STABILITY ANALYSIS AND CONTROL DESIGN OF LINEAR SYSTEMS WITH INPUT SATURATION USING MATRIX SUM OF SQUARES RELAXATION

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Abstract: In this paper, matrix sum of squares (SOS) relaxation is used for analysis of linear systems with input saturation. On the basis of a polytopic type saturation model, we consider a polynomial type saturation model which leads less conservative estimations of the domain of attraction. The idea is also applicable to design problems. Both analysis and design problems are reduced to robust linear matrix inequality problems that can be solved by using a matrix SOS technique. A numerical example shows the effectiveness for analysis problems.

Keywords: Sum of Squares, Input Saturation, Linear Systems, Stability

1. INTRODUCTION

There are systems whose actuator signals are limited by physical constraints. In control analysis and design, ignoring these limitations can be sources of closed-loop instability or performance degradation (Gutman and Hagander, 1985; Alvarez *et al.*, 1993). Many works have dealt with stability analysis or disturbance rejection design of linear systems with input saturation. In the past decade, linear matrix inequality (LMI) has been used for analysis or design of such systems in the framework of robust control (Pittet *et al.*, 1997; Hindi and Boyd, 1998; Kiyama *et al.*, 2001), and it enables us numerically tractable computations and less conservative results. In particular, a polytopic type saturation model has been introduced to and the polyhedral set condition has been reduced to LMI (da Silva Jr. *et al.*, 1997). Moreover, it has developed into the necessary and sufficient analysis of the domain of attraction for single input linear systems with state feedback (Hu and Lin, 2001). Thus, LMIs optimization is a standard tool for analysis and design of linear systems with input saturation.

On the other hand, recently, sum of squares (SOS) optimization(Parrilo, 2000; Kojima, 2003) has been proposed, which is a method to be able to handle non-convex polynomial programmings globally by using SOS polynomials and semi-definite programming (SDP). In particular, a matrix SOS technique(Hol and Scherer, 2004) has potentially a lot of applications in control engineering. One of the applications is relaxation of robust LMI problems. Consider a robust LMI problem

$$\min_{p} \ c^T p \ \text{s.t.} \tag{1}$$

$$F_k(\theta, p) \succeq 0, \ \forall \theta \in \Omega, \ (k = 1, \dots, m_F), \tag{2}$$

where $p \in \mathbb{R}^{n_p}$, $c \in \mathbb{R}^{n_p}$, $\theta^\alpha = \theta_1^{\alpha_1}\theta_2^{\alpha_2}\cdots\theta_m^{\alpha_m}$, $\sum_{i=1}^m \alpha_i \le d$, $F_k(\theta,p) = \sum_\alpha F_{k\alpha}(p)\theta^\alpha \in \mathbb{S}[\theta]^{n_k}$, $F_{k\alpha}(p) = F_{k\alpha}(0) + \sum_{j=1}^{n_p} C_{k\alpha}^{(j)} p_j$, $C_{k\alpha}^{(j)} \in \mathbb{S}^{n_k}$, $\Omega = \{\theta \in \mathbb{R}^m | g_i(\theta) \ge 0 \ (i = 1, \dots, m_g)\}$, and $g_i(\theta) \in \mathbb{R}[\theta]$. Each of $F_k(\theta,p)$ is a symmetric polynomial matrix of θ, whose coefficient matrix has the affine decision variables p. If $F_k(\theta,p)$ is strictly positive definite, we may consider $F_k(\theta,p) - \varepsilon I$ for $\varepsilon > 0$ instead of $F_k(\theta,p)$ in (2). Under a suitable SOS representation, exact relaxation of the robust LMI problem is performed (Scherer and Hol, 2004). Exactness is an advantage of this technique while the robust LMI is often referred to as the parameter-dependent LMI whose feasible solutions have been discussed (Gahinet *et al.*, 1996; Tuan and Apkarian, 1999).

In this paper, we propose an approach for stability analysis and design of linear systems with input saturation through a saturation model to estimate the domain of attraction of the closed-loop continuous-time systems. All problems are reduced to matrix SOS relaxation through robust LMI problems. In section 2, a vector saturation model is introduced, which represents each channel independently and confines the analysis region. The advantages of that model are also discussed for matrix SOS approaches over the convenient polytopic model. In section 3, a set invariant condition is considered for the saturation model of section 2. In section 4, the domain of attraction analysis and design are discussed under a shape reference set. In section 5, the robust LMI problem is relaxed to matrix SOS problems. Exactness condition is referred to this section. Numerical examples are shown in section 6. Conclusions are in section 7.

Notation: For the set of $n \times n$ symmetric matrices $\mathbb{S}^n$, $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n | X \succeq 0\}$. $X \succeq 0$ means that X is a positive semidefinite matrix. $\mathrm{He}\{A\}$ means $A + A^T$, and $\mathrm{co}\{\cdot\}$ means convex hull. For a vector $u \in \mathbb{R}^m$, a vector saturation function is defined by $\Phi(u) := [\phi(u_1), \cdots, \phi(u_m)]^T$, where

$$\phi(u_i) := \begin{cases} \mathrm{sign}(u_i), & |u_i| > 1 \\ u_i, & |u_i| \le 1. \end{cases}$$

For a polyhedron Ω, $\mathrm{vert}\,\Omega$ means the set of vertices of Ω. $\mathbb{R}[\theta]$ is the ring of real polynomials. $\mathbb{S}[\theta]^n$ is the ring of real $n \times n$ symmetric matrix polynomials. The set of matrix SOS polynomials is denoted by $\sum[\theta]^n$. Let $d_{[N]}(m) := \binom{m+N}{N}$, which is abbreviated as $d_{[N]}$, $z_{[N]}(\theta) \in \mathbb{R}^{d_{[N]}}$ is the monomial vector of θ whose maximum degree is N. For examples, in the case of $\theta = [\,\theta_1\ \theta_2\,]^T$ and $N = 2$, $z_{[N]}(\theta) = [\,1\ \theta_1\ \theta_2\ \theta_1^2\ \theta_1\theta_2\ \theta_2^2\,]^T$. $z_{[N]}(\theta)$ is also abbreviated as $z_{[N]}$. If $M \in \mathbb{R}^{nm \times nm}$ is partitioned into $n \times n$ blocks as $(M_{jk})_{j,k=1,\ldots,m}$ define

$$\mathrm{trace}_m(M) := \begin{bmatrix} \mathrm{trace}\, M_{11} & \cdots & \mathrm{trace}\, M_{1m} \\ \vdots & \ddots & \vdots \\ \mathrm{trace}\, M_{m1} & \cdots & \mathrm{trace}\, M_{mm} \end{bmatrix}$$

as well as a bilinear mapping $< \cdot, \cdot >_m : \mathbb{R}^{mm \times nm} \times \mathbb{R}^{mm \times nm} \to \mathbb{R}^{m \times m}$ as $< A, B >_m = \mathrm{trace}_m(A^T B)$.

2. MODELING OF VECTOR SATURATION FUNCTION

Let us consider the following continuous-time linear systems with input saturation

$$\dot{x} = Ax + B\Phi(u), \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $z \in \mathbb{R}^r$ is the performance output, and $\Phi(u)$ is the vector saturation function. For the system (3), a state feedback law

$$u = Fx \tag{4}$$

is provided, where $F \in \mathbb{R}^{m \times n}$ is a feedback matrix.

As a preliminary to modeling the vector saturation function, we remember a convex hull representation of the single saturation function $\phi(u_i)$.

Lemma 1. ((Hu and Lin, 2001)). Let $u_i, v_i \in \mathbb{R}$. Suppose that $|v_i| \le 1$, then $\phi(u_i) \in \mathrm{co}\{u_i, v_i\}$.

From Lemma 1, we can directly write a polytopic model of the vector saturation function.

Proposition 1. Suppose that $|v_i| \le 1$ $(i = 1, \ldots, m)$, then

$$\Phi(u) \in \big\{\Theta(\theta)u + (I - \Theta(\theta))v \ \big|\ \theta \in \Omega\big\}, \tag{5}$$

where $\theta = [\theta_1, \ldots, \theta_m]^T$, $\Theta(\theta) = \mathrm{diag}\,\theta^T$ and $\Omega = \{\theta \in \mathbb{R}^m |\ 0 \le \theta_i \le 1\ \ (i = 1, \ldots, m)\}$.

The polytopic model (Hu and Lin, 2001) of vector saturation function is sometimes written as

$$\Phi(u) \in \big\{ \textstyle\sum_{i=1}^{2^m} \lambda_i (E_i u + E_i^- v) \ \big|\ \lambda \in \Xi\big\}, \tag{6}$$

where $\Xi = \{\lambda \in \mathbb{R}^{2^m} | 0 \le \lambda_i \le 1\ (i = 1, \ldots, 2^m),\ \sum_{i=1}^{2^m} \lambda_i = 1\}$, E_i belongs to the $m \times m$ diagonal 2^m matrices set $\mathcal{D}$ whose diagonal elements are either 1 or 0, and has the relation $E_i = I - E_i^-$. The above expression (6) is the same as (5) in the sense of having 2^m vertices. The differences between (5) and (6) are the number of variables and constraints. The expression (5) has m variables and $2m$ constraints while (6) has 2^m variables and $2m+1$ constraints. These differences are no meaning when we adopt LMI based approaches. However, when we consider a matrix polynomial inequality based approach discussed in subsequent sections, these differences affect the amount of computation. Thus we adopt the expression (5) as a polytopic model of the vector saturation function.

On the polytopic saturation model (5), we use $u = Fx$ according to (4). On the other hand, we newly introduce $v = H(\theta)x$, where $H(\theta)$ is a $m \times n$ continuous real matrix function of θ. In this paper, we consider $H(\theta)$ as a real polynomial matrix, that is, $H(\theta) = \sum_\alpha H_\alpha \theta^\alpha \in \mathbb{R}[\theta]^{m\times n}$. For example, for $m = 2$ and $\sum_{i=1}^m \alpha_i \le 2$, $H(\theta)$ is

$$H_0 + H_{10}\theta_1 + H_{01}\theta_2 + H_{20}\theta_1^2 + H_{11}\theta_1\theta_2 + H_{02}\theta_2^2.$$

The vector saturation model in this paper is as follows:

Proposition 2. For given any $x \in \mathbb{R}^n$, assume that $|h_i(\theta)x| \le 1$ for all $\theta \in \Omega$ $(i = 1,\dots,m)$, then

$$\Phi(Fx) \in \left\{ \left[\Theta(\theta)F + (I - \Theta(\theta))H(\theta)\right]x \,\middle|\, \theta \in \Omega \right\}, \tag{7}$$

where $h_i(\theta)$ is the i-th row of $H(\theta) \in \mathbb{R}[\theta]^{m\times n}$, θ and Ω follow Proposition 1.

The assumptions say that the saturation model (7) is valid only for $x \in \mathbb{R}^n$ satisfying $|h_i(\theta)x| \le 1$ for all $\theta \in \Omega$. If we can confine the analysis region properly, we can analyze the system tightly. Introducing $H(\theta)$ makes the shape of the analysis region an intersection of curving lines $h_i(\theta)x = 1$ while constant H makes it an intersection of straight lines $h_i x = 1$. Thereby we can expect to reduce the conservativeness of analysis by introducing $H(\theta)$. However, in general, we can not adopt LMI based approaches any more.

3. A SET INVARIANCE CONDITION

In this section, we consider a set invariance condition for the saturation model (7). Ellipsoids are adopted to a candidate of the invariant sets, which is inside the domain of attraction $\mathcal{S}$, where $\mathcal{S} := \left\{ x_0 \in \mathbb{R}^n \middle| \lim_{t\to\infty} \phi(t, x_0) = 0 \right\}$.

Let us define ellipsoids as

$$\mathcal{E}(Q^{-1}) = \left\{ x \in \mathbb{R}^n \middle| x^T Q^{-1} x \le 1 \right\},$$

where $Q \in \mathbb{S}^n$ is a positive definite matrix. For the function $V(x) = x^T Q^{-1} x$, the ellipsoid $\mathcal{E}(Q^{-1})$ is contractive invariant if

$$\dot{V}(x) = 2x^T Q^{-1}(Ax + B\Phi(Fx)) < 0 \tag{8}$$

for all $x \in \mathcal{E}(Q^{-1}) \backslash \{0\}$. Then $V(x)$ is a Lyapunov function, and the contractive invariant set is included in the domain of attraction, i.e. $\mathcal{E}(Q^{-1}) \subset \mathcal{S}$.

Now we consider exploiting the saturation model (2), and define a region $\mathcal{L}(H(\theta))$ as

$$\left\{ x \in \mathbb{R}^n \middle| \, |h_i(\theta)x| \le 1, \;\; i = 1,\dots,m, \;\; \forall \theta \in \Omega \right\},$$

where $h_i(\theta)$ is the i-th row of $H(\theta)$. $H(\theta)$ has a decomposition such as $H(\theta) = T(\theta)Q^{-1}$. $\mathcal{L}(T(\theta)Q^{-1})$ is also defined by $\mathcal{L}_Q(T(\theta))$. Then we derive a set invariance condition.

Theorem 1. Given an ellipsoid $\mathcal{E}(Q^{-1})$. If there exists a polynomial matrix $T(\theta) \in \mathbb{R}[\theta]^{m\times n}$ such that for all $\theta \in \Omega$,

$$\mathrm{He}\Big\{ (A + B\Theta(\theta)F)Q + B\left(I - \Theta(\theta)\right)T(\theta) \Big\} \prec 0, \tag{9}$$

$$\mathcal{E}(Q^{-1}) \subset \mathcal{L}_Q(T(\theta)), \tag{10}$$

then $\mathcal{E}(Q^{-1})$ is a contractive invariant set.

Remark 1. Theorem 1 is similar to Theorem 7.4.1(Hu and Lin, 2001), which presents a sufficient stability condition by using the polytopic saturation model (6) under $v = Hx$, in particular, which also presents the necessary and sufficient condition for single input systems. If Theorem 1 is completely solved, the advantages could appear for multiple input systems not for single input systems.

4. ANALYSIS AND CONTROL DESIGN

In addition to the set invariance condition of Theorem 1, in this section, the largest one is chosen as the least conservative estimation of the domain of attraction on the basis of a measure of set size. This measure index is also applied to control design problems.

Let us consider shape reference sets $\mathcal{X}_R$(Hu and Lin, 2001). A typical type of $\mathcal{X}_R$ is a polyhedron $\mathcal{X}_R = \mathrm{co}\left\{ x^{(1)}, x^{(2)}, \dots, x^{(l)} \right\}$. For a set $\mathcal{E}(Q^{-1}) \subset \mathcal{S}$, define the size of $\mathcal{E}(Q^{-1})$ with respect to $\mathcal{X}_R$ by $\alpha > 0$ such that

$$\alpha \mathcal{X}_R \subset \mathcal{E}(Q^{-1}). \tag{11}$$

Both Theorem 1 and (11) are considered at the same time to choose a larger estimation of the domain of attraction.

We firstly estimate it under the reference $\mathcal{X}_R$.

Problem 1. Given F and $\mathcal{X}_R$, maximize α with respect to α, Q, and $T(\theta)$ subject to (9), (10), and (11).

Theorem 2. Given F and $\mathcal{X}_R$. If there exist $Q(\succ 0)$, $T(\theta)$, and $\gamma(> 0)$ satisfying for all $\theta \in \Omega$,

$$\mathrm{He}\Big\{ (A + B\Theta(\theta)F)Q + B\left(I - \Theta(\theta)\right)T(\theta) \Big\} \prec 0, \tag{12}$$

$$\begin{bmatrix} 1 & t_i(\theta) \\ t_i(\theta)^T & Q \end{bmatrix} \succeq 0 \quad (i = 1,\dots,m), \tag{13}$$

$$\begin{bmatrix} \gamma & x^{(j)T} \\ x^{(j)} & Q \end{bmatrix} \succeq 0 \quad (j = 1,\dots,l), \tag{14}$$

where $t_i(\theta)$ is i-th row of $T(\theta)$, and $\gamma = 1/\alpha^2$, then $\mathcal{E}(Q^{-1})$ is a domain of attraction of the closed-loop system (3) and (4).

proof: The condition (9) and (12) are the same, so that it is required to show that the polynomial matrix constraints (13) and (14) are equivalent to (10) and (11), respectively. The condition (10) for all $\theta \in \Omega$ is equivalent to

$$h_i(\theta)Qh_i(\theta)^T \le 1, \quad \forall\theta \in \Omega, \quad (i = 1, \dots, m).$$

Using Schur complements, and introducing $t_i(\theta)$, we have (13). The last constraint (11) is rewritten as

$$\alpha^2 x^{(j)T} Q^{-1} x^{(j)} \le 1, \quad \forall\theta \in \Omega, \quad (j = 1, \dots, l).$$

Using Schur complements, and introducing γ, we have (14). □

For the optimization scheme for choosing a larger estimation of the domain of attraction, we minimize γ satisfying conditions (12), (13), and (14) for all $\theta \in \Omega$. These conditions include symmetric matrix polynomial constraints whose coefficient matrices are affine dependent of decision variables such as (2) of the problem (1). That is, the decision variables p on (2) are correspond to γ, Q, and T_α on the conditions. It is possible to formulate this optimization scheme into a SOS optimization problem, and is also possible to require the decision variables by solving SDP. The discussion is in the subsequent section.

If we uses a constant matrix H_0 instead of $H(\theta)$ on the saturation model (7), we can see the pure advantages of introducing $H(\theta)$ to the saturation model. We have the following corollary.

Corollary 1. Given F and $\mathcal{X}_R$. If there exist $Q(\succ 0)$, $T(\theta)$, and $\gamma(> 0)$ satisfying for all $\theta \in \Omega$, (9), (10), and (11) with $T(\theta) = T_0$, where T_0 is a constant matrix, then $\mathcal{E}(Q^{-1})$ is a domain of attraction of the closed-loop system (3) and (4).

The optimization scheme under Corollary 1 can be solved by LMIs on vertices of Ω. That is the same condition with the article (Hu and Lin, 2001).

Secondly, we consider a design problem to enlarge the domain of attraction under $\mathcal{X}_R$.

Problem 2. Given $\mathcal{X}_R$, maximize α with respect to F, α, Q, and $T(\theta)$ subject to (9), (10), and (11).

Theorem 3. Given $\mathcal{X}_R$. If there exist K, $Q(\succ 0)$, $T(\theta)$, and $\gamma(> 0)$ satisfying for all $\theta \in \Omega$,

$$\mathrm{He}\Big\{AQ + B\Big[\Theta(\theta)K + (I - \Theta(\theta))\,T(\theta)\Big]\Big\} \prec 0, \tag{15}$$

(13) and (14), where $t_i(\theta)$ is i-th row of $T(\theta)$, and $\gamma = 1/\alpha^2$, then $F = KQ^{-1}$ is a state feedback gain and $\mathcal{E}(Q^{-1})$ is a domain of attraction of the closed-loop system (3) and (4).

The optimization scheme on Theorem 3 is considered as well as Theorem 1. However, in general, Theorem 3 can not improve its conservativeness compared with its corollary with a constant matrix $T(\theta) = T_0$. Because (15) satisfies negative definiteness only by choosing the matrix $Q(\succ 0)$ without choosing high-order part of the matrix polynomial $T(\theta)$. If we can introduce parameter-dependent Lyapunov functions such as $V(x, \theta) = x^T Q(\theta)^{-1} x$ to this design problem, there is a possibility to improve its conservativeness. That is a future target.

5. MATRIX SUM OF SQUARES RELAXATION

A matrix SOS technique is briefly summarized in this section in order that one may readily apply it to the problems of the previous section. We also show a relaxed version of Theorem 2.

The robust LMI problem (1) is relaxed to a sequence of SOS relaxation problems.

Lemma 2. Consider a sequence of SOS relaxation problems

$$\min_{p, \mathcal{Q}_k, \mathcal{R}_k^{(i)}} \ c^T p \ \text{ s.t. } \forall\theta \in \mathbb{R}^m, \tag{16}$$

$$F_k(\theta, p) - \sum_{i=1}^{m_g} g_i(\theta) S_{ki}(\theta) = S_{k0}(\theta), \tag{17}$$
$$(k = 1, \dots, m_F),$$

where $S_{k0}(\theta) = (I_{n_k} \otimes z_{[N_k]})^T \mathcal{Q}_k (I_{n_k} \otimes z_{[N_k]}) \in \sum[\theta]^{n_k}$, $\mathcal{Q}_k \in \mathbb{S}_+^{n_k d_{[N_k]}}$, $S_{ki}(\theta) = (I_{n_k} \otimes z_{[N_k - w_i]})^T \mathcal{R}_k^{(i)} (I_{n_k} \otimes z_{[N_k - w_i]}) \in \sum[\theta]^{n_k}$, $\mathcal{R}_k^{(i)} \in \mathbb{S}_+^{n_k d_{[N_k - w_i]}}$, and $w_i = \lceil\{\deg g_i(\theta)\}/2\rceil$. Then, each of the optimal solution of the problems (16) is a feasible solution of the problem (1).

proof: Suppose that p^* is the optimal solution of one of the problems (16). Then, the constraint (17) is rewritten as $F_k(\theta, p^*) = S_{k0}(\theta) + \sum_{i=1}^{m_g} g_i(\theta) S_{ki}(\theta)$. Since matrix SOS polynomials $S_{k0}(\theta)$ and $S_{ki}(\theta)$ are positive semidefinite for all $\theta \in \mathbb{R}^m$, and $g_i(\theta)$ are also positive semidefinite for all $\theta \in \Omega$, thus, the right side is positive semidefinite for all $\theta \in \Omega$, i.e, the constraints (2) are consisted. Consequently, p^* is a feasible solution of the problem (1). □

Assigning the degree N_k of each matrix SOS polynomials appropriately, we have one SOS relaxation problem. If the constraints (2) are LMI, i.e., $F_k(p) \succeq 0$, the right side is replaced with a matrix SOS polynomial whose monomials order N_k is 0, i.e., positive semidefinite symmetric matrix. On the basis of the result (Scherer and Hol, 2004), the sequence of SOS relaxation problems (16) can be completely solved numerically by the next lemma.

Lemma 3. Consider a sequence of SDP

$$\begin{aligned}&\min_{p,\mathcal{Q}_k,\mathcal{R}_k^{(i)}} \ c^T p \text{ s.t.}\\ &< I_{n_k} \otimes A_{k\alpha}, \mathcal{Q}_k >_{n_k} + \sum_{i=1}^{m_g} < B_{k\alpha}^{(i)}, \mathcal{R}_k^{(i)} >_{n_k}\\ &- \sum_{j=1}^{n_p} C_{k\alpha}^{(j)} p_j = F_{k\alpha}(0) \ ,\forall\alpha, \ (k = 1,\dots,m_F),\end{aligned} \quad (18)$$

where $A_{k\alpha} \in \mathbb{S}^{d_{[N_k]}}$ and $B_{k\alpha}^{(i)} \in \mathbb{S}^{n_k d_{[N_k - w_i]}}$ satisfy $z_{[N_k]} z_{[N_k]}^T = \sum_\alpha A_{k\alpha}\theta^\alpha$ and $g_i(\theta)S_{ki}(\theta) = \sum_\alpha < B_{k\alpha}^{(i)}, \mathcal{R}_k^{(i)} >_{n_k} \theta^\alpha$, respectively. Then, each of the optimal value of the SDP (18) is equivalent to that of the corresponding problems (16).

Although the optimal solution of the problems (16) can be obtained by solving the SDP (18), Lemma 2 says that the optimal value of the problems (16) are upper bounds of the original problem (1). On the other hand, in some cases, asymptotically exactness of this relaxation has been discussed as follows:

Lemma 4. ((Scherer and Hol, 2004)). Assume that functions $g_i(\theta)$ $(i = 1,\dots,m_g)$ be affine, $\Omega = \{\theta \in \mathbb{R}^m | g_i(\theta) \geq 0 \ (i = 1,\dots,m_g)\}$ is a compact set, and $F(\theta)$ is positive semidefinite for all $\theta \in \Omega$, then there exist matrix SOS polynomials $S_0(\theta)$, $S_i(\theta)$ $(i = 1,\dots,m_g)$ such that $F(\theta) - \sum_{i=1}^{m_g} g_i(\theta)S_i(\theta) = S_0(\theta)$ for all $\theta \in \mathbb{R}^m$.

This lemma means that there exists a finite N_k on (17) for each k such that the optimal value of the corresponding problem (16) is equivalent to that of the problem (1).

Now we go back to the problems of the previous sections. The region Ω in the saturation model (7) is composed of affine functions of θ, and is a compact set, and hence satisfies the assumptions of Lemma 4. Thus we can expect the exact relaxation, for example, of Theorem 2 as follows:

Theorem 4. Given F and $\mathcal{X}_R$. If there exist $Q(\succ 0)$, $T(\theta)$, $\gamma(> 0)$, $\mathcal{Q}_1 \in \mathbb{S}_+^{nd_{[N_1]}}$, $\mathcal{Q}_2^i \in \mathbb{S}_+^{(n+1)d_{[N_2]}}$, $\mathcal{Q}_3^j \in \mathbb{S}_+^{n+1}$, $\mathcal{R}_1^{(k)} \in \mathbb{S}_+^{nd_{[N_1-1]}}$, $\mathcal{R}_2^{i(k)} \in \mathbb{S}_+^{(n+1)d_{[N_2-1]}}$, and $\varepsilon(> 0)$ satisfying for all $\theta \in \Omega$,

$$\begin{aligned}-\left\{\text{the left of (12)}\right\} - \varepsilon I - \sum_{k=1}^{2m} g_k(\theta)S_{1k}(\theta)\\ = S_{10}(\theta),\end{aligned} \quad (19)$$

$$\begin{aligned}\left\{\text{the left of (13)}\right\} - \sum_{k=1}^{2m} g_k(\theta)S_{2k}^i(\theta)\\ = S_{20}^i(\theta), \quad (i = 1,\cdots,m),\end{aligned} \quad (20)$$

$$\left\{\text{the left of (14)}\right\} = \mathcal{Q}_3^j, \quad (j = 1,\cdots,l), \quad (21)$$

where

$$g_k(\theta) = \begin{cases} 1 - \theta_k \ (k = 1,\dots,m), \\ \theta_{k-m} \ \ (k = m+1,\dots,2m), \end{cases}$$

then $\mathcal{E}(Q^{-1})$ is a domain of attraction of the closed-loop system (3) and (4).

For one pair of (N_1, N_2), we have one relaxed problem. The optimization scheme is performed for each relaxed problem. (12) and (13) are relaxed to (19) and (20) by using matrix SOS polynomials. On the other hand, it is always enough to relax (14) to (21) by using positive semidefinite matrices because (14) are normal LMI conditions, and are not influenced by $\theta \in \Omega$. $\mathcal{Q}_3^j$ means $S_{30}^j(\theta)$ for $N_3 = 0$. If $h_i(\theta)$ on (13) is θ-affine matrix polynomial, it is possible to consider LMIs on vertices on Ω in stead of matrix SOS relaxation. For non-convex conditions, in general, high-order matrix SOS polynomials are recommended for less conservative relaxation. However, hight-order relaxation sometimes causes the computation burden heavily. If we assume that the burden mainly depends on the size of positive definite matrices $\mathcal{Q}_k$ in the SDP (18), we can say that the order of matrix polynomials $H(\theta)$ and the dimension of the system (3) have an effect on it.

6. A NUMERICAL EXAMPLE

In this section, an analysis example is shown to confirm the effectiveness of our approach. The example is computed by using Matlab and SeDuMi (Strum, 1998) on PC (Pentium 4 processor, 1 GB memory).

For the closed-loop system (3) and (4) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 5 & 2 \end{bmatrix}, \ F = \begin{bmatrix} -2 & -1 \\ 0.5 & -0.3 \end{bmatrix},$$

the shape reference set $\mathcal{X}_R = \text{co}\left\{x^{(1)}, x^{(2)}\right\}$ with $x^{(1)} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $x^{(2)} = \begin{bmatrix} -0.1 & 0.8 \end{bmatrix}^T$, the domain of attraction is estimated by using Corollary 1 and Theorem 2. Corollary 1 is computed by the LMI based approach (Hu and Lin, 2001). Theorem 2 is computed by the SOS based approach discussed in the previous sections. As we see in Theorem 4, the constraints (12), (13), and (14) are correspond to (17) with $m_F = 3$. In (12), $T(\theta)$ with $\sum_{i=1}^m \alpha_i \leq 1$ is adopted. The half degree of $S_{k0}(\theta)$, N_k, should be assigned in advance for $k = 1, 2$, and 3. $(N_1, N_2, N_3) = (1, 1, 0)$ and $(2, 1, 0)$ are tested.

The results are shown in Table 1, Figure 1 and 2. In Table 1, Theorem 2 with $(2, 1, 0)$ is superior to with $(1, 0, 0)$, and is also superior to Corollary 1. Each of the average CPU times of computing the SDP (18) is also shown in this Table. In this example, there is not any improvement of α_{max} even if we use larger number of (N_1, N_2, N_3) than $(2, 1, 0)$, and there is not dramatically improvement of high-order $T(\theta)$ such as with $\sum_{i=1}^m \alpha_i \leq 2$. In Figure 1, we can find that Theorem 2 with $(2, 1, 0)$ enlarges the estimated region compared to Corollary 1. The state trajectories in the estimated region by Theorem 2 with $(2, 1, 0)$ are

Table 1. Maximum α for (N_1, N_2, N_3)

method	Corollary 1	Theorem 2	Theorem 2
$\alpha_{\max}$	5.9180	5.8458	6.2048
(N_1, N_2, N_3)	-	$(1, 1, 0)$	$(2, 1, 0)$
CPU time [**s**]	0.2300	0.3000	0.3640

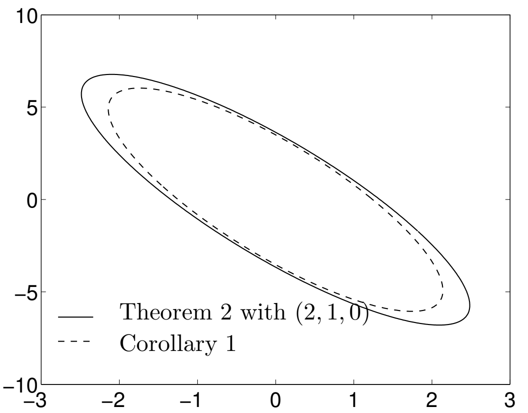

Fig. 1. Estimation of the domain of attraction with different methods

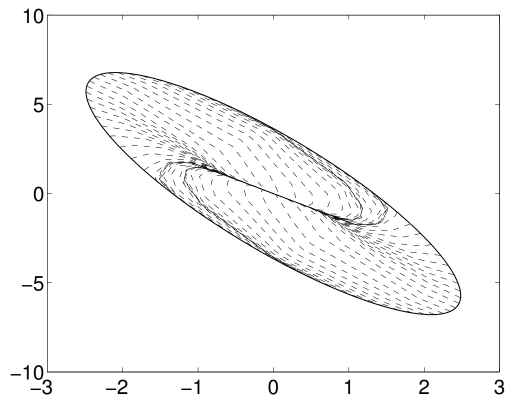

Fig. 2. State trajectories in the estimated region

shown in Figure 2. All trajectories converge to the origin.

7. CONCLUSIONS

In this paper, we have proposed an approach for analysis and design of linear systems with input saturation by using matrix SOS relaxation. Introducing matrix polynomially $v = H(\theta)x$ on the saturation model leads less conservative estimation of the domain of attraction for multiple input systems. This idea is also applicable to design problems. Both analysis and design problems are reduced to robust LMI problems that is solved by the matrix SOS technique. A numerical example for analyzing the closed-loop system shows the effectiveness of the matrix SOS approach.

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