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Geometric Control of Input Saturated Systems with Guaranteed Closed-Loop Performance and Stability

Sairam Valluri[1] and Vikram Kapila[2]

[1]GSE Systems, 7133 Rutherford Road, Baltimore, MD 21244
[2]Department of Mechanical, Aerospace, and Manufacturing Engineering, Polytechnic University, Brooklyn, NY 11201

Abstract

In this paper we study the issue of stability of input constrained closed-loop systems with a class of differential geometric controllers. Specifically, a Lyapunov function-based method is presented for determining the closed-loop stability under recently developed geometric control laws for single-input single-output systems. This method provides a subset of the domain of attraction for guaranteed stability of the closed-loop system. Furthermore, it offers flexibility in adjusting the closed-loop performance while ensuring closed-loop stability in the presence of input saturation. Hence, by using this framework the designer can *a posteriori* verify overall stability of the closed-loop system for a desired closed-loop performance.

1. Introduction

Many engineering systems exhibit complex nonlinear behavior that motivate the design of nonlinear controllers. In addition, physical constraints on the system variables introduce additional nonlinearities, for instance, in the form of frequently encountered actuator saturation nonlinearities. Neglecting actuator saturation may severely degrade the closed-loop performance and may even drive the system to instability. The undesirable performance degradation due to actuator saturation is traditionally referred to as windup.

In recent years, saturation control has received considerable attention. In particular, two main approaches can be distinguished for feedback controller design for systems with saturating inputs. The first approach is a two-step method which first involves the design of a controller for unconstrained process and then accounts for the input constraints through a suitable anti-windup modification so as to attenuate the windup related adverse affects [1,3]. These anti-windup controllers may be appealing as they preserve the original structure of the controller designed for unconstrained system. But many of these anti-windup controllers lack comprehensive theoretical results in regard to the closed-loop stability. In contrast, the second approach is characterized as a direct control design methodology, which simultaneously accounts for closed-loop performance and stability in the face of active input constraints (e.g., [4]). However, these controllers are usually much different from those that are prevalent in practice. For linear systems, the problems of input constraint handling and/or closed-loop stability *via* both approaches have been well addressed [12].

In regard to saturation control of nonlinear processes, there are several approaches reported in the literature for the differential geometric controllers. These approaches include *i)* conditional integration for input-output linearizing controller [8] *ii)* input-output linearizing control with constraint mapping of [2] together with linear model predictive control [7], *iii)* input-output linearizing control for input constrained systems [6] with constraint mapping together with the modified linear internal model control [11], *iv)* model predictive control formulation of input-output linearization [9], and *v)* an optimization-based method for the synthesis of geometric controllers [10].

The main contribution of this paper is the development of a methodology for ascertaining the closed-loop stability under a class of differential geometric controllers that were recently derived in [10]. Specifically, we address the stability problem by pursuing the Lyapunov function based saturation control design framework of [4,5]. All theorem *proofs* are omitted due to the page limitation.

Nomenclature

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r$	–real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$
$L_f h(x)$	–Lie derivative of scalar field $h(x)$ wrt. vector field $f(x)$; $L_f^1 h(x) \triangleq \sum_{i=1}^n \frac{\partial h(x)}{\partial x_i} f_i(x)$
$L_f^{i+1} h(x)$	–Lie derivative of scalar field $L_f^i h(x)$ wrt. vector field $f(x)$
$L_g L_f^i h(x)$	–Lie derivative of scalar field $L_f^i h(x)$ wrt. vector field $g(x)$
$\|\|\omega(t)\|\|_p$	–the p-function norm of $\omega(t)$ over $[a,\ b]$, $p \geq 1$ $\|\|\omega(t)\|\|_p \triangleq \left(\int_a^b \|\omega(t)\|^p dt\right)^{1/p}$
$\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$	$-r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \leq Z_2$	$-Z_2 - Z_1 \in \mathbb{N}^r; Z_1, Z_2 \in \mathbb{S}^r$
$Z_1 < Z_2$	$-Z_2 - Z_1 \in \mathbb{P}^r; Z_1, Z_2 \in \mathbb{S}^r$
$n, r, \tilde{n}$	–positive integers; $1 \leq r \leq n$; $\tilde{n} = n + 2r$

2. Stability Analysis of Linear Systems with Input Nonlinearities

In this section we consider the problem of stability analysis for linear systems with a class Φ of time-invariant feedback input nonlinearities $\phi : \mathbb{R} \to \mathbb{R}$.

Specifically, given $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, we provide conditions that guarantee global asymptotic stability of the feedback interconnection of $G(s)$ and $\phi(\cdot)$ for all $\phi \in \Phi$. Note that the feedback interconnection of $G(s)$ and $\phi(\cdot)$ has the state-space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\phi(y(t)), \quad x(0) = x_0, \quad t \geq 0, && (1) \\ y(t) &= Cx(t). && (2) \end{aligned}$$

To state the main result of this section the following definitions are needed. First, define the set Φ of input nonlinearity $\phi(\cdot)$ by

$$\Phi \triangleq \{\phi : \mathbb{R} \to \mathbb{R} : M_1 y^2 \leq \phi(y)y \leq M_2 y^2, y \in \mathbb{R}\}, \quad (3)$$

where $M_2 > M_1 \geq 0$ are given scalars. Next, we also consider the case where the input nonlinearity $\phi(\cdot)$ is contained in Φ for a finite or semi-infinite range of its arguments, i.e.,

$$\Phi_{\mathrm{b}} \triangleq \{\phi : \mathbb{R} \to \mathbb{R}: M_1 y^2 \leq \phi(y)y \leq M_2 y^2, \underline{y} \leq y \leq \overline{y}\}, \quad (4)$$

where $\underline{y} < 0$ and $\overline{y} > 0$ are given. Furthermore, let H be a given positive scalar, $M \triangleq M_2 - M_1$, $R_0 \triangleq 2HM^{-1}$, and $R \in \mathbb{N}^n$. Finally, define

$$\begin{aligned} \psi(\alpha) &\triangleq \frac{\alpha^2}{CP^{-1}C^T}, && (5) \\ V^+ &\triangleq \psi(\overline{y}), \ V^- \triangleq \psi(\underline{y}), \ V_{\mathrm{S}} \triangleq \min(V^+, V^-), && (6) \\ \mathcal{D}_{\mathrm{A}} &\triangleq \{x \in \mathbb{R}^n : V(x) < V_{\mathrm{S}}\}, && (7) \end{aligned}$$

where $V(\cdot)$ is a Lyapunov function for (1), (2) and $P \in \mathbb{P}^n$ satisfies

$$\begin{aligned} 0 = &\ (A + BM_1C)^T P + P(A + BM_1C) \\ &+ (HC + B^T P)^T R_0^{-1} (HC + B^T P) + \epsilon P + R, \quad (8) \end{aligned}$$

with $\epsilon > 0$.

Next, we state a key theorem from [4] establishing the asymptotic stability of the feedback interconnection of $G(s)$ and $\phi(\cdot)$ for all $\phi(\cdot) \in \Phi$. Note that the following theoretical result is fundamental and forms the basis for all later developments.

Theorem 2.1 [4]. Let H, M be positive scalars. Suppose there exist $P \in \mathbb{P}^n$ and a scalar $\epsilon > 0$ satisfying (8). Then the function

$$V(x) = x^T P x, \quad (9)$$

is a Lyapunov function that guarantees that the feedback interconnection of $G(s)$ and $\phi(\cdot)$ for all $\phi \in \Phi$ is globally asymptotically stable. Furthermore, if $\phi \in \Phi_{\mathrm{b}}$ then the feedback interconnection of $G(s)$ and $\phi(\cdot)$ is locally asymptotically stable and $\mathcal{D}_{\mathrm{A}}$ defined by (7) is a subset of the domain of attraction for (1), (2).

Remark 2.1. A key application of Theorem 2.1 is the case in which $\phi(y)$ represents a time-invariant saturation nonlinearity. Specifically, let $\phi(y(t))$, $t \geq 0$, be characterized by

$$\begin{aligned} \phi(y(t)) &= y(t), && |y(t)| \leq a, \\ \phi(y(t)) &= a \operatorname{sgn}(y(t)), && |y(t)| > a. \quad (10) \end{aligned}$$

In this case, Theorem 2.1 can be used to guarantee asymptotic stability of the system (1), (2) for all $\phi(\cdot)$ satisfying (10) with a guaranteed domain of attraction. Specifically, if $M_1 > 0$ and $M_2 = 1 \geq M_1 > 0$ and there exists a positive-definite matrix P satisfying (8), then take $\overline{y} = -\underline{y} = \dfrac{a}{M_1}$ in (6).

3. Differential Geometric Control of Linear Systems with Input Saturation

In this section we consider the differential geometric control of input constrained SISO linear systems with full state measurements. Specifically, we provide sufficient conditions for the closed-loop stability with the geometric control laws for linear time-invariant systems presented in [10] using Theorem 2.1.

3.1. Mixed Error- and State-Feedback Control Problem

Given the n^{th}-order stabilizable, minimum-phase plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\phi(u(t)), \ x(0) = x_0, \ t \geq 0, && (11) \\ y(t) &= Cx(t), && (12) \end{aligned}$$

where $u(t)$ is the scalar manipulated input, $y(t)$ is the scalar controlled output, and input nonlinearity $\phi(\cdot)$ is characterized by the saturation function (10), determine the $2r^{\text{th}}$-order dynamic feedback controller

$$\begin{aligned} \dot{\eta}(t) &= A_c \eta(t) + B_1 x(t) + B_2 \phi(u(t)), \eta(0) = \eta_0, && (13) \\ \dot{\xi}(t) &= A_c^* \xi(t) + B_1^* \eta(t) + B_2^* y(t), \xi(0) = \xi_0, && (14) \\ u(t) &= \tilde{K} \begin{bmatrix} x(t) \\ \eta(t) \\ \xi(t) \end{bmatrix}, && (15) \end{aligned}$$

that satisfies the following design criteria:

i) the closed-loop system (11)–(15) is asymptotically stable for $\phi(\cdot)$ given by (10);

ii) the controller (13)–(15) in the absence of input constraints induces input-output closed-loop response of the form of

$$y^*(t) + \gamma_1 \frac{dy^*(t)}{dt} + \cdots + \gamma_r \frac{d^r y^*(t)}{dt^r} = 0, \quad (16)$$

where r is relative order of the system i.e., the smallest integer for which $CA^{r-1}B \neq 0$;

iii) the performance functional

$$\min_{u(t)} \left\{ ||\hat{y}(\tau) - \tilde{y}^*(\tau)||_p^2 \right\}, \tag{17}$$

subject to constraints

$$\underline{u} \le u \le \overline{u}, \tag{18}$$

where *a)* t represents the present time; *b)* $||\omega(\tau)||_p$ denotes the p-function norm of the scalar function $\omega(t)$ over the finite time-interval $[t,\ t+T_h]$ where T_h is a sufficiently short time horizon into the future; *c)* $\hat{y}(\tau)$ is the predicted value of the output $y(t)$ when constraints are present;

$$\hat{y}(\tau) \approx \sum_{\ell=0}^{r} CA^\ell x(t)\tfrac{(\tau-t)^\ell}{\ell!} + CA^{r-1}B \bullet u(t)\tfrac{(\tau-t)^r}{r!}, \tag{19}$$

and *d)* $\tilde{y}^*(\tau)$ is related to $y^*(\tau)$ described by (16) as given below:

$$\tilde{y}^*(\tau) + \beta_1 \frac{\tilde{y}^*(\tau)}{d\tau} + \cdots + \beta_r \frac{d^r \tilde{y}^*(\tau)}{d\tau^r} = \tilde{y}(t), \tag{20}$$

with initial conditions

$$\tilde{y}^*(t) = Cx(t), \frac{d^i \tilde{y}^*(t)}{dt^i} = CA^i x(t), i = 1, \ldots, r-1,$$

and

$$\tilde{y}(t) = (1 - \tfrac{\beta_r}{\gamma_r})y^*(t) + (\beta_1 - \gamma_1 \tfrac{\beta_r}{\gamma_r})\tfrac{dy^*(t)}{dt} + \cdots + (\beta_{r-1} - \gamma_{r-1}\tfrac{\beta_r}{\gamma_r})\tfrac{dy^{*r-1}(t)}{dt^{r-1}},$$

specifically, for the sufficiently short time $(\tau - t)$, the solution of (20) can be approximated by

$$\tilde{y}^*(\tau) \approx Cx(t) + \textstyle\sum_{\ell=1}^{r-1} CA^\ell x(t)\tfrac{(\tau-t)^\ell}{\ell!} + \tfrac{\tilde{y} - Cx(t) - \sum_{\ell=1}^{r-1}\beta_\ell CA^\ell Bx(t)}{\beta_r}\tfrac{(\tau-t)^r}{r!}, \tag{21}$$

iv) the controller (13)–(15) minimizes the mismatch between the constrained and unconstrained controlled outputs at an adjustable decay rate which is governed by

$$[y(t) - y^*(t)] + \beta_1 \frac{d[y(t) - y^*(t)]}{dt} + \cdots + \beta_r \frac{d^r[y(t) - y^*(t)]}{dt^r} = 0, \tag{22}$$

v) the controller (13)–(15) has integral action in order to ensure offset free response in the presence of unmeasurable disturbances and model errors.

The following theorem provides a sufficient condition for the asymptotic stability of the closed-loop system (11)–(15) under the gemoetric control law of [10]. For the statement of this result let $\tilde{R} \in \mathbb{N}^{\tilde{n}}$ and define

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ \eta(t) \\ \xi(t) \end{bmatrix}, \tilde{A} \triangleq \begin{bmatrix} A & 0 & 0 \\ B_1 & A_c & 0 \\ B_2^* C & B_1^* & A_c^* \end{bmatrix}, \tilde{B} \triangleq \begin{bmatrix} B \\ B_2 \\ 0 \end{bmatrix},$$

$$B_c \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\beta_r} \end{bmatrix}^T, B_c^* \triangleq \begin{bmatrix} 0 & 0 \cdots & 0 & \frac{1}{\gamma_r} \end{bmatrix}^T,$$

$$C_c \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, P_c \triangleq [C + \beta_1 CA + \cdots + \beta_r CA^r],$$

$$C_c^* \triangleq \left[\left(\gamma_r \tfrac{1}{\beta_r} - 1\right) \left(\gamma_r \tfrac{\beta_1}{\beta_r} - \gamma_1\right) \cdots \left(\gamma_r \tfrac{\beta_{r-1}}{\beta_r} - \gamma_{r-1}\right) \right].$$

Finally, let

$$\tilde{\psi}(\alpha) \triangleq \frac{\alpha^2}{\tilde{K}\tilde{P}^{-1}\tilde{K}^T},$$

$$V^+ \triangleq \psi(\overline{y}), \qquad V^- \triangleq \psi(\underline{y}), \qquad V_S \triangleq \min(V^+, V^-),$$

$$\mathcal{D}_A \triangleq \{\tilde{x} \in \mathbb{R}^{\tilde{n}} : V(\tilde{x}) < V_S\}, \tag{23}$$

where $V(\cdot)$ is a control Lyapunov function for (11)–(15) and $\tilde{P} \in \mathbb{P}^{\tilde{n}}$ satisfies

$$0 = (\tilde{A} + \tilde{B}M_1\tilde{K})^T\tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}M_1\tilde{K}) + (H\tilde{K} + \tilde{B}^T\tilde{P})^T R_0^{-1}(H\tilde{K} + \tilde{B}^T\tilde{P}) + \epsilon\tilde{P} + \tilde{R}, \tag{24}$$

with $\epsilon > 0$.

Theorem 3.1. Let $A_c, A_c^*, B_1, B_1^*, B_2, B_2^*$, and $\tilde{K}$ be given by

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\beta_r} & -\frac{\beta_1}{\beta_r} & \cdots & -\frac{\beta_{r-1}}{\beta_r} \end{bmatrix},$$

$$A_c^* = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\gamma_r} & -\frac{\gamma_1}{\gamma_r} & \cdots & -\frac{\gamma_{r-1}}{\gamma_r} \end{bmatrix},$$

$$B_1 = B_c P_c, \qquad B_1^* = B_c^* C_c,$$

$$B_2 = \beta_r B_c CA^{r-1}B, \qquad B_2^* = -B_c^*,$$

$$\tilde{K} = \frac{1}{CA^{r-1}B}\left[-\frac{1}{\beta_r}P_c - \frac{1}{\gamma_r}C\,\frac{1}{\gamma_r}C_c\,\frac{1}{\gamma_r}C_c^* \right]. \tag{25}$$

Furthermore, let $H > 0$, $R_0 > 0$, and suppose there exists an $\tilde{n} \times \tilde{n}$ positive definite matrix $\tilde{P}$ and a scalar $\epsilon > 0$ satisfying (24), with $M_1 = 0$. Then the function $V(\tilde{x}) = \tilde{x}^T\tilde{P}\tilde{x}$ is a Lyapunov function that guarantees that the closed-loop system consisting of (11)–(15) is globally asymptotically stable for $\phi(\cdot)$ characterized by (10). In the case when positive definite matrix $\tilde{P}$ and scalar $\epsilon > 0$ satisfying (24), exist only when $M_1 > 0$, it follows that the asymptotic stability of (11)–(15) can be established only for $\phi \in \Phi_b$. Thus in this case the closed-loop system (11)–(15) is locally asymptotically stable and $\mathcal{D}_A$ defined by (23) is a subset of the domain of attraction for system (11)-(15).

Remark 3.1. Note that it has been shown in [10] that the dynamic controller (13)–(15) with controller gain (25) minimizes the performance index (17) subject to (18) and satisfies the design criteria *ii)*, *vi)*, and *v)*.

4. Stability Analysis of Nonlinear Systems with Input Nonlinearities

In this section we consider the problem of stability analysis of nonlinear affine systems with input nonlinearities $\phi \in \Phi$. Specifically, consider

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))\phi(y(t)),\ x(0) = x_0,\ t \geq 0, \quad (26)\\ y(t) &= h(x(t)), \quad (27) \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$, and $\phi \in \Phi$. For the statement of the main result of this section, let

$$\Gamma^+ \triangleq \{x \in \mathbb{R}^n : \phi(y) = \overline{y}\},\ \ \Gamma^- \triangleq \{x \in \mathbb{R}^n : \phi(y) = \underline{y}\},$$

with associated minimum Lyapunov values, respectively,

$$V_{\Gamma_+} \triangleq \begin{cases} \min_{x\in\Gamma^+} V(x), & \Gamma^+ \neq \emptyset \\ \infty, & \Gamma^+ = \emptyset \end{cases} \quad V_{\Gamma^-} \triangleq \begin{cases} \min_{x\in\Gamma^-} V(x), & \Gamma^- \neq \emptyset \\ \infty, & \Gamma^- = \emptyset \end{cases}$$

where $V(x)$ is a Lyapunov function for (4.1), (4.2). Furthermore, let

$$\begin{aligned} V_S &\triangleq \min(V_{\Gamma^+}, V_{\Gamma^-}),\\ \mathcal{D}_A &\triangleq \{x \in \mathbb{R}^n : V(x) < V_S\}. \quad (28) \end{aligned}$$

Next, we invoke a specialization of Corollary 3.1 of [5] to establish the asymptotic stability of the nonlinear system (26), (27) with feedback nonlinearity $\phi(\cdot) \in \Phi$. The following result is fundamental and forms the basis for the *a posteriori* closed-loop stability analysis of the input constrained nonlinear systems with nonlinear differential geometric controllers.

Theorem 4.1. Consider the system (26), (27). Suppose there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0,\ \ V(x) > 0,\ \ x \in \mathbb{R}^n,\ \ \lim_{x\to\infty} V(x) \to \infty, \quad (29)$$

$$V'(x)f(x) + V'(x)g(x)M_1h(x) + \tfrac{1}{4}\left[Mh(x) + g^T V'^T(x)\right]^2 < 0,\ \ x \in \mathbb{R}^n,\quad x \neq 0. \quad (30)$$

Then the solution $x(t) = 0$, $t \geq 0$, of the nonlinear system (26), (27) is globally asymptotically stable for all $\phi \in \Phi$. In addition, if $\phi \in \Phi_b$ then the nonlinear system is locally asymptotically stable and $\mathcal{D}_A$ defined by (28) is a subset of domain of the attraction for system (26), (27).

Remark 4.1. Note that in the spirit of Remark 2.1, Theorem 4.1 can be directly utilized for the stability analysis of systems with time-invariant saturation nonlinearity.

5. Geometric Control of Nonlinear Systems with Input Saturation

In this section we consider the differential geometric control of input constrained SISO nonlinear systems with full state measurements. Specifically, we provide sufficient conditions for the closed-loop stability of input constrained nonlinear systems with the generalized, mixed error- and state-feedback nonlinear control law of [10] using Theorem 4.1.

5.1. Mixed Error- and State-Feedback Nonlinear Control Problem

Given the n^{th}-order stabilizable, minimum phase plant

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))\phi(u(t)),\ x(0) = x_0,\ t \geq 0, \quad (31)\\ y(t) &= h(x(t)), \quad (32) \end{aligned}$$

where $u(t)$ is the scalar manipulated input, $y(t)$ is the scalar controlled output, and input nonlinearity $\phi(\cdot)$ is characterized by the saturation function (10), determine the $2r^{\text{th}}$-order dynamic feedback controller

$$\begin{aligned} \dot{\eta}(t) &= A_c\eta(t) + B_1 l(x(t)) \quad (33)\\ &\quad + B_2 m(x(t))\phi(u(t)),\ \eta(0) = \eta_0,\ t \geq 0, \quad (34)\\ \dot{\xi}(t) &= A_c^*\xi(t) + B_1^*\eta(t) + B_2^* y(t),\ \xi(0) = \xi_0,\ t \geq 0, \quad (35)\\ u(t) &= \Psi(x(t), \eta(t), \xi(t)), \quad (36) \end{aligned}$$

that satisfies the following design criteria:

i) the closed-loop system (31)–(36) is asymptotically stable for $\phi(\cdot)$ given by (10);

ii) the controller (33)–(36) in the absence of input constraints induces a linear input-output closed-loop response of the form of (17) where r is relative order of the system i.e., the smallest integer for which $L_g L_f^{r-1} h(x(t)) \neq 0$;

iii) the performance functional (17) subject to (18) is minimized where $\hat{y}(\tau)$ and $\tilde{y}^*(\tau)$ are given, respectively, by

$$\hat{y}(\tau) \approx \sum_{\ell=0}^{r} L_f^\ell h(x(t))\tfrac{(\tau-t)^\ell}{\ell!} + L_g L_f^{r-1} h(x(t)) \bullet u(t)\tfrac{(\tau-t)^r}{r!}, \quad (37)$$

$$\tilde{y}^*(\tau) \approx h(x(t)) + \sum_{\ell=1}^{r-1} L_f^\ell h(x(t))\frac{(\tau-t)^\ell}{\ell!} + \frac{\tilde{y} - h(x(t)) - \sum_{\ell=1}^{r-1}\beta_\ell L_f^\ell h(x(t))}{\beta_r}\frac{(\tau-t)^r}{r!}, \quad (38)$$

$$\tilde{y}^*(t) = h(x(t)),\ \frac{d^i\tilde{y}^*(t)}{dt^i} = L_f^i x(t),\ i = 1,\ldots,r-1,$$

iv) the controller (33)–(36) minimizes mismatch between the constrained and unconstrained controlled outputs at an adjustable decay rate which is governed by (22); and

v) the controller (33)–(36) has integral action in order to ensure offset free response in the presence of unmeasurable disturbances and model errors.

For the statement of the next result let $\tilde{R} \in \mathbb{N}^{\tilde{n}}$ and define

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ \eta(t) \\ \xi(t) \end{bmatrix}, \tilde{f}(\tilde{x}(t)) \triangleq \begin{bmatrix} f(x(t)) \\ A_c\eta(t) + B_1 l(x(t)) \\ A_c^*\xi(t) + B_1^*\eta(t) + B_2^* y(t) \end{bmatrix},$$

$$\tilde{g}(\tilde{x}(t)) \triangleq \begin{bmatrix} g(x(t)) \\ B_2 m(x(t)) \\ 0 \end{bmatrix}.$$

Furthermore, recall the notation for B_c, B_c^*, C_c, and C_c^*.

Theorem 5.1. Let A_c, A_c^*, B_2, and B_2^* be given by (25) and let $l(x(t))$, $m(x(t))$, B_1, B_1^*, and $\Psi(x(t), \eta(t), \xi(t))$ be given by

$$\begin{aligned} l(x(t)) &= h(x(t)) + \sum_{\ell=1}^{r} L_f^{\ell} h(x(t)), B_1 = B_c, \\ m(x(t)) &= L_g L_f^{r-1} h(x(t)), B_1^* = B_c^* C_c, \end{aligned}$$

$$\Psi(x(t), \eta(t), \xi(t)) = \tag{39}$$

$$\frac{C_c^*\xi(t) + \eta_1(t) - \frac{\gamma_r}{\beta_r} h(x(t)) - \sum_{\ell=1}^{r} \gamma_r \frac{\beta_\ell}{\beta_r} L_f^{\ell} h(x(t))}{\gamma_r L_g L_f^{r-1} h(x(t))}.$$

Suppose there exists a C^1 function $V(\tilde{x})$ satisfying (29)–(30) with $M_1 = 0$. Then, the function $V(\tilde{x})$ is a Lyapunov function that guarantees that the closed-loop system (31)–(35) is globally asymptotically stable for $\phi(\cdot)$ characterized by (10). In the case when $V(\tilde{x})$ satisfies (29)–(30) only when $M_1 > 0$, it follows that $\phi(\cdot)$ given by (10) is such that $\phi \in \Phi_b$. Thus in this case the closed-loop system (31)–(35) is locally asymptotically stable and $\mathcal{D}_A$ defined by (4.4) is a subset of the domain of attraction for (31)–(35).

6. Conclusions

The closed-loop stability under a class of differential geometric controllers is analyzed *via* a Lyapunov function-based framework. Specifically, this framework led to the development of sufficient conditions for closed-loop stability under geometric control laws of [10] in the face of input saturation. The developed stability analysis method not only allows the designer to calculate a subset of the domain of attraction for guaranteed stability but also enables the designer to obtain a balance between the size of the stability region and performance. The closed-loop performance and stability under the control laws are investigated by means of linear and nonlinear examples. The simulation results are omitted due to the space limitation.

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