Stability analysis and stabilization for quadratic discrete-time systems with actuator saturation

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Abstract: In this paper, we deal with the stability and stabilization problems for quadratic discrete-time systems with input saturation. By introducing an auxiliary matrix and a particular representation for the quadratic terms, sufficient conditions for stability and stabilization for quadratic discrete-time systems are derived in terms of "quasi"-linear matrix inequalities (LMIs). Then, LMIs-based optimization problems are presented to maximize the region of attraction of corresponding systems. Simulation examples are provided to demonstrate the effectiveness of the obtained results.

Key Words: Stability, Stabilization, Quadratic systems, Discrete-time systems, Actuator saturation

1 INTRODUCTION

The problems of stability and stabilization for linear systems subject to actuator saturation have attracted considerable attention in science and engineering during the past several decades because saturation nonlinearity exists in many real control systems as a practical physical actuator can only generate bounded signals. Generally speaking, stability for systems subject to actuator saturation can be classified into three categories, namely the global, the semi-global, and the local stability. Many approaches based on the concept of Lyapunov level set have been proposed; see [1]-[9] and the references therein.

On the other hand, many nonlinear processes, such as electric power systems, robots in the applications of engineering and biology, and other areas can be characterized by nonlinear quadratic systems; see [10]-[11] and the references therein. Moreover, lots of nonlinear systems are often investigated by considering their linearly approximating systems and then applying well-established linear control tools. However, it should be pointed out that in the case when nonlinearities of a system become significant, the linearly approximating system can not adequately represent the considered nonlinear system. In such a case, one can consider the second order approximation. Over the last years, the issue of analysis and design of quadratic systems has attracted increasing interest of many researchers. In [12], an ellipsoidal estimate of the region of attraction (RA) of second-order nonlinear systems involving either linear and quadratic or linear and cubic terms is calculated by using a Lyapunov-based procedure. In [13] and [14], the problem of determining whether a given polytope containing the origin of the state space belongs to the RA of the equilibrium was considered by using quadratic Lyapunov functions. Later, the problem studied in [13] and [14] was further investigated in [15] by using polyhedral Lyapunov functions and less conservative results were obtained. In [16] and [17], the problem of controller synthesis for quadratic nonlinear systems with saturation inputs was considered, where LMIs-based optimization problems were provided to obtain the state feedback gains maximizing the region of attraction of the equilibrium of the closed-loop system. In [18], the stability problem of nonlinear autonomous quadratic discrete-time systems in the critical case was investigated, where the case that the spectral radius of a linearized model around the equilibrium point is equivalent to one is considered. However, to the best of our knowledge, there have not been any available results on the stability and stabilization of quadratic discrete-time systems with actuator saturation.

In this paper, we consider the stability and stabilization problems for input-saturating quadratic discrete-time systems. By introducing an auxiliary matrix and a particular representation for the quadratic terms, sufficient conditions for stability and stabilization for quadratic discrete-time systems are derived in terms of "quasi"-linear matrix inequalities. Then, LMIs-based optimization problems are presented to maximize the region of attraction of corresponding systems.

Notations: In this paper, matrices are assumed to have compatible dimensions. For two symmetric matrices A and B, the notation $A > B (A \geq B)$ means that $A - B$ is positive definite (positive semi-definite). I denotes the identity matrix with properly compatible dimension. $\otimes$ is used to denote the Kronecker product. For $u \in R^m, sat(u) : R^m \rightarrow R^m$, denotes the standard saturation function defined as $sat(u) = [sat(u_1), sat(u_2), \cdots, sat(u_m)]^T$, where, $sat(u_i) = sign(u_i) min\{1, \|u_i\|\}$.

2 PROBLEM STATEMENT AND PRELIMINARIES

Consider an input-saturating quadratic discrete-time system given by the following state equation:

$$x(k+1) = Ax(k) + C(x(k)) + Bsat(u(k)), \qquad (1)$$

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where $x(k) \in R^n$ is the state; $u(k)$ is the control input;

$$C(x(k)) = \begin{bmatrix} x(k)^T A_{q1} x(k) \\ x(k)^T A_{q2} x(k) \\ \vdots \\ x(k)^T A_{qn} x(k) \end{bmatrix}. \quad (2)$$

Now, for later development, we revisit the definition on region of attraction of the origin for system (1).

Definition 1. For $x(0) = x_0$, denote the solution of quadratic discrete-time system (1) as $x(k, x(0))$. The region of attraction of the origin is the set

$$\mathscr{D} = \{x(0) \in R^n : \lim_{k\to\infty} ||x(k, x(0))|| = 0\}.$$

For the input-saturating discrete-time system (1), the problems dealt with in this note can be stated as follows:

Problem 1. Finding an estimate of the region of attraction of the origin of system (1) with $u(k) = 0$.

Problem 2. Finding an estimate of the region of attraction of the origin of system (1) under the following nonlinear control law

$$u(k) = Kx(k) + \begin{bmatrix} x(k)^T K_{q1} x(k) \\ x(k)^T K_{q2} x(k) \\ \vdots \\ x(k)^T K_{qn} x(k) \end{bmatrix}, \quad (3)$$

where $K, K_{qi}, i = 1, 2, \cdots, n$ are to be designed.

Remark 1. In this note, we deal with the stability and stabilization of the zero equilibrium. The other equilibrium points of system (1) can be investigated by performing a coordinate change stated in [14].

Remark 2. The quadratic terms introduced into the control law (3) is to counteract the effects of the quadratic terms of the system.

Before concluding this section, we introduce some lemmas which will be used to obtain our main results.

For a matrix $H \in R^{m\times n}$, denote

$$\mathscr{L}(H) = \{x \in R^n : ||Hx||_\infty \le 1\}.$$

Let $\mathscr{D}$ be the set of $n \times n$ diagonal matrices whose diagonal elements are either 0 or 1. One can verify there are 2^n elements in $\mathscr{D}$ and denote the j-th element as $D_j, j \in [1, 2^n]$. Moreover denote $D_j^- = I - D_j$.

Lemma 1 [5]. Let $u, v \in R^n$. Suppose that $||v||_\infty \le 1$. Then $sat(u) \in co\{D_j u + D_j^- v : j \in [1, 2^n]\}$, where $co\{.\}$ denotes the convex hull of a set.

Lemma 2 [17]. Consider matrix $P > 0$ and a vector $||v||$ such that $||v|| = 1$. Every point on the boundary of an ellipsoid, $\partial\mathscr{E}(P) = \{x \in R^n : x^T P x = 1\}$, can be parameterized by $x = P^{-\frac{1}{2}} T v$, with $T^T T = I$.

Lemma 3 [19]. For matrices $A, P \in R^{n\times n}, D \in R^{n\times n_f}, E \in R^{n_f\times n}$ and $F \in R^{n_f\times n_f}$, with $P > 0, ||F|| \le 1$ and a scalar $\varepsilon > 0$, then the following inequality holds:

$$\begin{aligned} & If P - \varepsilon DD^T > 0, then \\ & (A + DF(t)E)^T P^{-1} (A + DF(t)E) \\ \le\ & A^T (P - \varepsilon DD^T)^{-1} A + \varepsilon^{-1} E^T E. \end{aligned}$$

3 MAIN RESULTS

3.1 Stability analysis

In this section, we will deal with Problem 1. For this purpose,Defining a matrix

$$A_q = \begin{bmatrix} A_{q1(1)} & A_{q1(2)} & \cdots & A_{q1(n)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{qn(1)} & A_{qn(2)} & \ldots & A_{qn(n)} \end{bmatrix}. \quad (4)$$

Then, system (1) with $u(k) = 0$ can be written as

$$x(k+1) = Ax(k) + A_q (x(k) \otimes x(k)). \quad (5)$$

Now, we are ready to state a result on stability criterion of system (1) with $u(k) = 0$.

Theorem 1. If there exist a matrix $P > 0$ and a positive scalar $\varepsilon > 0$ such that the following inequality holds:

$$\begin{bmatrix} -P + \varepsilon I & A^T P & 0 \\ * & -P & PA_q \\ * & * & -(\varepsilon I \otimes P) \end{bmatrix} < 0. \quad (6)$$

Then the region $\mathscr{E}(P) = \{x \in R^n : x^T P x \le 1\}$ is an estimate of attraction for system (1) with $u(k) = 0$.

Proof : Take the Lyapunov function for system (1) with $u(k) = 0$ as $V(x(k)) = x^T(k) P x(k)$ with $P > 0$. The forward difference of $V(x(k))$ along system (5) is given by

$$\begin{aligned} & \triangle V(x(k)) \\ = \ & x^T(k)(A + A_q(I \otimes x(k)))^T P (A + A_q(I \otimes x(k))) x(k) \\ & - x^T(k) P x(k). \end{aligned} \quad (7)$$

Noticing that $\partial\mathscr{E}(P)$ is an ellipsoid, thus, for every $x \in \mathscr{E}(P)$, there exist two points $x_i \in \partial\mathscr{E}(P)$ such that

$$x = \sum_{i=1}^{2} \lambda_i x_i, \quad (8)$$

where $\sum_{i=1}^{2} \lambda_i = 1, \lambda_i \ge 0, i = 1, 2.$

Using Lemma 3 from [20], we have

$$(\sum_{i=1}^{2} \lambda_i x_i)^T P (\sum_{i=1}^{2} \lambda_i x_i) \le \sum_{i=1}^{2} \lambda_i x_i^T P x_i. \quad (9)$$

By the parameterization stated in Lemma 2, for $i = 1, 2$, it can be seen that there exist matrices T_i satisfying $T_i^T T_i = I$ and vectors v_i satisfying $||v_i|| = 1$ such that

$$x_i = P^{-\frac{1}{2}} T_i v_i. \quad (10)$$

From (7)-(10), for every $x \in \mathscr{E}(P)$, the forward difference of $V(x(k))$ is given by

$$\triangle V(x(k))$$

$$
\begin{aligned}
&= x^T(k)(\sum_{i=1}^{2}\lambda_i(A+A_q(I\otimes P^{-\frac{1}{2}}\otimes T_i\otimes v_i)))^T \\
&\quad \times P(\sum_{i=1}^{2}\lambda_i(A+A_q(I\otimes P^{-\frac{1}{2}}\otimes T_i\otimes v_i)))x(k) \\
&\quad -x^T(k)Px(k) \\
&\leq \sum_{i=1}^{2}\lambda_i x^T(k)(A+A_q(I\otimes P^{-\frac{1}{2}}\otimes T_i\otimes v_i)))^T \\
&\quad \times P(A+A_q(I\otimes P^{-\frac{1}{2}}\otimes T_i\otimes v_i)))x(k) \\
&\quad -x^T(k)Px(k). \qquad (11)
\end{aligned}
$$

By Lemma 3, one can easily verify

$$
\begin{aligned}
&(A+A_q(I\otimes P^{-\frac{1}{2}}\otimes T_i\otimes v_i)))^T \\
&\times P(A+A_q(I\otimes P^{-\frac{1}{2}}\otimes T_i\otimes v_i))) \\
&\leq A^T(P^{-1}-\varepsilon^{-1}A_q(I\otimes P)^{-1}A_q^T)^{-1}A+\varepsilon I,\ i=1,2 \qquad (12)
\end{aligned}
$$

are satisfied for any positive scalar ε satisfying

$$
P^{-1}-\varepsilon^{-1}A_q(I\otimes P)^{-1}A_q^T>0.
$$

Applying (12) to (11) yields

$$
\triangle V(x(k))\leq x^T(k)\Omega_1 x(k), \qquad (13)
$$

where $\Omega_1=A^T(P^{-1}-\varepsilon^{-1}A_q(I\otimes P)^{-1}A_q^T)^{-1}A+\varepsilon I-P$. By using Schur complement, we have that (6) is equivalent to $\Omega_1<0$. The ellipsoid $\mathscr{E}(P)$ is thus a region of stability for the system (1) with $u(k)=0$. This completes the proof. Theorem 1 gives a sufficient condition to judge if the region $\mathscr{E}(P)$ is an estimate of the region of attraction of system (1) with $u(k)=0$, In general, it is expected to make the estimated region $\mathscr{E}(P)$ as large as possible. To this end, we consider the following optimization problem:

$$
\mathbf{OP1}:\quad \min_{P,\,\varepsilon}\ Trace(P) \quad s.t.(6). \qquad (14)
$$

3.2 Stabilization analysis

In this section, we will deal with Problem 2. To this end, we define a matrix $K_q\in R^{n\times n^2}$ as

$$
K_q=\begin{bmatrix} K_{q1(1)} & K_{q1(2)} & \cdots & K_{q1(n)} \\ \vdots & \vdots & \ddots & \vdots \\ K_{qm(1)} & K_{qm(2)} & \ldots & K_{m(n)} \end{bmatrix}. \qquad (15)
$$

By using (4) and (15), the resulting closed-loop system is given by

$$
\begin{aligned}
x(k+1) &= Ax(k)+A_q(x(k)\otimes x(k))+ \\
&\quad Bsat(Kx(k)+K_q(x(k)\otimes x(k))). \qquad (16)
\end{aligned}
$$

Now, we are ready to state a result on stabilization criterion of system (1).

Theorem 2. If there exist matrices $S>0,L,L_q,H$ and a scalar $\varepsilon>0$ such that the following inequalities hold:

$$
\begin{bmatrix} -S & \Phi_1 & 0 & \varepsilon S \\ * & -S & \Phi_2 & 0 \\ * & * & -(\varepsilon I\otimes S) & 0 \\ * & * & * & -\varepsilon I \end{bmatrix}<0,\ i\in[1,2^m], \qquad (17)
$$

$$
\mathcal{E}(S^{-1})\subseteq\mathscr{L}(H) \qquad (18)
$$

where

$$
\begin{aligned}
\Phi_1 &= SA^T+L^TD_iB^T+SH^TD_i^-B^T, \\
\Phi_2 &= A_q(I\otimes S)+BD_iLq.
\end{aligned}
$$

Then, for every initial condition belonging to $\mathcal{E}(S^{-1})=\{x\in R^n: x^TS^{-1}x\leq 1\}$, matrix gains $K=LS^{-1}$ and $K_q=L_q(I\otimes S)^{-1}$ stabilize system (1).

Proof. Let $P=S^{-1}$. Pre- and post-multiplying (17) by $diag(P,I,\underbrace{P,\cdots,P}_{n},I)$, and then using Schur complement, we have

$$
\begin{aligned}
&(A+BD_iK+BD_i^-H)^T\times \\
&(P^{-1}-\varepsilon^{-1}(A_q+BD_iK_q)(I\otimes P)^{-1}(A_q+BD_iK_q)^T))^{-1} \\
&\times(A+BD_iK+BD_i^-H)-P+\varepsilon I<0, i\in[1,2^m]. \qquad (19)
\end{aligned}
$$

Take the Lyapunov function for the closed-loop system (16) as $V(x(k))=x(k)^TPx(k)$. Then, the forward difference of $V(x(k))$ along system (16) can be calculated as

$$
\begin{aligned}
&\triangle V(x(k))=[Ax(k)+A_q(x(k)\otimes x(k))+Bsat(Kx(k) \\
&+K_q(x(k)\otimes x(k))]^TP\times[Ax(k)+A_q(x(k)\otimes x(k)) \\
&+Bsat(Kx(k)+K_q(x(k)\otimes x(k))]-x^T(k)Px(k). \qquad (20)
\end{aligned}
$$

Noticing that $\mathcal{E}(S^{-1})\subseteq\mathscr{L}(H)$ for all $x\in\mathscr{E}(P)$, and using Lemma 1, for every $x\in\mathscr{E}(P)$, we have

$$
\begin{aligned}
&sat(Kx(k)+K_q(x(k)\otimes x(k))) \\
&\in co\{D_i(Kx(k)+K_q(x(k)\otimes x(k)))+D_i^-Hx(k) \\
&: i\in[1,2^m]\}. \qquad (21)
\end{aligned}
$$

From (21), the forward difference of $V(x(k))$ can be rewritten as

$$
\begin{aligned}
&\triangle V(x(k)) \\
&=x(k)^T(\sum_{i=1}^{2^m}\alpha_i((A+BD_iK+BD_i^-H)+(A_q+BD_iK_q) \\
&\times(I\otimes x(k))))^TP\times(\sum_{i=1}^{2^m}\alpha_i((A+BD_iK+BD_i^-H) \\
&+(A_q+BD_iK_q)(I\otimes x(k))))x(k)-x(k)^TPx(k),
\end{aligned}
$$

where $\sum_{i=1}^{2^m}\alpha_i=1,\alpha_i\geq 0, i=1,2,\cdots 2^m$.

From Lemma 2 stated in [20], it follows that

$$
\begin{aligned}
&\triangle V(x(k)) \\
&\leq \max_{i\in[1,2^m]} x(k)^T((A+BD_iK+BD_i^-H)+ \\
&\quad (A_q+BD_iK_q)(I\otimes x(k)))^TP \\
&\quad \times((A+BD_iK+BD_i^-H)+ \\
&\quad (A_q+BD_iK_q)(I\otimes x(k)))x(k)-x(k)^TPx(k). \qquad (22)
\end{aligned}
$$

As shown in the proof of Theorem 1, for every $x \in \mathscr{E}(P)$, we can verify

$$\begin{aligned} & \triangle V(x(k)) \\ \leq & \max_{i\in[1,2^m]} x(k)^T[(A+BD_iK+BD_i^-H)^T \times \\ & (P^{-1}-\varepsilon^{-1}(A_q+BD_iK_q)(I\otimes P)^{-1} \times \\ & (A_q+BD_iK_q)^T)^{-1}(A+BD_iK+BD_i^-H) \\ & -P+\varepsilon I]x(k). \end{aligned} \tag{23}$$

Substituting (19) into the above inequality yields

$$\triangle V(x(k)) < 0. \tag{24}$$

The ellipsoid $\mathscr{E}(P)$ is therefore a region of stability for the closed-loop system (16). This completes the proof.
Next, we can consider the following optimization problem to maximize the region $\mathscr{E}(P)$:

$$\begin{aligned} \textbf{OP2}: \quad & \max_{S,L,L_q,H,\varepsilon} trace(S) \\ & s.t.\ (17)\ and\ (18). \end{aligned} \tag{25}$$

Let $Z = HS$, then constraint (17) is equivalent to

$$\begin{bmatrix} -S & \Phi_1' & 0 & \varepsilon S \\ * & -S & \Phi_2 & 0 \\ * & * & -\varepsilon(I\otimes S) & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0,\ i\in[1,2^m] \tag{26}$$

and constraint (18) is equivalent to

$$\begin{bmatrix} 1 & Z_i \\ * & S \end{bmatrix} > 0,\ i = 1,\cdots,m, \tag{27}$$

where

$$\Psi_1' \;=\; SA^T + L^TD_iB^T + Z^TD_i^-B^T$$

and Z_i denotes the i-th row of matrix Z.
Thus, optimization problem (25) can be converted into

$$\begin{aligned} & \max_{S,L,L_q,H,\varepsilon} trace(S) \\ & s.t.\ (26)\ and\ (27). \end{aligned} \tag{28}$$

Remark 4. If ε in (6) or (26) is fixed, the inequalities become LMIs; optimization problems **OP1** and **OP2** can be solved by a line search on the parameter ε.

4 SIMULATION EXAMPLES

In this section, we will give two examples to illustrate the effectiveness of our results.
Example 1. Consider the following quadratic discrete-time system in the form of (1) with $u(k) = 0$. Matrices A and A_q are defined as

$$\begin{aligned} A &= \begin{bmatrix} -0.5 & 2 & 0 \\ -0.35 & 0.7 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \\ A_q &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.3 & 0 & 0 & 0 & -0.3 & 0 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{29}$$

The region of attraction of the origin derived by applying (OP1) is given by

$$P = \begin{bmatrix} 2.3372 & -2.2953 & 0 \\ -2.2953 & 8.0098 & 0 \\ 0 & 0 & 4.1682 \end{bmatrix}$$

corresponding to the value $\varepsilon = 0.914$. Ellipsoid $\mathscr{E}(P)$ is shown in Figure 1; the trajectories of system (29) with initial condition $x_{10} = [0.5\ 0.4\ \ -0.3]^T$ is plotted in Figure 2.

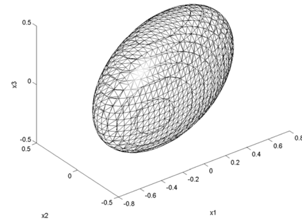

Figure 1: Estimation of the region of attraction for system (29).

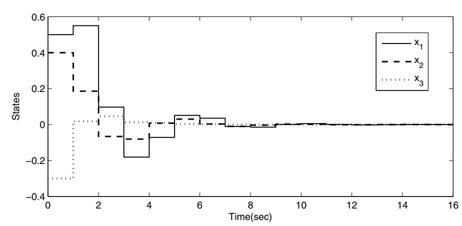

Figure 2: The trajectories for system (29) with initial condition x_{10}

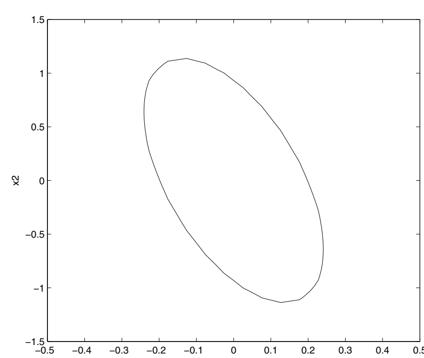

Figure 3: Estimation of the region of attraction for system (30).

Example 2. Consider a quadratic discrete-time system in the form of (1) described by

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1.2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

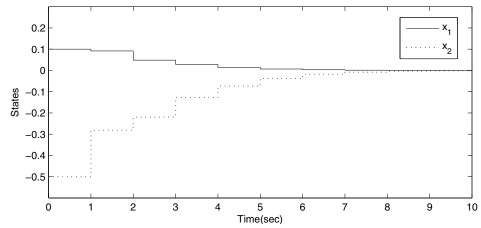

Figure 4: The trajectories for system (30) with initial condition x_{20}.

$$Aq = \left[\begin{array}{cccc} 0 & 0 & 0 & 0.1 \\ 1.2 & 0.6 & 0.6 & 0 \end{array} \right]. \qquad (30)$$

The region of attraction of the origin and the gains obtained by using **OP2** are given by

$$\begin{aligned} S^{-1} &= \left[\begin{array}{cc} 25.0597 & 3.0169 \\ 3.0169 & 1.1347 \end{array} \right], \\ K &= \left[\begin{array}{cc} -1.7424 & -1.0258 \end{array} \right], \\ Kq &= \left[\begin{array}{cccc} -0.1546 & -0.0773 & -0.0773 & 0.0871 \end{array} \right]. \end{aligned}$$

Then, ellipsoid $\mathscr{E}(S^{-1})$ is plotted in Figure 3; the trajectories of system (30) with initial condition $x_{20} = [0.1\ -0.5]^T$ is shown in Figure 4.

5 CONCLUSION

The problems of the stability and stabilization for quadratic discrete-time systems have been studied with actuator saturation. For the quadratic discrete-time system, sufficient conditions for the solvability of the problems have been derived in terms of "quasi"-LMIs. Then, LMI-based optimization problems are presented to estimate the region of attraction of corresponding systems. Two examples have been provided to verify the effectiveness of the proposed approach.

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