# Theoretical analysis on Bond Automated Market Making

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## 1 Notations and concepts

- **Asset:** can be yield generating asset or not. Typical example is DAI (or cDAI). [However this does not add value]
- **Bond:** Unlike existing protocols where the underlying liability is a debt note (from which bonds are derived), we start with zero coupon bonds right off the bat. [This relates to the burning mechanism]
- **Maturity:** Unlike existing protocols that consider only short term bond maturities (3 month, 6 month and 1 year), we consider monthly-expired bonds with maturities up to 25 years. [To allow leverage]
- **Reference rate:** This is the annualized rate determined from supply and demand. If rate is $r$ then the $t$-year bond price will be $P = e^{-rt}$.
- **Yield curve:** In practice, (annualized) yield curves depend on maturities and typically are increasing functions of maturities. In V1, we consider only flat yield curve, i.e. reference rate is used for all maturities.
- **Liquidity provider (LP):** LPs provide liquidity which is a pair of assets and bonds. Asset is deposited first, then bonds is minted in proportion such that the reference rate remains the same as before providing liquidity. When LPs withdraw liquidity, bonds and assets are withdrawn in proportion so that the reference rate is unchanged too.
- **Accumulated lending:** Just like AAVE, this represents the total net present value of **active bonds** that have been minted up from the pool at the current time (net minted).
- **Accumulated borrowing:** Just like AAVE, this represents the total net present value of **active bonds** that have been sold to the pool current time (net purchase).
- **Bid-Ask spread:** In current AMM models, bid-ask spread is zero but for bond trading, spreads need to exist to cover permanent losses for liquidity providers. Note that this is different from price slippage which is caused by the presence of large orders.
- **Bond AMM (AYC):** the reference rate is determined by a deterministic function of pool states. Pool states include: asset amount, bond amount, accumulated lending and accumulated borrowing. Trading rates are different from reference rates where there are slippage and bid-ask spread involved.
- **Bond virtualization:** Bonds in the pools are virtual in the sense that they can be converted from one maturity to another to meet demand from traders, or can be negative (budget deficit).

- **Bond burning:** Bonds are burnt gradually in such a way that no trading does not cause the reference to change. This is different from Notional Finance.

# 2 Mechanism design and formulas

## 2.1 The state space

The state space of our model is a quintuplet $(x, y, r, B, L)$, where

- $x$ denotes the amount of bonds, $x = (x_1, \ldots, x_n)$ is a vector of bonds with $n$ different maturities $T_1, \ldots, T_n$. We may assume that $T_1 < \ldots < T_n$.
- $y$ denotes the amount of asset (DAI) in the pool. As 1 DAI is worth roughly 1 USD, the total value at time $t$ is $Y_t = y$.
- $w$ denotes the weights of each bond traded in a transaction, $w = (w_1, \ldots, w_n)$. For example, in the case when only the bond with maturity $T_i$ is traded, then $w_i = 1$ and $w_j = 0, \forall j \neq i$, and in the case which the asset is allocated equally among $n$ bonds, then $w_i = \frac{1}{n}, \forall i \in \{1, \ldots, n\}$.
- $r$ denotes the reference interest rate.
- $X$ denotes the net present value (NPV) of all bonds in the pool. The value of $X$ at time $t \leqslant \min\{T_i\}$ with reference $r_t$ is given by

$$X_t = \sum_{i=1}^{n} \left( x_i \cdot e^{-r_t \cdot (T_i - t)} \right) \tag{1}$$

- $B$ denotes the NPV of *active bonds minted to the pool by borrowers (or bond issuers).* By active here we also mean that we have to subtract the amount of bonds that have been burnt by bond issuers for early payment purposes.

$$B_t = \sum_{i=1}^{n} \left( (\mathbf{mint}_i - \mathbf{burnt}_i) \cdot e^{-r_t \cdot (T_i - t)} \right) \tag{2}$$

- $L$ denotes the NPV of *active bonds bought from the protocol.*

$$L_t = \sum_{i=1}^{n} \left( (\mathbf{buy}_i - \mathbf{sell}_i) \cdot e^{-r_t \cdot (T_i - t)} \right) \tag{3}$$

  Here for each maturity, we calculate the cumulative amount of bonds taken out of the pool by buyers and subtract that sold back to the pool by sellers to get the net purchase.

- We also introduce LPs' liquidity contribution,

$$L_{LP} := X - (L - B) \tag{4}$$
$$B_{LP} := Y + (L - B) \tag{5}$$

  This $L_{LP}$ is not the same as the initial liquidity contribution of all LPs because it includes also trading fees and impermanent losses caused by fluctuation of interest rates.

## 2.2 Build interest rates function.

The interest rates function has three major components:

- **Reference rate.** For simplicity, we assume that the reference rate is just a deterministic function of the bond proportion:

$$r = R(p), p = \frac{X}{X+Y} \in [0,1]. \tag{6}$$

The function $R$ should be non-decreasing to reflect the fact that more bond sellers will cause bonds to be more expensive, or rates to be low. ***We may assume that** $R$ **is strictly increasing function with respect to the bond proportion** $p$. A natural choice for $R$ is a linear function:*

$$R(p) = r_{\min} + (r_{\max} - r_{\min}) \cdot p \tag{7}$$

We are to choose the suitable function $R$ later.

- **Bid-ask spread.** In our analysis, we assume that the bid-ask spread is zero.

- **Slippage.** Unlike the case of an AMM, our model requires to solve a system of partial differential equations (PDE) in general case. If the transaction relates to only one maturity, the we only have to solve an ordinary differential equation (ODE) of one variable.

At time $t$, the fixed point equation $r = R(p)$ can be rewritten as

$$\begin{aligned} r &= R\left(\frac{X}{X+Y}\right) \\ \Leftrightarrow r_t &= R\left(\frac{\sum_{i=1}^{n}\left(x_i \cdot e^{-r_t\cdot(T_i-t)}\right)}{\sum_{i=1}^{n}\left(x_i \cdot e^{-r_t\cdot(T_i-t)}\right)+y}\right) \end{aligned} \tag{8}$$

The spot price of a bond with respect to maturity $T_i$ is $e^{-r_t\cdot(T_i-t)}, \forall i \in \{1,\ldots,n\}$. From here we get

$$-w_i \cdot dy = e^{-r_t\cdot(T_i-t)} \cdot dx_i, \forall i \in \{1,\ldots,n\}. \tag{9}$$

**We first consider the case when $n = 1$, and $R$ is a linear function $R(p) = r_{\text{min}} + (r_{\text{max}} - r_{\text{min}}) \cdot p$.** Let $\tau = T - t$. Then we have

$$
\begin{aligned}
r = R\left(\frac{x \cdot e^{-r\cdot\tau}}{x \cdot e^{-r\cdot\tau} + y}\right) &= r_{\text{min}} + (r_{\text{max}} - r_{\text{min}}) \cdot \frac{x \cdot e^{-r\cdot\tau}}{x \cdot e^{-r\cdot\tau} + y} \\
\Leftrightarrow \frac{r - r_{\text{min}}}{r_{\text{max}} - r_{\text{min}}} = \frac{x \cdot e^{-r\cdot\tau}}{x \cdot e^{-r\cdot\tau} + y} &\Leftrightarrow y = \frac{r_{\text{max}} - r}{r - r_{\text{min}}} \cdot x \cdot e^{-r\cdot\tau} \qquad (10)
\end{aligned}
$$

Here we can look at $y$ as a function of $x$ and $r$, $y = y(x, r)$

$$
\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{r_{\text{max}} - r}{r - r_{\text{min}}} \cdot e^{-r\cdot\tau} \qquad (11) \\
\frac{\partial y}{\partial r} &= x \cdot \left(\frac{r_{\text{max}} - r}{r - r_{\text{min}}} \cdot (-\tau) \cdot e^{-r\cdot\tau} - \frac{r_{\text{max}} - r_{\text{min}}}{(r - r_{\text{min}})^2} \cdot e^{-r\cdot\tau}\right) \\
&= -x \cdot \left(\frac{\tau \cdot (r_{\text{max}} - r)}{r - r_{\text{min}}} + \frac{r_{\text{max}} - r_{\text{min}}}{(r - r_{\text{min}})^2}\right) \cdot e^{-r\cdot\tau} \qquad (12)
\end{aligned}
$$

The differential of $y$ is

$$dy = \frac{\partial y}{\partial x} \cdot dx + \frac{\partial y}{\partial r} \cdot dr \qquad (13)$$

We also have the formula for the spot price of the bond

$$-\frac{dy}{dx} = e^{-r\cdot\tau} \qquad (14)$$

From the last four equations, we get

$$
\begin{aligned}
-e^{-r\cdot\tau} \cdot dx &= \frac{r_{\text{max}} - r}{r - r_{\text{min}}} \cdot e^{-r\cdot\tau} \cdot dx - x \cdot \left(\frac{\tau \cdot (r_{\text{max}} - r)}{r - r_{\text{min}}} + \frac{r_{\text{max}} - r_{\text{min}}}{(r - r_{\text{min}})^2}\right) \cdot e^{-r\cdot\tau} \cdot dr \\
\Rightarrow -dx &= \frac{r_{\text{max}} - r}{r - r_{\text{min}}} \cdot dx - x \cdot \left(\frac{\tau \cdot (r_{\text{max}} - r)}{r - r_{\text{min}}} + \frac{r_{\text{max}} - r_{\text{min}}}{(r - r_{\text{min}})^2}\right) \cdot dr
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow x \cdot \left(\frac{\tau \cdot (r_{\text{max}} - r)}{r - r_{\text{min}}} + \frac{r_{\text{max}} - r_{\text{min}}}{(r - r_{\text{min}})^2}\right) \cdot dr &= \frac{r_{\text{max}} - r_{\text{min}}}{r - r_{\text{min}}} \cdot dx \\
\Rightarrow x \cdot \left(\frac{\tau \cdot (r_{\text{max}} - r)}{r_{\text{max}} - r_{\text{min}}} + \frac{1}{r - r_{\text{min}}}\right) \cdot dr &= dx \qquad (15) \\
\Rightarrow \left(\frac{\tau \cdot (r_{\text{max}} - r)}{r_{\text{max}} - r_{\text{min}}} + \frac{1}{r - r_{\text{min}}}\right) \cdot dr &= \frac{dx}{x} \\
\Rightarrow -\frac{\tau \cdot (r_{\text{max}} - r)^2}{2(r_{\text{max}} - r_{\text{min}})} + \ln(r - r_{\text{min}}) + C &= \ln x \\
\Rightarrow x = A \cdot (r - r_{\text{min}}) \cdot e^{-\frac{(r_{\text{max}} - r)^2}{2(r_{\text{max}} - r_{\text{min}})} \cdot \tau} &=: x(r) \qquad (16) \\
\Rightarrow y = A \cdot (r_{\text{max}} - r) \cdot e^{-\frac{(r_{\text{max}} - r)^2}{2(r_{\text{max}} - r_{\text{min}})} \cdot \tau - r\cdot\tau} &=: y(r) \qquad (17)
\end{aligned}
$$

Now, we come back to our general case. From the fixed point equation, if we define $\tau_i = T_i - t$, then at time $t$, we have

$$y = \frac{1 - R^{-1}(r)}{R^{-1}(r)} \cdot \sum_{i=1}^{n} \left( x_i \cdot e^{-r \cdot \tau_i} \right) \tag{18}$$

Define $U(r) := \frac{1-R^{-1}(r)}{R^{-1}(r)}$.

We can consider $y$ as a function of $n+1$ variables $x_1, \dots, x_n, r$: $y = y(x_1, \dots, x_n, r)$.

We have

$$\frac{\partial y}{\partial x_i} = U(r) \cdot e^{-r \cdot \tau_i} \tag{19}$$

$$\begin{aligned} \frac{\partial y}{\partial r} &= U'(r) \cdot \sum_{i=1}^{n} \left( x_i \cdot e^{-r \cdot \tau_i} \right) + U(r) \cdot \sum_{i=1}^{n} \left( -\tau_i \cdot x_i \cdot e^{-r \cdot \tau_i} \right) \\ &= \sum_{i=1}^{n} \left[ \left( U'(r) - \tau_i \cdot U(r) \right) \cdot x_i \cdot e^{-r \cdot \tau_i} \right] \end{aligned} \tag{20}$$

The differential of $y$ is

$$dy = \sum_{i=1}^{n} \left( \frac{\partial y}{\partial x_i} \cdot dx_i \right) + \frac{\partial y}{\partial r} \cdot dr \tag{21}$$

$$\Rightarrow \frac{dy}{dx_i} = \frac{\partial y}{\partial x_i} + \frac{\partial y}{\partial r} \cdot \frac{dr}{dx_i}, \forall i \in \{1, \dots, n\}. \tag{22}$$

From

$$\begin{aligned} -w_i \cdot dy &= e^{-r \cdot \tau_i} \cdot dx_i \\ \Rightarrow w_i \cdot \left( \frac{\partial y}{\partial x_i} \cdot dx_i + \frac{\partial y}{\partial r} \cdot dr \right) &= -e^{-r \cdot \tau_i} \cdot dx_i \\ \Rightarrow w_i \cdot \left( U(r) \cdot e^{-r \cdot \tau_i} \cdot dx_i + \sum_{j=1}^{n} \left[ \left( U'(r) - \tau_j \cdot U(r) \right) \cdot x_j \cdot e^{-r \cdot \tau_j} \right] \cdot dr \right) &= -e^{-r \cdot \tau_i} \cdot dx_i \\ \Rightarrow \sum_{j=1}^{n} \left[ \left( U'(r) - \tau_j \cdot U(r) \right) \cdot x_j \cdot e^{-r \cdot \tau_j} \right] \cdot dr &= -(1 + w_i \cdot U(r)) \cdot e^{-r \cdot \tau_i} \cdot dx_i \\ \Rightarrow -\sum_{j=1}^{n} \left[ \frac{U'(r) - \tau_j \cdot U(r)}{1 + w_i \cdot U(r)} \cdot x_j \cdot e^{-r \cdot (\tau_j - \tau_i)} \right] &= \frac{dx_i}{dr} \\ \Rightarrow \frac{dx_i}{dr} = -\sum_{j=1}^{n} \left[ \left( \frac{U'(r) - \tau_j \cdot U(r)}{1 + w_i \cdot U(r)} \cdot e^{-r \cdot (T_j - T_i)} \right) \cdot x_j \right] \end{aligned}$$

Define

$$f_{ij}(r) := -\frac{U'(r) - \tau_j \cdot U(r)}{1 + w_i \cdot U(r)} \cdot e^{-r \cdot (T_j - T_i)}, \forall i, j \in \{1, \dots, n\} \tag{23}$$

Then

$$\frac{dx_i}{dr} = \sum_{j=1}^{n} f_{ij}(r) \cdot x_j$$

$$\Rightarrow \frac{dx}{dr} = A \cdot x, \text{ where } x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \text{ and } A \text{ is a } n \times n \text{ matrix}, A_{ij} = f_{ij}(r). \tag{24}$$

***Consider the case when one of the $w_i$s is 1 and others are 0. Assume that $w_i = 1$ and $w_j = 0, \forall j \neq i$, i.e. we only perform trade only on the bond with maturity $T_i$.***

Then we have the following equations:

$$dx_j = 0, \forall j \neq i \tag{25}$$

$$dy = \sum_{j=1}^{n} \left( \frac{\partial y}{\partial x_j} \cdot dx_j \right) + \frac{\partial y}{\partial r} \cdot dr = \frac{\partial y}{\partial x_i} \cdot dx_i + \frac{\partial y}{\partial r} \cdot dr \tag{26}$$

$$\frac{\partial y}{\partial x_i} = U(r) \cdot e^{-r \cdot \tau_i} \tag{27}$$

$$\frac{\partial y}{\partial r} = \sum_j \left[ (U' - \tau_j \cdot U) \cdot e^{-r \cdot \tau_j} \cdot x_j \right] \tag{28}$$

$$dy = -\sum_j \left( w_j \cdot e^{-r \cdot \tau_j} \cdot dx_j \right) = -e^{-r \cdot \tau_i} \cdot dx_i \tag{29}$$

Note that

$$U(r) = \frac{1 - R^{-1}(r)}{R^{-1}(r)} = \frac{1}{R^{-1}(r)} - 1 \geqslant \frac{1}{1} - 1 = 0$$

$$U'(r) = \frac{-[R^{-1}]'(r)}{(R^{-1}(r))^2} \leqslant 0$$

(note that $R$ is an increasing function, so is $R^{-1}$, hence $[R^{-1}]' > 0$)

Define $V_j := \tau_j \cdot U - U'$, we have $V_j > 0$. From here,

$$\frac{\partial y}{\partial r} < 0, \forall r; \frac{\partial y}{\partial r} = -\sum_j V_j \cdot e^{-r \cdot \tau_j} \cdot x_j < 0, \tag{30}$$

$$\tag{31}$$

Combining (26), (27), (29), (30), we get

$$-e^{-r \cdot \tau_i} \cdot dx_i = U \cdot e^{-r \cdot \tau_i} \cdot dx_i - \sum_j V_j \cdot e^{-r \cdot \tau_j} \cdot x_j$$

$$\Rightarrow \frac{dx_i}{dr} = \frac{\sum_j V_j(r) \cdot e^{r \cdot (\tau_i - \tau_j)} \cdot x_j}{1 + U(r)} \tag{32}$$

$$= \sum_{j \neq i} \frac{V_j(r)}{1 + U(r)} \cdot e^{r \cdot (\tau_i - \tau_j)} \cdot x_j + \frac{V_i(r)}{1 + U(r)} \cdot x_i \tag{33}$$

$$= A_i(r) \cdot x_i + B_i(r), \tag{34}$$

$$\text{with } A_i(r) = \frac{V_i(r)}{1 + U(r)}, B_i(r) = \sum_{j \neq i} \frac{V_j(r)}{1 + U(r)} \cdot e^{r \cdot (\tau_i - \tau_j)} \cdot x_j$$

Its solution is

$$x_i = e^{\int A_i(r) dr} \cdot \left[ \int \left( e^{-\int A_i(r) dr} \cdot B_i(r) \right) dr + C \right] \tag{35}$$

With

$$
\begin{aligned}
R(p) &= r_{\min} + (r_{\max} - r_{\min}) \cdot p \\
\Rightarrow R^{-1}(r) &= \frac{r - r_{\min}}{r_{\max} - r_{\min}} = \frac{r - r_{\min}}{\Delta r}, \\
\text{with } \Delta r &:= r_{\max} - r_{\min} \\
\Rightarrow 1 - R^{-1}(r) &= \frac{r_{\max} - r}{\Delta r}, \\
\Rightarrow U(r) &= \frac{r_{\max} - r}{r - r_{\min}} = \frac{\Delta r}{r - r_{\min}} - 1, \\
\Rightarrow U'(r) &= -\frac{\Delta r}{(r - r_{\min})^2}, \\
\Rightarrow V_j &= \tau_j \cdot U - U' = \tau_j \cdot \left( \frac{\Delta r}{r - r_{\min}} - 1 \right) + \frac{\Delta r}{(r - r_{\min})^2} \\
\Rightarrow \frac{V_j}{1+U} &= \frac{\tau_j \cdot \left( \frac{\Delta r}{r - r_{\min}} - 1 \right) + \frac{\Delta r}{(r - r_{\min})^2}}{\frac{\Delta r}{r - r_{\min}}} = \tau_j - \tau_j \cdot \frac{r - r_{\min}}{\Delta r} + \frac{1}{r - r_{\min}} \qquad (36) \\
\Rightarrow A_i &= \frac{V_i}{1+U} = \tau_i - \tau_i \cdot \frac{r - r_{\min}}{\Delta r} + \frac{1}{r - r_{\min}} \qquad (37) \\
\Rightarrow \int A_i(r) dr &= \tau_i \cdot r - \tau_i \cdot \frac{(r - r_{\min})^2}{2\Delta r} + \ln(r - r_{\min}) + C_1, \text{let } C_1 = 0 \\
\Rightarrow e^{-\int A_i(r) dr} &= e^{-\tau_i \cdot r + \tau_i \cdot \frac{(r - r_{\min})^2}{2\Delta r} - \ln(r - r_{\min})} \\
\Rightarrow B_i(r) \cdot e^{-\int A_i(r) dr} &= \sum_{j \neq i} \frac{V_j(r)}{1 + U(r)} \cdot e^{r \cdot (\tau_i - \tau_j)} \cdot x_j \cdot e^{-\tau_i \cdot r + \tau_i \cdot \frac{(r - r_{\min})^2}{2\Delta r} - \ln(r - r_{\min})} \\
&= \sum_{j \neq i} \left( \tau_j - \tau_j \cdot \frac{r - r_{\min}}{\Delta r} + \frac{1}{r - r_{\min}} \right) \cdot e^{-r \cdot \tau_j + \tau_i \cdot \frac{(r - r_{\min})^2}{2\Delta r} - \ln(r - r_{\min})} \cdot x_j \\
&= \sum_{j \neq i} \left( \frac{\tau_i - \tau_j}{\Delta r} \cdot (r - r_{\min}) + f'_{ij}(r) \right) \cdot e^{-f_{ij}(r)} \cdot x_j, \qquad (38) \\
\text{with } f_{ij}(r) &:= r \cdot \tau_j - \tau_i \cdot \frac{(r - r_{\min})^2}{2\Delta r} + \ln(r - r_{\min}) \\
\Rightarrow B_i(r) \cdot e^{-\int A_i(r) dr} &= \sum_{j \neq i} \left( \frac{\tau_i - \tau_j}{\Delta r} \cdot e^{-r \cdot \tau_j + \tau_i \cdot \frac{(r - r_{\min})^2}{2\Delta r}} \cdot x_j + f'_{ij}(r) \cdot e^{-f_{ij}(r)} \cdot x_j \right),
\end{aligned}
$$

$$
\Rightarrow \int \left[ B_i(r) \cdot e^{-\int A_i(r) dr} \right] dr = \sum_{j \neq i} x_j \cdot \left[ -e^{-f_{ij}(r)} + \frac{\tau_i - \tau_j}{\Delta r} \cdot \int_{r_0}^{r} e^{-\tau_j \cdot r + \frac{\tau_i}{2\Delta r} \cdot (r - r_{\min})^2} dr \right],
$$

with $\int_{r_0}^{r} e^{-\tau_j \cdot r + \frac{\tau_i}{2\Delta r} \cdot (r - r_{\min})^2} dr$ is an integral of a Gaussian function.

**We are to deal with the following questions:**

- **What is the financial basis of this system of differential equations when we perform a transaction with multiple bonds at the same time, what is the relationship between these rates?**
- **Can we get a solution which leads to a path-independent curve?**
- **Can we perform batching multiple transactions together?**

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