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A STABILIZING MODEL PREDICTIVE CONTROL FOR LINEAR SYSTEMS WITH INPUT SATURATION

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Abstract:

A stabilizing model predictive control (MPC) strategy is proposed for linear systems with input saturation. A saturated linear feedback controller is selected as the local stabilizing controller. The terminal constraint set and terminal cost function can be computed by solving a corresponding semi-definite programming (SDP) problem. Then the control action is obtained by solving a second order cone programming (SOCP) problem on-line. Feasibility of SOCP problem implies the stability of the closed-loop system. Simulation shows that the proposed algorithm has a larger region of attraction than existed stabilizing MPC algorithms.

Keywords:

Input saturation; Model predictive control; Stability; Set invariance

1. Introduction

Model Predictive Control (MPC), which is also known as receding horizon control, has captured much attention of both academic and industrial control communities in the last two decades because of its widely successful applications in the process industries. Recently, a comprehensive review of its theoretical research results is provided in [1], and a survey of industrial MPC technology used in process control practice is presented in [2]. The former shows that its theoretical study has achieved relatively mature stage and the latter shows its enormous development and broad future in industrial application field.

One of the most important reasons for its popularity is that MPC provides a systematic method to handle the constraints of control system. Its control action is computed by solving an optimization problem that minimizes a pre-determined objective function in each sampling period. Typically a sequence of predicted control moves will be calculated, but only the first one is implemented. At the next sampling time, the optimization problem is solved again with new measurements, and control input is updated. Thus the constraints can be tackled naturally in optimization problem in each sampling step. However, under the presence of the constraints, both the control action and the closed loop system display non-linearity even if the controlled plant is linear. Then how to guarantee stability of closed loop system was a difficult problem of MPC control structure. For this reason MPC had once been criticized by many researchers.

A stabilizing MPC algorithm is presented in [3], in which a terminal equality constraint is enforced and the value function is first employed as Lyapunov function. Hereafter, the 1990's saw the emergence of most stabilizing MPC algorithms [4]-[8]. In these algorithms, three ingredients to guarantee stability of closed-loop system are necessary: a local stabilizing controller, a terminal cost function, a terminal constraint set.

On the other hand, two related tools, set invariance [9] and semi-definite programming (SDP) [10] play an increasing important role in the design of stabilizing MPC. For the former the terminal constraint set must be positively invariant no matter how one chooses it. And for the latter SDP can be introduced in not only computing the control action in each sample time naturally [11][12], but also determining the terminal constraint set and terminal cost function [7][8].

In this paper, we propose a stabilizing MPC algorithm for linear system with input saturation. We use a saturated linear feedback control law as local stabilizing controller. The terminal constraint set and terminal weighting matrix can be obtained by solving the corresponding SDP problem. And the control action can be computed by solving an SOCP problem on-line. Feasibility of the optimization problem implies the stability of the closed loop system. Simulation results show that the proposed method has a larger terminal constraint set and then broadens the region of attraction.larger terminal constraint set and then broadens the region of attraction.

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2. Stabilizing MPC algorithm

Consider the following discrete-time linear system:

$$x(k+1) = Ax(k) + Bu(k) \quad x(k) \in R^n, u(k) \in R^m \tag{1}$$

where (A, B) is assumed to be stabilizable. Without loss of generality, input constraint is assumed to be:

$$\| u(k) \|_\infty \leq 1 \tag{2}$$

Note that this particular choice is not restrictive since a rescaling of the input matrix B is always possible. The initial state x(0)=x0, and the control objective is to regulate the initial state to origin.

And consider the cost function at time k with state and input weighting matrix Q>0, R>0 of adequate dimension as follows:

$$J(x(k),U(k)) = \sum_{i=0}^{N-1}[x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)] + x^T(k+N|k)\Psi x(k+N|k) \tag{3}$$

where U(k)=[u(k|k),u(k+1|k),…,u(k+N-1|k)]T, N>0 is control horizon or prediction horizon and Ψ>0 is terminal weighting matrix.

In order to construct a stabilizing MPC controller, we need to determine a local stabilizing controller K(x) and a terminal constraint set X firstly [1]. In previous MPC methods [5]-[8], the local stabilizing controller is selected as a linear feedback controller, i.e. K(x)=Fx, where F is usually an LQR controller. In this paper, a saturated linear feedback controller is used as follows:

$$K(x)=Sat(Fx) \tag{4}$$

in which Sat(.) is the saturation function of appropriate dimensions, i.e.,

$$Sat(u)=[sat(u_1), sat(u_2),\ldots, sat(u_m)]^T \tag{5}$$

$$sat(u_j) = \begin{cases} u_j & -1 \leq u_j \leq 1 \\ 1 & u_j > 1 \\ -1 & u_j < -1 \end{cases}, j = 1,\cdots,m \tag{6}$$

The terminal constraint set X is an invariant set for the local stabilizing controller $K(x)$. And for a saturation feedback controller, we have the follow lemmas:

Let D be the set comprised of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. Obviously there are 2^m elements in D. Suppose that each element of D is labelled as D_i, i=1,2,…, 2^m. Then $D = \{D_i : i \in [1, 2^m]\}$. Define $D_i^- = I - D_i$. Clearly D_i^- is also an element of D. Let Co denotes the convex hull. In other words, for a group of points, $p_1, p_2, \ldots, p_q$, their convex hull is defined as:

$$Co\{p_i : i \in [1,q]\} = \left\{ \sum_{i=1}^{q} \alpha_i p_i : \sum_{i=1}^{q} \alpha_i = 1, \alpha_i \geq 0 \right\} \tag{7}$$

Given any matrix H, h_i denote the ith row of H. Then we have the following linear polytopic representation for saturated linear feedback $Sat(Fx)$.

Lemma 1 [13] Given $F, H \in R^{m\times n}$. For $x \in R^n$, if |hix|$\leq$1, for all i=1,2,…,m , then

$$Sat(Fx) = Co\{(D_i F + D_i^- H)x : i \in [1, 2^m]\} \tag{8}$$

Let $P \in R^{n\times n}$ be a positive definite matrix, an ellipsoidal set can be described as:

$$E_P = \{x \in R^n, x^T Px \leq 1\} \tag{9}$$

Then we have the following result.

Lemma 2 [13] Given an ellipsoid E_P and a saturated linear feedback law u= Sat(Fx) for the system (1), if there exists an $H \in R^{m\times n}$ such that |hix|$\leq$1, $\forall$ x$\in$ EP, i$\in$ [1, m], and

$$(A + B(D_i F + D_i^- H))^T P(A + B(D_i F + D_i^- H)) - P \leq 0, \forall i \in [1, 2^m] \tag{10}$$

then EP is an invariant set for closed loop system.

Then in terms of lemma 2, EP is an appropriate terminal constraint set for local stabilizing controller u=Sat(Fx). Obviously a set as large as possible is a natural choice. Let P=W-1 and Z=HW, the optimization problem of maximizing the area of the ellipsoid EP can be formulated as:

$$\min_{W,Z} \; -\log\det(W) \tag{11}$$

$$\text{subject to} \begin{bmatrix} W & * \\ A + BD_i FW + BD_i Z & W \end{bmatrix} \geq 0, i \in [1, 2^m] \tag{12}$$

$$\begin{bmatrix} 1 & * \\ z_i^T & W \end{bmatrix} \geq 0, i = [1, m] \tag{13}$$

where '*' in the matrix is used to induce the symmetric structure. Thus we can obtain a terminal constraint set:

$$E_P=\{x\in R^n, x^T Px \leq 1, P=W^{-1}\} \tag{14}$$

Furthermore, in a stabilizing MPC algorithm the terminal weighting matrix needs to satisfy the following condition [1]:

$$(Ax + BK(x))^T \Psi(Ax + BK(x)) - x^T\Psi x + x^T Qx + K(x)^T RK(x) \leq 0, \forall x \in X \tag{15}$$

For $K(x)=sat(Fx)$, using the linear polytopic description (8), (15) can be transformed into $|h_i x| \leq 1$and

$$(A + B(D_i F + D_i^- H))^T \Psi(A + B(D_i F + D_i^- H)) - \Psi + Q + (D_i F + D_i^- H)^T R(D_i F + D_i^- H) \leq 0 \tag{16}$$

Let $Y = \Psi^{-1}$ and $Z=HY$, we can obtain the following LMIs:

$$\begin{bmatrix} Y & * & * & * \\ A+BD_iFY+BD_i^-Z & Y & * & * \\ Q^{1/2}Y & 0 & I & * \\ R^{1/2}(D_iFY+D_i^-Z) & 0 & 0 & I \end{bmatrix} \geq 0, i \in [1,2^m] \quad (17)$$

$$\begin{bmatrix} 1 & * \\ z_i^T & Y \end{bmatrix} \geq 0, i=[1,m] \quad (18)$$

Then an appropriate Ψ =Y-1 can be computed by solving the following SDP problem:

$$\min_{Y,Z} \quad -\log\det(Y) \quad (19)$$

subject to (17)(18)

After obtaining Ψ and EP, the on-line optimization problem of MPC can be represented as:

$$\min_{U(k)} J(x(k),U(k)) \quad (20)$$

subject to (1), $x(k+N|k) \in X$, $\| U(k) \|_\infty \leq 1$

However, in the optimization problem (20), x(k+N|k)$\in$ EP is a nonlinear constraint, and conventional quadratic programming cannot handle it. Since EP is an ellipsoid, we can transform the optimization problem (20) into an SOCP problem:

$$\min_{U(k),\gamma} \quad \gamma \quad (21)$$

subject to (1) and

$$|u_j(k+i\text{-}1|k)| \leq 1,\ i\in [1, N], j\in [1, m] \quad (22)$$

$$\| P^{1/2}(A^N x(k)+B_2U(k)) \| \leq 1 \quad (23)$$

$$\left\| \begin{bmatrix} 2R_1^{1/2}U(k) \\ 2Q_1^{1/2}(A_1x(k)+B_1U(k)) \\ 1-\gamma \end{bmatrix} \right\| \leq 1+\gamma \quad (24)$$

where B2 =[AN-1B, AN-2B, …, B] and

$$A_1 = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} B_1 = \begin{bmatrix} B & 0 & 0 & 0 \\ AB & B & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} \quad (25)$$

$$Q_1 = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \Psi \end{bmatrix} R_1 = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & R \end{bmatrix} \quad (26)$$

The inequality (23) shows x(k+N|k)$\in$ EP and (24) shows that γ is upper bound of J(x(k),U(k)).

The proposed stabilizing MPC algorithm can be represented as:

Algorithm 1:

Off-line:

(1) Design K(x) using (4);

(2) Obtain X by solving (11) ;

(3) Obtain Ψ by solving (19);

On-line:

(1) Measure x(k) ;

(2) Let x(k|k)=x(k), Obtain U(k) by solving (21);

(3) Apply u(k)=u(k|k) to the system (1);

(4) Return (1).

3. Stability and region of attraction

Theorem 1: For the closed system composed of the Algorithm 1 and the system (1), feasibility of the optimization problem (21) at initial time k=0 guarantees its asymptotic stability.

Proof: Assume at time k the optimal cost function is J*(k), and the related control and state sequences are respectively presented as U*(k)=[u*(k|k), u*(k+1|k), …, u*(k+N-1|k)]T and x*(k)=[x*(k|k), x*(k+1|k), …, x*(k+N|k)]T. This implies x*(k+N|k)$\in$ X. In terms of X is an invariant set for u(k)=Sat(Fx(k)), Ax*(k+N|k)+BSat(Fx*(k+N|k)) $\in$ X. Then a feasible solution of optimization problem (21) at k+1 time is U(k+1)= [u*(k+1|k), …, u*(k+N-1|k), Sat(Fx*(k+N|k))] T. And its related state sequence is x(k+1)=[x*(k+1|k), …, x*(k+N|k) Ax*(k+N|k)+BSat(Fx*(k+N|k))]T and the cost function is:

$$\begin{aligned} J(k+1) &= \sum_{i=1}^{N-1}[x^{*T}(k+i|k)Qx(k+i|k)+u^{*T}(k+i|k)Ru(k+i|k)] \\ &+x^{*T}(k+N|k)Qx^*(k+N|k)+Sat(Fx^*(k+N|k))^T RK(x^*(k+N|k)) \\ &+(Ax^*(k+N|k)+BSat(Fx^*(k+N|k)))^T\Psi(Ax^*(k+N|k)+BSat(Fx^*(k+N|k))) \\ &= J^*(k)-x^{*T}(k|k)Qx^*(k|k)-u^{*T}(k|k)Ru^*(k|k) \\ &+(Ax^*(k+N|k)+BSat(Fx^*(k+N|k)))^T\Psi(Ax^*(k+N|k)+BSat(Fx^*(k+N|k))) \\ &-x^{*T}(k+N|k)\Psi x^*(k+N|k) \\ &+x^{*T}(k+N|k)Qx^*(k+N|k)+Sat(Fx^*(k+N|k))^T RSat(Fx^*(k+N|k)) \end{aligned} \quad (27)$$

According to (15), we can obtain trivially

$$J(k+1) \leq J^*(k)-x^{*T}(k|k)Qx^*(k|k)-u^{*T}(k|k)Ru^*(k|k) \quad (28)$$

Since the value of cost function of feasible solution is larger than that of optimal solution, i.e., J(k+1) $\geq$J*(k+1), we have

$$J^*(k+1) \leq J^*(k)-x^{*T}(k|k)Qx^*(k|k)-u^{*T}(k|k)Ru^*(k|k) \quad (29)$$

This shows convergence of the closed loop system, that is

$$x(k) \to 0 \text{ and } u(k) \to 0 \text{ as } k \to \infty$$

So it can be observed that J*(k) can serve as a Lyapunov function for asymptotical stability.

Region of attraction is a region in state space, in which MPC algorithm can drive each state to origin.

Theorem 2: If the initial state x0 ∈ EP for the system (1), then the problem (21) is feasible at time k=0 for any N ≥ 1, that is to say, EP is a trivial region of attraction for proposed MPC algorithm independent of the control horizon N.

Proof: For any x0 ∈ EP and any N ≥ 1, we can find a feasible solution U(0)=[sat(Fx(0|0)), sat(Fx(1|0)), …, sat(Fx(N-1|0))] guaranteeing x(N|0) ∈ EP at time k=0. Hence, the optimization problem (21) is always feasible for any x0 ∈ EP at time k=0 and then EP is naturally a region of attraction for proposed MPC in terms of Theorem 1.

Note that by increasing N, MPC algorithm has a larger region of attraction in state space. It will be shown that EP is much larger than the terminal invariant set used by other methods in section 4. Then for same control horizon, the proposed algorithm has the larger region of attraction.

4. Simulation results

In this section, we present an example that illustrates the implementation of the proposed stabilizing MPC algorithm. Consider the double integrators system sampled at a frequency of 2 Hz, for which

$$A = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0.125 \end{bmatrix}$$

Let Q=I, R=I. We can obtain an LQR controller $F = [-1.3142, -0.6514]$. The terminal constraint ellipsoid and the terminal weighting matrix are respectively computed by solving (11) and (19) and are shown as

$$E_P = \left\{ x : x^T P x \le 1, P = \begin{bmatrix} 0.0807 & 0.0217 \\ 0.0217 & 0.0164 \end{bmatrix} \right\},$$

$$\Psi = \begin{bmatrix} 445.6 & 107.7 \\ 107.7 & 96.7 \end{bmatrix}$$

We compare EP with other three the terminal constraint sets used in [5] [7] [6] in Fig. 1. In these methods, the former two use an invariant ellipsoidal set EC and EL respectively, and the last uses an invariant polyhedral set, i.e., the maximal output admissible set O_∞ [14]. Obviously, the proposed terminal constraint set contains the ones used in other methods. Thus if using the same control horizon N for all methods, the proposed strategy will have the largest region of attraction.

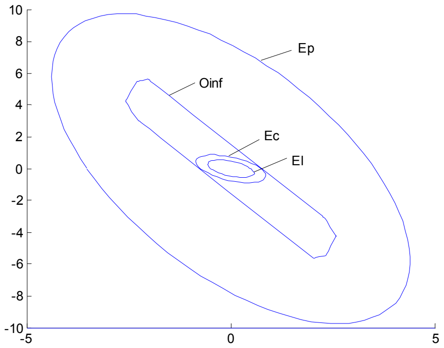

Figure 1. Comparison of terminal constraint sets

Let N=10, Figure2 shows the trajectory of state from different initial states and Figure 3,4 respectively shows state and input response at x0 =[-3,10] and x0 =[5,-5].

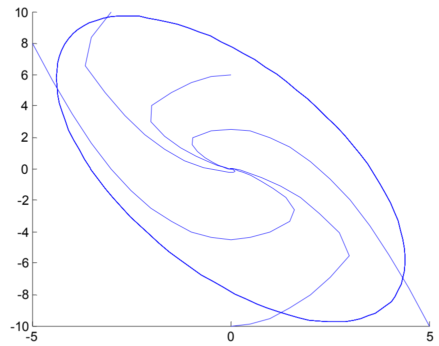

Figure 2. The trajectory of state from different x_0

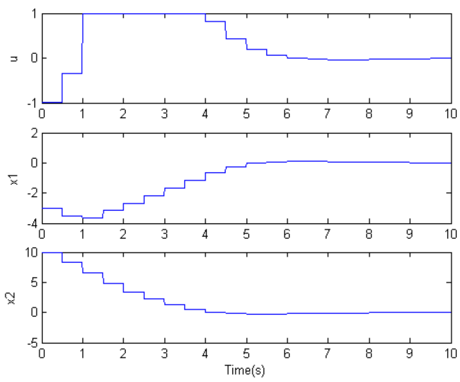

Figure 3. State and input response at x_0=[-3, 10]

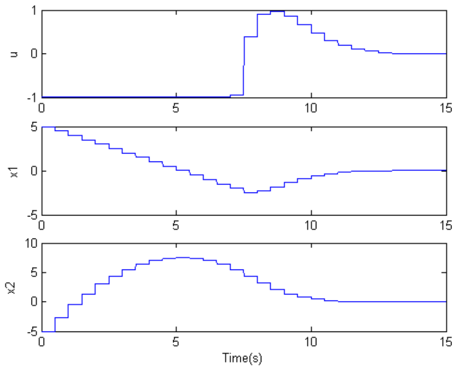

Figure 4. State and input response at x_0=[5, -5]

5. Conclusion

In this paper a stabilizing MPC controller is proposed for linear systems with input saturation. This controller uses a saturated linear feedback controller as local stabilizing controller and has a larger region of attraction than existed MPC algorithms. The control law at each sample time can be obtained by solving an SOCP problem. A simulation example shows its effectiveness.

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