Constrained Optimal Coverage Control of Multi-Unicycle Systems

Michael Shell, *Member, IEEE,* John Doe, *Fellow, OSA,* and Jane Doe, *Life Fellow, IEEE*

***Abstract*—The abstract goes here.**

***Index Terms*—IEEE, IEEEtran, journal, LaTeX, paper, template.**

I. Introduction

VORONOI coverage control is a particular problem of importance [1]. The problem considers a network of multiple autonomous robots, tasked with optimally covering a large area. For example, events of interest may occur randomly in the area, and the robots are equipped with sensors and deployed in a spatial pattern as to provide optimal sensing of the events occurring.

It is not difficult to appreciate that a coordinated group of mobile robots can provide coverage over a large area better than a single complex robot. Applications of coverage control involving multiple networked mobile robots include search and rescue operations, surveillance, environmental monitoring and exploration of hazardous/inaccessible regions. In many of these applications, an appropriate choice of mobile robot is a *fixed-wing* Unmanned Aerial Vehicle (UAV). Fixed-wing UAVs can have great endurance in both flight time and range, stay in operation for a number of hours, and can be equipped with a range of different sensors. These are all of benefit to the surveillance, monitoring and search applications discussed above. In contrast, quadrotor UAVs are severely limited in terms of the time it can remain airborne, its range, and its sensor payload. Moreover, fixed-wing UAVs can operate at a height which provides better line-of-sight over tall terrain or obstacles when compared to ground-based robotic vehicles.

Under the assumption that each robot is mobile, then the dynamic model of the robots must be considered when designing a distributed control algorithm to solve the coverage control problem. The pioneering work [1] considered integrator-type dynamics. Similar dynamic concerns can also be found in [2]. However, if the robots are with complex dynamics, e.g. fixed-wing UAVs as motivated above, it may not be sufficient to use integrator-type models to capture the dynamics. Along this streamline, our previous work [3] studies the problem of optimal coverage control over a convex bounded region by a group of unicycle-type mobile robots with constant cruising speeds. Two controllers were designed to asymptotically drive the virtual rotation centres to a Centroidal Voronoi configuration, thus achieving the optimal coverage objective. At the Centroidal Voronoi configuration, each unicycle executes steady-state circular orbit about its virtual centre.

Motivated from the above-mentioned control policy, the main contribution of this study is the consideration of the state and input constraints from a practical aspect. Constraints are ubiquitous in real life application. For instance, fixed-wing UAV operate with a constant heading velocity is energy-efficient, or the input saturation constraints since vehicle's velocity are bounded. Furthermore, operation under a specific formulation requires agents to be responsible for their position, which is considered as a state constraint. The highlight of this study is the design of a controller that considers multiple constraints, while ensures the success of the coverage problem.

The paper is organized as follows. Section II introduces the mathematical notations used in this paper, the problem of coverage control, system dynamics and the problem statements of this study. Section III provides the solution based on the theorem of Barrier-Lyapunov function that can handle both input and state constraints. Simulation results and the evaluation are presented in section IV. Section V concludes the study.

II. Preliminaries

In this section, we introduce the essential mathematical concepts and tools used in this paper. The agents are represented as unicycle models, featured with constant linear velocity and controllable steering angles. The optimal coverage of the agents over a convex region is depicted by a certain Voronoi Configuration. The communication topology among the agents is modeled by Graph Theory. The methods of Barrier Lyapunov Functions (BLF) are used to confine the virtual centers of the agents within the confined covering region.

A. Mathematical notation

$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of non-negative real numbers
$\mathbb{C}$	Set of complex numbers
$i=\sqrt{-1}$	The imaginary unit
$\Re(z)$	Real part of $z\in\mathbb{C}$
$\Im(z)$	Imaginary part of $z\in\mathbb{C}$
$\bar{z}$	The complex conjugate of $z\in\mathbb{C}$
$\langle z_1,z_2\rangle=\Re(\bar{z}_1 z_2)$	Scalar product of $z_1,z_2\in\mathbb{C}$

B. Optimal Coverage and Voronoi Configuration

In coverage control, a group of agent is deployed to monitor a predefined region. Let there be a set $Z:\{z_k\in\mathbb{R}^2,$

M. Shell was with the Department of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA, 30332 USA e-mail: (see http://www.michaelshell.org/contact.html).

J. Doe and J. Doe are with Anonymous University.

Manuscript received April 19, 2005; revised August 26, 2015.

$k \in \mathcal{N} = \{1, 2, \cdots, N\}\}$ that indicates agents' position. The bounded convex polygonal coverage region is defined as $\Omega \in \mathbb{R}^2$, which is be divided into sub-regions $\{\Omega_k | \Omega = \bigcup_{k=1}^{n} \Omega_k,\ k \in \{1, 2, \cdots, N\}\}$. Each agent z_k is responsible for events occurred in their own partition Ω_k, therefore the sensing cost function of one agent is determined as

$$H_k(z_k, \Omega_k) = \int_{\Omega_k} f(q, z_k)\Phi(q)\mathrm{d}q$$

Here, q denotes the events inside the coverage region, which has the sensing expense $f(q, z_k)$ and the density function $\phi(q)$. Motivated from the working principle of multiple sensors, the sensing cost $f(q, z_k)$ can be defined as $\|q - z_k\|^2$, which is strictly increasing, proportional to the sensing range. $\phi(q)$ is the distribution density function used as a weight factor of events. For instance, during an operation to monitor an area, a local region crowded by many people is considered more important, therefore the weight function for this location has higher magnitude. The total sensing cost of n agents is then defined as

$$H(Z, \Omega) = \sum_{k=1}^{n} H_k(z_k, \Omega_k) = \sum_{k=1}^{n} \int_{\Omega_k} \|q - z_k\|^2 \Phi(q)\mathrm{d}q \quad (1)$$

According to (1), for a particular definition of how the coverage region Ω is divided into n partitions Ω_k, the cost function is minimized by agents' position z_k. One meaningful definition of Ω_k is the Voronoi partition. The Voronoi partition indicates that each agent takes responsible only for the events that are nearer to its positions than to any other agents. The mathematical definition of the Voronoi partition is

$$\Omega_k = \{q \in \Omega | \|q - z_k\| \leq \|q - z_i\|,\ \forall i \neq k\}$$

From the definition of Voronoi parition, each agent shares the boundary lines of its partition with the adjacent agents. These agents are defined as the Voronoi neighbors. The events within the internal of the partition Ω_k is monitored only by agent z_k, while the events on the boundary lines are observed both by itself and its neighbors. This property shows that the sensing partitions depend only on agents' position $z_k \in Z$, e.g the partition is written as $\Omega_k(Z)$. For simplicity, each partition is written as $\Omega_k(Z)$ but it does not depend on the position of non-adjacent agent. Moreover, this implies that each agent requires only the information from its Voronoi neighbors, thus the cost function is optimized distributively.

The optimal coverage control investigates the control method that drives agents to the optimal position, which minimizes the sensing function $H(Z)$

$$\min_{z_k \in \Omega, k \in \mathcal{N}} H(Z) = \min_{z_k \in \Omega, k \in \mathcal{N}} \sum_{k=1}^{n} \int_{\Omega_k} \|q - z_k\|^2 \Phi(q)\mathrm{d}q$$

The gradient of $H(Z)$ is introduced in [2] by Schwager as

Lemma 1: (Lemma 2.1, Schwager)

$$\frac{\partial H(Z)}{\partial z_k} = \int_{\Omega_k} \frac{\partial}{\partial z_k} \|q - z_k\|^2 \Phi(q)\mathrm{d}q$$

Let the mass and the centroid of a Voronoi partition is defined as

$$\begin{aligned} M_{\Omega_k} &= \int_{\Omega_k} \Phi(q)\mathrm{d}q \\ C_k &= \frac{1}{M_{\Omega_k}} \int_{\Omega_k} q\Phi(q)\mathrm{d}q \end{aligned} \quad (2)$$

The partial derivative of $H(Z)$ is then computed from (1) as

$$\begin{aligned} \frac{\partial H(Z)}{\partial z_k} &= \int_{\Omega_k} \frac{\partial}{\partial z_k} \|q - z_k\|^2 \Phi(q)\mathrm{d}q \\ &= 2z_k \int_{\Omega_k} \Phi(q)\mathrm{d}q - 2\int_{\Omega_k} q\Phi(q)\mathrm{d}q \\ &= 2M_{\Omega_k}(z_k - C_k) \end{aligned} \quad (3)$$

The cost function $H(Z)$ has the local optimum at $z_k = C_k,\ \forall k \in \mathcal{N}$. From the introduction of Voronoi partition and its centroid in (2), each partition Ω_k and the centroid C_k depend only on Z. The local minimum of $H(Z)$ is then rewritten as

$$z_k = C_k(Z),\ \forall k \in \mathcal{N} \quad (4)$$

Let's $C(Z) = \{C_k(Z),\ \forall k \in \mathcal{N}\}$ be the set of centroidal Voronoi tessellation (CVT), the problem of coverage control is solved when agents' position converge on this set, e.g agents establish the Voronoi tessellations and be the centre of each cell at the same time.

C. The Unicycle model

A unicycle model is the prototype of a class of nonholonomic systems that are used to depict the kinematics of two-wheel ground vehicles and fixed-wing UAVs. In this paper, we consider the coverage of the convex region Q using a team of N unicycle agents. The kinematic model of each agent k, $k = 1, 2, \cdots, N$, is described by

$$\begin{aligned} \dot{x}_k &= v_k \cos(\theta_k) \\ \dot{y}_k &= v_k \sin(\theta_k) \\ \dot{\theta}_k &= u_k, \end{aligned} \quad (5)$$

where $(x_k, y_k) \in \mathbb{R}^2$ is the coordinate of agent k in Cartesian coordinate, $\theta_k \in [0, 2\pi)$ is the heading angle of agent k. With a predefined constant linear velocity $v_k \in \mathbb{R}^+$ for each agent k, (5) renders an underactuated system with the state $[x_k, y_k, \theta_k]^\top \in \mathbb{R}^3$ and the control input is the angular velocity $u_k \in \mathbb{R}$. Note that, in this paper, we regulate $u_k < 0$ as the agent k rotates in a clockwise manner, $u_k > 0$ as it acts in the anticlockwise direction, and $u_k = 0$ as the agent moves along a straight line.

For brevity, we represent the coordinate of agent i in the complex domain, i.e., $r_k = x_k + jy_k \in \mathbb{C}$. In this sense, the agent kinematics can be reformulated as

$$\begin{aligned} \dot{r}_k &= v_k e^{j\theta_k} \\ \dot{\theta}_k &= u_k. \end{aligned} \quad (6)$$

We define the virtual center of agent k, $z_k \in \mathbb{C}$, as

$$z_k = r_k + \frac{v_k}{\omega_k} j e^{j\theta_k}, \tag{7}$$

where $\omega_{k_0} \in \mathbb{R}^+$ is a constant scalar for each agent k. Taking the derivative of (??), we obtain the dynamics of the virtual center as

$$\dot{z}_k = v_k e^{j\theta_k} - \frac{v_k}{\omega_{k_0}} e^{j\theta_k} u_k, \ k = 1, 2, \cdots, N. \tag{8}$$

To achieve effective coverage of region Ω, all the agents are expected to orbit their virtual centers $z_k \in \mathbb{R}^2$, while these virtual centers are driven to converge on the set of the optimal position in the coverage region, which is the set of centroidal Voronoi tessellations.

Remark 1: Properties of the control input

- If $u_k(t) = 0$, the virtual mass is driven in a straight line.
- If $u_k(t) = \omega_{k_0}$, the agent orbits the current virtual mass. Moreover, the virtual mass maintains unchanged as $\dot{z}_k = 0$

D. Distributed Computation of Voronoi Tessellation

Agents during their operations transmit the necessary information, e.g its position, current virtual mass's position, etc. Motivated from the practical scenario, suppose that these information are transmitted radially within a predefined radius. This implies all agents, which stay inside this radius, will receive and interpret the information to determine its Voronoi tessellation as well as the true adjacent agents. Note that the term "adjacent agents" implies the neighbor agents that share the common boundary lines of the Voronoi tessellation.

The study in [4] and [5] introduces the method to compute the CVT distributively and the procedure to determine the Voronoi neighbors.

E. Lyapunov-Barrier Function

In [6], Tee introduced the theorem of Barrier Lyapunov function (BLF) as

Lemma 2: For any positive constants k_{b_i}, $i \in \{1, 2, ..., n\}$, let $Z := \{z \in \mathbb{R}^n : \|z_i\| < k_{b_i}, \ \forall i\}$ be an open set. Consider the system

$$\dot{z} = h(t, z)$$

where $h : \mathbb{R}_+ \times N \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in z, uniformly in t, on $\mathbb{R}_+ \times Z$. Suppose that there exist a positive definite function $V : Z \to \mathbb{R}_+$, continuously differentiable on Z, and they satisfy

$$V(Z) \to \infty \Leftrightarrow \exists \|z_i\| \to k_{b_i}$$

if the inequality holds:

$$\dot{V} = \frac{\partial V}{\partial z} h \leq 0$$

then $z_i(t) \in Z, \ \forall t \in [0, \infty), \forall i$

F. Problem statement

In the previous studies of coverage control [] , [] [], z_k often represents the position of system that has simple dynamic model, e.g single integrator model. To drive these systems to the optimal position, the proportional control input u_k can be obtained directly from (3), e.g

$$\dot{z}_k = u_k = -(z_k - C_k) \tag{9}$$

The time derivative of the Lyapunov-based cost function $H(Z)$ is

$$\dot{H}(Z) = \sum_{k=1}^{n} \frac{\partial H}{\partial z_k} \dot{z}_k = -2M_{\Omega_k}(z_k - C_k)^2 \leq 0$$

Furthermore, [2] shows that the proportional control input (9) ensures the asymptotic stability of this problem. However, the coverage control problem becomes more complicated under consideration of complex system, such as underactuated vehicles, e.g fixed-wing UAVs, system with saturated control input, etc. For instance, in [3], the model of unicycles that have constant heading velocity were used to execute the coverage task. Motivated by the operation of fixed wings UAV, each agent orbits a virtual centers and these virtual centers are driven to converge on the set of CVT. From the definition of the virtual centre and its dynamics in (7), (8), it can theoretically be driven to achieve the local minimum by applying the control input similar to (9). Nevertheless, the input constraints impair the stability of the coverage, i.e by introducing an input saturation, the result for the control input from (9) becomes infeasible and the negative time derivative of $H(Z)$ is no longer guaranteed. Furthermore, the control input must ensure that the virtual masses stay within the coverage region Ω, else the Voronoi paritions are not proper defined. This introduce the state constraints, that the virtual centers are not allowed to cross the boundary lines of $Omega$. This study investigates the control input that considers both the input saturation and the state constraint.

Given n uni-cycle agents, each agent has a non-identical constant translation velocity v_k. For a predefined convex region $\Omega = \{q \in \mathbb{C} | \ \langle q, a_j \rangle \leq b_j, \ \forall m\}$, where $a_j \in \mathbb{C}, \ b_j \in \mathbb{R}$, m is the amount of boundary lines that define the coverage region Ω, find the control law u_k for each agent to orbit the set of CVT, e.g there virtual masses converge on the set of CVT in Ω, while satisfy the following constraints:

- The control input must be distributed, e.g it is determined only by the information received from the neighbors within the range of communication.
- The heading velocity of each agent is constant and can be nonidentical.

$$v_k = const$$

- The angular velocity is bounded.

$$u_k(t) \in (-w_m, w_m), \ \forall k \in \mathcal{N}$$

- Agents drive the virtual masses to converge on the set of centroidal Voronoi tessellation.

$$z_k \to C_k(Z), \ \forall k \in \mathcal{N}$$

- Each virtual center is not allowed to cross the boundary lines of the dominate region Ω.

$$\langle z_k(t), a_j \rangle \leq b_j, \ \forall m, \ \forall t, \ \forall k \in \mathcal{N}$$

III. Main Results

A. Proposed Lyapunov function for the coverage problem

We introduce the Barrier Lyapunov function $V(Z) : \mathbb{C}^n \to \mathbb{R}_+$ as follows

$$V(Z) = \sum_{k=1}^{n} V_k(Z) = \sum_{k=1}^{n}\sum_{j=1}^{m} \frac{1}{2}\frac{\langle z_k - C_k(Z), z_k - C_k(Z)\rangle}{b_j - \langle z_k, a_j\rangle} \tag{10}$$

where

$$V_k(Z) = \sum_{j=1}^{m} \frac{1}{2}\frac{\langle z_k - C_k(Z), z_k - C_k(Z)\rangle}{b_j - \langle z_k, a_j\rangle}$$

$$C_k(Z) = \frac{\int_{\Omega_k(Z)} q\rho(q)dq}{\int_{\Omega_k(Z)} \rho(q)dq}$$

and $a_j \in \mathbb{C}$, $b_j \in \mathbb{R}$ are constants used to formulate the coverage region Ω defined by m boundary lines. Furthermore, each agent is responsible for their own coverage sub-region Ω_k. In other words

$$\Omega : \{q \in \mathbb{C} | \langle q, a_j \rangle \leq b_j, \forall j \in \{1, ..., m\}\} = \bigcup_{k=1}^{n} \Omega_k(Z)$$

The proposed function $V(Z)$ has the following properties:

1) $V(Z)$ is positive definite
Proof.
$$V_k(Z) \geq 0 \iff z_k \in \Omega, \ \forall k \in \{1, \cdots, n\}$$
$$V_k(Z) = 0 \iff z_k \to C_k(Z), \ \forall k \in \{1, \cdots, n\}$$
2) $V(Z)$ grows to infinity if and only if at least one agent crosses the boundary of the coverage region
Proof.
$$\begin{aligned} V(Z) \to \infty &\iff \exists k : V_k(Z) \to \infty \\ &\iff \exists k, j : b_j - \langle z_k, a_j \rangle \to 0^+ \\ &\iff \exists k : z_k \to \partial\Omega \end{aligned}$$

According to lemma (2), it is shown that $V(Z)$ is a feasible BLF candidate.

B. Control Law

The following control law ensures the state and input feasibility of this coverage problem.

$$u_k = \omega_{k_0} + \gamma_k \text{sign}(\omega_{k_0}) \frac{\left\langle \sum_{i \in \{\tilde{K} \cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, e^{i\theta_k} \right\rangle}{\left\| \left\langle \sum_{i \in \{\tilde{K} \cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, e^{i\theta_k} \right\rangle \right\|} \tag{11}$$

where

ω_{k_0} : Desired orbiting velocity
$\tilde{K}$: $\{j \in 1, \cdots, n | \Omega_k \cup \Omega_j \neq \emptyset\}$: Set of adjacent agents
γ_k : Positive control gain

By using the proposed control law in (11), the dynamic of each agent's virtual mass is described as

$$\begin{aligned} \dot{z}_k &= v_k e^{i\theta_k} - \frac{v_k}{w_{k_0}} e^{i\theta_k} u_k \\ &= -\frac{\gamma_k v_k}{\|\omega_{k_0}\|} \frac{\left\langle \sum_{i \in \{\tilde{K} \cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, e^{i\theta_k} \right\rangle}{\left\| \left\langle \sum_{i \in \{\tilde{K} \cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, e^{i\theta_k} \right\rangle \right\|} e^{i\theta_k} \end{aligned} \tag{12}$$

The proposed control law u_k has two essential characteristics.

1) u_k is distributed, e.g agent k requires only the information from its adjacent agents
Proof.
The control law (11) requires each agent the term $\frac{\partial V_i}{\partial z_k}$ from its adjacent agent. The computation of this term is complicated due to the term $\frac{\partial C_i}{\partial z_k}$, where $C_i(Z)$ is the centroid of the adjacent agent i's tessellation. As can be seen from appendix [A], the region Ω_i is monitored by agent i and is defined only by the virtual masses z_i and the virtual masses of its adjacent agents z_k. Therefore, the partial derivative only depends on these variables. In other words,
$$\frac{\partial V_i}{\partial z_k} = \begin{cases} \frac{\partial V_k}{\partial V_k} & i = k \\ \frac{\partial V_i}{\partial z_k} & i \in \tilde{K} \\ 0 & i \notin \tilde{K} \end{cases} \tag{13}$$
Since agent i is able to compute the term $\frac{\partial V_i}{\partial z_k}$ from its adjacent agents, the result will then be transmitted to agent k so that agent k can compute its desired control input. This implies that each agent does not require any further information from their non-adjacent agents. The control policy is concluded to be distributed.
2) u_k ensures the stability of the coverage problem
Proof.
$$\begin{aligned} \dot{V}(Z) &= \sum_{k=1}^{n} \dot{V}_k(Z) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} \left\langle \frac{\partial V_k(Z)}{\partial z_i}, \dot{z}_i \right\rangle \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} \left\langle \frac{\partial V_i(Z)}{\partial z_k}, \dot{z}_k \right\rangle \\ &= \sum_{k=1}^{n} \left\langle \sum_{i=1}^{n} \frac{\partial V_i(Z)}{\partial z_k}, \dot{z}_k \right\rangle \end{aligned}$$
From (A), for $i \notin \tilde{K}$
$$\frac{\partial V_i(Z)}{\partial z_k} = 0$$
Therefore
$$\dot{V}(Z) = \sum_{k=1}^{n} \left\langle \sum_{i \in \{\tilde{K} \cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, \dot{z}_k \right\rangle$$

Substitute $\dot{z}_k$ from (12), we have

$$\dot{V}(Z) = -\sum_{k=1}^{n} \frac{\gamma_k v_k}{\|\omega_{k_0}\|} \frac{\left\langle \sum_{i\in\{\tilde{K}\cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, e^{i\theta_k} \right\rangle^2}{\left\| \left\langle \sum_{i\in\{\tilde{K}\cup k\}} \frac{\partial V_i(Z)}{\partial z_k}, e^{i\theta_k} \right\rangle \right\|} \leq 0 \tag{14}$$

The time derivative of the introduced BLF $V(Z)$ from (10) is shown to be non-positive, which concludes the stability of the coverage problem.

C. Asymptotic stability from LaSalle's principle

This section indicates that all virtual masses converge on the set of Centroidal Voronoi Tessellation (CVT). Define Z^* is the invariant set of CVT as follows $Z^* := \{Z \in \Omega | Z = C(Z)\}$. The BLF $V(Z)$ is positive definite and is equal to 0 if and only if $z_k = C_k(Z), \forall k$. This property implies that Z^* is the set of global minimum of $V(Z)$, which means

$$\nabla V(Z) = 0 \iff Z \in Z^* \tag{15}$$

Note that the solution of Z^* is not unique because there exist various CVT configurations that all virtual masses may converge on. Since the time derivative of $V(Z)$ is non-positive, the controller (11) drives all agent to an equilibrium point $\tilde{Z}$. Assume that this equilibrium point $\tilde{Z}$ does not establish centroidal Voronoi tessellations. In other words, $\tilde{Z} \notin Z^*$ and $\dot{V}(Z)|_{Z=\tilde{Z}} = 0$. From (14), we have

$$\begin{aligned}
&\dot{V}(Z)|_{Z=\tilde{Z}} = 0 \\
\iff &\sum_{i\in\{\tilde{K}\cup k\}} \frac{\partial V_i(Z)}{\partial z_k}\Bigg|_{Z=\tilde{Z}} = \sum_{i=1}^{n} \frac{\partial V_i(Z)}{\partial z_k}\Bigg|_{Z=\tilde{Z}} = 0, \forall k \\
\implies &\sum_{k=1}^{n}\sum_{i=1}^{n} \frac{\partial V_i(Z)}{\partial z_k}\Bigg|_{Z=\tilde{Z}} = 0 \\
\implies &\nabla V(Z)|_{Z=\tilde{Z}} = 0 \\
(15) \implies &\tilde{Z} \in Z^*
\end{aligned}$$

This contradicts the assumption that the equilibrium position of virtual masses does not create centroidal Voronoi tessellations. In other words, CVT is the only invariant set and LaSalle's principle concludes that all virtual masses converge on the set of CVT asymptotically.

D. Problem of state constraint

From (14), the introduced control law u_k ensures that the time derivative of the BLF $V(Z)$ is non-positive. According to Lemma (2), BLF $V(Z)$ is always bounded and all virtual masses will maintain inside the interior of the coverage region Ω. Therefore, if all virtual masses are initialized inside the coverage region, the state constraint is not violated.

E. Problem of input constraint

The second term of the control law (11) was normalized so that the positive control gain γ_k is used to monitor the magnitude of the control input.

$$(11) \implies u_k = \omega_{k_0} \pm \gamma_k$$

From the input constraint, since $u_k \in \{-w_m, w_m\}$, this implies the positive control gain γ_k must satisfy

$$-w_m \leq \omega_{k_0} - \gamma_k \leq \omega_{k_0} + \gamma_k \leq w_m$$

Since ω_{k_0} can either be positive or negative, therefore

$$\gamma_k \leq w_m - \|\omega_{k_0}\| \tag{16}$$

IV. Simulation

In this section, we provide the simulation of the coverage problem using three agents to monitor a convex region. Figure (1) depicts the convergence of the non-increasing BLF V_K, which implies that all virtual masses converge on the set of CVT and the state constraints are never violated, e.g virtual masses never cross the boundary lines.

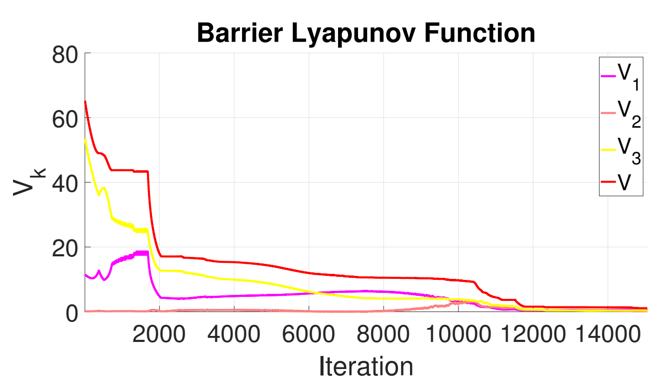

Fig. 1: Convergence of the non-increasing BLF $V(Z)$ and the sub-BLF of each agent $V_k(Z)$

As can be seen from figure (1), even though each BLF V_k does not have a non-positive time derivative, they are bounded by $V(Z)$. Figure (2) depicts the final CVT achieved by the distributed control input. All virtual masses establishes a desired CVT. Furthermore, figure (3) shows the trajectories of the virtual masses, which proves that they never cross the boundary lines.

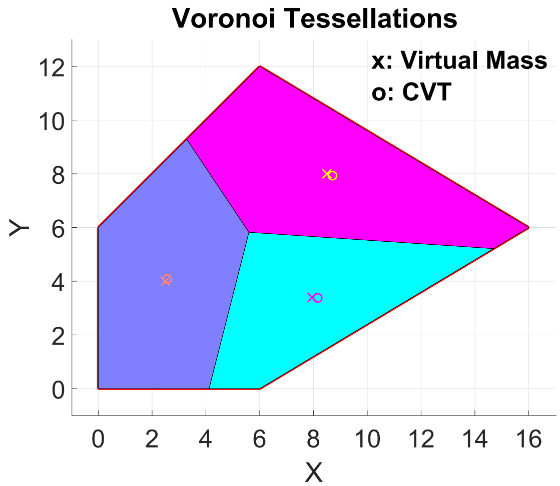

Fig. 2: Centroidal Voronoi tessellation established by virtual masses

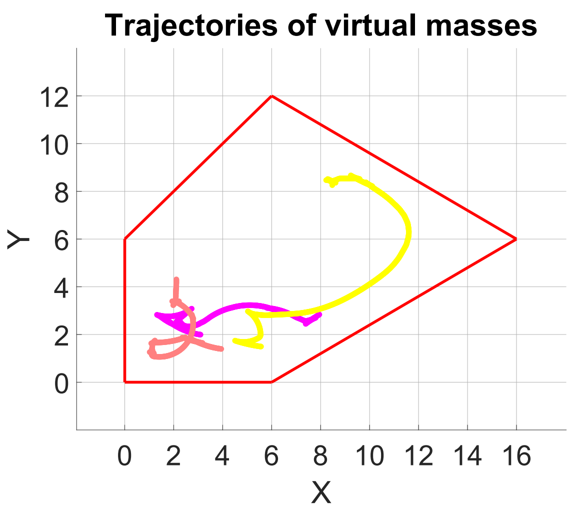

Fig. 3: Trajectories of virtual masses

Figure (4) shows that the control input of agents are bounded between the predefined maximum angular velocity $(-0.5, 0.5)$ rad/s

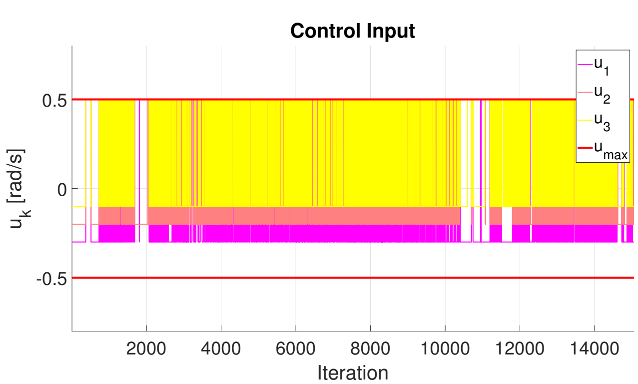

Fig. 4: Control input of each agent

V. Conclusion

Appendix A
Partial derivative of BLF

Remind the sub BLF of the adjacent agent i is $V_i(Z) : \mathbb{C}^n \to \mathbb{R}_+$, which is defined as

$$V_i(Z) = \sum_{j=1}^{m} \frac{1}{2} \frac{\langle z_i - C_i(Z), z_i - C_i(Z)\rangle}{b_j - \langle z_i, a_j\rangle}$$

In [7], Du formulated the term $\frac{\partial C_i}{\partial z_k}$ in $\mathbb{R}^2$ as

$$\frac{\partial C_i^{(a)}}{\partial z_k{}^{(b)}} = \frac{\int_{\partial\Omega_{i,k}} \rho(q) q^{(a)} \frac{q^{(b)} - z_k^{(b)}}{\|z_k - z_i\|} dq}{m_i} - \frac{(\int_{\partial\Omega_{i,k}} \rho(q) \frac{q^{(b)} - z_k{}^{(b)}}{\|z_k - z_i\|} dq)(\int_{\Omega_i(Z)} \rho(q) q^{(a)} dq)}{m_i^2}$$

where

$$a, b \in \{x, y\}$$
$$m_i = \int_{\Omega_i(Z)} \rho(q) dq$$

Note that in the introduced BLF $V_i(Z)$, even though the term $C_i(Z)$ are defined in the complex domain, its partial derivative $\frac{\partial C_i}{\partial z_k}$ is computed in the real planar as a Jacobian matrix

$$\frac{\partial C_i}{\partial z_k} = \begin{bmatrix} \frac{\partial C_{ix}}{\partial z_{kx}} & \frac{\partial C_{ix}}{\partial z_{ky}} \\ \frac{\partial C_{iy}}{\partial z_{kx}} & \frac{\partial C_{iy}}{\partial z_{ky}} \end{bmatrix} \in \mathbb{R}^2$$

where

$$C_i = [C_{ix}, C_{iy}]^T$$
$$z_k = [z_{kx}, z_{ky}]^T$$

It is concluded that the partial derivative of $C_i(Z)$ does not depend on the non-adjacent agent, this implies

$$\frac{\partial C_i}{\partial z_k} = \begin{cases} \frac{\partial C_k}{\partial z_k} & i = k \\ \frac{\partial C_i}{\partial z_k} & i \in \tilde{K} \\ 0 & i \notin \tilde{K} \end{cases}$$

Furthermore, the above-mentioned Jacobian matrix will occur in the inner product as a result of the applied derivation rule

$$\frac{d}{dz} \langle f, g\rangle = \left\langle \frac{d}{dz} f, g \right\rangle + \left\langle f, \frac{d}{dz} g \right\rangle$$

In order to strictly follow the definition of the inner product, we introduce the new opearator $\odot : \mathbb{R}^{2\times 2} \times \mathbb{C} \to \mathbb{C}$ to represent the intuitive meaning of partial derivative in $\mathbb{C}$ plane

$$\langle A \odot z\rangle = A \begin{bmatrix} \Re(z) \\ \Im(z) \end{bmatrix}$$

where the right equation is the matrix multiplication between $A \in \mathbb{R}^{2\times 2}$ and a vector in $\mathbb{R}^2$. The partial derivative of $V_i(Z)$ is then computed as

- $i = k$

$$\frac{\partial V_k(Z)}{\partial z_k} = \left\langle I_{2\times 2} - \frac{\partial C_k(Z)}{\partial z_k} \odot z_k - C_k(Z) \right\rangle \sum_{j=1}^{m} \frac{1}{b_j - \langle z_k, a_j\rangle} - \frac{\langle z_k - C_k(Z), z_k - C_k(Z)\rangle}{2} \sum_{j=1}^{m} \frac{a_j}{\left(b_j - \langle z_k, a_j\rangle\right)^2}$$

Note: $I_{2\times 2}$ is the identity matrix

- $i \in \tilde{K}$

$$\frac{\partial V_i(Z)}{\partial z_k} = \left\langle -\frac{\partial C_i(Z)}{\partial z_k} \odot z_i - C_i(Z) \right\rangle \sum_{j=1}^{m} \frac{1}{b_j - \langle z_i, a_j \rangle}$$

- $i \notin \tilde{K}$

$$\frac{\partial V_i(Z)}{\partial z_k} = 0$$

It is concluded that the term $V_i(Z)$ does not depend on the non-adjacent agent, which allow each agent to compute from the limited information of the adjacent within the communication range.

Acknowledgment

The authors would like to thank...

References

[1] Jorge Cortes, Sonia Martinez, Timur Karatas, Francesco Bullo and *Member IEEE*. *Coverage control for mobile sensing networks*. IEEE Transactions on Robotics and Automation.

[2] Mac Schwager. *A Gradient Optimization Approach to Adaptive Multi-Robot Control*. Ph.D. thesis, Massachusetts Institute of Technology.

[3] Qingchen Liu, Mengbin Ye, Zhiyong Sun, Jiahu Qin, and Changbin Yu. *Coverage control of unicycle agents under constant speed constraints*. IFAC-PapersOnLine, 50(1):24712476, 2017.

[4] Christoforos N Hadjicostis and Min Cao. *Distributed algorithms for Voronoi diagrams and applications in ad-hoc networks* Technical report, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 2003.

[5] Matthew L. Elwin, Randy A. Freeman, and Kevin M. Lynch. *Distributed Voronoi Neighbor Identification From Inter-Robot Distances* IEEE ROBOTICS AND AUTOMATION LETTERS, VOL. 2, NO. 3, JULY 2017

[6] Keng Peng Tee, Shuzhi Sam Gea, Eng Hock Tay. *Barrier Lyapunov Functions for the control of output-constrained nonlinear systems* Automatica, Volume 45, Issue 4, 2009, Pages 918-927, ISSN 0005-1098, https://doi.org/10.1016/j.automatica.2008.11.017.

[7] Qiang Du and Maria Emelianenko2. *Acceleration Schemes for Computing Centroidal Voronoi Tessellations*. Numerical Linear Algebra with Applications. Numer. Linear Algebra Appl. 2006, 13 173 - 192

PLACE
PHOTO
HERE

Michael Shell Biography text here.

John Doe Biography text here.

Jane Doe Biography text here.