# Constrained Optimal Coverage Control of a Multi-Unicycle System

Scientific thesis for the procurance of the degree B.Sc.
from the Department of Electrical and Computer Engineering at the
Technical University of Munich.


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# BACHELOR THESIS

## Constrained Optimal Coverage Control of a Multi-Agent System

Problem description:

Coverage control of a multi-agent system is targeted to achieve an optimal coverage in a convex bounded region using several mobile robots [1]. This project intends to design an optimal coverage control scheme with input constraints and evaluate [2] its performance on a set of mobile robots. The project includes both theoretical syntheses, design of the coverage control method, and practical implementation, an experimental demonstration of the proposed method. The highlight of this project is the distributive structure, both the control method and the hardware implementation, and the integration of input saturation constraints. Therefore, the student is firstly expected to design a mobile robot based on the unicycle prototype, which is supposed to execute the distributed computation routine and exchange sensory information with other agents. Agile motion and compact structure are required for the mobile robot platform. Then, the student should design a optimal coverage control with input constraints and validate it on the robot systems.

Tasks:

- 1st-2nd months: Design and validation of the prototype of the mobile robot;
- 3rd-4th : Formulation and analysis of the optimal coverage control scheme;
- 5th week: Conduct the experiment and evaluate the results.
- 6th month: Writing the report.

Bibliography:


[1] Qingchen Liu, Mengbin Ye, Zhiyong Sun, Jiahu Qin, and Changbin Yu. Coverage control of unicycle agents under constant speed constraints. *IFAC-PapersOnLine*, 50(1):2471–2476, 2017.

[2] Cheng Song, Lu Liu, and Gang Feng. Coverage control for mobile sensor networks with input saturation. *Unmanned Systems*, 4(01):15–21, 2016.


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|---|---|
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(S. Hirche)
Univ.-Professor

DECLARATION

I declare that the work in this bachelor's thesis in electrical and computer engineering is my own work and I have documented all sources and material used.

**Abstract**

This thesis presents a reliable control method for a group of Wheeled Mobile Robots (WMRs) to cover a specific region. There are many useful applications of coverage control such as surveillance, exploration, and rescue operation. The motivation of the proposed control method is to ensure the performance of coverage control while being able to handle a given constraint set. There are three constraints considered in this work. The state constraint requires all agents to maintain in a specific range of a coverage region to ensure the communication ability. The input constraint is the bounded rotation velocity of a WMR due to the limits of its hardware components. The third constraint is the constant heading velocity, which makes the design of the controller challenging due to the underactuation.

Input constraints are ubiquitous and inevitable in practical. For this reason, this thesis is motivated to find a possible control law that is able to handle many constraints during the operation. In this project, we use the nonlinear anti-windup technique and the theorem of barrier Lyapunov function to deal with the above-mentioned constraints. To test the feasibility and evaluate the performance of the control strategy, we create a platform and simulate the coverage control. The proposed controller is proven and tested under different scenarios, such as the number of agents or regions with varying complexity. The proposed controller is distributed, feasible, and applicable.

ACKNOWLEDGMENTS

I would like to express my sincere thanks to my supervisors, Dr. Qingchen Liu and Mr. Zengjie Zhang, for their guidances through each stage of the process. Their enthusiasm has always been inspiring me to identify my research interests.
I am also grateful to Professor Sandra Hirche and all of the Department faculty lecturers for their knowledge and motivation that creates the love of Control Theory in me.

# Contents

# Chapter 1

# Introduction

In recent decades, autonomous systems have been playing increasingly essential roles in industry, healthcare, and our daily life. Particularly, they help humans reduce the heavy workload and perform tasks effectively. Motivated by this, automation scientists and engineers are always seeking reliable and effective solutions for various systems to solve any specific tasks automatically. Many automation ideas have been already proposed and applied in the industry - automotive, manufacturing, etc. The applications produce great results and improve the quality of human life. Nevertheless, there are still many complicated tasks that require sophisticated solutions. For instance, tasks such as area sensing, surveillance, and rescue operations can not be solved by a individual system. This motivates researchers to design working strategies for Multi-Agent-Systems.
The word *Agent* refers to any kind of individual system with its own dynamics and characteristics, for example, wheeled mobile robots, unmanned aerial vehicles, or micro mouses.
The terminology *Multi-Agent System - MAS* refers to an autonomous system that consists of multiple agents with specific characteristics. In comparison to a single-agent-system (SAS), an agent in a MAS just owns a limited amount of information and uses a decentralized control law during the operation. The total workload is divided into many agents so a MAS achieves higher performance than a SAS in many tasks. On the other hand, designing a control strategy for a MAS is challenging due to the property of distributed control method and the cooperation between many agents.

## 1.1 Motivation

In this thesis, we focus on the problem of coverage control executed by a MAS. According to the definition, coverage control implies a task, in which a group of many agents tries to cover the desired region for some specific purposes, such as exploration, data collection, or rescue operation. For instance, a group of aerial vehicle covers a predefined region for monitoring. The problem is complicated because it requires agents to cooperate with each other to find a solution themselves. Furthermore, the limited communication, e.g due to the long-distance, does not allow an agent to obtain the information of all other agents. This also enhances the complexity of the task. The pioneering works in [2],[3] introduce the concept of coverage control by using a simple model to perform the task. [1],[3],[5],[6],[9] study the problem of coverage using models with higher complexity such as Unmanned Aerial Vehicles or WMR. Motivated by the application of MAS from the practical aspect, this thesis considers the problem of constraints related to WMR in the coverage control. For instance, a WMR must maintain a specific range in relation to the region to guarantee a reliable connection, this refers to the state constraint of the system. Furthermore, due to the physical restriction of the hardware components, the velocity of WMR is also limited. This refers to the input constraint. These constraints either exist due to the nature of the problem or they are intentionally imposed by humans to enhance the reliability and compatibility of the MAS. Indeed, constraints are ubiquitous in real situations so we are interested in this research direction.

## 1.2 Contributions

In general, the main theoretical contribution of this thesis is to study the abovementioned constraints of WMR in coverage problems and propose a control method to handle the constraints. We note that there are some researches also considers constraints in coverage control. In [6],[7],[15],[16],[17],[18] the input saturation is considered for a group of agent operating coverage with different formulation. Respectively, [11],[20],[21] propose techniques to handle state constraints. The study in [1] proposes a switching control law to ensure an agent does not leave the coverage region while executing the task. This work is motivated to study the existence of a control law that can satisfy many constraints at the same time and still ensure the operational performance. Indeed, additionally considering a constraint is challenging and the existence of a solution is not always guaranteed. For instance, there might exist conflicts between the condition of constraints, so a controller that can handle input saturation is not able to ensure the state feasibility and vice versa. Moreover, due to the requirement that the heading velocity must be constant, it implies that a WMR is underactuated. The highlight of this thesis is the design of a controller that can find a compromise between many constraints and is therefore applicable in real situations. Furthermore, the theoretical and analytical derivation used to handle various constraints at the same time can be useful for many related researches.

## 1.3 Organization

Chapter 2 introduces the necessary preliminaries for the whole thesis. Chapter 3 studies the problems of the constraints and theoretically proposes a control law to overcome these problems. In chapter 4, we present the simulation platform that we create to simulate a group of WMR executing the coverage task. The performance of the proposed controller is also evaluated in this chapter. Chapter 5 describes the limitations of the proposed control method. In chapter 6, we conclude the thesis and propose the future work.

# Chapter 2

# Preliminaries

In this chapter, we introduce the mathematical notations and preliminaries used for the whole thesis. Firstly, the coverage control is formulated analytically to demonstrate the goal of the control method. The agents considered in this thesis are Wheeled Mobile Robots (WMRs), we also introduce the standard nonlinear dynamic of WMR afterwards.

## 2.1 Notations

| | |
|---|---|
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{R}_+$ | Set of non-negative real numbers |
| $\mathbb{C}$ | Set of complex numbers |
| $i = \sqrt{-1}$ | The imaginary unit |
| $\Re(z)$ | Real part of $z \in \mathbb{C}$ |
| $\Im(z)$ | Imaginary part of $z \in \mathbb{C}$ |
| $\bar{z}$ | The complex conjugate of $z \in \mathbb{C}$ |
| $\langle z_1, z_2 \rangle = \Re(\bar{z}_1 z_2)$ | Scalar product of $z_1, z_2 \in \mathbb{C}$ |

## 2.2 Voronoi Coverage Control

The problem of coverage control is to deploy a group of multiple agents to cover a specific area. This general concept is well defined in the pioneering works [2], [3]. In the above-mentioned publications, MAS use the coverage control method to obtain an optimal sensing capability, which means they can robustly capture any events within a specific region. There are many kinds of formation which depends on the the art of coverage. Firstly, this section will introduce the Voronoi tessellations and later on will depict the relation between of this topology and the sensing performance of agents.

### 2.2.1 Voronoi Tessellations

Define a region $Q : \{q \in \mathbb{R}^2\}$ and $n$ arbitrary points $Z = \{z_1, z_2, ..., z_n\} \in \mathbb{R}^2$ inside this region. These points are called the generator of Voronoi tesselation and they divide the region $Q$ into n partitions $V_k$ , $k \in \{1, ..., n\}$ that satisfy

$$\begin{aligned} &V_k = \{q \in Q \mid \|q - z_k\| \leq \|q - z_j\|\} \\ &Q = V_1 \cup V_2 \cup ... \cup V_n \end{aligned} \tag{2.1}$$

Figure 2.1 depicts the Voronoi partitions constructed inside a bounded region.

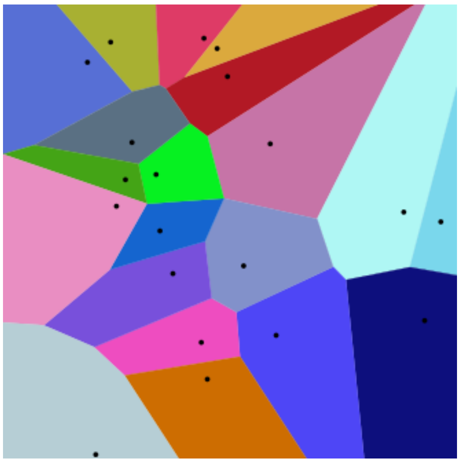

Figure 2.1: Voronoi Tessellations

$source : https : //en.wikipedia.org/wiki/Voronoi\_diagram$

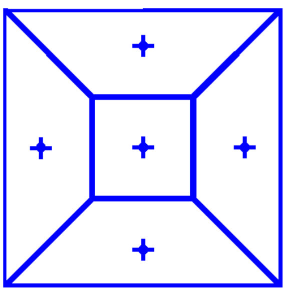

Figure 2.2: Centroidal Voronoi Tessellations

$source : https : //en.wikipedia.org/wiki/Centroidal\_Voronoi\_tessellation$

### 2.2.2 Centroidal Voronoi Tessellations

A special type of Voronoi tessellation is a centroidal Voronoi tessellation (CVT). This configuration has the characteristic that the generator points are also the central mass of each partition. Figure 2.2 shows an example of a CVT.

### 2.2.3 Optimization of Sensing Capability

In coverage control, a group of $n$ agents with onboard sensors has to cover a region $Q \in \mathbb{R}^2$. Define a distribution density function $\phi(q) : Q \to \mathbb{R}_+$, this function determines the chance that an event would happen at a specific region in Q. The location of agents is described as vector notation $Z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$. The performance of the attached sensors on each agent relates to the physical characteristic, such as ultrasonic or light sensors. The sensing capability is inverse proportional to the distance between sensor location and the events in the detectable area. This means the closer one sensor is located to the events that tend happen, the more efficient the performance will be. Therefore, we define a positive, increasing function $f(\|q - z_k\|)$ to represent how poor the performance of sensors in relation to the distance of measurement. The standard cost function proposed in [2],[3] for the sensing performance is considered as

$$H(Z) = \int_Q \underset{k\in 1,\ldots,n}{min} f(\|q - z_k\|)\phi(q)\mathrm{d}q \tag{2.2}$$

From the introduction of Voronoi Tessellations in the previous subsection, we can see from the cost function that the performance of the whole system depends on the position of each agent to maximize the sensing capability within its own region. The cost function is reformulated as

$$H(Z) = \sum_{k=1}^{n} \int_{V_k} f(\|q - z_k\|)\phi(q)\mathrm{d}q \tag{2.3}$$

In order to maximize the sensing capability of MAS, we minimize the cost function (2.3). Let $f(\|q - z_k\|) = \frac{1}{2}\|q - z_k\|^2$ as a strictly increasing, differentiable function. We determine the optimal point of (2.3) through gradient of H(Z) as following:

**Lemma 1. (Lemma 2.1, Schwager (2009))**

$$\begin{aligned}\frac{\partial H(Z)}{\partial z_k} &= \sum_{k=1}^{n} \int_{V_k} \frac{\partial}{\partial z_k} \frac{1}{2} \|q - z_k\|^2 \phi(q)\mathrm{d}q \\ &= \sum_{k=1}^{n} (z_k \int_{V_k} \phi(q)\mathrm{d}q - \int_{V_k} q\phi(q)\mathrm{d}q)\end{aligned}$$

By defining

$$M_{V_k} = \int_{V_k} q\phi(q)\mathrm{d}q$$

$$C_{V_k} = \frac{1}{M_{V_k}} \int_{V_k} q\phi(q)\mathrm{d}q$$

we have

$$\frac{\partial H(Z)}{\partial z_k} = \sum_{k=1}^{n} M_{V_k}(z_k - C_{V_k}) \tag{2.4}$$

The gradient of (2.3) is thought to be complex because the boundaries of each region are also related to the adjacent regions. What Schwager did in his Lemma 2.1 pointed out that the complex terms of the boundary points eliminate each other and result in a simple function that the optimum location of each agent $z_k$ only depends on the geometry of the region $V_k$ . Interested readers may refer to [3] to understand more about the way Schwager performed the derivative under the integral sign. Moreover, the dynamics of Voronoi partitions are studied in previous works [12],[13],[14],[19]. From (2.4), the cost function achieves its local optimum if and only if all generators converge on the set of centroidal Voronoi configuration. This motivates us to design the control policy for agents to approach this configuration.

## 2.3 Non-holonomic Wheeled Mobile Robot Model

The agents considered in this thesis are non-holonomic wheeled mobile robots (WMRs) that have non-linear dynamic. Each agent has a non-identical fixed translation velocity $v_k \in \mathbb{R}_+$ and a rotation velocity $u_k$. When a WMR moves, we define a unique virtual mass which is related to the control input and its actual states. We formulate the dynamic of a WMR and its virtual mass as follows

• Dynamics of WMR

$$\begin{aligned} \dot{x}_k &= v_k cos(\theta_k) \\ \dot{y}_k &= v_k cos(\theta_k) \\ \dot{\theta}_k &= u_k \end{aligned} \tag{2.5}$$

Let $r_k \in \mathbb{C} : \ r_k = x_k + iy_k$, the above dynamic can be formulated in complex notation as

$$\begin{aligned} \dot{r}_k &= v_k e^{i\theta_k} \\ \dot{\theta}_k &= u_k \end{aligned}$$

The complex notation $z_k$ determines the virtual mass of an agent as

$$z_k = r_k + \frac{v_k}{w_0} i e^{i\theta_k}$$

• Dynamic of WMR's virtual mass

$$\dot{z}_k = v_k e^{i\theta_k} - \frac{v_k}{w_0} e^{i\theta_k} u_k \tag{2.6}$$

## 2.4 Problem Statement

Given $n$ wheeled mobile robots with vector $Z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ denotes the location of each agent's virtual mass. Each agent has a non-identical constant translation velocity $v_k$. For a predefined convex region $Q = \{q \in \mathbb{R}^2 \mid Aq \leq b\}$, $A \in \mathbb{R}^{m\times 2}$ , $b \in \mathbb{R}^m$, where $m$ is the amount of boundary lines that cover the region $Q$, find the control law $U = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ for all agents to orbit the set of Centroidal voronoi Configuration in $Q$ and always satisfy the hard constraints:

$$A[\Re(z_k(t)) \;\; \Im(z_k(t))]^T \leq b \; , \; \forall k \in \{1, ..., n\} \; , \; \forall t$$
$$u_k(t) \in [-U_{k_{low}} \;\; U_{k_{up}}] \; , \; \forall k \in \{1, ..., n\} \; , \; w_{k_0} \in [-U_{k_{low}} \;\; U_{k_{up}}], U_{k_{low}}, U_{k_{up}} \in \mathbb{R}_+$$

**Remark**

Motivated by the practical implementation of the system, we design the control law for agents to execute the coverage task, which must be feasible and ensure that the virtual mass of agents do not leave the bounded region. The criteria of the control input are represented as follows

• Performance Requirement

Every agent orbits the centroidal Voronoi configurations with a constant predefined rotation velocity $w_{k_0}$ , $k \in \{1, ..., n\}$.

• State Constraint

Every virtual mass must always stay within the region $Q$

• Input Constraint

The translation velocity of every agent are constant and can be nonidentical. The rotation velocity is bounded.

The problem is complicated because the existence of a possible solution is not guaranteed. The main challenge is to find a compromise between the above-mentioned three criteria. In the next chapter, we study the constraints analytically to determine when an agent tends to violate it. From the analysis, we design a feasible solution to handle each constraint step by step.

# Chapter 3

# Problem Analysis and Controller Design

## 3.1 Stability of Coverage Control

From (2.4), we have

$$\begin{aligned} \dot{H}(Z) &= \sum_{k=1}^{n} \frac{\partial H(Z)}{\partial z_k} \dot{z}_k \\ &= \sum_{k=1}^{n} M_{V_k} \langle z_k - C_{V_k}, \dot{z}_k \rangle \end{aligned} \tag{3.1}$$

By designing a proper feedback control law, we can ensure the time derivative of coverage cost function to be negative and therefore the coverage control is asymptotic stable. In [1], Liu already implemented a control strategy that satisfies the above-mentioned condition. By using this controller, WMR can change its heading orientation and keep moving forwards to orbit any arbitrary center points.

$$u_k = w_{k_0} + \gamma_k w_{k_0} \langle z_k - C_{V_k}, v_k e^{i\theta_k} \rangle \tag{3.2}$$

where

$$\gamma_k \in \mathbb{R}_+$$

$$C_{V_k} \in \mathbb{C}\ ,\ C_{V_k} = C_{kx}\ +\ iC_{ky}$$

**Proof.**

$$(2.6), (3.2) \implies \dot{z}_k = -\gamma_k v_k e^{i\theta_k} \langle z_k - C_{V_k}, v_k e^{i\theta_k} \rangle\ ,\ \gamma \in \mathbb{R}_+ \tag{3.3}$$

Substitute (3.3) into (3.1)

$$\dot{H}(Z) = -\gamma_k M_{V_k} \langle z_k - C_{V_k}, v_k e^{i\theta_k} \rangle^2 \leq 0 \tag{3.4}$$

The time derivative of $H(Z)$ is always non-positive and is equal to zero only when the virtual mass converge to the centroidal of the Voronoi partition. The asymptotic stability of the coverage problem motivates us to design a controller according to the non-negativity of $H(Z)$ from (3.3) and the dynamic of virtual mass from (2.6). However, the stability of the coverage control is only guaranteed whenever the rotation velocity is feasible. Because of the input constraints, the control law from (3.2) might saturate and impair the negativity of $H(Z)$. In the next part, we analyze the behavior of the proposed control method to determine potential issues caused by the input saturation. Afterwards, a modified control method is proposed to overcome the problem. The following derivation will point out the reasons that trigger input saturation and propose a method to overcome it.

## 3.2 Problem of Input Constraint

Recall the control strategy from (3.2)

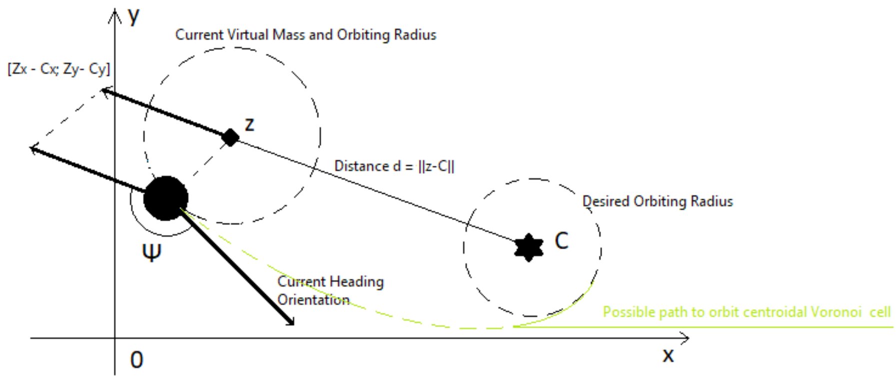


Figure 3.1: Illustration of $\psi_k$

We reformulate the control law from (3.2) as

$$u_k = w_{k_0} + \gamma_k w_{k_0} v_k \left\| z_k - C_{V_k} \right\| cos(\psi_k)$$

where

$$\gamma_k \in \mathbb{R}_+$$
$$w_{k_0}, v_k = const \;,\; v_k \in \mathbb{R}_+$$
$$\psi_k = \angle(z_k - C_{V_k}, v_k e^{i\theta_k})$$

Figure 3.1 depicts the relation between desired control input and actual state of each WMR. We observe that the norm of control input depends on two factors:

- $d_k(z_k) = \left\| z_k - C_{V_k} \right\| = \sqrt{(z_{kx} - C_{kx})^2 + (z_{ky} - C_{ky})^2}$

• $cos(\psi_k)$ , $\psi_k = \angle(z_k - C_{V_k}, v_k e^{i\theta_k})$

**Proposition 1:**
By using the following control law, there always exist a positive control gain $\mu_k(\psi_k)$ for all agent that make the their virtual masses converge the set of centroidal Voronoi configuration and always satisfies the feasibility of control input.

$$u_k = w_{k_0} + \mu_k(\psi_k) sign(w_{k_0}) cos(\psi_k) \ , \ \mu_k(\psi_k) \in \mathbb{R}_+ \tag{3.5}$$

**Proof.**
• *Asymptotic Stability of Coverage Control*
With the proposed controller in (3.5), the dynamic of virtual mass $z_k$ from (2.6) is

$$\begin{aligned} \dot{z}_k &= (v_k - \frac{v_k}{w_{k_0}} u_k) e^{i\theta_k} \\ &= -\mu_k(\psi_k) \frac{v_k}{\|w_{k_0}\|} cos(\psi_k) e^{i\theta_k} \end{aligned} \tag{3.6}$$

Substitute (3.6) into (3.1)

$$\begin{aligned} \dot{H}(Z) &= \sum_{k=1}^{n} M_{V_k} \langle z_k - C_{V_k}, \dot{z}_k \rangle \\ &= \sum_{k=1}^{n} M_{V_k} \langle z_k - C_{V_k}, -\mu_k(\psi_k) \frac{v_k}{\|w_0\|} cos(\psi_k) e^{i\theta_k} \rangle \\ &= -\sum_{k=1}^{n} \frac{\mu_k(\psi_k) M_{V_k} v_k}{\|w_0\|} \langle z_k - C_{V_k}, cos(\psi_k) e^{i\theta_k} \rangle \end{aligned}$$

From

$$\begin{aligned} \psi_k &= \angle(z_k - C_{V_k}, v_k e^{i\theta_k}) \\ \implies cos(\psi_k) &= \frac{\langle z_k - C_{V_k}, e^{i\theta_k} \rangle}{\|z_k - C_{V_k}\|} \end{aligned}$$

We have,

$$\dot{H}(Z) = -\sum_{k=1}^{n} \frac{\mu_k(\psi_k) M_{V_k} v_k}{\|w_0\| \|z_k - C_{V_k}\|} \langle z_k - C_{V_k}, e^{i\theta_k} \rangle^2$$

With $\mu_k(\psi_k) \in \mathbb{R}_+$, the time derivative of cost function is proven to be non-positive and this implies that the coverage control is stable. It is obvious that any equilibrium points $z_k \neq C_k$ are unstable and the system keep converging. Therefore, LaSalle's invariance principle shows that the system is asymptotic stable. (q.e.d)

• *Existence of the Control Gain* $\mu_k(\psi_k)$
The norm of the proposed control method depends on the adjustable control gain $\mu_k(\psi_k)$ and $\psi_k$. By designing an appropriate $\mu_k(\psi_k)$, we can achieve a reasonable control input. So long the control input is feasible, the coverage system maintains stable since $\mu_k \in \mathbb{R}_+$ as proven in the previous part.
Recall $u_k = w_{k_0} + \mu_k(\psi_k)sign(w_{k_0})cos(\psi_k)$ and the condition from the problem statement that $w_{k_0} \in [-U_{k_{low}} \;\; U_{k_{up}}]$. Because $cos(\psi_k) \in [-1 \;\; 1] \;\; \forall\psi_k$, it is obvious that there always exist a positive $\mu_k(\psi_k)$ so that $u_k \in [-U_{k_{low}} \;\; U_{k_{up}}]$. A more analytical analysis of $\mu_k$ will be shown later in the controller design section.

## 3.3 Problem of State Constraint

The control input must satisfy three constraints at the same time. These are the stability of coverage control, the input saturation and the state constraint. The first two constraints were already mentioned and the existence of the solution were proven in the previous subsection. However, additionally considering the state constraint requires us to approach the problem differently since there might be no analytical closed form control method that is able to fulfill all three requirements at the same time. For example, when an agent approaches the boundary of the dominating region, a feasible control input can steer it far away from the boundary but will impair the stability of coverage control. Or when an agent tries to steer away from the boundary, the control input is not feasible because of the input saturation, therefore it will leave the bounded region and violate the state constraints.
A possible solution is a kind of switching controller, which prioritizes the requirements and makes decision accordingly to the full state feedback. This approach requires a proper switching condition introduced in the next subsection.

### 3.3.1 Introduction of Barrier Lyapunov Function

In [4], the lemma of Barrier Lyapunov function was represented
**Lemma 2. (Lemma 1, [4] Keng (2018))**
For any positive constants $k_{b_i}$ , $i = 1, 2, ..., n$, let $Z := \{z \in \mathbb{R}^n : \|z_i\| < k_{b_i} \ , \ i = 1, .., n\}$ $\in \mathbb{R}^n$ be an open set. Consider the system

$$\dot{z} = h(t, z)$$

where $h : \mathbb{R}_+ \times N \to \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $z$, uniformly in $t$, on $\mathbb{R}_+ \times Z$. Let $Z_i := \{z_i \in \mathbb{R} : \|z_i\| < k_{b_i}\} \in \mathbb{R}$. Suppose that there exist a positive definite function $V_i : Z_i \to \mathbb{R}_+ (i = 1, ..., n)$, continuously differentiable on $Z_i$, and they satisfy

$$V_i(z_i) \to \infty \Leftrightarrow \|z_i\| \to k_{b_i}$$

if the inequality holds:

$$\dot{V} = \frac{\partial V}{\partial z} h \leq 0$$

then $z_i(t) \in Z \forall t \in [0, \infty)$

### 3.3.2 Design of Barrier Lyapunov Function for Coverage Control with State Constraint

For a convex region $Q = \{Q^\circ \cup \partial Q\} = \{q \in \mathbb{R}^2 \mid Aq < b\} \cup \{q \in \mathbb{R}^2 \mid Aq = b\}$, the controllers must satisfy $z_k(t) \in Q^\circ$, or $Az_k(t) \leq b$ , $\forall k \in \{1, ..., n\}$ , $\forall t \in [t_0, \infty)$ respectively. Matrix $A$ and vector $b$ determine a specific coverage region and every agent has this information.

$$A = \begin{pmatrix} a_{1x} & a_{1y} \\ a_{2x} & a_{2y} \\ \vdots & \vdots \\ a_{mx} & a_{my} \end{pmatrix} \in \mathbb{R}^{m \times 2}$$

$$b = \begin{pmatrix} b_1 & b_2 & \cdots & b_m \end{pmatrix}^T \in \mathbb{R}^m$$

and the position of $k$-th in $n$ agents and its time varying target at time $t \in [t_0, \infty)$ in Cartesian coordinate is presented as

$$z_k(t) = [z_{kx}(t) \;\; z_{ky}(t)]^T \;,\; C_k(t) = [C_{kx}(t) \;\; C_{ky}(t)]^T \;,\; k \in \{1, ..., n\}$$

The dynamic of virtual mass $z_k$ from (3.6) refers to

$$\dot{z}_k(t) = [\dot{z}_{kx}(t) \;\; \dot{z}_{ky}(t)]^T = -\mu_k(\psi_k(t)) \frac{v_k}{\|w_0\|} cos(\psi_k(t))[cos(\theta_k(t)) \;\; sin(\theta_k(t))]^T \quad (3.7)$$

One BLF candidate $V(Z) : Q \to \mathbb{R}$ for the whole system of $n$ agents:

$$\begin{aligned} V(Z(t)) &= \sum_{k=1}^{n} V_k(z_k(t)) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} [ln(\frac{b_j - a_{jx}C_{kx}(t) - a_{jy}C_{ky}(t)}{b_j - a_{jx}z_{kx}(t) - a_{jy}z_{ky}(t)})]^2 \end{aligned} \quad (3.8)$$

The proposed BLF is always positive whenever all agents are inside the interior $Q^\circ$ and grows to infinity when at least one agent reaches the boundary $\partial Q$. Note that the center masses of Voronoi partition are always determined inside $Q^\circ$ and we only define this BLF on the domain $Q \to \mathbb{R}$ so the logarithm term is always well defined.
**Proof:**

- $\forall t \in [t_0 \;\; \infty)$ , $\forall j \in \{1, ..., m\}$ , $\forall k \in \{1, ..., n\}$

$$\begin{aligned} z_k(t) \in Q^\circ &\implies b_j - a_{jx} z_{kx}(t) - a_{jy} z_{ky}(t) > 0 \\ C_k(t) \in Q^\circ &\implies b_j - a_{jx} C_{kx}(t) - a_{jy} C_{ky}(t) > 0 \\ &\implies 0 < \frac{b_j - a_{jx} C_{kx}(t) - a_{jy} C_{ky}(t)}{b_j - a_{jx} z_{kx}(t) - a_{jy} z_{ky}(t)} < \infty \\ &\implies 0 < V_k(z_k(t)) < \infty \implies 0 < V_(Z(t)) < \infty \end{aligned}$$

- $\exists k \in \{1, ..., n\}$ , $\exists t_1 \in [t_0 \;\; \infty) : z_k(t_1) \to \partial Q$

$$\implies \exists j \in \{1, ..., m\} : \;\; b_j - a_{jx} z_{kx}(t_1) - a_{jy} z_{ky}(t_1) = 0$$

$$\implies V_k(z_k(t_1)) \to \infty \implies V(Z(t_1)) \to \infty$$

From the above statements, (3.8) is a suitable candidate to analyze the state constraints.

### 3.3.3 Analysis of Time Derivative of Barrier Lyapunov Function - Extension of Lemma 2

**Observation 1:** In (3.8), the center mass $C_k(t)$ is defined by the virtual mass's position of agent $z_k(t)$ and its $l$ neighbors $z_i(t)$ , $i \in \{1, ..., l\}$. This means the dynamic of $C_k(t)$ depends only on the movement of the above-mentioned agents. Thus, the time derivative of $V_k(t)$ can be written as following:

$$\dot{V}_k(z_k) = \frac{\partial V_k(z_k)}{dz_k} \dot{z}_k + \sum_{i=1}^{l} \frac{\partial V_k(z_k)}{dz_i} \dot{z}_i \tag{3.9}$$

Lemma 2 concludes that if we can ensure $\dot{V}_k \leq 0 \;\; \forall t$, the state constraints are never violated. Unfortunately this condition can not always be fulfilled for equation (3.9). Because $\dot{V}_k$ also depends on the dynamic of the adjacent agents, no closed form control law for agent $k$ can be concluded. For this reason, we extend the Lemma 2 to show that even $\dot{V}_k > 0$, the states $z_k$, under a specific condition of control input $\dot{z}_k$, are still feasible.

From the proposed BLF (3.8), it is not only positive definite but always have a local minimum at $C_k(t)$ over domain $Q^\circ$. Furthermore, it is designed to be monotonically increasing in every direction outwards from the minimum. Thus, we observe that the BLF is well defined as long as $C_k \in Q^\circ$.

**Proposition 2:** If $z_k(t) \in Q^\circ$ and the agent $k$ follows the descent direction of BLF in relation to $z_k$, which means

$$\frac{\partial V_k(Z)}{dz_k} \dot{z}_k \leq 0 \;\; \forall t$$

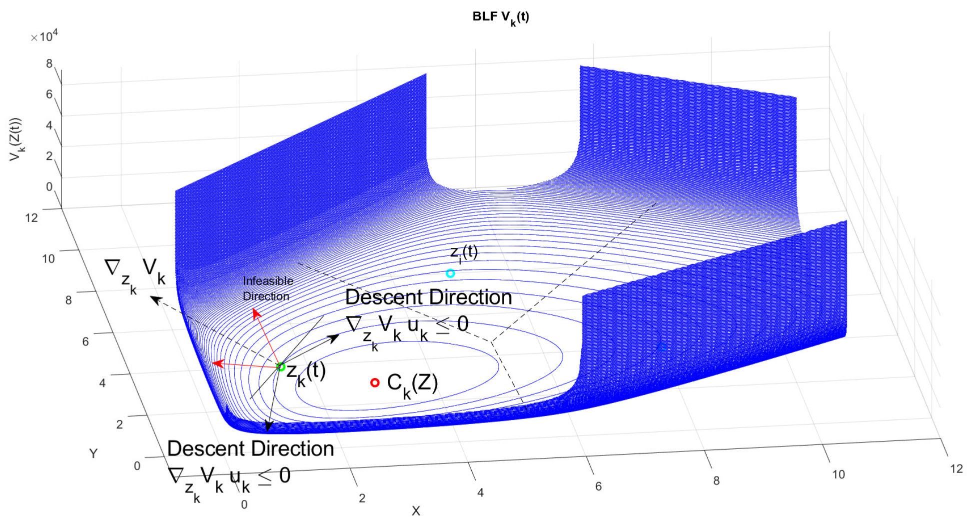


Figure 3.2: Barrier Lyapunov Function and The Feasible Direction of Agent $k$

It is sufficient to conclude that $z_k(t + \delta t) \in Q^\circ$.

Figure 3.2 demonstrates the concept of Proposition 2, there exists a feasible movement that ensures the state constraint.

**Proof.**
From the initial condition, agent $k$ and the center of Voronoi partition $C_k$ is inside the coverage region so the BLF is well defined. Because the proposed function is differentiable, the directional derivative delivers a descent and ascent direction of $V_k Z$ related to the variable $z_k$. Since the region $Q^\circ$ is convex and the BLF is designed so that it always has a local optimum at $C_k$, it implies that $z_k$ is feasible if it follows the descent direction.

Due to the complexity of this problem, we must analyze all of the necessary and sufficient conditions to ensure the feasibility for all agents. This will be presented with more details in the Discussion in chapter 5.

According to the proposition 2, we analyze $\frac{\partial V_k(Z)}{dz_k}\dot{z}_k$ to obtain a control law for agent $k$

$$
\begin{aligned}
&\frac{\partial V_k(Z)}{dz_k}\dot{z}_k \\
&= \frac{\partial V_k(Z)}{dz_{kx}}\dot{z}_{kx} + \frac{\partial V_k(Z)}{dz_{ky}}\dot{z}_{ky} \\
&= \dot{z}_{kx}\sum_{j=1}^{m}\frac{\partial}{\partial z_{kx}}[ln(\frac{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}})]^2 + \dot{z}_{ky}\sum_{j=1}^{m}\frac{\partial}{\partial z_{ky}}[ln(\frac{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}})]^2 \\
&= \sum_{j=1}^{m} 2ln(\frac{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}(t)}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}})(\frac{-a_{jx}\frac{\partial C_{kx}}{\partial z_{kx}}\dot{z}_{kx} - a_{jy}\frac{\partial C_{ky}}{\partial z_{kx}}\dot{z}_{ky}}{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}} + \frac{-a_{jx}\dot{z}_{kx} - a_{jy}\dot{z}_{ky}\dot{z}_{ky}}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}})
\end{aligned}
$$

Substitute the partial derivative of virtual mass $z_k$ from (3.7) into $\dot{V}(Z)$, we obtain

$$
\begin{aligned}
&\frac{\partial V_k(Z)}{dz_k}\dot{z}_k = -\mu_k(\psi_k)\frac{v_k}{\|w_{k_0}\|}. \\
&\underbrace{cos(\psi_k)\sum_{j=1}^{m} 2ln(\frac{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}})(\frac{a_{jx}cos(\theta_k) + a_{jy}sin(\theta_k)}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}} - \frac{a_{jx}\frac{\partial C_{kx}}{\partial z_{kx}}cos(\theta_k) + a_{jy}\frac{\partial C_{ky}}{\partial z_{kx}}sin(\theta_k)}{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}})}_{M_k(t)}
\end{aligned}
$$

The complicated $M_k(t)$ can be obtained by the full state feedback at time t, which are $z_k(t), \theta_k(t), C_k(t)$ and constant $A$ , $b$. Note that the gradient $[\frac{\partial C_{kx}}{\partial z_{kx}} \;\; \frac{\partial C_{ky}}{\partial z_{ky}}]$ is complicated but there already exists an analytical formulation. These terms will be introduced later in the controller design section. We obtain

$$
\frac{\partial V_k(Z)}{dz_k} = -\mu_k(t)\frac{v_k}{\|w_{k_0}\|}M_k(t) \tag{3.10}
$$

**Intuitive Explanation of the Switching Control Law:**
Note that we have two Lyapunov functions in our derivation, which are $H(Z(t))$ and $V(Z(t))$. While the negative time derivative of $H(Z(t))$ ensures the asymptotic stability of the coverage control, it does not reflect the feasibility of the state. On the other hand, from proposition 2, there exist a feasible movement of agent $z_k$, but does not guarantee asymptotic stability of coverage control because its time derivative is not always non-positive.
The idea behind is to use the property of Proposition 2 that whenever $\frac{\partial V_k(Z)}{dz_k}\dot{z}_k \leq 0$, $z_k \in Q^\circ$, the virtual mass $z_k$ is allowed to move. As long as there exists a control input with a positive control gain $\mu_k$ that satisfies this condition, it is shown that $\dot{H}(Z(t)) < 0$ and the coverage control converge asymptotically.
Additionally, $\mu_k \in \mathbb{R}_+$ must satisfy the condition obtained from section 3.2 to handle the input saturation.
**Proposition 3:**
The following switching condition ensures that all agents' virtual mass always stay

inside the interior of region Q and asymptotically converge to the set of centroidal Voronoi configuration.

$$\mu_k(t) = \begin{cases} \mu_k(t) \in \mathbb{R}_+ & , M_k(t) \geq 0 \\ 0 & , M_k(t) < 0 \end{cases} \tag{3.11}$$

**Proof.**

• ***Feasibility of States***

$\underline{M_k(t) \geq 0 \implies \mu_k(t) \in \mathbb{R}_+}$

$$\begin{aligned} \frac{\partial V_k(Z)}{dz_k} &= -\mu_k(t)\frac{v_k}{\|w_{k_0}\|}M_k(t) \\ &= -\big\|\mu_k(t)\big\|\frac{v_k}{\|w_{k_0}\|}\big\|M_k(t)\big\| \leq 0 \end{aligned}$$

$$\text{Proposition 2} \implies z_k(t) \in Q^\circ \; as \; t \to \infty$$

$\underline{M_k(t) < 0 \implies \mu_k(t) = 0}$

$$\exists t \in [t_0 \;\; \infty) \;\; z_k(t) \in Q^\circ$$

$$\mu_k(t) = 0 \implies \dot{z}_k(t) = 0 \implies z_k(t + \delta t) = z_k(t) \in Q^\circ \;\; for \;\; \delta t \to 0$$

Intuitively, if the virtual mass of an agent is not allowed to move due to state feasibility, the control input forces it to stay there by keep the agent rotate at the current virtual mass.

(3.11) fulfills state constraints (q.e.d).

• ***Asymptotic Convergence of Virtual Masses on Centroidal Voronoi Configuration***

$\underline{M_k(t) \geq 0 \implies \mu_k(t) \in \mathbb{R}_+}$

$$\mu_k(t) \in \mathbb{R}_+ \implies \dot{H}(Z) = -\sum_{k=1}^{n} \frac{\mu_k(\psi_k)M_{V_k}v_k}{\|w_0\|\|z_k - C_{V_k}\|}\langle z_k - C_{V_k}, e^{i\theta_k}\rangle^2 \leq 0$$

From Proposition 1: LaSalle' invariance principle implies that system approaches the unique stable equilibrium points if and only if $z_k = C_{V_k} \forall k \in \{1, ..., n\}$

$\underline{M_k(t) < 0 \implies \mu_k(t) = 0}$

Assume model's dynamic approaches the set of stable equilibrium points at time $t \in [t_0 \;\; \infty)$

This set is defined as $\Omega = \{Z \in Q^\circ | \dot{H}(Z(t')) = 0 \;\; \forall t' \in [t \;\; \infty)\}$. We have

$$\dot{H}(Z(t')) = 0 \;\; \forall t' \in [t \;\; \infty)$$

$$\Leftrightarrow \mu_k(t') = 0 \;\; \forall k \in \{1, ..., n\} \;\; \forall t' \in [t \;\; \infty)$$

$$\Leftrightarrow M_k(t') < 0 \;\; \forall k \in \{1, ..., n\} \;\; \forall t' \in [t \;\; \infty) \tag{P3.1}$$

If every agent rotates around its unchanged virtual mass, the center of each Voronoi cell remains constant

$$\begin{aligned} \mu_k(t') = 0 \implies \dot{z}_k(t') = 0 \implies z_k(t') = z_k(t) \;\; \forall k \in \{1, ..., n\} \;\; \forall t' \in [t \;\; \infty) \\ \implies C_k(t') = C_k(t) = const \;\; \forall k \in \{1, ..., n\} \;\; \forall t' \in [t \;\; \infty) \end{aligned}$$

Then from the definition, $\psi_k$ has the same dynamic with $\theta_k$

$$\begin{aligned} \mu_k(t') = 0 \implies u_k(t') = w_{k_0} \;\text{ from (3.5)} \;\; \forall t' \in [t \;\; \infty) \\ \psi_k = \angle(z_k - C_{V_k}, v_k e^{i\theta_k}) \;,\; C_k(t') = const \;,\; \dot{z}_k(t) = 0 \implies \dot{\psi}_k = \dot{\theta}_k = w_{k_0} \end{aligned}$$

It is shown that if these equilibrium points are not the CVT by the following contradiction

$$\begin{aligned} \dot{\psi}_k = w_{k_0} \implies \exists t' \in [t \;\; \infty) : cos(\psi_k(t')) = 0 \\ \implies \exists t' \in [t \;\; \infty) : M_k(t') = 0 \end{aligned}$$

This contradicts the statement (P3.1) and consequently these equilibrium points are unstable. Only when the virtual masses of every agent coincide the set of centroidal Voronoi configuration, $H(Z) = 0$ , $\dot{H}(Z) = 0$. The theorem of Barbashin states that the system is asymptotic stable. (q.e.d)

## 3.4 Controller Design

### 3.4.1 Scaling Factor for Input Saturation

In **Proposition 1** (Subsection 3.2. Problem of Input Constraints) we pointed out the existence of an feasible positive control gain $\mu_k(\psi_k)$. The following derivation determines $\mu_k(\psi_k) : [0 \;\; 2\pi] \rightarrow \mathbb{R}_+$ in detail.

**• Problem statement**

For a given reference $C_{V_k} = [C_{kx} \;\; C_{ky}]^T \in \mathbb{R}^2$, upper and lower input saturation $U_{up}, U_{low} > 0$, desired orbiting velocity $w_{k_0} \in [-U_{low} \;\; U_{up}]$. From the measurement of virtual mass's current position $z_k = [z_{kx} \;\; z_{ky}]^T \in \mathbb{R}^2$ and heading orientation $\theta_k \in [0, 2\pi]$. By defining $\psi_k = \angle(z_k - C_{V_k}, v_k e^{i\theta_k}) \in [0 \;\; 2\pi]$, find $\mu_k(\psi_k) : [0 \;\; 2\pi] \rightarrow \mathbb{R}_+$ to keep the control input $u_k = w_{k_0} + \mu_k(\psi_k) sign(w_{k_0}) cos(\psi_k)$ always stay in feasible region, which means $u_k \in [-U_{low} \;\; U_{up}]$.

Task: Find

$$\mu_k(\psi_k) \in \mathbb{R}_+ \;,\; \forall \psi_k \in [0 \;\; 2\pi]$$

so that

$$\begin{aligned} u_k = w_{k_0} + \mu_k(\psi_k) sign(w_{k_0}) cos(\psi_k) \\ u_k \in [-U_{low} \;\; U_{up}] \;,\; \forall \psi_k \in [0 \;\; 2\pi] \end{aligned} \tag{3.12}$$

• **Solution**
For any $C_{V_k}$, a desired orbiting angular velocity is $w_{k_0}$ known in advance. From these factors, we propose a control gain that depends from actual $\psi_k$ and $w_{k_0}$ as following

■ If $w_{k_0} > 0$

$$\begin{aligned} & \mu_k(\psi_k) sign(w_{k_0}) \\ & = \mu_k(\psi_k) \\ & = \begin{cases} k_1 \in \mathbb{R} \ , \ 0 < k_1 \leq U_{up} - \|w_{k_0}\| & for \ \psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}] \\ k_2 \in \mathbb{R} \ , \ 0 < k_2 \leq U_{low} + \|w_{k_0}\| & for \ \psi_k \in [0 \ \ \frac{\pi}{2}] \cup [\frac{3\pi}{2} \ \ 2\pi] \end{cases} \end{aligned} \tag{3.13}$$

■ If $w_{k_0} < 0$

$$\begin{aligned} & \mu_k(\psi_k) sign(w_{k_0}) \\ & = -\mu_k(\psi_k) \\ & = \begin{cases} -k_1 \in \mathbb{R} \ , \ 0 < k_1 \leq U_{low} - \|w_{k_0}\| & for \ \psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}] \\ -k_2 \in \mathbb{R} \ , \ 0 < k_2 \leq U_{up} + \|w_{k_0}\| & for \ \psi_k \in [0 \ \ \frac{\pi}{2}] \cup [\frac{3\pi}{2} \ \ 2\pi] \end{cases} \end{aligned} \tag{3.14}$$

• **Proof.** The feasibility of control input
■ For predefined $w_{k_0} > 0$

$$\begin{aligned} u_k &= w_{k_0} + \mu_k(\psi_k) sign(w_{k_0}) cos(\psi_k) \\ &= \|w_{k_0}\| + \mu_k(\psi_k) cos(\psi_k) \\ (3.13) \implies & \begin{cases} u_k = \|w_{k_0}\| + k_1 cos(\psi_k) & for \ \ 0 < k_1 \leq U_{up} - \|w_{k_0}\| \ , \ \psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}] \\ u_k = \|w_{k_0}\| + k_2 cos(\psi_k) & for \ \ 0 < k_2 \leq U_{low} + \|w_{k_0}\| \ , \ \psi_k \in [0 \ \ \frac{\pi}{2}] \cup [\frac{3\pi}{2} \ \ 2\pi] \end{cases} \\ \implies & \begin{cases} u_k \leq \|w_{k_0}\| + (U_{up} - \|w_{k_0}\|) & for \ \psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}] \\ u_k \geq \|w_{k_0}\| + (U_{low} + \|w_{k_0}\|) & for \ \psi_k \in [0 \ \ \frac{\pi}{2}] \cup [\frac{3\pi}{2} \ \ 2\pi] \end{cases} \\ \implies & u_k \in [-U_{low} \ \ U_{up}] \ \ \forall \psi_k \in [0 \ \ 2\pi] (q.e.d) \end{aligned}$$

■ Analog for $w_{k_0} < 0$

$$\begin{aligned} u_k &= w_{k_0} + \mu_k(\psi_k) sign(w_{k_0}) cos(\psi_k) \\ &= -\|w_{k_0}\| - \mu_k(\psi_k) cos(\psi_k) \\ (3.14) \implies & \begin{cases} u_k = -\|w_{k_0}\| - k_1 cos(\psi_k) & for \ \ 0 < k_1 \leq U_{low} - \|w_{k_0}\| \ , \ \psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}] \\ u_k = -\|w_{k_0}\| - k_2 cos(\psi_k) & for \ \ 0 < k_2 \leq U_{up} + \|w_{k_0}\| \ , \ \psi_k \in [0 \ \ \frac{\pi}{2}] \cup [\frac{3\pi}{2} \ \ 2\pi] \end{cases} \\ \implies & \begin{cases} u_k \geq -\|w_{k_0}\| - (U_{low} - \|w_{k_0}\|) & for \ \psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}] \\ u_k \leq -\|w_{k_0}\| + (U_{up} + \|w_{k_0}\|) & for \ \psi_k \in [0 \ \ \frac{\pi}{2}] \cup [\frac{3\pi}{2} \ \ 2\pi] \end{cases} \\ \implies & u_k \in [-U_{low} \ \ U_{up}] \ \ \forall \psi_k \in [0 \ \ 2\pi] (q.e.d) \end{aligned}$$

### 3.4.2 Model Based Control Design for State Constraint

Recall

$$
\begin{aligned}
M_k(t) =& cos(\psi_k)\sum_{j=1}^{m} 2ln(\frac{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}}) \\
& (\frac{a_{jx}cos(\theta_k) + a_{jy}sin(\theta_k)}{b_j - a_{jx}z_{kx} - a_{jy}z_{ky}} - \frac{a_{jx}\frac{\partial C_{kx}}{\partial z_{kx}}cos(\theta_k) + a_{jy}\frac{\partial C_{ky}}{\partial z_{kx}}sin(\theta_k)}{b_j - a_{jx}C_{kx} - a_{jy}C_{ky}})
\end{aligned}
\tag{3.15}
$$

The switching controller law from (3.11) is used to consider whether each agent moves or rotates around its virtual mass through the control gain as follows

$$
\mu_k(t) = \begin{cases} \mu_k(t) \in \mathbb{R}_+ & M_k(t) \geq 0 \\ 0 & M_k(t) < 0 \end{cases}
$$

In [5], Appendix A, Lee already formulated the gradient of $C_k = [\frac{\partial C_{kx}}{\partial z_{kx}} \quad \frac{\partial C_{ky}}{\partial z_{ky}}]$ as

$$
\frac{\partial C_k^{(a)}}{\partial z_k^{(b)}} = \frac{(\int_{\partial V_{i,j}} \phi(q)q^{(a)}\frac{q^{(b)}-z_k^{(b)}}{\|z_j - z_i\|}dq)}{m_k} - \frac{(\int_{\partial V_{i,j}} \phi(q)\frac{q^{(b)}-z_k{}^{(b)}}{\|z_j - z_i\|}dq)(\int_{V_k(Z)} \phi(q)q^{(a)}dq)}{m_k^2}
\tag{3.16}
$$

where $a, b \in x, y$ and $m_k = \int_{V_k(Z)} \phi(q)dq$.
With all of the necessary state feedback, the implementation of a controller for $k-th$ agent is presented as follows

**Algorithm 1** Computation of Control Input for n agents

**Data:**
- Information of dominating region $Q$: $A \in \mathbb{R}^{m\times 2}$ , $b \in \mathbb{R}^m$.
- Input limits: $[-U_{k_{low}} \ \ U_{k_{up}}]$ , $\forall k$ , $U_{k_{low}}, U_{k_{up}} > 0$
- Desired orbiting velocity: $w_{k_0} \in [-U_{k_{low}} \ \ U_{k_{up}}]$
- Constant heading velocity: $v_k \in \mathbb{R}_+$

**Result:**
- Control input for each agent: $u_k$

• **Initialization**
- Choose feasible control gain: (3.13),(3.14) $\implies \mu_k \in \{k_1, k_2\} \ \ \forall k \in \{1, ..., n\}$
- Initialize all agents position that satisfy: $z_k(t_0) \in Q^\circ \ \ \forall k \in \{1, ..., n\}$

• **Loop**

**for** k = 1 **to** n **do**

$\psi_k \leftarrow arccos(\frac{\langle z_k - C_{V_k}, e^{i\theta_k}\rangle}{\left\| z_k - C_{V_k}\right\|})$

**if** $\psi_k \in [\frac{\pi}{2} \ \ \frac{3\pi}{2}]$ **then**
$\mu_k \leftarrow k_1$
**else**
$\mu_k \leftarrow k_2$
**end if**

$(\frac{\partial C_{kx}}{\partial z_{kx}} \ \ \frac{\partial C_{ky}}{\partial z_{ky}}) \leftarrow (3.16)$

$M_k \leftarrow (3.15)$
**if** $M_k \geq 0$ **then**
$\mu_k \leftarrow \mu_k$
**else**
$\mu_k \leftarrow 0$
**end if**

$u_k \leftarrow w_{k_0} + \mu_k sign(w_{k_0}) cos(\psi_k)$
**end for**

**Return:** $u_k \ \ \forall k \in \{1, ..., n\}$

# Chapter 4

# Evaluation

In order to evaluate the proposed control method, we create the simulation platform using MATLAB and VREP (Coppelia Sim). The problem configuration is a group of five unicycle-type agents cover a convex bounded region with a constant heading velocity. They are not allowed to get out of the region during the operation and the rotation velocity is limited. Figure 4.1 depicts the scenario of this coverage problem.

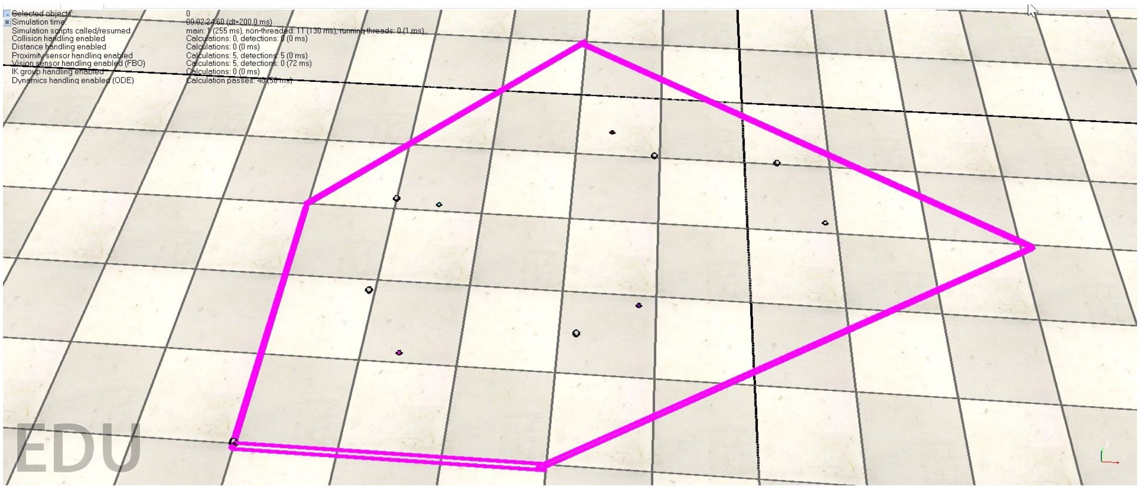


Figure 4.1: 5 Agents - Convex region

During the operation, all data of agents are logged for the evaluation. Data such as virtual mass's position and control input, are plotted intuitively. Figure 4.2 demonstrates the trajectories of five agents created throughout the operation. They never cross the boundary lines, this implies the state constraint is not violated.

Figure 4.3 and 4.4 depicts the final state of the coverage problem. It can be seen that all agents are orbiting their virtual masses, which converge to the set of centroidal Voronoi configuration.

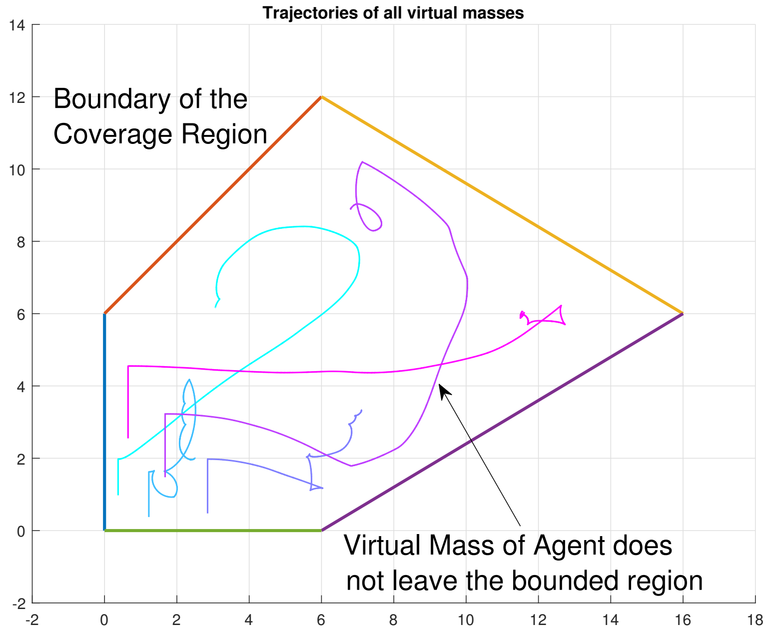


Figure 4.2: Feasibility of States

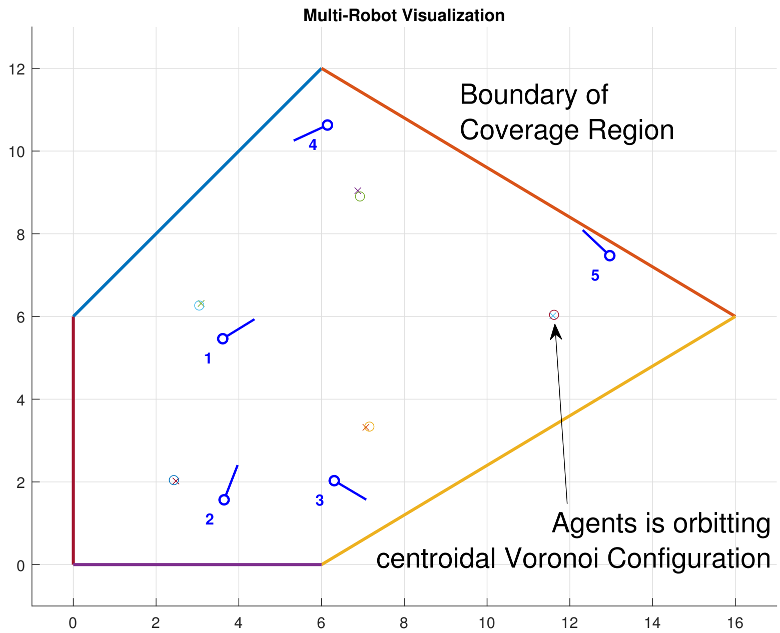


Figure 4.3: Final State of the Coverage Control

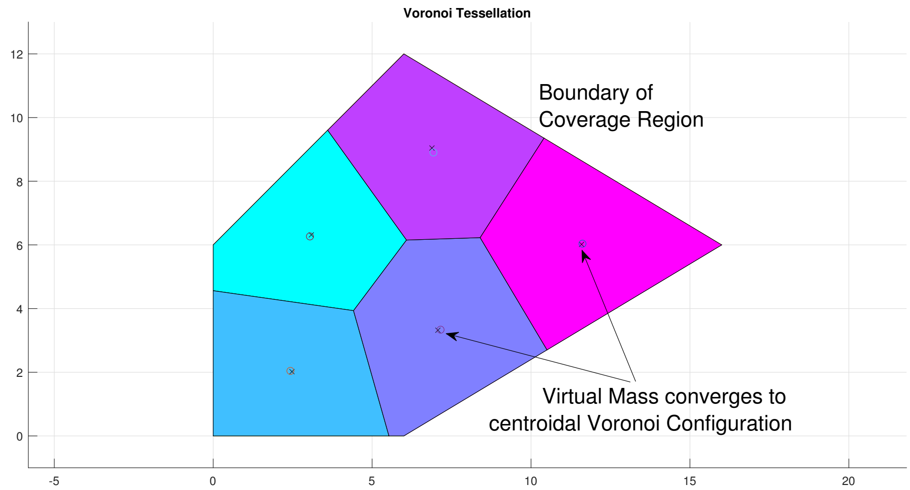


Figure 4.4: Agents Orbit the Centroidal Voronoi Configuration

The maximal rotation velocity of agents are configured to be -0.5 rad/s and 0.5 rad/s. As can be seen in Figure 4.5 that the control input is always inside the red bounded lines, this indicates that the input constraint are never violated.

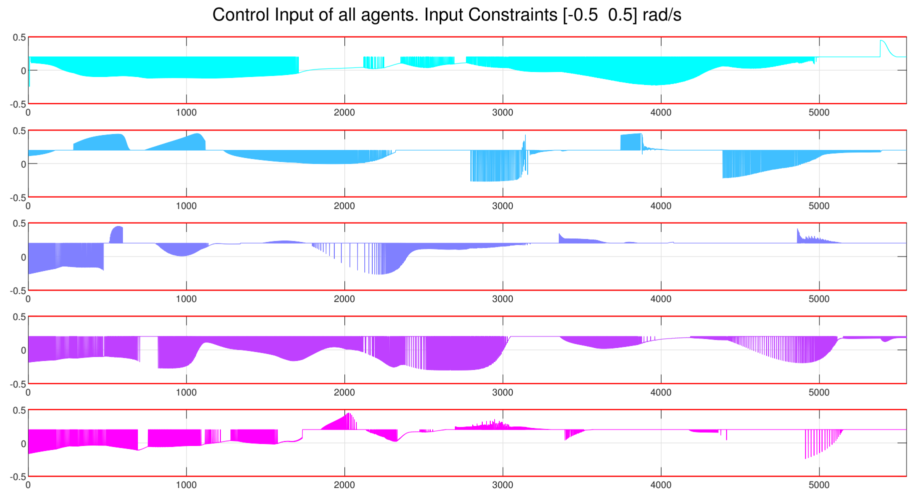


Figure 4.5: Feasible Control Input

Using this simulation environment, we also evaluate the coverage problem with different convex regions such as triangle or rectangle form. The results depicts the reliability and feasibility of the proposed control method.

# Chapter 5

# Discussion

In the analysis of the state constraint, we introduce a Barrier Lyapunov Function and use the proposition 2 to show the feasibility of agent $z_k$. However, this method is yet still not proven under a mathematical analysis. There might be some sufficient and necessary conditions that must be fulfilled. In this chapter, we discuss about this challenge with more details and propose related future work in the conclusion.

We introduce the coverage control in one dimension to demonstrate intuitively the concept of proposition 2. Figure 5.1 depicts the schematic of the 1D coverage problem, in which we have two agents trying to approach the center masses. Intuitively, the boundary of the Voronoi cell between these 2 agents is the perpendicular bisector, which divides the coverage region into two bounded sub-regions. Obviously, the center mass is always the middle point of one region.

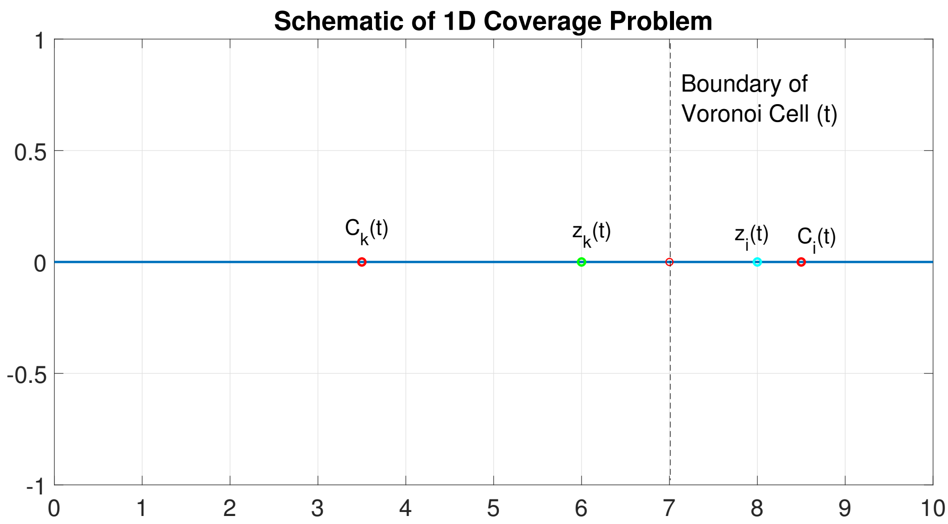


Figure 5.1: Schematic of 1D Coverage

Since the boundaries are fixed, the center mass $C_k$ depends only on the position of two agents. From the BLF in 2D Coverage Problem, we apply it for the 1D scenario.

$$V_k(Z(t)) = \sum_j ln(\frac{b_j - a_j C_k(t)}{b_j - a_j z_k(t)})^2 \tag{5.1}$$

where $j$ denotes the boundaries. From the definition of our Barrier Lyapunov Function, it has the following properties:
• $C_k(Z)$ is always the local optimum of the BLF.
• $C_k(Z)$ and $V_k(Z)$ are always well defined if the agents are always inside the interior of the coverage region.
There exists a descent direction of $V_k(Z)$ related to $z_k$, we use the convexity of the region to show that $z_k$ is still feasible if it follows this direction. Figure 5.2 illustrates the descent direction of $V_k$. Note that the plot shows the contour of $V_k(Z)$, which depends on the position of both two agents and the green point is a vector notation that represents the position of two agents at any time.

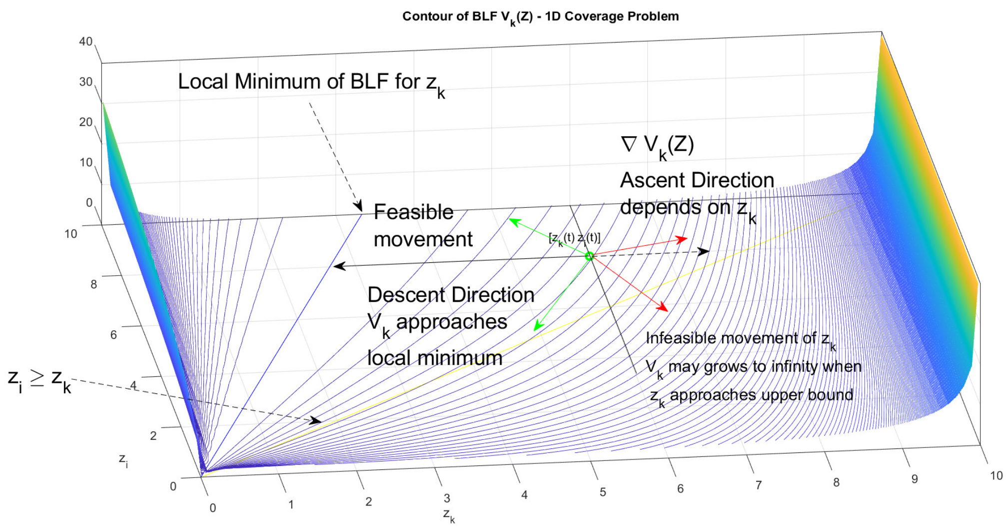


Figure 5.2: Contour of BLF in 1D Coverage Problem

We observe that if $z_k$ always approaches $C_k$ by following the descent direction, it maintains inside the coverage region because the region is convex and $C_k$ is the local minimum of BLF.
Figure 5.3 shows the time evolution of $z_k$. Even $z_k$ (x Axis) moves in the feasible direction, the BLF can still increase because the neighbor (y Axis) moves in the ascent direction related to $z_i$. Consequently, using this BLF alone can not solve the problem of coverage control because of the non-negative time derivative. However, what we need is just a feasible direction that guarantee the state constraints, and as long as all of the virtual masses keep moving, they converge to the set of centroidal Voronoi asymptotically.

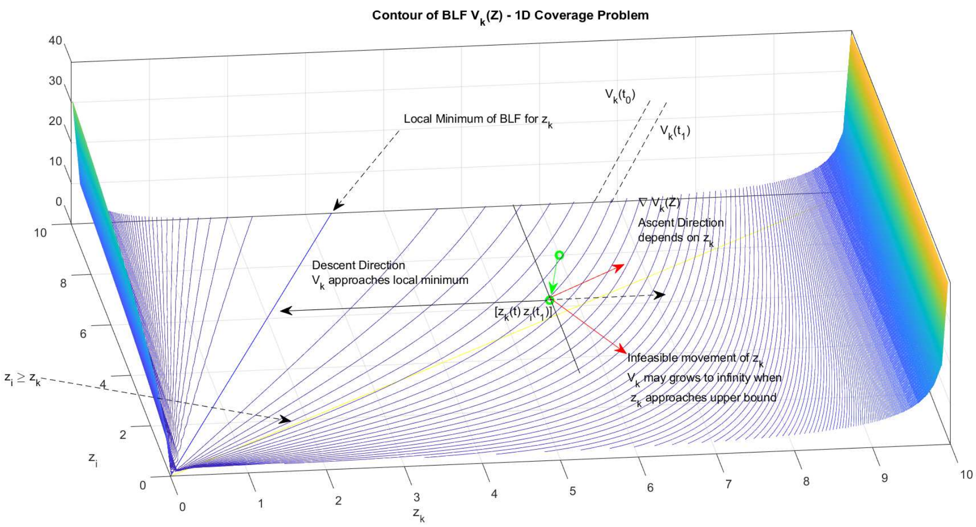


Figure 5.3: Feasibility of Agent k in 1D Coverage Problem

# Chapter 6

# Conclusion and Future Work

This thesis studies the problem of coverage control executed by a group of Wheeled Mobile Robots (WMR) under the state and input constraint. By applying a non-linear scaling factor, the control law can handle the input saturation to ensure the stability of the coverage problem. Besides, the thesis also proposes a switching condition, based on the theorem of barrier Lyapunov function, to guarantee that the state constraints are never violated. The method is proven and simulated under many scenarios with varying parameters and complexity. From a practical point of view, this control law is decentralized, applicable for any hardware specification of WMR, and for any convex region. This corresponds to the motivation of the thesis that a control law can find a compromise to deal with all constraints at the same time and ensure the operational performance.

During the project, we note some challenges that open the potential research directions as future work. The first one refers directly to the proposed control law. By means of applying a switching condition, we do not consider the adjacent agents. Indeed, the movement of the neighbor agents is what makes the guarantee of the state feasibility challenging. Since the barrier Lyapunov function also depends on the position of these agents, we were not able to analyze the function analytically. This motivates us to find strategies that consider the uncertainties due to the neighbor agents to ensure all sufficient conditions of the proposition 2.

One of the most important future work is conducting experiments to assess the control method. Since all the constraints considered in this project are strongly related to real situations, we are looking forward to evaluating its performance and reliability from the practical aspect.

Furthermore, the proposed controller has a limitation that it is only applicable for convex regions. Therefore, we are inspired to find a control method for a non-convex coverage control that can ensure all of the constraints.

# List of Figures

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