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Barrier Lyapunov Functions for the control of output-constrained nonlinear systems[☆]

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ABSTRACT

In this paper, we present control designs for single-input single-output (SISO) nonlinear systems in strict feedback form with an output constraint. To prevent constraint violation, we employ a Barrier Lyapunov Function, which grows to infinity when its arguments approach some limits. By ensuring boundedness of the Barrier Lyapunov Function in the closed loop, we ensure that those limits are not transgressed. Besides the nominal case where full knowledge of the plant is available, we also tackle scenarios wherein parametric uncertainties are present. Asymptotic tracking is achieved without violation of the constraint, and all closed loop signals remain bounded, under a mild condition on the initial output. Furthermore, we explore the use of an Asymmetric Barrier Lyapunov Function as a generalized approach that relaxes the requirements on the initial conditions. We also compare our control with one that is based on a Quadratic Lyapunov Function, and we show that our control requires less restrictive initial conditions. A numerical example is provided to illustrate the performance of the proposed control.

1. Introduction

Lyapunov's direct method provides a means of determining stability without explicit knowledge of system solutions (Slotine & Li, 1991). Besides the analysis of system stability, it is also employed to design stable controllers via the concept of Control Lyapunov Functions (CLFs). For simplicity, quadratic functions of the form $V(z) = \frac{1}{2}z^T Pz$ are often proposed as CLF candidates. Although they are sufficient to solve a large variety of control problems, some difficult problems call for more sophisticated forms of Lyapunov functions. Novel Lyapunov functions have been introduced to handle unknown virtual control coefficients and nonlinearly parameterized functions (Ge,Hang, & Zhang, 1999a,b). For practical systems, it is well known that physical insight and intuition can reveal ways of constructing suitable Lyapunov functions to yield stable control design, as demonstrated in numerous works in robotics and mechatronics systems (Ge, Lee, & Harris, 1998; Lewis, Jagannathan, & Yesildirek, 1999; Ortega, Schaft, van der Mareels, & Maschke, 2001; Slotine & Li, 1991). In this paper, we adopt this approach of tailoring the Lyapunov function to the needs of the problem. In particular, we tackle the tracking problem for nonlinear systems in strict feedback form with an output constraint, motivated by the fact that many practical systems are subjected to constraints in the form of physical stoppages, saturation, or performance and safety specifications.

Existing methods to handle constraints include model predictive control (Allgöwer, Findeisen, & Ebenbauer, 2003; Mayne, Rawlings, Rao, & Scokaert, 2000), reference governors (Bemporad, 1998; Gilbert & Kolmanovsky, 2002), and the use of set invariance notions (Hu & Lin, 2001; Liu & Michel, 1994). Besides these, Barrier Lyapunov Functions have been employed to handle constraints for systems in the Brunovsky form (Ngo, Mahony, & Jiang, 2005). Such a function yields a value that approaches infinity whenever its arguments approach some limits. Inspired by this idea, our previous work has presented an Asymmetric Barrier Lyapunov Function for the control of electrostatic parallel plate microactuators with guaranteed non-contact between the movable and fixed electrodes (Tee, Ge, & Tay, in press).

Adaptive backstepping yields a means of applying adaptive control to parametric-uncertain systems with non-matching conditions (Krstic, Kanellakopoulos, & Kokotovic, 1995; Marino & Tomei, 1995), as well as systems with uncertain functions (Farrell & Polycarpou, 2006; Ge, Hang, Lee, & Zhang, 2001; Lewis et al., 1999).

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Despite its maturity, the explicit consideration of constraints within this framework has received little attention, with a few exceptions. In Krstic and Bement (2006), backstepping has been employed to achieve nonovershooting tracking response for strict feedback systems, by appropriately choosing the control gains such that the initial error variables are negative. Another work (Li & Krstic, 1997) has presented a modified backstepping design based on positively invariant feasibility regions for nonlinear systems with control singularities.

The current work investigates the use of Barrier Lyapunov Functions for SISO nonlinear systems in strict feedback form with an output constraint. By designing the control to render the time derivative of the Barrier Lyapunov Function negative semidefinite, we keep the Barrier Lyapunov Function bounded in the closed loop and ensure that the constraints are not transgressed. Post-design analysis reveals that the stabilizing functions and the control signal remain bounded. Further contributions include the design of adaptive controllers to handle parametric uncertainties while simultaneously preventing constraints from being violated. We also propose novel Asymmetrical Barrier Lyapunov Functions, which add flexibility in control design and relaxes the restriction on initial conditions. Last, but not least, we show that Barrier Lyapunov Functions may yield less restrictive requirements on the initial conditions than Quadratic Lyapunov Functions.

The paper is organized as follows. Section 2 introduces the key technicalities underlying the use of Barrier Lyapunov Functions for constraint satisfaction. Section 3 elucidates the design on a class of second-order nonlinear systems. In Sections 4 and 5, we extend the designs based on Symmetric and Asymmetric Barrier Lyapunov Functions to systems with arbitrary order, and take into account parametric uncertainty in the systems. A comparison study with Quadratic Lyapunov Functions is shown in Section 6, and a numerical example in Section 7.

2. Problem formulation and preliminaries

Throughout this paper, we denote by $\mathbb{R}_+$ the set of nonnegative real numbers, $\|\bullet\|$ the Euclidean vector norm in $\mathbb{R}^m$, and $\lambda_{\max}(\bullet)$ and $\lambda_{\min}(\bullet)$ the maximum and minimum eigenvalues of $\bullet$, respectively. We also denote $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T$, $\bar{z}_i = [z_1, z_2, \ldots, z_i]^T$, $z_{i:j} = [z_i, z_{i+1}, \ldots, z_j]^T$ and $\bar{y}_{d_i} = [y_d^{(1)}, y_d^{(2)}, \ldots, y_d^{(i)}]^T$, for positive integers i, j.

Consider the strict feedback nonlinear system:

$$\begin{aligned}
\dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, 2, \ldots, n-1 \\
\dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u \\
y &= x_1
\end{aligned} \tag{1}$$

where $f_1, \ldots, f_n, g_1, \ldots, g_n$ are smooth functions, $x_1, \ldots, x_n$ are the states, u and y are the input and output respectively. The output $y(t)$ is required to remain in the set $|y| \le k_{c_1} \forall t \ge 0$, where k_{c_1} is a positive constant.

Assumption 1. For any $k_{c_1} > 0$, there exist positive constants $\underline{Y}_0$, $\overline{Y}_0$, A_0, $Y_1, Y_2, \ldots, Y_n$ satisfying $\max\{\underline{Y}_0, \overline{Y}_0\} \le A_0 < k_{c_1}$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $-\underline{Y}_0 \le y_d(t) \le \overline{Y}_0$, $|\dot{y}_d(t)| < Y_1$, $|\ddot{y}_d(t)| < Y_2, \ldots, |y_d^{(n)}(t)| < Y_n$, $\forall t \ge 0$.

Assumption 2. The functions $g_i(\bar{x}_i)$, $i = 1, 2, \ldots, n$, are known, and there exists a positive constant g_0 such that $0 < g_0 \le |g_i(\bar{x}_i)|$ for $|x_1| < k_{c_1}$. Without loss of generality, we further assume that the $g_i(\bar{x}_i)$ are all positive for $|x_1| < k_{c_1}$.

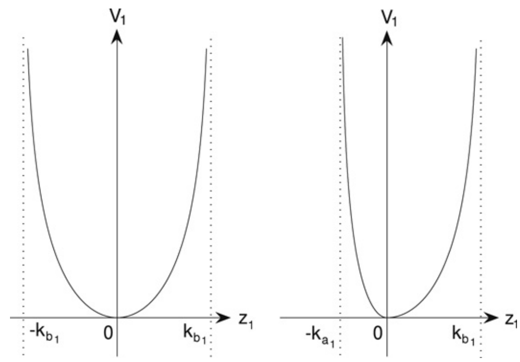

Fig. 1. Schematic illustration of a Symmetric (left) and an Asymmetric (right) Barrier Lyapunov Function.

Definition 1 (Krantz & Parks, 1999). A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be continuously differentiable of order k, or C^k, if

$$D^a f := \frac{\partial^{a_1}}{\partial x_1^{a_1}} \frac{\partial^{a_2}}{\partial x_2^{a_2}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} f$$

exists and is continuous, for all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, and all nonnegative integers $a_1, a_2, \ldots, a_n$ satisfying $\sum_{i=1}^n a_i \le k$. A smooth, or C^∞, function $f : \mathbb{R}^n \to \mathbb{R}$ is one that is C^k for every positive integer k.

The nonlinear functions $f_i(\bar{x}_i)$ may be uncertain, in which case they satisfy the following linear-in-the-parameters (LIP) condition:

$$f_i(\bar{x}_i) = \theta^T \psi_i(\bar{x}_i), \quad i = 1, \ldots, n \tag{2}$$

where $\psi_1, \ldots, \psi_n$ are smooth functions, and $\theta \in \mathbb{R}^l$ is a vector of uncertain parameters satisfying $\|\theta\| \le \theta_M$ with θ_M a positive constant.

The control objective is to track a desired trajectory $y_d(t)$ while ensuring that all closed loop signals are bounded and that the *output constraint is not violated*. To prevent the output from violating the constraint, we employ a Barrier Lyapunov Function, defined as follows.

Definition 2. A Barrier Lyapunov Function is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region $\mathcal{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{D}$, has the property $V(x) \to \infty$ as x approaches the boundary of $\mathcal{D}$, and satisfies $V(x(t)) \le b \; \forall t \ge 0$ along the solution of $\dot{x} = f(x)$ for $x(0) \in \mathcal{D}$ and some positive constant b.

A Barrier Lyapunov Function may be symmetric or asymmetric, as illustrated in Fig. 1. The following lemma formalizes the result for general forms of barrier functions and is used in the control design and analysis for strict feedback system (1) to ensure that output or state constraints are not violated.

Lemma 1. *For any positive constants k_{a_1}, k_{b_1}, let $\mathcal{Z}_1 := \{z_1 \in \mathbb{R} : -k_{a_1} < z_1 < k_{b_1}\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$ be open sets. Consider the system*

$$\dot{\eta} = h(t, \eta) \tag{3}$$

where $\eta := [w, z_1]^T \in \mathcal{N}$, and $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in t and locally Lipschitz in z, uniformly in t, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose that there exist functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_1 : \mathcal{Z}_1 \to \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains, such that

$$V_1(z_1) \to \infty \quad \text{as } z_1 \to -k_{a_1} \quad \text{or} \quad z_1 \to k_{b_1} \tag{4}$$

$$\gamma_1(\|w\|) \le U(w) \le \gamma_2(\|w\|) \tag{5}$$

where γ_1 and γ_2 are class K_∞ functions. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0)$ belong to the set $z_1 \in (-k_{a_1}, k_{b_1})$. If the inequality holds:

$$\dot{V} = \frac{\partial V}{\partial \eta} h \leq 0 \tag{6}$$

then $z_1(t)$ remains in the open set $z_1 \in (-k_{a_1}, k_{b_1})$ $\forall t \in [0, \infty)$.

Proof. The conditions on h ensure the existence and uniqueness of a maximal solution $\eta(t)$ on the time interval $[0, \tau_{\max})$, according to Sontag (1998, p. 476 Theorem 54). This implies that $V(\eta(t))$ exists for all $t \in [0, \tau_{\max})$.

Since $V(\eta)$ is positive definite and $\dot{V} \leq 0$, we know that $V(\eta(t)) \leq V(\eta(0))$ for all $t \in [0, \tau_{\max})$. From $V(\eta) := V_1(z_1) + U(w)$ and the fact that $V_1(z_1)$ and $U(w)$ are positive functions, it is clear that $V_1(z_1(t))$ is also bounded for all $t \in [0, \tau_{\max})$. Consequently, we know, from (4), that $|z_i| \neq k_{b_1}$ and $|z_i| \neq -k_{a_1}$. Given that $-k_{a_1} < z_1(0) < k_{b_1}$, we infer that $z_1(t)$ remains in the set $-k_{a_1} < z_1 < k_{b_1}$ for all $t \in [0, \tau_{\max})$.

Therefore, there is a compact subset $K \subseteq \mathcal{N}$ such that the maximal solution of (3) satisfies $\eta(t) \in K$ for all $t \in [0, \tau_{\max})$. As a direct consequence of Sontag (1998, p. 481 Proposition C.3.6), we have that $\eta(t)$ is defined for all $t \in [0, \infty)$. It follows that $z_1(t) \in (-k_{a_1}, k_{b_1}) \forall t \in [0, \infty)$. ■

Remark 1. In Lemma 1, we split the state space into z_1 and w, where z_1 is the state to be constrained, and w the free states. The constrained state z_1 requires the barrier function V_1 to prevent it from reaching the limits $-k_{a_1}$ and k_{b_1}, while the free states may involve quadratic functions.

3. Control design with a Barrier Lyapunov Function: A motivating example

For illustration purposes, we consider a class of second-order strict feedback systems, and outline the control design based on a Barrier Lyapunov Function to ensure that the output constraint is not violated. For simplicity, we suppose that the system is known; the treatment of system uncertainty is postponed to Sections 4 and 5, where we detail adaptive control designs for uncertain high-order systems. Consider the system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned} \tag{7}$$

where $f_1(x_1)$, $f_2(x_1, x_2)$, $g_1(x_1)$ and $g_2(x_1, x_2)$ are smooth functions, $u \in \mathbb{R}$ the control input, and $x_1, x_2 \in \mathbb{R}$ the states, with x_1 required to satisfy $|x_1(t)| < k_{c_1} \forall t \geq 0$, with k_{c_1} being a positive constant. We employ backstepping design as follows:

Step 1 Let $z_1 := x_1 - y_d$ and $z_2 := x_2 - \alpha_1$, where α_1 is a stabilizing function to be designed. Choose the following Symmetric Barrier Lyapunov Function candidate, originally proposed in Ngo et al. (2005):

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2}$$

where $\log(\bullet)$ denotes the natural logarithm of $\bullet$, and $k_{b_1} = k_{c_1} - A_0$ the constraint on z_1, that is, we require $|z_1| < k_{b_1}$. It can be shown that V_1 is positive definite and C^1 continuous in the set $|z_1| < k_{b_1}$, and thus a valid Lyapunov function candidate. The derivative of V_1 is given by

$$\dot{V}_1 = \frac{z_1 \dot{z}_1}{k_{b_1}^2 - z_1^2} = \frac{z_1(f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d)}{k_{b_1}^2 - z_1^2}. \tag{8}$$

Design the stabilizing function α_1 as:

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{9}$$

where $\kappa_1 > 0$ is a constant. Substituting (9) into (8) yields

$$\dot{V}_1 = -\kappa_1 z_1^2 + \frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} \tag{10}$$

where the coupling term $g_1 z_1 z_2 / (k_{b_1}^2 - z_1^2)$ is canceled in the subsequent step.

Step 2 Since x_2 does not need to be constrained, we choose a Lyapunov function candidate by augmenting V_1 with a quadratic function:

$$V_2 = V_1 + \frac{1}{2} z_2^2. \tag{11}$$

The time derivative of V_2 is given by

$$\dot{V}_2 = -\kappa_1 z_1^2 + \frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} + z_2(f_2 + g_2 u - \dot{\alpha}_1). \tag{12}$$

The control law is designed as

$$u = \frac{1}{g_2}\left(-f_2 + \dot{\alpha}_1 - \kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2}\right) \tag{13}$$

where $\kappa_2 > 0$ is constant, and the last term on the right-hand side is to cancel the residual coupling term $g_1 z_1 z_2 / (k_{b_1}^2 - z_1^2)$ from the first step. Substituting (13) into (11) yields $\dot{V}_2 = -\sum_{i=1}^{2} \kappa_i z_i^2$.

According to Lemma 1, we have $|z_1(t)| < k_{b_1} \forall t > 0$, provided that the initial conditions satisfy

$$|z_1(0)| < k_{b_1}. \tag{14}$$

Then, it is straightforward to show, from $y(t) = z_1(t) + y_d(t)$, $|z_1(t)| < k_{b_1}$, and $|y_d(t)| \leq A_0$, that $|y(t)| < k_{b_1} + A_0 = k_{c_1}$. Thus, the output constraint will never be violated.

From (13), there is a concern of $u(t)$ becoming unbounded whenever $|z_1(t)| = k_{b_1}$. However, we have established that, in the closed loop, the error signal $|z_1(t)|$ never reaches $k_{b_1} \forall t \geq 0$. As a result, the control $u(t)$ will not become unbounded because of the presence of terms comprising $(k_{b_1}^2 - z_1^2(t))$ in the denominator.

4. Control design for higher-order systems

In this section, control design and analysis are presented for systems with order greater than two. We first consider the case where the system model is known, and extend the techniques introduced in Section 3. Subsequently, we deal with the presence of parametric uncertainty, and show that, by incorporating a barrier function in adaptive backstepping design, the output constraint is not violated at any time, including the transient phase of adaptation.

4.1. Known case

First, we consider the case where the functions $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ are known. The control design is based on backstepping, with a Barrier Lyapunov Function candidate employed in the first step, and Quadratic Lyapunov Function candidates in the remaining steps.

Since backstepping design has been well studied and mature, we omit the details of the procedure. Let $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$, $i = 2, \ldots, n$. The first two steps of backstepping design are similar to that presented in Section 3 for second-order strict feedback systems. From Step 3 onwards, the design procedure is identical to standard backstepping using Quadratic Lyapunov Functions. Consider the Lyapunov function candidates:

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \tag{15}$$

$$V_i = V_{i-1} + \frac{1}{2} z_i^2, \quad i = 2, \ldots, n \tag{16}$$

where $k_{b_1} = k_{c_1} - A_0$. Design the stabilizing functions and control law as

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{17}$$

$$\alpha_2 = \frac{1}{g_2}\left(-f_2 + \dot{\alpha}_1 - \kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2}\right) \tag{18}$$

$$\alpha_i = \frac{1}{g_i}(-f_i + \dot{\alpha}_{i-1} - \kappa_i z_i - g_{i-1} z_{i-1}), \quad i = 3, \ldots, n \tag{19}$$

$$u = \alpha_n \tag{20}$$

where $\kappa_i > 0$ is constant, and $\dot{\alpha}_{i-1}$ is given by

$$\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(f_j + g_j x_{j+1}) + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \tag{21}$$

for $i = 2, \ldots, n$. The closed loop system is obtained as

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + g_1 z_2 \tag{22}$$

$$\dot{z}_2 = -\kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} + g_2 z_3 \tag{23}$$

$$\dot{z}_i = -\kappa_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1}, \quad i = 3, \ldots, n-1 \tag{24}$$

$$\dot{z}_n = -\kappa_n z_n - g_{n-1} z_{n-1}. \tag{25}$$

Then, the time derivative of V_n along (22)–(25) can be written as

$$\dot{V}_n = -\sum_{j=1}^{n} \kappa_j z_j^2 \le 0. \tag{26}$$

According to Lemma 1, the error signal z_1 is ensured to satisfy $|z_1| < k_{b_1}$, provided that $|z_1(0)| < k_{b_1}$.

Theorem 1. *Consider the closed loop system (1), (17)–(20) under Assumptions 1 and 2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_1| < k_{b_1}\}$, then the following properties hold.*

(i) *The signals $z_i(t)$, $i = 1, 2, \ldots, n$, remain in the compact set defined by*

$$\Omega_z = \left\{\bar{z}_n \in \mathbb{R}^n : |z_1| \le D_{z_1}, \|z_{2:n}\| \le \sqrt{2V_n(0)}\right\} \tag{27}$$

$$D_{z_1} = k_{b_1}\sqrt{1 - \mathrm{e}^{-2V_n(0)}} \tag{28}$$

where V_n is the overall Lyapunov function candidate obtained from (15)–(16).

(ii) *The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : |y| \le D_{z_1} + A_0 < k_{c_1}\}$ $\forall t \ge 0$, i.e. the output constraint is never violated.*

(iii) *All closed loop signals are bounded.*

(iv) *The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

Proof. (i) From $\dot{V}_n \le 0$, it follows that $V_n(t) \le V_n(0)$. For $|z_1(0)| < k_{b_1}$, Lemma 1 yields $|z_1(t)| < k_{b_1} \forall\, t > 0$. Thus, we infer that

$$\frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2(t)} \le V_n(0). \tag{29}$$

Taking exponentials on both sides of the inequality and rearranging, we obtain $k_{b_1}^2 \le \mathrm{e}^{2V_n(0)}(k_{b_1}^2 - z_1^2(t))$. Then, the following inequality can be obtained:

$$|z_1(t)| \le k_{b_1}\sqrt{1 - \mathrm{e}^{-2V_n(0)}}. \tag{30}$$

Similarly, from the fact that $(1/2)\sum_{j=2}^{n} z_j^2(t) \le V_n(0)$, we can show that $\|z_{2:n}(t)\| \le \sqrt{2V_n(0)}$. Therefore, $z_i(t)$ remains in the compact set $\Omega_z \forall\, t$.

(ii) Since $y(t) = z_1(t) + y_d(t)$, $|z_1(t)| \le D_{z_1} < k_{b_1}$, and $|y_d(t)| \le A_0$, we infer that $|y(t)| \le D_{z_1} + A_0 < k_{b_1} + A_0 = k_{c_1}$. Hence, we can conclude that $y(t) \in \Omega_y \forall t \ge 0$.

(iii) From (i), we know that the error signals $z_1(t), \ldots, z_n(t)$ are bounded. The boundedness of $z_1(t)$ and $y_d(t)$ implies that the state $x_1(t)$ is bounded. Together with the fact that $\dot{y}_d(t)$ is bounded from Assumption 1, it is clear, from (17), that the stabilizing function $\alpha_1(t)$ is also bounded. This leads to the boundedness of $x_2(t)$, since $x_2 = z_2 + \alpha_1$. From (27), we have that $|z_1(t)| < k_{b_1}$. Since α_2 is a continuous function of the bounded signals $\bar{x}_2(t), \bar{z}_2(t)$, and $\bar{y}_{d_2}(t)$ in the set $|z_1| < k_{b_1}$, we know that $\alpha_2(t)$ is bounded. This leads to the boundedness of state $x_3(t)$, since $x_3 = z_3 + \alpha_2$. Following this line of argument, we can progressively show that each $\alpha_i(t)$, for $i = 3, \ldots, n-1$, is bounded, since it is a continuous function of the bounded signals $\bar{x}_i(t), \bar{z}_i(t)$, and $\bar{y}_{d_i}(t)$ in the set $z_1 \in (-k_{b_1}, k_{b_1})$. Thus, the boundedness of state $x_{i+1}(t)$ can be shown. With $\bar{x}_n(t)$, $\bar{z}_n(t)$ bounded, and $|z_1(t)| < k_{b_1} \forall t > 0$, we conclude that the control $u(t)$ is bounded. Hence, all closed loop signals are bounded.

(iv) From the fact that $x_i(t), z_i(t)$, $i = 1, \ldots, n$, are bounded, particularly with $|z_1(t)| < k_{b_1} \forall t \ge 0$, it can be shown, from (22)–(25), that $\ddot{V}_n(t)$ is bounded, which means that $\dot{V}_n(t)$ is uniformly continuous. Then, by Barbalat's Lemma (Slotine & Li, 1991), we obtain that $z_i(t) \to 0$ as $t \to \infty$, for $i = 1, \ldots, n$. ■

4.2. Uncertain case

In this section, we consider the system (1) in which the nonlinear functions $f_i(\bar{x}_i)$ are uncertain, and satisfy the LIP condition (2). Since adaptive backstepping design is mature, we omit the details. Interested readers are referred to Krstic et al. (1995). Denote $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$, $i = 2, \ldots, n$. Consider the Lyapunov function candidates:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta} \tag{31}$$

$$V_i = V_{i-1} + \frac{1}{2}z_i^2, \quad i = 2, \ldots, n \tag{32}$$

where $k_{b_1} = k_{c_1} - A_0$, $\Gamma_1 = \Gamma_1^T > 0$ is constant matrix, and $\tilde{\theta} := \hat{\theta} - \theta$ is the error between θ and its estimate, $\hat{\theta}$. It can be shown that V_1 is positive definite and continuously differentiable in the set $|z_1| < k_{b_1}$, and thus a valid Lyapunov function candidate. The adaptive backstepping control is designed as follows:

$$\alpha_1 = \frac{1}{g_1}(-\hat{\theta}^T w_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{33}$$

$$\alpha_2 = \frac{1}{g_2}\left(-\hat{\theta}^T w_2 - \kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} + \frac{\partial \alpha_1}{\partial x_1} g_1 x_2 + \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma \tau_2\right) \tag{34}$$

$$\alpha_i = \frac{1}{g_i}\left(-\hat{\theta}^T w_i - \kappa_i z_i - g_{i-1} z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} g_j x_{j+1} + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\Gamma \tau_i + \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\Gamma w_i z_j\right)$$

$$i = 3, \ldots, n \tag{35}$$

$$w_1 = \psi_1(x_1), \qquad w_i = \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\psi_j(\bar{x}_j) \tag{36}$$

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$$\tau_1 = \frac{w_1 z_1}{k_{b_1}^2 - z_1^2}, \qquad \tau_i = \tau_{i-1} + w_i z_i, \quad i = 2, \ldots, n \tag{37}$$

$$u = \alpha_n \tag{38}$$

$$\dot{\hat{\theta}} = \Gamma \tau_n \tag{39}$$

which yields the closed loop system

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + g_1 z_2 - \tilde{\theta}^T w_1 \tag{40}$$

$$\dot{z}_2 = -\kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} + g_2 z_3 - \tilde{\theta}^T w_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma \tau_2 - \dot{\hat{\theta}}) \tag{41}$$

$$\begin{aligned}\dot{z}_i &= -\kappa_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1} - \tilde{\theta}^T w_i \\ &\quad + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}(\Gamma \tau_i - \dot{\hat{\theta}}) + \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_i z_j\end{aligned} \tag{42}$$

$$\begin{aligned}\dot{z}_n &= -\kappa_n z_n - g_{n-1} z_{n-1} - \tilde{\theta}^T w_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}(\Gamma \tau_n - \dot{\hat{\theta}}) \\ &\quad + \sum_{j=2}^{n-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_n z_j\end{aligned} \tag{43}$$

along with (39). The time derivative of V_n along (39)–(43) is given by $\dot{V}_n = -\sum_{j=1}^{n} \kappa_j z_j^2$. According to Lemma 1, the error signal z_1 satisfies $|z_1(t)| < k_{b_1} \forall t > 0$, provided that $|z_1(0)| < k_{b_1}$.

Theorem 2. *Consider the closed loop system consisting of (1), (33)–(39) under Assumptions 1 and 2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_1| < k_{b_1}\}$, then the following properties hold.*

(i) *The signals $z_i(t)$, $i = 1, 2, \ldots, n$, and $\hat{\theta}(t)$ remain in the compact sets defined by*

$$\Omega_z = \left\{\bar{z}_n \in \mathbb{R}^n : |z_1| \le D_{z_1}, \|z_{2:n}\| \le \sqrt{2\bar{V}_n}\right\} \tag{44}$$

$$\Omega_{\hat{\theta}} = \left\{\hat{\theta} \in \mathbb{R}^l : \|\hat{\theta}\| \le \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}}\right\}$$

$$D_{z_1} = k_{b_1}\sqrt{1 - \mathrm{e}^{-2\bar{V}_n}}$$

$$\begin{aligned}\bar{V}_n &= \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2(0)} + \frac{1}{2}\sum_{j=2}^{n} z_j^2(0) \\ &\quad + \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2\end{aligned}$$

(ii) *The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : |y| \le D_{z_1} + A_0 < k_{c_1}\}$ $\forall t \ge 0$, i.e. the output constraint is never violated.*

(iii) *All closed loop signals are bounded.*

(iv) *The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

Proof. (i) From $V_n(t) \le V_n(0)$ and $\|\theta\| \le \theta_M$, we know that $V_n(0) \le \bar{V}_n$. For $|z_1(0)| < k_{b_1}$, we have, from Lemma 1, that $|z_1(t)| < k_{b_1} \forall\, t > 0$. These imply that $k_{b_1}^2/(k_{b_1}^2 - z_1^2(t)) \le \mathrm{e}^{2\bar{V}_n}$. Then, we follow a similar approach as the proof of Theorem 1(i) to show that $\bar{z}_n(t) \in \Omega_z \ \forall\, t > 0$. Furthermore, from $V_n(t) \le V_n(0) \le \bar{V}_n$, it follows that $\lambda_{\min}(\Gamma^{-1})\|\hat{\theta} - \theta\|^2 \le 2\bar{V}_n$, and hence $\hat{\theta}(t) \in \Omega_{\hat{\theta}} \forall\, t > 0$.

(ii) Since $y(t) = z_1(t) + y_d(t)$, and $|z_1(t)| \le D_{z_1} < k_{b_1}$, $|y_d(t)| \le A_0$, we conclude that $y(t) \in \Omega_y \forall t \ge 0$.

(iii) From $\dot{V}_n \le 0$ and Lemma 1, the error signals $z_1(t), \ldots, z_n(t)$, $\tilde{\theta}(t)$ are bounded. Since θ is constant, we have that $\hat{\theta}(t)$ is bounded. Similar to the proof of Theorem 1(iii), we can show that $\alpha_i(t)$, for $i = 1, \ldots, n-1$, $\bar{x}_n(t)$ and $u(t)$ are bounded. Hence, all closed loop signals are bounded.

(iv) Since $\hat{\theta}(t)$, $\bar{x}_n(t)$, $\bar{z}_n(t)$ are bounded, particularly with $|z_1(t)| \le k_{b_1} \forall t \ge 0$, it can be shown that $\omega_i(t)$, $\tau_i(t)$, $i = 1, \ldots, n$, are bounded. Then, from the closed loop system dynamics (40)–(43), $\ddot{V}_n(t)$ is bounded. By Barbalat's Lemma, $z_i(t) \to 0$ as $t \to \infty$ for $i = 1, \ldots, n$. ■

5. Asymmetric Barrier Lyapunov Function

Asymmetric barrier functions include symmetric ones as a special class, and are thus, more general. The additional parameter k_{a_1} can be set independently of k_{b_1}, subject to the upper and lower bounds of the desired trajectory y_d. As such, an Asymmetric Barrier Lyapunov Function may afford greater flexibility in control design and relax the conditions on starting values of the output.

In the following, we first present the design procedure and results for the case of known system, and then state only the results for the adaptive part.

5.1. Known case

Step 1

Denote $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where α_1 is a stabilizing function to be designed. Choose an Asymmetric Barrier Lyapunov Function candidate as

$$V_1 = \frac{1}{p} q(z_1) \log\frac{k_{b_1}^p}{k_{b_1}^p - z_1^p} + \frac{1}{p}(1 - q(z_1)) \log\frac{k_{a_1}^p}{k_{a_1}^p - z_1^p} \tag{45}$$

where p is an even integer satisfying $p \ge n$, and

$$q(\bullet) := \begin{cases} 1, & \text{if } \bullet > 0 \\ 0, & \text{if } \bullet \le 0 \end{cases} \tag{46}$$

$$k_{a_1} = k_{c_1} - \underline{Y}_0, \qquad k_{b_1} = k_{c_1} - \overline{Y}_0. \tag{47}$$

Throughout this paper, for ease of notation, we abbreviate $q(z_1)$ by q, unless otherwise stated.

Remark 2. For the Symmetric Barrier Lyapunov Function candidates considered in the previous sections, $p = 2$ is sufficient. However, for asymmetric ones, we need an even integer $p \ge n$. The reason is apparent in Step 2, where the stabilizing function α_2 needs to cancel the residual coupling term from the first step. Following the backstepping procedure, to ensure that α_2 is $n-1$ times differentiable, we choose $p \ge n$.

Lemma 2. *The Lyapunov function candidate $V_1(z_1)$ in (45) is positive definite and C^1 in the set $z_1 \in (-k_{a_1}, k_{b_1})$.*

Proof. For ease of analysis, we rewrite V_1 as

$$V_1(z_1) = \begin{cases} \dfrac{1}{p}\log\dfrac{k_{b_1}^p}{k_{b_1}^p - z_1^p}, & 0 < z_1 < k_{b_1} \\ \dfrac{1}{p}\log\dfrac{k_{a_1}^p}{k_{a_1}^p - z_1^p}, & -k_{a_1} < z_1 \le 0. \end{cases} \tag{48}$$

For $-k_{a_1} < z_1 < k_{b_1}$, we have that $V_1(z_1) \ge 0$ and that $V_1(z_1) = 0$ if and only if $z_1 = 0$, thus implying that $V_1(z_1)$ is positive definite.

Additionally, V_1 is piecewise smooth within each of the two intervals $z_1 \in (-k_{a_1}, 0]$ and $z_1 \in (0, k_{b_1})$. Together with the fact that $\lim_{z_1 \to 0^+} \mathrm{d}V_1/\mathrm{d}z_1 = \lim_{z_1 \to 0^-} \mathrm{d}V_1/\mathrm{d}z_1 = 0$, we conclude that $V_1(z_1)$ is C^1. ■

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Choose the stabilizing function as

$$\alpha_1 = \frac{1}{g_1}[-f_1 + \dot{y}_d - (q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p))\kappa_1 z_1^m] \quad (49)$$

where κ_1 is a positive constant, and m any odd integer satisfying

$$m \geq \max\{3, n\}. \quad (50)$$

The integer m has to be odd in order to yield a negative semidefinite term $-\kappa_1 z_1^{m+p-1}$ in $\dot{V}_1$. Since $n \geq 2$ for system (1) considered in this paper ($n = 1$ is trivial), it means that m must be at least 3.

Substituting (49) into $\dot{z}_1 = \dot{x}_1 - \dot{y}_d$ yields

$$\dot{z}_1 = -\left(q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p)\right)\kappa_1 z_1^m + g_1 z_2. \quad (51)$$

Then, the derivative of V_1 along (51) is

$$\dot{V}_1 = -\kappa_1 z_1^{m+p-1} + \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) g_1 z_1^{p-1} z_2$$

Step i ($i = 2, \ldots, n$)
Denote $z_{i+1} = x_{i+1} - \alpha_i$, where α_i is a stabilizing function. Choose the Lyapunov function candidates:

$$V_i = V_{i-1} + \frac{1}{2} z_i^2, \quad i = 2, 3, \ldots, n. \quad (52)$$

The stabilizing functions and control are designed as:

$$\alpha_2 = \frac{1}{g_2}\left[\dot{\alpha}_1 - f_2 - \kappa_2 z_2 - \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) g_1 z_1^{p-1}\right] \quad (53)$$

$$\alpha_i = \frac{1}{g_i}(\dot{\alpha}_{i-1} - f_i - \kappa_i z_i - g_{i-1} z_{i-1}), \quad i = 3, \ldots, n \quad (54)$$

$$u = \alpha_n \quad (55)$$

which yields the closed loop dynamics:

$$\dot{z}_2 = -\kappa_2 z_2 - \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) g_1 z_1^{p-1} + g_2 z_3 \quad (56)$$

$$\dot{z}_i = -\kappa_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1}, \quad i = 3, \ldots, n-1 \quad (57)$$

$$\dot{z}_n = -\kappa_n z_n - g_{n-1} z_{n-1} \quad (58)$$

along with (51). The derivative of V_n along (51), (56)–(58) is

$$\dot{V}_n = -\kappa_1 z_1^{m+p-1} - \sum_{j=2}^{n} \kappa_j z_j^2.$$

As a result, based on Lemma 1, we know that the error signal z_1 remains in the interval $-k_{a_1} < z_1(t) < k_{b_1} \forall t > 0$, provided that $-k_{a_1} < z_1(0) < k_{b_1}$.

According to the backstepping methodology, α_1 is differentiated $n-1$ times before appearing in the final control law. In general, α_i needs to be differentiable at least $n - i$ times. A further requirement is that $\dot{\alpha}_{n-1}$ is continuous, so as to preserve the continuity of the control and closed loop signals. As such, α_1 must be at least C^{n-1}. Due to the presence of the switching function $q(z_1)$, the stabilizing function α_1 in (49) is designed to contain the mth power of z_1, where $m \geq \max\{3, n\}$, so as to ensure that its derivatives $\alpha_1^{(1)}, \ldots, \alpha_1^{(n-1)}$, which are used in the design of the control law, are continuous, as will be shown in Lemma 3.

In Step 2 of backstepping, we have seen that α_2 also contains the switching function $q(z_1)$. As a result, it is essential that the associated z_1 term has an order of at least $n - 1$ to ensure that α_2 is C^{n-2}. If $p = 2$, then the resulting stabilizing function

$$\alpha_2 = \frac{1}{g_2}\left[-f_2 - \kappa_2 z_2 + \dot{\alpha}_1 - \left(\frac{q}{k_{b_1}^2 - z_1^2} + \frac{1-q}{k_{a_1}^2 - z_1^2}\right) g_1 z_1\right]$$

is not even C^1. However, with $p \geq n$, α_2 is at least C^{n-2} in the interval $z_1 \in (-k_{a_1}, k_{b_1})$, as will be shown shortly. The remaining stabilizing functions $\alpha_3, \ldots, \alpha_{n-1}$ are in standard form as derived from backstepping, and are C^{n-i} provided that α_1 and α_2 are, respectively, C^{n-1} and C^{n-2} in $z_1 \in (-k_{a_1}, k_{b_1})$. The following lemma provides a formal treatment of this point.

Lemma 3. *Each stabilizing function $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$, $i = 1, \ldots, n-1$, as described in (49), (53) and (54), is at least C^{n-i} in the set $z_1 \in (-k_{a_1}, k_{b_1})$.*

Proof. First, we establish that α_1 and α_2 are, respectively, at least C^{n-1} and C^{n-2} in $z_1 \in (-k_{a_1}, k_{b_1})$. Then, these 2 facts imply that α_i is at least C^{n-i} in $z_1 \in (-k_{a_1}, k_{b_1})$. In the rest of the proof, it is understood that C^{n-i} refers to C^{n-i} in $z_1 \in (-k_{a_1}, k_{b_1})$.

To prove that $\alpha_1(x_1, z_1, \dot{y}_d)$ is C^{n-1}, we need to prove that the $(n-1)$th order partial derivatives exist, and are continuous. Note that (49) can be split into 3 parts as follows

$$\alpha_1(x_1, z_1, \dot{y}_d) = \alpha_{1,a}(x_1) + \alpha_{1,b_1}(x_1)\alpha_{1,b_2}(z_1) + \alpha_{1,c}(\dot{y}_d) \quad (59)$$

where $\alpha_{1,a} := -f_1/g_1$, $\alpha_{1,b_1} := -\kappa_1/g_1$, $\alpha_{1,c} := \dot{y}_d/g_1$, and $\alpha_{1,b_2} := [q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p)]z_1^m$. Since $\alpha_{1,a}(x_1)$, $\alpha_{1,b_1}(x_1)$ and $\alpha_{1,c}(\dot{y}_d)$ are obviously C^{n-1} functions, our task is reduced to proving that $\alpha_{1,b_2}(z_1)$ is C^{n-1}. To this end, we note that

$$\alpha_{1,b_2} = \begin{cases} (k_{b_1}^p - z_1^p) z_1^m, & 0 < z_1 < k_{b_1} \\ (k_{a_1}^p - z_1^p) z_1^m, & -k_{a_1} < z_1 \leq 0. \end{cases} \quad (60)$$

The function α_{1,b_2} is piecewise C^{n-1} with respect to z_1 over the two intervals $z_1 \in (-k_{a_1}, 0)$ and $z_1 \in (0, k_{b_1})$. Thus, to show it is C^{m-1} for $-k_{a_1} < z_1 < k_{b_1}$, we need only to show that

$$\lim_{z_1 \to 0^+} \frac{\mathrm{d}^{m-1}\alpha_{1,b_2}}{\mathrm{d}z_1^{m-1}} = \lim_{z_1 \to 0^-} \frac{\mathrm{d}^{m-1}\alpha_{1,b_2}}{\mathrm{d}z_1^{m-1}}. \quad (61)$$

To achieve this, we express the piecewise derivative of (60) in the following form:

$$\frac{\mathrm{d}^{m-1}\alpha_{1,b_2}}{\mathrm{d}z_1^{m-1}} = \begin{cases} \left(m! k_{b_1} - \dfrac{(p+m)!}{(p+1)!} z_1^p\right) z_1, & 0 < z_1 < k_{b_1} \\ \left(m! k_{a_1} - \dfrac{(p+m)!}{(p+1)!} z_1^p\right) z_1, & -k_{a_1} < z_1 \leq 0 \end{cases}$$

where "!" denotes the factorial operator. Then, by inspection of the above, it is clear that the equality (61) holds, and thus, $\alpha_{1,b_2}(z_1)$ is C^{m-1}. Based on the structure of $\alpha_1(x_1, z_1, \dot{y}_d)$ in (59), and with $m \geq n$, it follows that $\alpha_1(x_1, z_1, \dot{y}_d)$ is at least C^{n-1}.

Following a similar approach as the above, we can show that the term $(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}) g_1 z_1^{p-1}$ from (53) is C^{p-2} in the interval $z_1 \in (-k_{a_1}, k_{b_1})$, by analyzing the limits of the derivative from both sides of 0. Due to the fact that α_1 is C^{n-1}, as established above, it follows that $\dot{\alpha}_1(\bar{x}_2, \bar{z}_2, \bar{y}_{d_2})$ is C^{n-2}. Furthermore, $p \geq n$ implies that α_2 is C^{n-2} in the interval $z_1 \in (-k_{a_1}, k_{b_1})$.

From the remaining stabilizing functions (54), we see that α_i is C^{n-i} if $\dot{\alpha}_{i-1}(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ is C^{n-i}. Following the fact that α_2 is C^{n-2}, as established above, it can be shown that $\dot{\alpha}_2(\bar{x}_3, \bar{z}_3, \bar{y}_{d_3})$ is C^{n-3}, which further implies that α_3 is C^{n-3}. By iterating this procedure, we can eventually show that every α_i is at least C^{n-i} in $z_1 \in (-k_{a_1}, k_{b_1})$. ■

Theorem 3. *Consider the closed loop system (1), (49), (53)–(55) under Assumptions 1 and 2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : -k_{a_1} < z_1 < k_{b_1}\}$, then the following properties hold.*

(i) *The signals $z_i(t)$, $i = 1, 2, \ldots, n$, remain in the compact set*

$$\Omega_z = \{\bar{z}_n \in \mathbb{R}^n : -\underline{D}_{z_1} \le z_1 \le \overline{D}_{z_1}, \|z_{2:n}\| \le \sqrt{2V_n(0)}\}$$

$$\overline{D}_{z_1} = k_{b_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}} \tag{62}$$

$$\underline{D}_{z_1} = k_{a_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}} \tag{63}$$

where V_n is the overall Lyapunov function candidate obtained from (45) and (52).

(ii) *The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : -k_{c_1} < -\underline{D}_{z_1} - \underline{Y}_0 \le y \le \overline{D}_{z_1} + \overline{Y}_0 < k_{c_1}\}$ $\forall t \ge 0$, i.e. the output constraint is never violated.*

(iii) *All closed loop signals are bounded.*

(iv) *the output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

Proof. (i) From the result $\dot{V}_n \le 0$, it follows that $V_n(t) \le V_n(0)$. For $-k_{a_1} < z_1(0) < k_{b_1}$, we have, from Lemma 1, that $-k_{a_1} < z_1(t) < k_{b_1} \forall t > 0$. These imply that

$$V_n(0) \ge \begin{cases} \frac{1}{p}\log\frac{k_{b_1}^p}{k_{b_1}^p - z_1^p(t)}, & 0 < z_1(t) < k_{b_1} \\ \frac{1}{p}\log\frac{k_{a_1}^p}{k_{a_1}^p - z_1^p(t)}, & -k_{a_1} < z_1(t) \le 0. \end{cases}$$

Taking exponentials on both sides of the inequality, we can rearrange the above inequality to yield

$$z_1^p(t) \le \begin{cases} k_{b_1}^p(1 - e^{-pV_n(0)}), & 0 < z_1(t) < k_{b_1} \\ k_{a_1}^p(1 - e^{-pV_n(0)}), & -k_{a_1} < z_1(t) \le 0. \end{cases}$$

By taking the pth root on both sides of the inequality, we obtain that $z_1(t) \le k_{b_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}}$ for positive $z_1(t)$, and that $z_1(t) \ge -k_{a_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}}$ for negative $z_1(t)$. Combining both cases, it is obvious that $-\underline{D}_{z_1} \le z_1(t) \le \overline{D}_{z_1} \forall t \ge 0$.

Similarly, from the fact that $\frac{1}{2}\sum_{j=2}^n z_j^2 \le V_n(0)$, we can easily show that $\|z_{2:n}\| \le \sqrt{2V_n(0)}$. Therefore, we obtain that $z_i(t)$, $i = 1, 2, \ldots, n$, remains in the compact set $\Omega_z \forall t \ge 0$.

(ii) From $y(t) = z_1(t) + y_d(t)$, $-\underline{D}_{z_1} \le z_1(t) \le \overline{D}_{z_1}$, and $-\underline{Y}_0 \le y_d(t) \le \overline{Y}_0$, it can be shown that

$$-\underline{D}_{z_1} - \underline{Y}_0 \le y(t) \le \overline{D}_{z_1} + \overline{Y}_0. \tag{64}$$

Since $\underline{D}_{z_1} < k_{a_1}$ and $\overline{D}_{z_1} < k_{b_1}$, we know that

$$\overline{D}_{z_1} + \overline{Y}_0 < k_{b_1} + \overline{Y}_0 = k_{c_1}$$
$$\underline{D}_{z_1} + \underline{Y}_0 < k_{a_1} + \underline{Y}_0 = k_{c_1}. \tag{65}$$

Hence, we can conclude that $y(t) \in \Omega_y \forall t \ge 0$.

(iii) We follow a similar approach of signal chasing as described in Theorem 1. The difference in analysis is that the stabilizing functions α_i, $i = 1, \ldots, n-1$ are now C^{n-i} instead of C^∞, for $z_1 \in (-k_{a_1}, k_{b_1})$.

We have shown in (i) that $\bar{z}_n(t) \in \Omega_z$, and in (ii) that $y(t) \in \Omega_y$. Thus, from (49), $\alpha_1(t)$ is bounded, which, in turn, ensures that $x_2(t)$ is bounded. Based on Lemma 3, α_1 is a C^{n-1} function of $x_1, z_1, \dot{y}_d$, and $\dot{\alpha}_1$ is a C^{n-2} function of $\bar{x}_2, \bar{z}_2, \bar{y}_{d_2}$. Then, the boundedness of $\bar{x}_2(t), \bar{z}_2(t), \bar{y}_{d2}(t)$, particularly with $z_1(t) \in (-k_{a_1}, k_{b_1}) \forall t \ge 0$, implies that $\dot{\alpha}_1$ is bounded.

For $i = 2, \ldots, n-1$, since $\dot{\alpha}_{i-1}(t)$ is bounded, together with the fact that $z_1(t) \in (-k_{a_1}, k_{b_1})$ $\forall t \ge 0$, we can conclude that $\alpha_i(t)$ from (54) is also bounded, which, in turn, implies the boundedness of $x_{i+1}(t)$. From Lemma 3, α_i is a C^{n-i} function of $\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}$ in the set $z_1 \in (-k_{a_1}, k_{b_1})$, and $\dot{\alpha}_i$ is a C^{n-i-1} function of $\bar{x}_{i+1}, \bar{z}_{i+1}, \bar{y}_{d_{i+1}}$ in the set $z_1 \in (-k_{a_1}, k_{b_1})$. Then, the boundedness of $\bar{x}_i(t), \bar{z}_i(t), \bar{y}_{d_i}(t)$, particularly with $z_1(t) \in (-k_{a_1}, k_{b_1}) \forall t \ge 0$, implies that $\dot{\alpha}_i(t)$ is bounded. Following this line of argument, it is straightforward to show the boundedness of the states $x_1(t), \ldots, x_n(t)$, stabilizing functions $\alpha_1(t), \ldots, \alpha_{n-1}(t)$, and control $u(t)$. Hence, all closed loop signals are bounded.

(iv) Similar to the proof of Theorem 1(iv), we show that $\ddot{V}_n(t)$, along (51), (56)–(58), is bounded. Then, by Barbalat's Lemma, $z_i(t) \to 0$ as $t \to \infty$, for $i = 1, \ldots, n$. ■

5.2. Uncertain case

Using a design methodology similar to that in Section 5.1, the results for the uncertain case can be derived. Since the corresponding proofs follow the same lines of argument from the preceding Theorems 2 and 3, they are omitted. The adaptive control is designed as:

$$\alpha_1 = \frac{1}{g_1}[-\hat{\theta}^T w_1 + \dot{y}_d - (q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p))\kappa_1 z_1^m] \tag{66}$$

$$\alpha_2 = \frac{1}{g_2}\left[-\hat{\theta}^T w_2 - \kappa_2 z_2 - \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) g_1 z_1^{p-1} + \frac{\partial\alpha_1}{\partial x_1} g_1 x_2 + \sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\tau_2\right] \tag{67}$$

$$\alpha_i = \frac{1}{g_i}\left[-\hat{\theta}^T w_i - \kappa_i z_i - g_{i-1} z_{i-1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{x_j} g_j x_{j+1} + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{j=2}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma w_i z_j\right]$$
$$(i = 3, \ldots, n-1) \tag{68}$$

$$w_1 = \psi_1(x_1), \qquad \tau_1 = \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) w_1 z_1$$

$$w_i = \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j(\bar{x}_j), \quad i = 2, \ldots, n$$

$$\tau_i = \tau_{i-1} + w_i z_i, \quad i = 2, \ldots, n$$

$$u = \alpha_n, \qquad \dot{\hat{\theta}} = \Gamma\tau_n. \tag{69}$$

Theorem 4. *Consider closed loop system (1), (66)–(69) under Assumptions 1 and 2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : -k_{a_1} < z_1 < k_{b_1}\}$, then the following properties hold.*

(i) *The signals $z_i(t)$, $i = 1, 2, \ldots, n$, and $\hat{\theta}(t)$ remain in the compact sets defined by*

$$\Omega_z = \{\bar{z}_n \in \mathbb{R}^n : -\underline{D}_{z_1} \le z_1 \le \overline{D}_{z_1}, \|z_{2:n}\| \le \sqrt{2\bar{V}_n}\}$$

$$\Omega_{\hat{\theta}} = \left\{\hat{\theta} \in \mathbb{R}^l : \|\hat{\theta}\| \le \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}}\right\}$$

$$\overline{D}_{z_1} = k_{b_1}(1 - e^{-p\bar{V}_n})^{\frac{1}{p}}$$

$$\underline{D}_{z_1} = k_{a_1}(1 - e^{-p\bar{V}_n})^{\frac{1}{p}}$$

$$\bar{V}_n = \frac{q(z_1(0))}{p}\log\frac{k_{b_1}^p}{k_{b_1}^p - z_1^p(0)} + \frac{1-q(z_1(0))}{p}\log\frac{k_{a_1}^p}{k_{a_1}^p - z_1^p(0)} + \frac{1}{2}\sum_{j=2}^{n} z_j^2(0) + \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2.$$

(ii) *The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : -k_{c_1} < -\underline{D}_{z_1} - \underline{Y}_0 \le y \le \overline{D}_{z_1} + \overline{Y}_0 < k_{c_1}\}$ $\forall t \ge 0$, i.e. the output constraint is never violated.*
(iii) *All closed loop signals are bounded.*
(iv) *The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

Remark 3. In this paper, we focus on the output constraint problem without explicit consideration of input saturation. The reason is that provision for a potentially large control effort is key to safeguarding against any constraint transgression. This stems from the use of Barrier Lyapunov Functions that grow rapidly when the states approach the boundaries of the constrained region. Nevertheless, the control signal remains bounded for all time. By careful selection of control parameters, we can limit the control signal within a desirable operating range.

6. Comparison with Quadratic Lyapunov Functions

If the initial conditions belong to certain sets, it is possible for backstepping control based on Quadratic Lyapunov Functions to ensure that the output does not violate its constraint. The question that naturally arises is whether or not these initial condition requirements are more relaxed than those arising from the use of Barrier Lyapunov Functions?

For the known system (1), consider the Quadratic Lyapunov Function candidates:

$$V_1 = \frac{1}{2}z_1^2, \qquad V_i = V_{i-1} + \frac{1}{2}z_i^2, \quad i = 2, \ldots, n \tag{70}$$

and the standard backstepping control laws

$$\alpha_1 = \frac{1}{g_1}(-f_1 - \kappa_1 z_1 + \dot{y}_d)$$
$$\alpha_i = \frac{1}{g_i}(-f_i - \kappa_i z_i - g_{i-1}z_{i-1} + \dot{\alpha}_{i-1}), \quad i = 2, \ldots, n$$
$$u = \alpha_n \tag{71}$$

where $\kappa_1, \ldots, \kappa_n$ are positive constants. It can be shown that $\dot{V}_n \le -\rho V_n$, where $\rho = 2\min\{\kappa_1, \ldots, \kappa_n\}$, which leads to the fact that $|z_i(t)| \le \|\bar{z}_n(0)\|$ for $t \ge 0$. A sufficient condition to ensure that $|z_1(t)| < k_{b_1}$ is that $\bar{z}_n(0) \in \Omega_0$, where

$$\Omega_0 = \{\bar{z}_n \in \mathbb{R}^n : \|\bar{z}_n\| < k_{b_1}\}. \tag{72}$$

This is more restrictive than the condition $|z_1(0)| < k_{b_1}$ required when using Barrier Lyapunov Function.

In the presence of parametric uncertainty, consider the following augmented Lyapunov function candidates:

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}_1^T \Gamma_1^{-1}\tilde{\theta}_1$$

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i, \quad i = 2, \ldots, n \tag{73}$$

where $\theta_i = \theta$, $\Gamma_i = \Gamma_i^T > 0$, and $\tilde{\theta}_i := \hat{\theta}_i - \theta$ is the error between θ and the estimate $\hat{\theta}_i$.

Based on standard adaptive backstepping control (Krstic et al., 1995), it can be shown that $\dot{V}_n = -\sum_{i=1}^n \kappa_i z_i^2$, from which we know that $V_n(t) \le V_n(0 \le \bar{V}_n)$, where $\bar{V}_n$ is the upper bound for the initial value of the Lyapunov function, defined by

$$\bar{V}_n := \frac{1}{2}\sum_{i=1}^{n}[z_i^2(0) + \lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}_i(0)\| + \theta_M)^2].$$

This yields $|z_i(t)| \le \sqrt{2\bar{V}_n}$. A sufficient condition to ensure that $|z_1(t)| < k_{b_1}$ is that $\sqrt{2\bar{V}_n} < k_{b_1}$, which leads to

$$\|\bar{z}_n(0)\| < \sqrt{k_{b_1}^2 - \lambda_{\max}(\Gamma^{-1})\sum_{i=1}^{n}(\|\hat{\theta}_i(0)\| + \theta_M)^2}. \tag{74}$$

Note that the additional condition

$$k_{b_1}^2 > \lambda_{\max}(\Gamma^{-1})\sum_{i=1}^{n}(\|\hat{\theta}_i(0)\| + \theta_M)^2 \tag{75}$$

needs to be satisfied. Again, these conditions are more restrictive than $|z_1(0)| < k_{b_1}$ arising from the use of a Barrier Lyapunov Function.

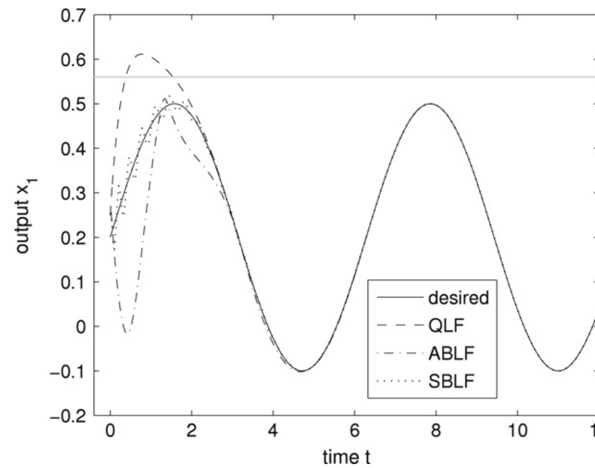

Fig. 2. Output tracking for different controllers based on QLF, SBLF, and ABLF.

7. Numerical example

Consider the second-order nonlinear system

$$\dot{x}_1 = \theta_1 x_1^2 + x_2$$
$$\dot{x}_2 = \theta_2 x_1 x_2 + \theta_3 x_1 + (1 + x_1^2)u$$

where $\theta_1 = 0.1$, $\theta_2 = 0.1$, and $\theta_3 = -0.2$. The objective is for x_1 to track the desired trajectory $y_d = 0.2 + 0.3\sin t$, subject to the output constraint $|x_1| < k_{c_1} = 0.56$. The initial conditions are $x_1(0) = 0.25$ and $x_2(0) = 1.5$, and the control gains $\kappa_1 = \kappa_2 = 2.0$. Since $-0.1 \le y_d \le 0.5$, we have, from (47), that $k_{b_1} = 0.56 - 0.5 = 0.06$, and $k_{a_1} = 0.56 - 0.1 = 0.46$.

From Fig. 2, it can be seen that asymptotic tracking performance is achieved. The output $x_1(t)$ stays strictly within the set $|x_1| < 0.56$ when the Symmetric Barrier Lyapunov Function (SBLF) and the Asymmetric Barrier Lyapunov Function (ABLF) are used. That $|x_1(t)| < 0.56$ is ensured by the fact that the tracking error $z_1(t)$ remains in the set $z_1 \in (-0.06, 0.06)$ when the SBLF is used, and the set $z_1 \in (-0.46, 0.06)$ when the ABLF is used. However, when the Quadratic Lyapunov Function (QLF) is used under the same initial conditions, the output constraint is violated.

The phase portraits of $z_1(t)$ and $z_2(t)$ are shown in Figs. 3 and 4. The error $z_1(t)$ does not transgress its barriers as long as its initial value satisfies $|z_1(0)| < 0.06$ when the SBLF is used, or $-0.46 < z_1(0) < 0.06$ when the ABLF is used. In other words, the

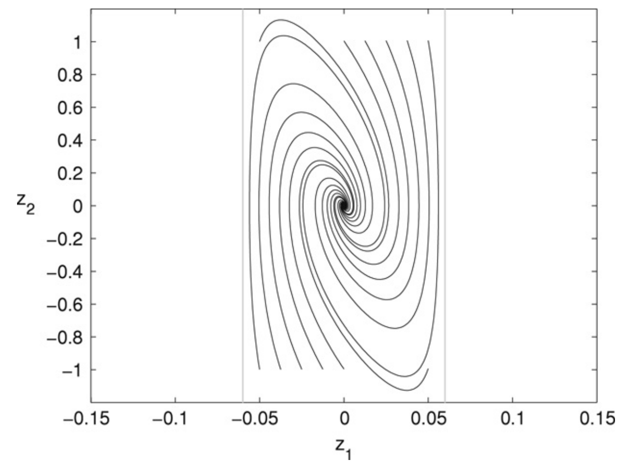

Fig. 3. Phase portrait of z_1, z_2 for the closed loop system when SBLF is used.

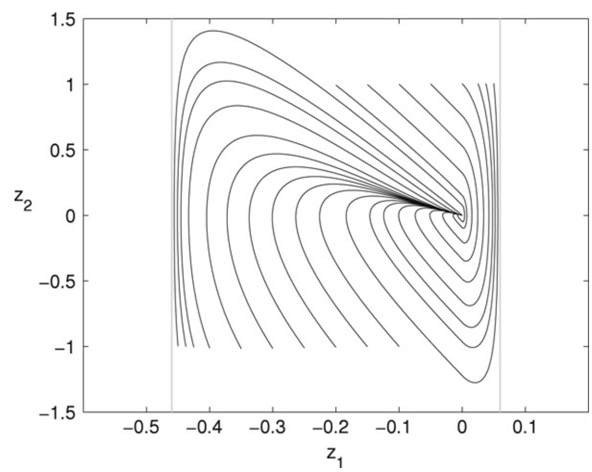

Fig. 4. Phase portrait of z_1 and z_2 for the closed loop system when ABLF is used.

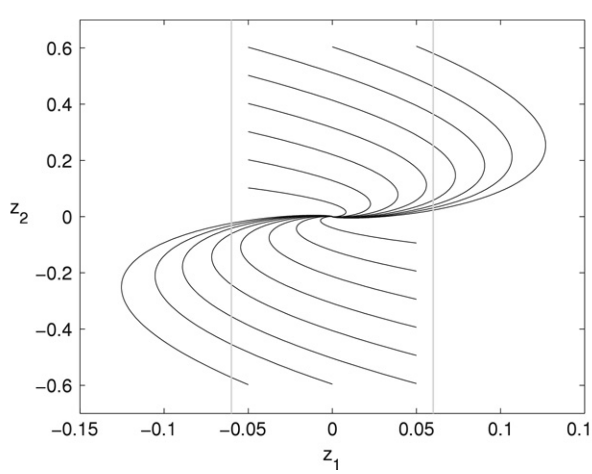

Fig. 5. Phase portrait of z_1 and z_2 for the closed loop system when QLF is used.

region between the barriers is positively invariant. In contrast, with the QLF, the region $|z_1(0)| < 0.06$ is not positively invariant, as witnessed in Fig. 5. Even though all these cases exhibit convergence of $(z_1(t), z_2(t))$ to 0, the set of admissible initial values of (z_1, z_2) that guarantees output constraint satisfaction is largest for the ABLF, followed by the SBLF, and finally the QLF.

With various control gains κ_1 and κ_2, the error z_1 does not transgress its barriers, as seen in Figs. 6 and 7. As the control gain increases, the tracking error $z_1(t)$ converges to 0 at a faster

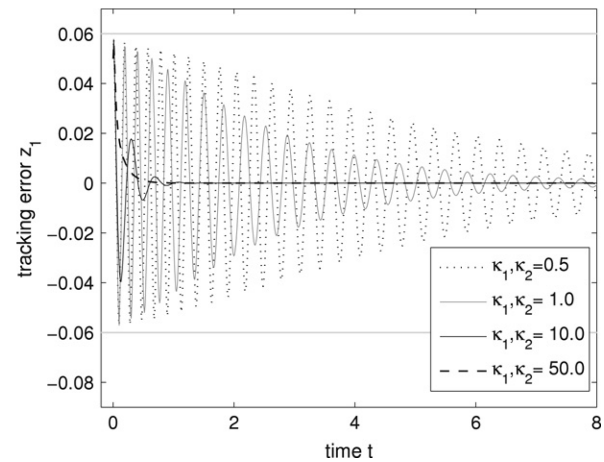

Fig. 6. Tracking error z_1 for various values of control parameters κ_1 and κ_2, when SBLF is used.

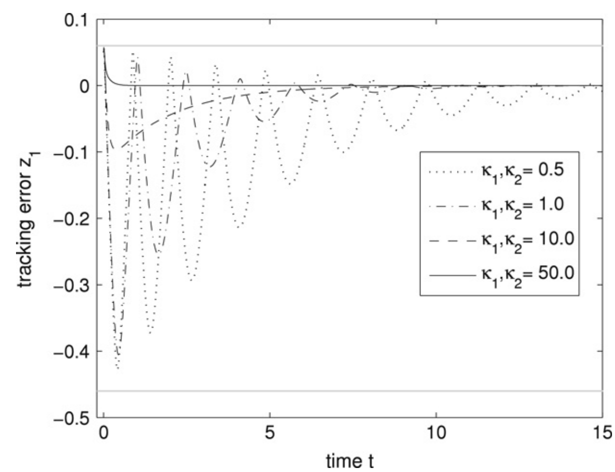

Fig. 7. Tracking error z_1 for various values of control parameters κ_1 and κ_2, when ABLF is used.

rate and with less oscillations. An interesting contrast between the ABLF and the SBLF is that, in the former, $z_1(t)$ exhibits asymmetric behavior, with a greater tendency to be negative, while in the latter, $z_1(t)$ has a symmetric behavior.

To gain some insights on how the SBLF-based control operates in keeping the output constrained, we observe, from the control law (13), that the nonlinear gain term $g_1 z_1/(k_{b_1}^2 - z_1^2)$ is responsible for ensuring that the constraint on the output is satisfied. Whenever $z_1(t)$ approaches the barriers at $z_1 = \pm 0.06$, the gain term grows rapidly and provides a large control action that repels $z_1(t)$ from the barriers. This effect is observed in Fig. 8, where the control input $u(t)$, based on the SBLF, peaks when the tracking error $z_1(t) \to \pm 0.06$. Similarly, the ABLF-based control pulls $z_1(t)$ away from the barriers with a control input $u(t)$ that grows rapidly when $z_1(t) \to 0.06$ or $z_1(t) \to -0.46$, as seen in Fig. 9. Interestingly, the negative peaks in $u(t)$, corresponding to $z_1(t) \to 0.06$, are larger than the positive peaks that correspond to $z_1(t) \to -0.46$. This is due to the fact that, with a smaller allowable positive range for $z_1(t)$, the control $u(t)$ needs to grow at a faster rate to ensure that the barrier $z_1 = 0.06$ is not reached. In avoiding the barriers in the z_1 dimension, the control action can cause large excursions in the z_2 dimension, as seen in Figs. 3 and 4 for SBLF and ABLF respectively.

8. Conclusions

In this paper, we have presented the control design for strict feedback systems with an output constraint, based on the use

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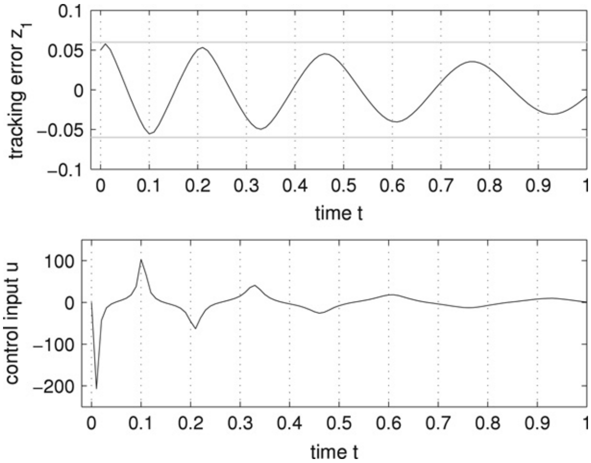

Fig. 8. Control input u when SBLF is used.

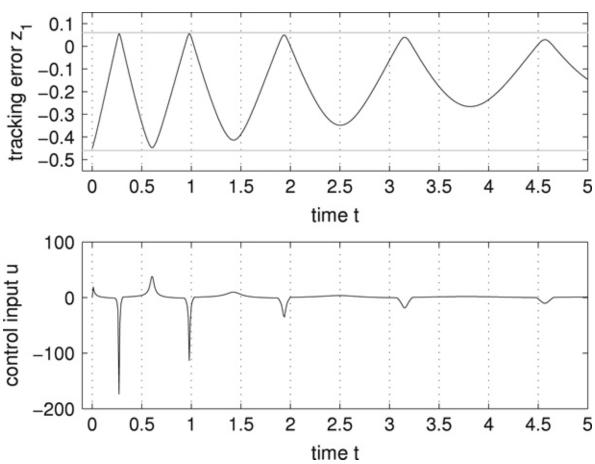

Fig. 9. Control input u when ABLF is used.

of Barrier Lyapunov Functions. We have shown that asymptotic tracking is achieved without violation of the constraint, and that all closed loop signals remain bounded, under a mild requirement on the initial conditions. Furthermore, we have explored the use of an Asymmetric Barrier Lyapunov Function, which provides greater design flexibility and relaxes the requirement on the initial conditions. The use of Quadratic Lyapunov Functions in handling output constraint has also been investigated, and our study suggests that the initial conditions are more restrictive. Finally, the performance of the proposed control has been illustrated through a numerical example.

References

Allgöwer, F., Findeisen, R., & Ebenbauer, C. (2003). Nonlinear model predictive control. *Encyclopedia of Life Support Systems (EOLSS)*, article contribution 6.43.16.2.

Bemporad, A. (1998). Reference governor for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, *43*(3), 415–419.

Farrell, J., & Polycarpou, M. M. (2006). *Adaptive approximation based control: Unifying neural, fuzzy and traditional adaptive approximation approaches*. John Wiley.

Ge, S. S., Hang, C. C., Lee, T. H., & Zhang, T. (2001). *Stable adaptive neural network control*. Boston, MA: Kluwer Academic.

Ge, S. S., Hang, C. C., & Zhang, T. (1999a). A direct adaptive controller for dynamic systems with a class of nonlinear parameterizations. *Automatica*, *35*, 741–747.

Ge, S. S., Hang, C. C., & Zhang, T. (1999b). A direct method for robust adaptive nonlinear control with guaranteed transient performance. *Systems and Control Letters*, *37*, 275–284.

Ge, S. S., Lee, T. H., & Harris, C. J. (1998). *Adaptive neural network control of robotic manipulators*. London: World Scientific.

Gilbert, E. G., & Kolmanovsky, I. (2002). Nonlinear tracking control in the presence of state and control constraints: A generalized reference governor. *Automatica*, *38*, 2063–2073.

Hu, T., & Lin, Z. (2001). *Control systems with actuator saturation: Analysis and design*. Boston, MA: Birkhäuser.

Krantz, S. G., & Parks, H. R. (1999). *The geometry of domains in space*. Boston: Birkhäuser.

Krstic, M., & Bement, M. (2006). Nonovershooting control of strict-feedback nonlinear systems. *IEEE Transactions on Automatic Control*, *51*(12), 1938–1943.

Krstic, M., Kanellakopoulos, I., & Kokotovic, P. V. (1995). *Nonlinear and adaptive control design*. New York: Wiley and Sons.

Lewis, F. L., Jagannathan, S., & Yesildirek, A. (1999). *Neural network control of robot manipulators and nonlinear systems*. London: Taylor & Francis.

Li, Z. H., & Krstic, M. (1997). Maximizing regions of attraction via backstepping and CLFs with singularities. *Systems & Control Letters*, *30*, 195–207.

Liu, D., & Michel, A. N. (1994). *Dynamical systems with saturation nonlinearities*. London, UK: Springer-Verlag.

Marino, R., & Tomei, P. (1995). *Nonlinear adaptive design: Geometric, adaptive, and robust*. London: Prentice Hall International (UK) Limited.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, *36*, 789–814.

Ngo, K.B., Mahony, R., & Jiang, Z.P. (2005). Integrator backstepping using barrier functions for systems with multiple state constraints. In *Proc. 44th IEEE conf. decision & control* (pp. 8306–8312).

Ortega, R., Schaft, A., van der Mareels, I., & Maschke, B. (2001). Putting energy back in control. *IEEE Control Systems Magazine*, *21*(2), 18–33.

Slotine, J. E., & Li, W. (1991). *Applied nonlinear control*. Englewood Cliff, NJ: Prentice-Hall.

Sontag, E. D. (1998). Mathematical control theory. In *Texts in Applied Mathematics*: *Vol. 6*. *Deterministic finite-dimensional systems* (2nd ed.). New York: Springer-Verlag.

Tee, K.P., Ge, S.S., & Tay, E.H. (2009). Adaptive control of electrostatic microactuators with bi-directional drive. *IEEE Transactions on Control Systems Technology*, in press (doi:10.1109/TCST.2008.2000981).

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