$

Example 2. The boiling cauldron

Now, let's fill this cauldron with water, and light a fire under it to get the water to boil (at 100^oC). Let's say it's a cold day: the temperature of the air outside the cauldron is 0^oC. How much energy does it take to boil this water, i.e. to raise the water's temperature from 0^oC to 100^oC? Assume the
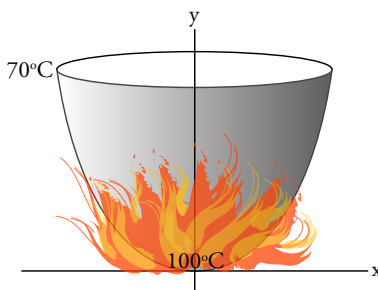

Figure 5: The boiling cauldron ($y = a = 1$ meter.)

temperature decreases linearly between the top and the bottom ($y = 0$) of the cauldron:

$$
T = 100 - 30y \quad \text{(degrees Celsius)}
$$

Use the method of disks, because the water's temperature is constant over each horizontal disk. The total heat required is

$$
\begin{aligned}
H &= \int_0^1 T(\pi x^2)dy \quad \text{(units are (degree)(cubic meters))} \\
&= \int_0^1 (100 - 30y)(\pi y)dy \\
&= \pi \int_0^1 (100y - 30y^2)dy = \pi(50y^2 - 10y^3)\Big|_0^1 = 40\pi \text{ (deg.)m}^3
\end{aligned}
$$

How many calories is that?

$$
\# \text{ of calories} = \frac{1 \text{ cal}}{\text{cm}^3 \cdot \text{deg}} (40\pi) \left(\frac{100 \text{ cm}}{1 \text{ m}} \right)^3 = (40\pi)(10^6) \text{ cal} = 125 \times 10^3 \text{ kcal}
$$

There are about 250 kcals in a candy bar, so there are about

$$
\# \text{ of calories} = \left(\frac{1}{2} \text{ candy bar} \right) \times 10^3 \approx 500 \text{ candy bars}
$$

So, it takes about 500 candy bars' worth of energy to boil the water.
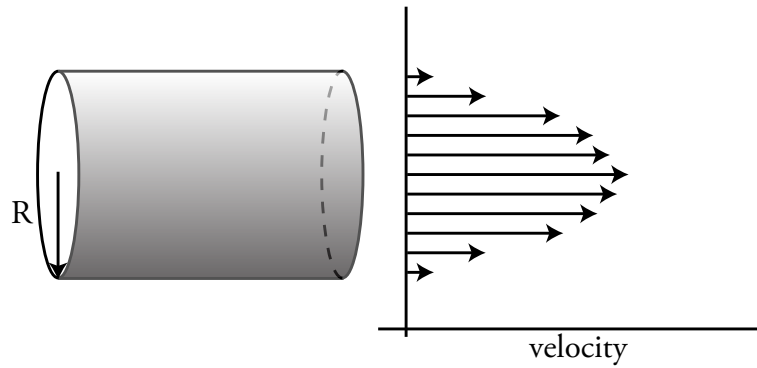
Figure 6: Flow is faster in the center of the pipe. It slows– "sticks"– at the edges (i.e. the inner surface of the pipe.)

Example 3. Pipe flow
 Poiseuille was the first person to study fluid flow in pipes (arteries, capillaries). He figured out the velocity profile for fluid flowing in pipes is:

$$
\begin{aligned}
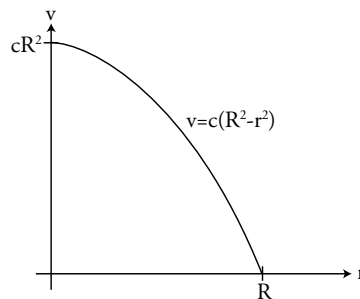v &= c(R^2 - r^2) \\
v &= \text{speed} = \frac{\text{distance}}{\text{time}}
\end{aligned}
$$

Figure 7: The velocity of fluid flow vs. distance from the center of a pipe of radius R.

The flow through the "annulus" (a.k.a ring) is (area of ring)(flow rate)

$$\text{area of ring} = 2\pi r dr \quad \text{(See Fig. 8: circumference } 2\pi r, \text{ thickness } dr)$$
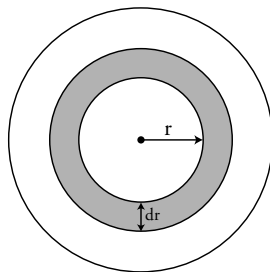
v is analogous to the height of the shell.

Figure 8: Cross-section of the pipe.

$$\text{total flow through pipe} \quad = \quad \int_0^R v(2\pi r\, dr) = c\int_0^R (R^2 - r^2)2\pi r\, dr$$

$$= \quad 2\pi c\int_0^R (R^2 r - r^3)dr = 2\pi c\left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Big|_0^R$$

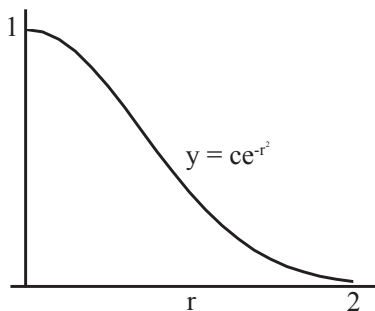$$\text{flow through pipe} \quad = \quad \frac{\pi}{2}cR^4$$

Notice that the flow is proportional to R^4. This means there's a big advantage to having thick pipes.

Example 4. Dart board
You aim for the center of the board, but your aim's not always perfect. Your number of hits, N, at radius r is proportional to e^{-r^2}.

$$N = ce^{-r^2}$$

This looks like:

Figure 9: This graph shows how likely you are to hit the dart board at some distance r from its center.

The number of hits within a given ring with $r_1 < r < r_2$ is

$$c\int_{r_1}^{r_2} e^{-r^2}(2\pi r\, dr)$$

We will examine this problem more in the next lecture.

Lecture 23: Work, Average Value, Probability

Application of Integration to Average Value

You already know how to take the average of a set of discrete numbers:

$$\frac{a_1 + a_2}{2} \quad \text{or} \quad \frac{a_1 + a_2 + a_3}{3}$$
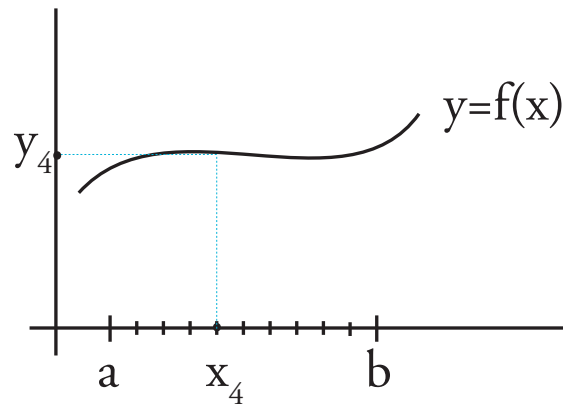
Now, we want to find the average of a continuum.

Figure 1: Discrete approximation to $y = f(x)$ on $a \leq x \leq b$.

$$\text{Average} \approx \frac{y_1 + y_2 + ... + y_n}{n}$$

where

$$a = x_0 < x_1 < \cdots x_n = b$$

$$y_0 = f(x_0), \ y_1 = f(x_1), \ \ldots y_n = f(x_n)$$

and

$$n(\Delta x) = b - a \quad \Longleftrightarrow \quad \Delta x = \frac{b - a}{n}$$

and

The limit of the Riemann Sums is

$$\lim_{n \to \infty} (y_1 + \cdots + y_n) \frac{b - a}{n} = \int_a^b f(x) \, dx$$

Divide by $b - a$ to get the continuous average

$$\lim_{n \to \infty} \frac{y_1 + \cdots + y_n}{n} = \frac{1}{b - a} \int_a^b f(x) \, dx$$
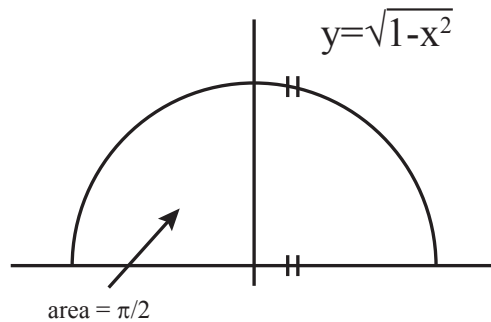
Figure 2: Average height of the semicircle.

Example 1. Find the average of $y = \sqrt{1 - x^2}$ on the interval $-1 \leq x \leq 1$. (See Figure 2)

$$\text{Average height} = \frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx = \frac{1}{2} \left(\frac{\pi}{2} \right) = \frac{\pi}{4}$$

Example 2. The average of a constant is the same constant

$$\frac{1}{b - a} \int_{a}^{b} 53 \, dx = 53$$

Example 3. Find the average height y on a semicircle, with respect to *arclength*. (Use $d\theta$ not dx. See Figure 3)
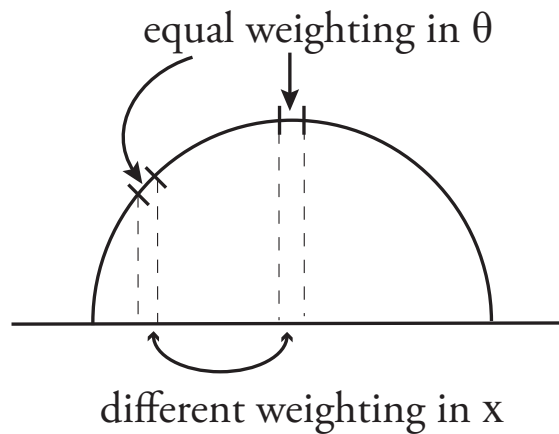

Figure 3: Different weighted averages.

$$y = \sin\theta$$

$$\text{Average} = \frac{1}{\pi}\int_0^\pi \sin\theta\, d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi = \frac{1}{\pi}(-\cos\pi - (-\cos 0)) = \frac{2}{\pi}$$

Example 4. Find the average temperature of water in the witches cauldron from last lecture. (See Figure 4).
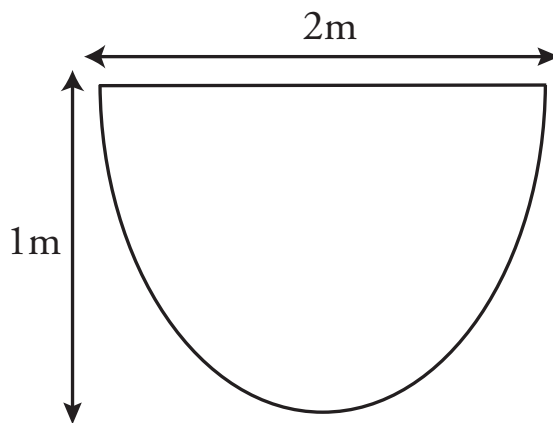

Figure 4: $y = x^2$, rotated about the y-axis.

First, recall how to find the volume of the solid of revolution by disks.

$$V = \int_0^1 (\pi x^2)\, dy = \int_0^1 \pi y\, dy = \frac{\pi y^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

Recall that $T(y) = 100 - 30y$ and $(T(0) = 100^o; T(1) = 70^o)$. The average temperature per unit volume is computed by giving an importance or "weighting" $w(y) = \pi y$ to the disk at height y.

$$\frac{\int_0^1 T(y)w(y)\, dy}{\int_0^1 w(y)\, dy}$$

The numerator is

$$\int_0^1 T\pi y\, dy = \pi\int_0^1 (100 - 30y)y\, dy = \pi(500y^2 - 10y^3)\Big|_0^1 = 40\pi$$

Thus the average temperature is:

$$\frac{40\pi}{\pi/2} = 80^o C$$

Compare this with the average taken with respect to height y:

$$\frac{1}{1}\int_0^1 T\, dy = \int_0^1 (100 - 30y)dy = (100y - 15y^2)\Big|_0^1 = 85^o C$$

T is linear. Largest $T = 100^o C$, smallest $T = 70^o C$, and the average of the two is

$$\frac{70 + 100}{2} = 85$$

The answer 85° is consistent with the ordinary average. The weighted average (integration with respect to $\pi y\, dy$) is lower (80°) because there is more water at cooler temperatures in the upper parts of the cauldron.

Dart board, revisited

Last time, we said that the accuracy of your aim at a dart board follows a "normal distribution":

$$ce^{-r^2}$$

Now, let's pretend someone – say, your little brother – foolishly decides to stand close to the dart board. What is the chance that he'll get hit by a stray dart?
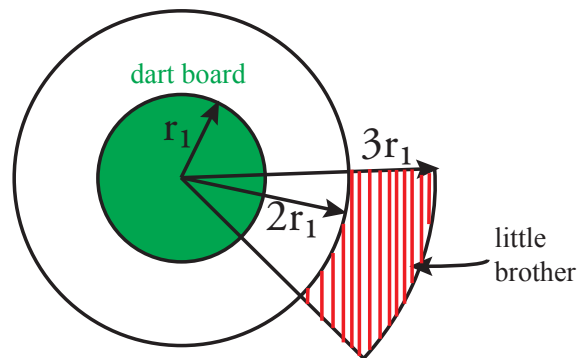

Figure 5: Shaded section is $2r_i < r < 3r_1$ between 3 and 5 o'clock.

To make our calculations easier, let's approximate your brother as a sector (the shaded region in Fig. 5). Your brother doesn't quite stand in front of the dart board. Let us say he stands at a distance r from the center where $2r_1 < r < 3r_1$ and r_1 is the radius of the dart board. Note that your brother doesn't surround the dart board. Let us say he covers the region between 3 o'clock and 5 o'clock, or $\dfrac{1}{6}$ of a ring.

Remember that

$$\boxed{\text{probability} \;=\; \frac{\text{part}}{\text{whole}}}$$
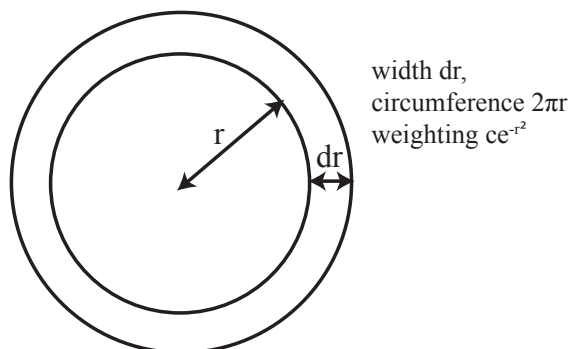
Figure 6: Integrating over rings.

The ring has weight $\left(ce^{-r^2}\right)(2\pi r)(dr)$ (see Figure 6). The probability of a dart hitting your brother is:

$$\frac{\frac{1}{6}\int_{2r_1}^{3r_1} ce^{-r^2} 2\pi r\, dr}{\int_0^\infty ce^{-r^2} 2\pi r\, dr}$$

Recall that $\dfrac{1}{6} = \dfrac{5-3}{12}$ is our approximation to the portion of the circumference where the little brother stands. (Note: $e^{-r^2} = e^{(-r^2)}$ not $(e^{-r})^2$)

$$\int_a^b re^{-r^2}\, dr = -\frac{1}{2}e^{-r^2}\Big|_a^b = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-a^2} \qquad \left(\frac{d}{dr}e^{-r^2} = -2re^{-r^2}\right)$$

Denominator:

$$\int_0^\infty e^{-r^2} r\, dr = -\frac{1}{2}e^{-r^2}\Big|_0^{R\to\infty} = -\frac{1}{2}e^{-R^2} + \frac{1}{2}e^{-0^2} = \frac{1}{2}$$

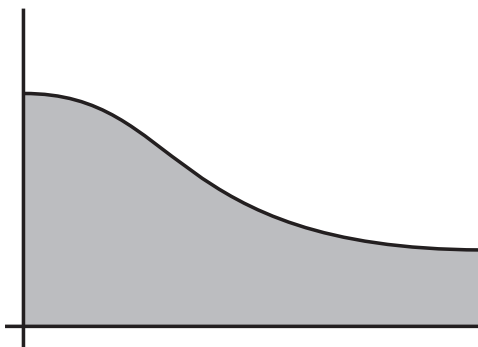(Note that $e^{-R^2} \to 0$ as $R \to \infty$.)

Figure 7: Normal Distribution.

$$\text{Probability} = \frac{\frac{1}{6}\int_{2r_1}^{3r_1} ce^{-r^2} 2\pi r\, dr}{\int_0^\infty ce^{-r^2} 2\pi r\, dr} = \frac{\frac{1}{6}\int_{2r_1}^{3r_1} e^{-r^2} r\, dr}{\int_0^\infty e^{-r^2} r\, dr} = \frac{1}{3}\int_{2r_1}^{3r_1} e^{-r^2} r\, dr = \frac{-e^{-r^2}}{6}\Big|_{2r_1}^{3r_1}$$

$$\text{Probability} = \frac{-e^{-9r_1^2} + e^{-4r_1^2}}{6}$$

Let's assume that the person throwing the darts hits the dartboard $0 \leq r \leq r_1$ about half the time. (Based on personal experience with 7-year-olds, this is realistic.)

$$P(0 \leq r \leq r_1) = \frac{1}{2} = \int_0^{r_1} 2e^{-r^2} r\, dr = -e^{-r_1^2} + 1 \implies e^{-r_1^2} = \frac{1}{2}$$

$$e^{-r_1^2} = \frac{1}{2}$$

$$e^{-9r_1^2} = \left(e^{-r_1^2}\right)^9 = \left(\frac{1}{2}\right)^9 \approx 0$$

$$e^{-4r_1^2} = \left(e^{-r_1^2}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

So, the probability that a stray dart will strike your little brother is

$$\left(\frac{1}{16}\right)\left(\frac{1}{6}\right) \approx \frac{1}{100}$$

In other words, there's about a 1% chance he'll get hit with each dart thrown.

Volume by Slices: An Important Example

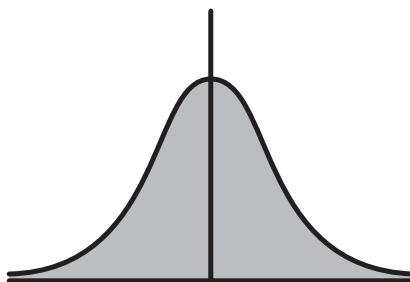Compute $Q = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$

Figure 8: $Q = $ Area under curve $e^{(-x^2)}$.

This is one of the most important integrals in all of calculus. It is especially important in probability and statistics. It's an improper integral, but don't let those ∞'s scare you. In this integral, they're actually easier to work with than finite numbers would be.

To find Q, we will first find a volume of revolution, namely,

$$V = \text{volume under } e^{-r^2} \quad (r = \sqrt{x^2 + y^2})$$

We find this volume by the method of shells, which leads to the same integral as in the last problem. The shell or cylinder under e^{-r^2} at radius r has circumference $2\pi r$, thickness dr; (see Figure 9). Therefore $dV = e^{-r^2} 2\pi r\, dr$. In the range $0 \le r \le R$,

$$\int_0^R e^{-r^2} 2\pi r\, dr = -\pi e^{-r^2} \Big|_0^R = -\pi e^{-R^2} + \pi$$

When $R \to \infty, e^{-R^2} \to 0$,

$$V = \int_0^{\infty} e^{-r^2} 2\pi r\, dr = \pi \qquad \text{(same as in the darts problem)}$$
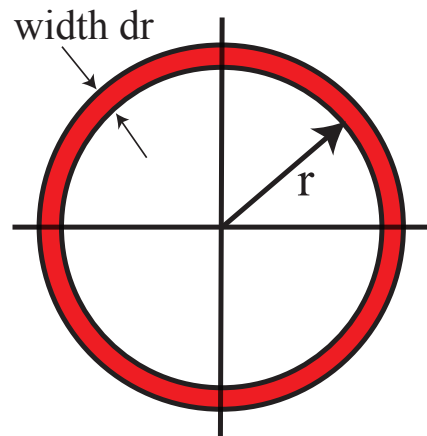
Figure 9: Area of annulus or ring, $(2\pi r)dr$.

Next, we will find V by a second method, the method of slices. Slice the solid along a plane where y is fixed. (See Figure 10). Call $A(y)$ the cross-sectional area. Since the thickness is dy (see Figure 11),
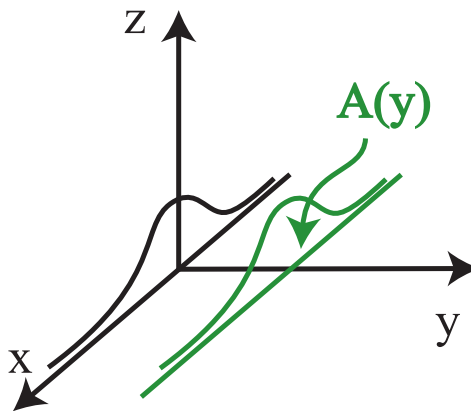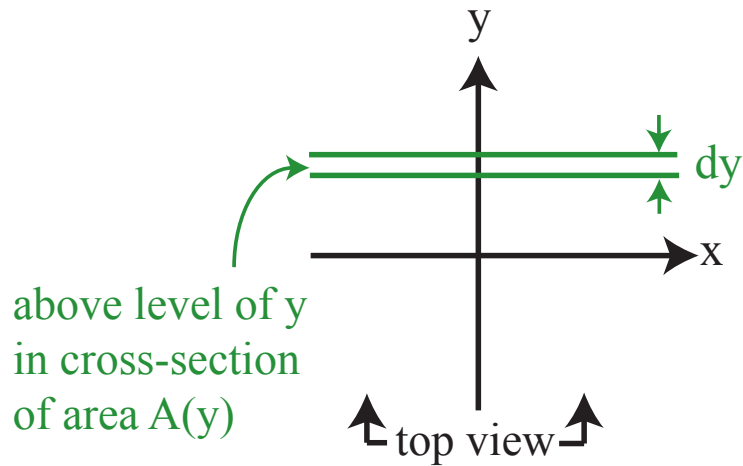
$$V = \int_{-\infty}^{\infty} A(y)\, dy$$

Figure 10: Slice $A(y)$.

Figure 11: Top view of $A(y)$ slice.

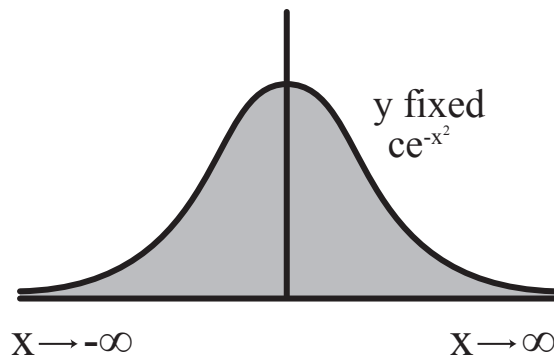To compute $A(y)$, note that it is an integral (with respect to dx)

$$A(y) = \int_{-\infty}^{\infty} e^{-r^2} \, dx = \int_{-\infty}^{\infty} e^{-x^2 - y^2} \, dx = e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2} \, dx = e^{-y^2} Q$$

Here, we have used $r^2 = x^2 + y^2$ and

$$e^{-x^2 - y^2} = e^{-x^2} e^{-y^2}$$

and the fact that y is a constant in the $A(y)$ slice (see Figure 12). In other words,

$$\int_{-\infty}^{\infty} c e^{-x^2} \, dx = c \int_{-\infty}^{\infty} e^{-x^2} \, dx \quad \text{with} \quad c = e^{-y^2}$$

Figure 12: Side view of $A(y)$ slice.

It follows that

$$V = \int_{-\infty}^{\infty} A(y)\, dy = \int_{-\infty}^{\infty} e^{-y^2} Q\, dy = Q \int_{-\infty}^{\infty} e^{-y^2}\, dy = Q^2$$

Indeed,

$$Q = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

because the name of the variable does not matter. To conclude the calculation read the equation backwards:

$$\pi = V = Q^2 \implies \boxed{Q = \sqrt{\pi}}$$

We can rewrite $Q = \sqrt{\pi}$ as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx = 1$$

An equivalent rescaled version of this formula (replacing x with $x/\sqrt{2}\sigma$) is used:

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx = 1$$

This formula is central to probability and statistics. The probability distribution $\dfrac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$ on $-\infty < x < \infty$ is known as the normal distribution, and $\sigma > 0$ is its standard deviation.

Lecture 24: Numerical Integration

Numerical Integration

We use numerical integration to find the definite integrals of expressions that look like:

$$\int_a^b (\text{a big mess})$$

We also resort to numerical integration when an integral has no elementary antiderivative. For instance, there is no formula for

$$\int_0^x \cos(t^2)dt \quad \text{or} \quad \int_0^3 e^{-x^2}dx$$

Numerical integration yields numbers rather than analytical expressions.

We'll talk about three techniques for numerical integration: Riemann sums, the trapezoidal rule, and Simpson's rule.
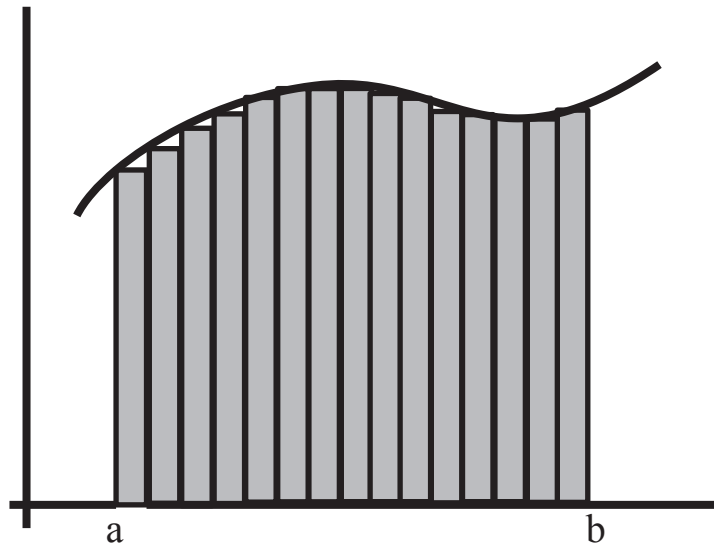
1. Riemann Sum

Figure 1: Riemann sum with left endpoints: $(y_0 + y_1 + \ldots + y_{n-1})\Delta x$

Here,

$$x_i - x_{i-1} = \Delta x$$

$$(\text{or, } x_i = x_{i-1} + \Delta x)$$

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

$$y_0 = f(x_0), \ y_1 = f(x_1), \ldots y_n = f(x_n)$$

2. Trapezoidal Rule

The trapezoidal rule divides up the area under the function into trapezoids, rather than rectangles. The area of a trapezoid is the height times the average of the parallel bases:

$$\text{Area} = \text{height} \left(\frac{\text{base } 1 + \text{base } 2}{2} \right) = \left(\frac{y_3 + y_4}{2} \right) \Delta x \quad \text{(See Figure 2)}$$
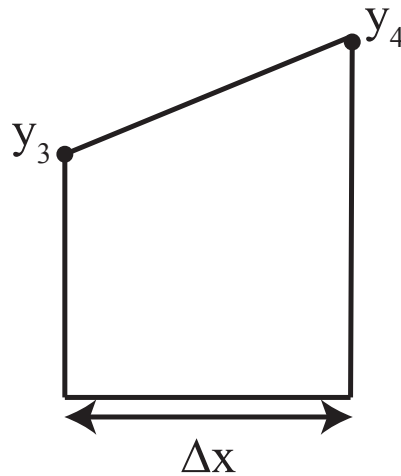

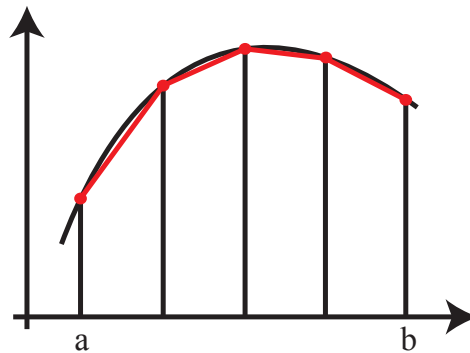Figure 2: $\text{Area} = \left(\dfrac{y_3 + y_4}{2} \right) \Delta x$

Figure 3: Trapezoidal rule $=$ sum of areas of trapezoids.

$$
\begin{aligned}
\text{Total Trapezoidal Area} \quad &= \quad \Delta x \left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + ... + \frac{y_{n-1} + y_n}{2} \right) \\
&= \quad \Delta x \left(\frac{y_0}{2} + y_1 + y_2 + ... + y_{n-1} + \frac{y_n}{2} \right)
\end{aligned}
$$

2

Note: The trapezoidal rule gives a more symmetric treatment of the two ends (a and b) than a Riemann sum does — the average of left and right Riemann sums.

3. Simpson's Rule

This approach often yields much more accurate results than the trapezoidal rule does. Here, we match quadratics (i.e. parabolas), instead of straight or slanted lines, to the graph. This approach requires an even number of intervals.
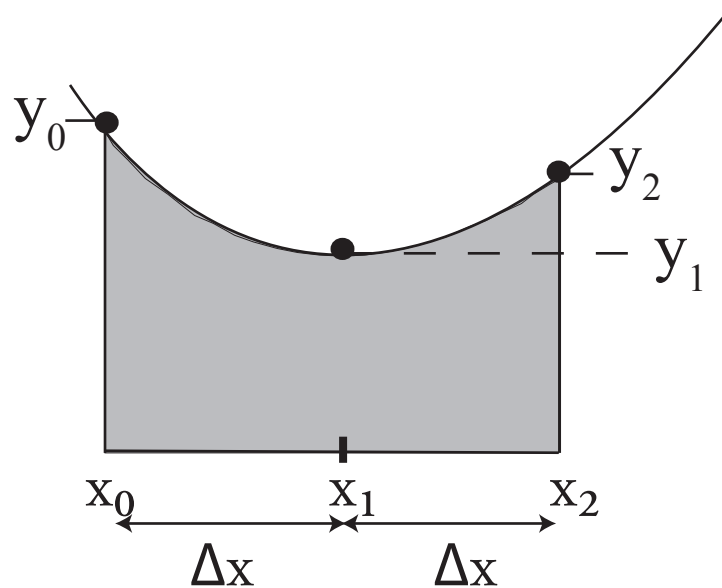

Figure 4: Area under a parabola.

$$\text{Area under parabola} = (\text{base})(\text{weighted average height}) = (2\Delta x)\left(\frac{y_0 + 4y_1 + y_2}{6}\right)$$

Simpson's rule for n intervals (n must be even!)

$$\text{Area} = (2\Delta x)\left(\frac{1}{6}\right)[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

Notice the following pattern in the coefficients:

$$\begin{array}{ccccccc} 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & \\ & & & & 1 & 4 & 1 \\ 1 & 4 & 2 & 4 & 2 & 4 & 1 \end{array}$$
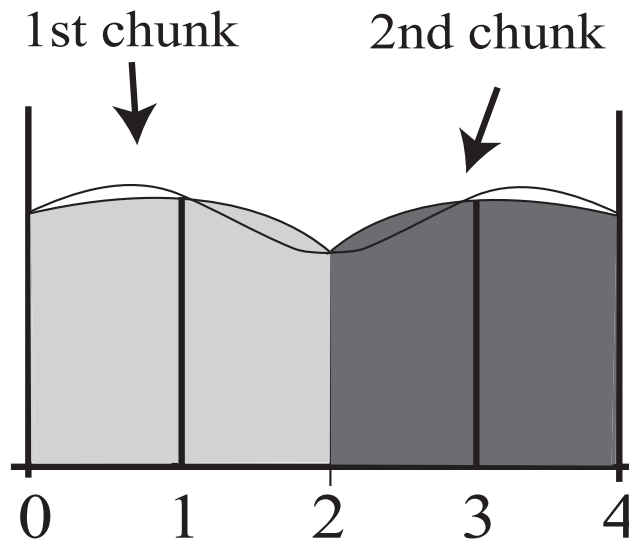
1st chunk   2nd chunk

Figure 5: Area given by Simpson's rule for four intervals

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

The pattern of coefficients in parentheses is:

$$
\begin{array}{ccccccccc}
 & & & 1 & 4 & 1 & & & = \quad \text{sum } 6 \\
 & & 1 & 4 & 2 & 4 & 1 & & = \quad \text{sum } 12 \\
 & 1 & 4 & 2 & 4 & 2 & 4 & 1 & = \quad \text{sum } 18
\end{array}
$$

To double check – plug in $f(x) = 1$ (n even!).

$$\frac{\Delta x}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 2 + 4 + 1) = \frac{\Delta x}{3}\left(1 + 1 + 4\left(\frac{n}{2}\right) + 2\left(\frac{n}{2} - 1\right)\right) = n\Delta x \quad (n \text{ even})$$

Example 1. Evaluate $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$ using two methods (trapezoidal and Simpson's) of numerical integration.
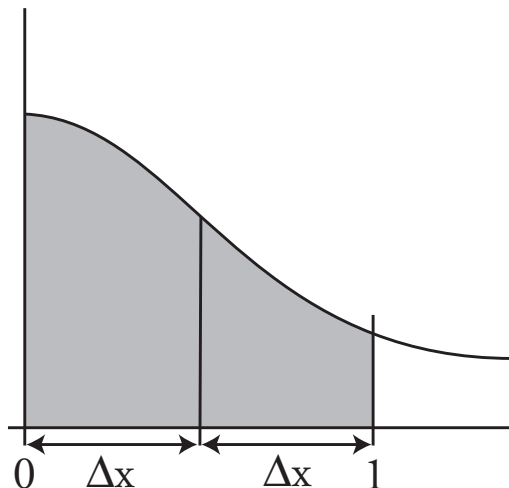

Figure 6: Area under $\frac{1}{(1+x^2)}$ above $[0,1]$.

x	$1/(1+x^2)$
0	1
$\frac{1}{2}$	$\frac{4}{5}$
1	$\frac{1}{2}$

By the trapezoidal rule:

$$\Delta x\left(\frac{1}{2}y_0 + y_1 + \frac{1}{2}y_2\right) = \frac{1}{2}\left(\frac{1}{2}(1) + \frac{4}{5} + \frac{1}{2}\left(\frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{4}{5} + \frac{1}{4}\right) = 0.775$$

By Simpson's rule:

$$\frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{1/2}{3}\left(1 + 4\left(\frac{4}{5} + \frac{1}{2}\right)\right) = 0.78333...$$

Exact answer:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4} \approx 0.785$$

Roughly speaking, the error, $|\text{ Simpson's} - \text{Exact }|$, has order of magnitude $(\Delta x)^4$.

Lecture 25: Exam 3 Review

Integration

1. Evaluate definite integrals. Substitution, first fundamental theorem of calculus (FTC 1), (and hints?)

2. FTC 2:

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(t)$$

   If $F(x) = \displaystyle\int_a^x f(t)\, dt$, find the graph of F, estimate F, and change variables.

3. Riemann sums; trapezoidal and Simpson's rules.

4. Areas, volumes.

5. Other cumulative sums: average value, probability, work, etc.

There are two types of volume problems:

1. solids of revolution

2. other (do by slices)

In these problems, there will be something you can draw in 2D, to be able to see what's going on in that one plane.

In solid of revolution problems, the solid is formed by revolution around the x-axis or the y-axis. You will have to decide how to chop up the solid: into shells or disks. Put another way, you must decide whether to integrate with dx or dy. After making that choice, the rest of the procedure is systematically determined. For example, consider a shape rotated *around the y-axis*.

- *Shells*: height $y_2 - y_1$, circumference $2\pi x$, thickness dx

- *Disks (washers)*: area πx^2 (or $\pi x_2^2 - \pi x_1^2$), thickness dy; integrate dy.

Work

$$\text{Work} = \text{Force} \cdot \text{Distance}$$

We need to use an integral if the force is variable.

Example 1: Pendulum. See Figure 1
Consider a pendulum of length L, with mass m at angle θ. The vertical force of gravity is mg (g = gravitational coefficient on Earth's surface)
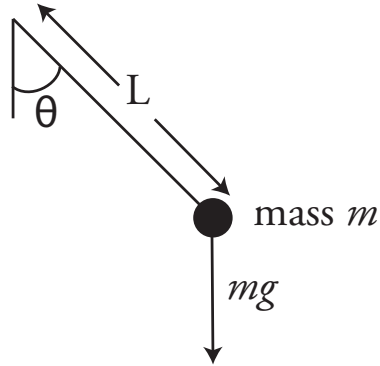

Figure 1: Pendulum.

In Figure 2, we find the component of gravitational force acting along the pendulum's path $F = mg \sin \theta$.
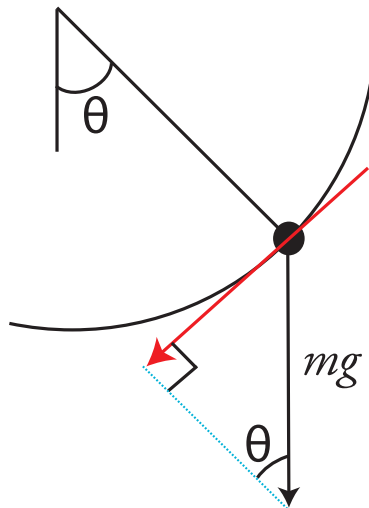

Figure 2: $F = mg \sin \theta$ (force tangent to path of motion).

Is it possible to build a perpetual motion machine? Let's think about a simple pendulum, and how much work gravity performs in pulling the pendulum from θ_0 to the bottom of the pendulum's arc.

Notice that F varies. That's why we have to use an integral for this problem.

$$W = \int_0^{\theta_0} (\text{Force}) \cdot (\text{Distance}) = \int_0^{\theta_0} (mg \sin \theta)(L \, d\theta)$$

$$W = -Lmg \cos \theta \Big|_0^{\theta_0} = -Lmg(\cos \theta_0 - 1) = mg \left[L(1 - \cos \theta_0) \right]$$

In Figure 3, we see that the work performed by gravity moving the pendulum down a distance $L(1 - \cos \theta)$ is the same as if it went straight down.
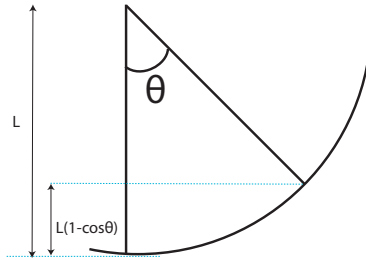

Figure 3: Effect of gravity on a pendulum.

In other words, the amount of work required depends only on how far down the pendulum goes. It doesn't matter what path it takes to get there. So, there's no free (energy) lunch, no perpetual motion machine.