Automated Yield Curve: A new design of decentralized bond exchange

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In this chapter, I propose a new design for bond exchange. This design aggregates both lending and trading functionalities into a unique flatform, which provides new instruments for asset management.

1 Introduction

In traditional finance, fixed income markets, though attract attention mainly from institutional traders, account for about three times in size as large as global equity markets. In DeFi however, the size of the lending sector is negligible compared to the the whole crypto market. The major reason is that crypto lending requires collateral, so the TVL of any lending protocol can not surpass the total market capitalization of the collateral assets. Another reason is, lending with fixed rates are not mature enough, and bond markets even have de facto not existed yet. Therefore, fixed income markets have not attracted enough trading volume.

Generally speaking, fixed rate lending protocols in DeFi are still badly designed. Let's take an example, AAVE has fixed rate borrowing (not lending), but the rate is prohibitively expensive, so fixed rate borrowing accounts for only less than 1% of total borrowing volume. Recently, Notional Finance has become top one fixed rate lending protocol on Ethereum, but due to their bad economical design, borrowing in Notional Finance is still more expensive than in AAVE or Compound Finance, which makes it less attractive to borrowers. At the other extreme, Yield Protocol, which is another important fixed rate protocol on Ethereum, offers cheaper lending rates compared to AAVE or Compound Finance, which makes it less attractive to lenders. These two extremes lower trading volume to the existing protocols and hence, lower revenues to LPs. Not only that, these protocols force the borrowing rates equal to lending rates, which will obviously cause losses to LPs. In reality, except when capital is fully utilized, otherwise bid-ask spreads always exist due to imbalance of capital supply and demand. This is a fundamental principle in building an economically sound and solid bond protocol. From banking practice, **no banks would survive if they charge borrowers the same rates they pay to lenders**. Even if we take lending/borrowing fees into account, these fees are still not enough to cover for the lending-borrowing imbalance and for the cost of inactive capital of LPs (just for comparison purpose: AAVE clearly obtains more fees while its bid-ask spreads still exist). To cover for these impermanent/

inactive cash losses, there have to be extra trading fees, which can only come from traders who use the protocol as a means of trading/investment and not just lending/borrowing. However, with the current design where only a few short term maturities are active (3 month, 6 month and 1 year- i.e. being less flexible than AAVE/Compound and lack of leverage), these protocols clearly cannot meet investment need from traders but rather a fixed-rates add-on to AAVE or Compound.

To be more precise, bond markets are different from lending markets in that they allow investors to speculate and bet on the future fluctuation of interest rates (or, equivalently, yield curve). Long duration bonds not only help investors lock in their future gains, but also help leverage their bets (as the price of long duration bonds is more sensitive with respect to the changes in yield than shorter counterparts. As a matter of fact, **a bond of 10Y maturity offers more than 10 times leverage compared to an 1Y bond due to the compound effect, which is not the case for futures**). These are exactly features we want our bond protocol to possess.

To summarize, there are several drawbacks of current fixed rate lending protocols.

- Permanent loss for LPs even if rates remain constant.
- Inactive assets (used for liquidity purposes) causes relative losses to LPs compared to lenders.
- Rates are not better than that on AAVE or Compound, either for lenders or borrowers.
- There's a lack of maturity flexibility which will lower LPs' revenues. Therefore, it does not even provide enough borrowing/lending service for users in need. Not only that, each maturity correspond to a pool, which might cause pools to behave independently.
- They do not offer inter-maturity bond trading features.

In this note, I propose a protocol that embraces both lending and trading functionalities. This can be considered to be a decentralized bond exchange (DBX). The closest to our idea is Notional Finance (NF). However, there are several key differences:

- NF serves as a lending protocol only, while we have bond trading functionality.
- Lending maturities in NF are currently short term (restricted to 3 month, 6 month and 1 year only), which makes it difficult for investors to leverage their investment. Our protocol allows long maturities up to 25 years and can be made flexible at will.
- In NF, lending and borrowing rates are equal by design (if price slippage is ignored), which causes LPs to suffer from losses due to lack of liquidity. In our protocol, this condition is relaxed.

2 Notations and concepts

- **Asset:** can be yield generating asset or not. Typical example is DAI (or cDAI). [However this does not add value]
- **Bond:** Unlike existing protocols where the underlying liability is a debt note (from which bonds are derived), we start with zero coupon bonds right off the bat. [This relates to the burning mechanism]
- **Maturity:** Unlike existing protocols that consider only short term bond maturities (3 month, 6 month and 1 year), we consider monthly-expired bonds with maturities up to 25 years. [To allow leverage]
- **Reference rate:** This is the annualized rate determined from supply and demand. If rate is r then the t-year bond price will be $P = e^{-rt}$.
- **Yield curve:** In practice, (annualized) yield curves depend on maturities and typically are increasing functions of maturities. In V1, we consider only flat yield curve, i.e. reference rate is used for all maturities.
- **Liquidity provider (LP):** LPs provide liquidity which is a pair of assets and bonds. Asset is deposited first, then bonds is minted in proportion such that the reference rate remains the same as before providing liquidity. When LPs withdraw liquidity, bonds and assets are withdrawn in proportion so that the reference rate is unchanged too.
- **Accumulated lending:** Just like AAVE, this represents the total net present value of **active bonds** that have been minted up from the pool at the current time (net minted).
- **Accumulated borrowing:** Just like AAVE, this represents the total net present value of **active bonds** that have been sold to the pool current time (net purchase).
- **Bid-Ask spread:** In current AMM models, bid-ask spread is zero but for bond trading, spreads need to exist to cover permanent losses for liquidity providers. Note that this is different from price slippage which is caused by the presence of large orders.
- **Bond AMM (AYC):** the reference rate is determined by a deterministic function of pool states. Pool states include: asset amount, bond amount, accumulated lending and accumulated borrowing. Trading rates are different from reference rates where there are slippage and bid-ask spread involved.
- **Bond virtualization:** Bonds in the pools are virtual in the sense that they can be converted from one maturity to another to meet demand from traders, or can be negative (budget deficit).
- **Bond burning:** Bonds are burnt gradually in such a way that no trading does not cause the reference to change. This is different from Notional Finance.

3 Existing models

Floating rates. The lending pool model proposed by AAVE and Compound Finance is not only the most important one in the lending space, but also the only one that relies on sound and solid economical reasoning. In this model, the borrowing interest rate is designed as an increasing, deterministic function of the capital utilization ratio, which reflects the supply and demand of the capital market. Recall that,

$$r_t^B = F(U_t) = F(\frac{B_t}{L_t}),$$

where B and L denote the total borrowing and lending volume of the pool at time t. If fees are ignored, then the conservation equation below that links lending and borrowing rates should satisfy all the time:

$$r_t^L = U_t r^B t,$$

The only drawback of this model is that rates fluctuate over time, while lenders and borrowers often requires certainty for their income inflow or expense outflow. This is the main reason of which many subsequent (fixed rate) lending protocols have born out.

Fixed rates AMM models. The major players in this space are Yield Protocol and Notional Finance. In this model, lenders are bond buyers, borrowers are bond issuers (backed by collateral). The lending pool now becomes a trading pool with an AMM that determines the interest rates in real time depending on the market suppy/demand of capital. Liquidity providers (LPs) deposit assets and bonds (actually not bonds but debt notes) and receive trading fees as rewards for their liquidity contribution.

The AMM model of Yield protocol has the form

$$x^{1-t} + y^{1-t} = C,$$

where x, y denote the total liquidity of assets and debt notes available in the trading pool, and t denote time to maturity of bonds (unique maturity). This equation is equivalent to $r = \frac{y}{x} - 1$.

The AMM model of Notional Finance has the form

$$r = R(\frac{y}{x+y}),$$

where R is a logistic-like function. We can see that the pricing mechanism of both Yield and Notional is somewhat similar to Uniswap V1. However, as we show in the sequel, **this model suffer from permanent loss** even if the interest rate stays constant due to the fact that LPs absorb all bid-ask spread. **In practice, no bank would survive if lending rates are equal to borrowing rates, therefore this model is a flaw if lenders are more populous than borrowers**.

Split principle and yield. There are many protocols in this vain: Pendle Finance, Element Finance, Sense protocol, Swivel, just to name a few. The central idea is that, given a yield-generating token (for example, debt note from Compound), one can split the token into two parts: principle and yield, and tokenize them. This means you can sell the future floating yield of your token for a fixed income, and this fixed yield is determined by supply and demand on the market, while the principle can be redeemed at maturity to receive an amount equal to the current asset value. The principle part can thus be viewed as a zero coupon bond. Due to the no arbitrage argument, the discount rate applied on the principle should be equal to the fixed rate to exchange for the floating yield. For example, if your one year floating yield is sold at 4%, then your principle will be priced at 4% discount.

Apparently, Pendle Finance et al. seem to offer a real bond trading flatform. But in fact, they don't. What they really offer is an interest rate swap service, where floating rates come from underlying lending protocols like Compound. Therefore, **they don't act as price discovery leader**. Plus, as all bonds are short term, they are essentially a means of lending/borrowing and not for trading/investment.

Structured products. There are BarnBridge and Tranche Finance in this category. If we view floating interest rates (like AAVE, Compound) as revenue flows of a company, then we can bring back the classical idea of corporate financing into the DeFi space, where investors are split into two tranches: seniors (or debt holders) and juniors (or equity holders). Seniors and Juniors deposit their money into a fund, and this fund will be deposited into AAVE or Compound. Depending on the proportion of each type of tranche in the fund, the fixed rate can be determined accordingly. Typically, debt holders enjoy low and fixed interest rates, while equity holders enjoy whatever is left on the table. From a mathematical point of view, equity holders take long position on a European option.

Capital efficiency. Morpho protocol (18M fund raising) is a successful lending app that tries to solve the capital efficiency problem. Essentially, they want borrowers and lenders to enjoy the same interest rate. To do that, they build an add-on dApp on top of Compound so that if users use Morpho, they have the chance to match each other (borrowers matching with lenders), and when they match, they enjoy the mid interest rate. If they do not find someone to match, they still enjoy the Compound rates by default.

Apparently, this sounds a smart idea. However, in the long run, it will destroy the underlying lending protocol. It is not difficult for lenders and borrowers to see the arbitrage opportunities between Morpho and Compound, so eventually users will register to Morpho as it costs virtually nothing to do so. Mathematically, if B_0, L_0 are potential borrowing and lending volume of the underlying protocol, and there is an amount of V matching volume using Morpho, then the utilization ratio will decay from $U_0 = \frac{B_0}{L_0}$ to

$$U_1 = \frac{B_0 - V}{L_0 - V} < U_0.$$

When $V \to B_0$, we have $U_1 \to 0$, i.e. the interest rate from the underlying protocol will eventually decay to zero, consequently, the mid-rate offered by Morpho converges to zero too. This means Morpho does more harm than good for the underlying protocol. This is a typical example of bad economic design which is **essentially a sort of parasite dApp and has no intrinsic value to the DeFi space**. But no doubt, this is a clever idea. It is somewhat similar to Ponzi scheme in the lending space, as early users take advantages from late users in terms of interest rates. Generally speaking, all sort of lending protocols that rely on an underlying protocol to increase capital efficiency will more or less do harm to that underlying protocol and to themselves too.

4 Mechanism design: an economic analysis

4.1 Permanent loss and consequences

To better understand the idea, let us start with an intuitive example. Assume Alice is the unique LP of the pool and initially, Alice deposits 100 DAI to the pool and mints 100 DAI worth of bonds to ensure the balance of the pool (just like Uniswap V1, however this assumption can be relaxed). Later, there are Bob who wants to buy 20 DAI worth of bonds, and Charlie who wants to sell 10 DAI worth of bonds. We assume that they come in at the same time and enjoy the same bond-implied interest rate (let's ignore price slippage for simplicity). After their trades, the positions in the pool now are 100+20-10=110 DAI in asset and 100-20+10=90 DAI worth of bonds. The net position will be 20-10=10 DAI, which is a credit to Alice, i.e. Alice has to pay out interest rates on this 10 DAI amount if the bid-ask spread on bond prices is zero. If the net position is stationary around the level 10 DAI, Alice will consistently loses money due to paying interest rates, even if the interest rate implied from the bond pool is roughly constant, for example, if the bond-implied interest rate is 10%, then Alice will lose 1 DAI per year. This loss is permanent to Alice. As a matter of fact, borrowing total amount tends to be smaller than lending total amount, for instance, AAVE and Commpound Finance maintains at roughly 0.5 for utilization ratio, which means liquidity providers can suffer from permanent loss even at typical conditions. This is very different from AMM models where impermanent loss is close to zero if the price process is mean-reverting or stationary.

From an economic equilibrium viewpoint, when LPs consistently suffer from permanent loss, it will discourage new users to become LPs when it is less attractive to do so, i.e. new users will choose to become bond buyers, and existing LPs will withdraw their liquidity from the pool, which causes the pool's liquidity to become thinner and thinner. Consequently, this behaviour will push the system fall into a very bad Nash equilibrium: net position (and hence debt position) increases to the point that the system might collapse. From the above example, if the total liquidation reduces to 10 DAI and the net position increases to 100 DAI, and if the rate is still 10% then all LPs will run out of capital after one year. In practice, the situation might not be that severe because lending rates tend to decrease when there are more bond buyers (or equivalently, lenders), but as the TVL of the bond pool is small due to the withdrawal of LPs, slippage will increase which will discourage users to come to the pool and consequently, low incentives are paid to LPs, which is also an unexpected outcome. This is exactly the case of Notional Finance currently, where incentives by native tokens will soon not be able to cover permanent losses for LPs.

However, the impact of net position is asymmetric. If the net position is negative, i.e. LPs receive interest rates from excessive borrowers. For instance, if Bob buys 10 DAI worth of bonds and Charlie sells 20 DAI worth of bonds at the same rates of 10% per annum, then Alice will receive 10*10%=1 DAI per annum from Charlie. In this case, LPs are profitable, they can be considered partial lenders and the system will fall into a robust Nash equilibrium, where new users will tend to become LPs until the average return of an LP becomes no longer more profitable than that of an average lender. This is exactly the equilibrium state that we want the system to attain.

To conclude, we want the system to attain the equilibrium state where bond issuers (or borrowers) are more populous than bond buyers (or lenders). But how can we achieve that when lenders tend to be more than borrowers in practice, as a matter of fact? The answer is, we need to make sure that LPs' revenue is so attractive that there would be fewer lenders (or bond buyers) than LPs in the pool. To remedy this, we need three things.

- First, permanent loss should not be significant to LPs (otherwise they will withdraw from the bond protocol).
- Second, there should be new sources of income from trading fees to incentivize more LPs.
- Third, extra incentives should be in place at the beginning, when the revenues are not enough.

As for the first item, we need to design a pricing mechanism so that users (or bond buyers/sellers) and LPs share the permanent losses.

As for the second item, we need to increase the trading volume. To do that, bonds should be traded as an investment instruments, not merely a lending tool like in existing fixed rates protocols. **If trading volume is comparable to that in a typical lending protocol, then there is no way to cover the permanent loss of LPs or the bid-ask spreads for users.**

As for the third item, we need to issue native tokens and incentivize LPs more instead of users in the bootstrapping phase of the protocol. These incentives should be enough to draw attention of capital suppliers towards becoming LPs rather than becoming bond buyers. **Thus incentives will be a non-decreasing function of the lenders-to-LPs ratio.**

4.2 Inter-markets Nash equilibrium

What would be the equilibrium state of the lending market when both bond market and AAVE-like lending market co-exist? Assume that at equilibrium, the borrowing rate and lending rate from the lending market are $r_b > r_l$. (Typically, $r_b = 2r_l$ as the utilization ratio is usually around 0.5). In this case, if the equilibrium rate from the bond market is $r > r_b$, then no or very few borrowers will be interested in the bond market as it is more expensive than the lending market. Low proportion of bond issuers will lower the bond rate until $r \leq r_b$. Symmetrically, if $r < r_b$ then bond buyers will find selling bond less profitable than lending. Hence, they will participate the lending market until $r \geq r_l$. Finally, $r \in [r_l, r_b]$. However, in this case, the bond market becomes more attractive than the lending market except the only one case where $r = r_b = r_l$. In this equilibrium, as $r_b = r_l$, the utilization ratio from the lending market is $U = 1$ (or close to one, as redundant liquidity is needed for withdrawal).

However, in practice the convergence of two markets will progress slowly as the existing lending market is much larger than a new bond market. Therefore, in the beginning phase, to make the bond protocol more attractive than the lending market, we can only ensure that $r \in [r_l, r_b]$, (though in practice r can be slightly higher than r_b or slightly smaller than r_l as lenders/borrowers usually appreciate certainty), then at the same time, bond buyers b enjoy higher rates than lenders (as $r > r_l$) and bond sellers s enjoy lower rates than borrowers (as $r < r_b$). The total surplus value (per unit of time) of the users, compared to the lending market, will be

$$S = b(r_l - r) + s(r_b - r) > 0$$

So the question is, where does this surplus value S come from? No free lunch principle in financial market makes sure that this surplus should come from new traders. Otherwise the bond market is just a rearrangement of the lending market with the loss absorbed by LPs. This is exactly the case of existing fixed rates lending protocols.

Now let's have some math to clarify the ideas. Assume that the capital demand/supply function are $\mathcal{B}(r)$ and $\mathcal{L}(r)$. In the case of lending market, there are two types of participants:

- Lenders with equilibrium lending rate of r_l.
- Borrowers with equilibrium borrowing rate of r_b.

The equilibrium rates are such that borrowers pay interest rates for lenders (for now we ignore fees for sake of simplicity).

$$r_b \mathcal{B}(r_b) = r_l \mathcal{L}(r_l) \to r_l = U(r_b, r_l) r_l.$$

The function U is commonly called utilization ratio and is defined by $U(x, y) = \frac{\mathcal{B}(x)}{\mathcal{L}(y)}$. For sake of simplicity, we denote $U(x) = U(x, x)$.

Now for the bond market, to make the two models comparable, let us assume that there are four types of participants:

- LPs: they are part of capital suppliers with proportion $\beta \in [0, 1] : L_{LP} = \beta \mathcal{L}(r)$. They mint the equal amount of bonds in the bond pool: $B_{LP} = \beta \mathcal{L}(r)$.
- Bond buyers, act as organic lenders: they are capital suppliers too. $L_L = \alpha \mathcal{L}(r), \alpha = 1 - \beta$.
- Bond issuers, act as organic borrowers. $B_L = \mathcal{B}(r)$.
- Traders: they are bond buyers and sellers and they trade to make profit from one another. We assume that at equilibrium the trade imbalance is zero, and at given point in time their trading volume is negligible compared to organic lenders/borrowers' volume, but in the long run their total trading volume is significant and their trading fees make up the main source of incomes for LPs. It is not a trivial task to separate borrowers/lenders from traders, but moving averages of net purchase can be used to identify borrowers/lenders' organic volumes. Moreover, traders tend to use bonds with long maturities for leverage purposes, which is different from crypto lenders or borrower whose needs are typically short term.

Denote r_{LP} the average return of an LP and r_F the average return collected from fees. At equilibrium state, capital supply will be allocated such that return of LPs and bond buyers (lenders) are identical, i.e. capital suppliers are indifferent of becoming LPs or lenders.

$$r_{LP} = r,$$

where

$$r_{LP} = r_F + \frac{\alpha - U(r)}{\beta} r.$$

The second term from the above equation is permanent loss to LP in terms of ROI, which comes from (per unit of time)

$$\textit{(bid-ask) Permanent Loss} = \frac{\alpha \mathcal{L}(r) r - \mathcal{B}(r) r}{\beta \mathcal{L}(r)}.$$

Note that this permanent loss is caused by bid-ask spread, it exists even when rates are constant. This is not the same as the common impermanent loss which is caused by the fluctuation of interest rates.

Note that if we account for trading fees of lenders and borrowers, then the real revenues of LPs should include a final term

$$\textit{Permanent Fees} = \epsilon \left[\frac{\alpha + U(r)}{\beta} \right] r.$$

However, these fees are small compared to r_F. **For existing fixed rate protocols, basically there are only permanent fees and no r_F from extra traders.**

At equilibrium, lenders have the same revenue as LPs, i.e. $r_{LP} = r$. Solve this equation for β we obtain.

$$\beta = \frac{[1 - U(r)] r}{2r - r_F}.$$

Under typical conditions, we have $\max_{r \in [r_l, r_b]} U(r) = U(r_l) < 1$. Hence from the above formulae, we can show that if r_F is large enough, for instance $r_F > 2r_b$ then $r_{LP} > r \, \forall \beta > 0$, which means that **if trading fees are large enough, then no users will choose to become lenders because being LP is more profitable**.

To find the equilibrium rate r, we need to design the rate function. Denote X, Y the total value of asset and bonds currently in the bond pool, respectively. We have

$$X = \beta \mathcal{L}(r) + \alpha \mathcal{L}(r) - \mathcal{B}(r) = \mathcal{L}(r) - \mathcal{B}(r),$$

$$Y = \beta \mathcal{L}(r) - \alpha \mathcal{L}(r) + \mathcal{B}(r) = 2\beta \mathcal{L}(r) - \mathcal{L}(r) + \mathcal{B}(r).$$

The bond proportion in the pool is defined by

$$p = \frac{Y}{X + Y}.$$

Using the definition of U and the formulae for β above, we can show that

$$p = 1 - \frac{1 - U(r)}{2\beta} = \frac{r_F}{2r}.$$

Let's take a simple linear interest rate model:

$$r = ap + b, a > 0, b > 0.$$

Solve this equation for r we obtain

$$r = \frac{b + \sqrt{b^2 + 2ar_F}}{2}.$$

What interesting is that, this equilibrium solution does not depend on the supply and demand function. It only depends on the protocol parameters and the revenues of LPs. We have some comments.

- The function $r_F \to r$ is increasing in r_F. Therefore, when LPs' revenues increase, it is better for lenders (bond buyers) and worse for borrowers (bond issuers).
- As $r \geq r_l > b$, there will be a lower bound for r_F below which r will fall outside $[r_l, r_b]$. More precisely

$$(r_F)_{\min} = \frac{2r_l(r_l - b)}{a} = \frac{2r_l(r_l - r_{\min})}{r_{\max} - r_{\min}}.$$

  This means that **if the revenues of LPs are small enough, the model will collapse.**
- It is straightforwards to prove that at equilibrium, we can consider β as a function of r_F (as $\beta = \beta(r, r_F)$ and $r = r(r_F)$), and that this function $r_F \to \beta$ is increasing. This means that if LPs' revenues increase, there will be more capital suppliers decide to become LPs. On the contrary case, **if revenues of LPs' drop, there will be existing LPs who withdraw liquidity from the pool to either become bond buyers or to quit the protocol.**

Optimal design. We want to know how to design the interest rate model to attain the maximal surplus value for the protocol. From the previous analysis, the total revenue of the LPs at equilibrium is $\beta\mathcal{L}(r)r$. As both β and $\mathcal{L}$ are increasing function of r, this total revenue is maximized when r is maximized. As $r \in [r_l, r_b]$, the optimal equilibrium rate is $r^* = r_b$, which theoretically maximizes the objective function $x\mathcal{B}(x)$. This result is consistent with the lending pool models. In practice, a good proxy for r_b is the equilibrium borrowing rate of top lending protocols as AAVE and Compound Finance. The coefficients a, b of the interest rate function should be such that

$$\frac{b + \sqrt{b^2 + 2ar_F}}{2} = r_b.$$

r_F can be estimated from real data and can be updated over time, while a, b are determined as $b = r_{\min}, a = r_{\max} - r_{\min}$ which are control parameters of the model.

Switching between two markets. An intuitive question is, when we know the proportion (and size) of lenders and borrowers from a lending market

(market 1) like AAVE, can we estimate the ones for a bond market (market 2) that we are building? We assume that AAVE has been designed optimally, i.e. the borrowing rate maximizes the revenues of the protocol.

$$r_b = \arg\max_r r\mathcal{B}(r).$$

The capital utilization ratio at equilibrium is supposed to be $U := U(r_b, r_l) < 1$. If the two markets co-exist and competitive, rates should be the same: $r = r_l = r_b$. Thus the capital demand/supply total volume of the two markets are the same too: $B_1 = B_2 = \mathcal{B}/2, L_1 = L_2 = \mathcal{L}/2 = \frac{\mathcal{B}}{2U}$ (as users are indifferent between the two markets). Now, excessive lenders move from the lending market to the bond market to ensure no bid-ask spreads for the lending market $\Delta L = \frac{\mathcal{L}-\mathcal{B}}{2}$. Therefore, the total capital supply in the bond market will be

$$L_2^{new} = L_2 + \Delta L = \mathcal{L} - \frac{\mathcal{B}}{2}.$$

If we want LPs to not suffer from permanent loss, then $L_2^{buyer} = B_2$, hence the total size of LPs should be

$$L_2^{LP} = L_2^{new} - L_2^{buyer} = \mathcal{L} - \mathcal{B}.$$

Consequently we have

$$\frac{\beta}{\alpha} = \frac{L_2^{LP}}{L_2^{buyer}} = \frac{\mathcal{L} - \mathcal{B}}{\mathcal{B}/2} = \frac{2}{U} - 2.$$

For instance, if $U = 1/2$, we have a ratio of 2:1, or 67% of capital suppliers decide to become LPs instead of bond buyers. So if the trading fees paid to LPs are about 2% compared to the rates (e.g. rate=5%, fee= 0.1%), we need the annualized trading volume to be 100 times (=2/2%) as large as the organic borrowing volume to bring LPs at par with lenders in terms of revenue. This is impossible with a typical fixed-rates protocol. However, if the average trading maturity is longer, let's say 10 years, then the multiplier now reduces to be 10 times. **This asserts the importance of leverage trading in bond markets via long term bonds**.

Capital efficiency. Many lending protocol founders state that by using yield-generating tokens rather than pure tokens, it will improve the capital efficiency as capital is utilized twice (lending pools and trading pools). There are Morpho, Notional Finance V2, Pendle Finance etc. just to name a few. This is indeed a smart strategy in the beginning when the revenue of LPs is small, but in the long term it is a flaw because of a simple principle in economics: No free lunch. In fact, when capital goes through a lending protocol before being deposited into a trading (bond) pool, it causes rate to fall from the lending market due to the redundancy of capital. As rates adjust and converge to the inter-market equilibrium state, this strategy is equivalent to non-connected market because lending interest rates will compensate for the drop in rates. I call this **low interest rate trap**, for the reason that when rates are low, bonds become expensive and their price will hardly move because of low volatility. As a result, it attract less and less trading volume to the bond market, which consequently decreases LPs' revenues. This capital-efficiency strategy thus does more harm than good in the long run.

5 Automated yield curve model

The most crucial task in building a bond exchange is to model the reference interest rate r and the yield curve. In this Version 1, we assume that yield curve is flat, so that we can focus on the interest modelling part. Dynamic yield curve will be experimented in Version 2. The essential difference between our model and existing fixed-rate ones is that we do not consider multiple bond pools for different maturities but rather aggregate them into unique bond pool. This requires and entails other important technical differences. For example bond conversion, bond burning and bond virtualization. More importantly, we allow flexible and long maturities so that traders can leverage their bet with bonds and consequently, LPs gain more from trading fees to compensate for the risks they take.

The fundamental instrument of this protocol is bond. A bond is a token that allows the buyer the right to redeem it at maturity for one DAI. The price of a bond with time-to-maturity t and implied interest rate r is given by the formulae $P = e^{-rt}$. It is important to stress that, in this bond protocol, bonds can be bought, sold, mint (short), and burnt (early payment) by users. Buying and selling activities are performed just like normal tokens. As for minting, a borrower has to stake their collateral and there will be a mechanism that estimates how much she can mint at maximum. Borrower can wait until the maturity is due and pay off their loan (if she fails to pay the loan then her collateral will be liquidated), or she can burn, whole or partially, the loan before maturity by buying back the same amount or just a part of bonds she issued and then burn them (this is equivalent to buying back the bonds and hold it until maturity, take the pay-off from the bond and then pay back the loan to the protocol, but burning just helps simplifying the whole process).

Unlike common lending protocols or fixed rate protocols, the state space of the model is a quintuplet (x, y, r, B, L). More precisely.

- x denotes the amount of bonds, $x = (x_1, .., x_n)$ is a vector of bonds with different maturities $T_1, .., T_n$.
- y denotes the amount of asset (DAI) in the pool. As 1 DAI is worth roughly 1 USD, the total value at time t is $Y_t = y$.
- r denotes the reference interest rate. The net present value (NPV) of all bonds in the pool with reference rate r_t at time t is given by
$$X_t = \Sigma_{i=1}^n x_i e^{-r_t(T_i - t)}, t < T_1.$$
- B denotes the the NPV of active bonds minted to the pool by borrowers (or bond issuers). By active here we also mean that we have to subtract the amount of bonds that have been burnt by bond issuers for early payment purposes.
$$B_t = \Sigma_{i=1}^n (\mathbf{mint}_i - \mathbf{burnt}_i) e^{-r_t(T_i - t)}.$$
- L denotes the NPV of active bonds bought from the protocol
$$L_t = \Sigma_{i=1}^n (\mathbf{buy}_i - \mathbf{sell}_i) e^{-r_t(T_i - t)}.$$

  Here for each maturity, we calculate the cumulative amount of bonds taken out of the pool by buyers and subtract that sold back to the pool by sellers to get the net purchase.

We also introduce LPs' liquidity contribution.

$$L_{LP} := X - (L - B),\, B_{LP} := Y + (L - B).$$

Note that L_{LP} is not the same as the initial liquidity contribution of all LPs because it includes also trading fees and impermanent losses caused by fluctuation of interest rates.

Let's take a simple example where Alice is the unique LP who deposits 100 DAI and mint 100 DAI worth of bonds into the pool (the maturities and allocation of bonds are not important here). At the same time, Bob, Charlie and David interact with the pool. Bob deposits collateral and mint 10 DAI bonds, while Charlie buys 20 DAI bonds and David sells 5 DAI bonds into the pool. In this case, $x = 105, y = 95, L = 15, B = 10$. The LPs' liquidity is $L_{LP} = B_{LP} = 100$.

Now we come to build interest rates function. It has three major components: reference rate (or implied rate), bid-ask spread and slippage. The second component is always ignored in existing fixed-rate protocols.

Reference rate. For simplicity, we assume that the reference rate is just a deterministic function of the bond proportion:

$$r = R(p), p = \frac{X}{X+Y} \in [0, 1].$$

This function should be non-decreasing to reflect the fact that more bond sellers will cause bonds to be more expensive, or rates to be low. A natural choice for R is a linear function:

$$R(p) = r_{\min} + (r_{max} - r_{min})p.$$

When $p = 0, r = r_{min}$ and when $p = 1, r = r_{\max}$.

Slippage. Unlike the case of an AMM, our model requires to solve a system of partial differential equations (PDE) in general case. If the transaction relates to only one maturity, then we only have to solve an ODE of one variable.

At time t, the fixed point equation $r = R(p)$ can be rewritten as

$$r = r_{\min} + (r_{\max} - r_{\min}) \frac{\Sigma_{i=1}^n x_i e^{-r_t(T_i - t)}}{\Sigma_{i=1}^n x_i e^{-r_t(T_i - t)} + y} = F(r, x, y),$$

and

$$y = \frac{r_t - r_{\min}}{r_{\max} - r_t} \Sigma_{i=1}^n x_i e^{-r_t(T_i - t)} = G(x, r).$$

The spot price of a bond with respect to maturity T_i is given by

$$-\frac{dy}{dx_i} = e^{-r_t(T_i - t)}\, \forall i.$$

If a transaction includes multiple bond positions then we need to replace the above ODEs by PDEs

$$\frac{\partial y}{\partial x_i} = -e^{-r_t(T_i - t)}\, \forall i.$$

The above system of n PDEs (plus one fixed point equation) allow to solve numerically for (x, y, r) by substitution. In practice, we can combine an iterative algorithm with an finite difference method to solve for (x, y, r) simultaneously. Even if a transaction involves multiple maturities, it should not impose any challenge.

Assume that the current (basic) state of the system is $(x^{start}, y^{start}, r^{start})$ and a trader comes to the pool to make a transaction $\Delta x := x^{end} - x^{start}$. We need to answer three questions:

- What is the post-trade reference rate r^{end}?
- What is the total cost of the transaction? i.e. $\Delta y := y^{end} - y^{start}$. (note that $\Delta X \Delta y \leq 0$).
- What is the average reference rate of the transaction?

The third question is easiest to answer. Once we know the answer to the first and second questions, then the average rate will be solution to the fixed point equation

$$\Delta y = -\Sigma_{i=1}^{N} \Delta x_i e^{-\bar{r}(T_i - t)} = H(\Delta x, \bar{r}).$$

The algorithm 1 uses the finite difference scheme to solve the PDEs. We devise the input vector Δx into smaller uniform slices and update the interest rate and the pool's money gradually. In practice, to speed up the algorithm and reduce the error, the slices are not necessarily uniform. The step size δx_i is defined in such a way that it equalizes all the partial step changes initially, and is proportional to the pool size.

$$\delta x_i = \epsilon (X^{start} + y^{start}) e^{r^{start}(T_i - t)}.$$

If the input is y^{end} instead of x^{end}, i.e. traders enter the amount of money instead of the number of bonds, then we have to solve a **dual problem** to the previous one. In this case, the trader buys only only one kind of bond. The dual algorithm is similar: we have to devise Δy instead of Δx and approximate the ODE by means of finite difference method.

Algorithm 1 Slippage estimate (primal problem)

procedure FINITE DIFFERENCE$(x^{start}, y^{start}, r^{start}, x^{end})$
    Assign $y^{end} = y^{start}, r^{end} = r^{start}$
    for $i = 1..n$ **do**
        Calculate $y^{end} = y^{end} + H(\frac{1}{n}\Delta x, r^{end})$.
        Calculate $r^{end} = F(x^{end}, y^{end}, r^{end})$.
    return y^{end}, r^{end}

Bid-ask spread. Now it's time to get involved with the (bid-ask) permanent loss issue. Recall that if the bid-ask spread is zero, then the LPs experience an PL (per unit of time-if PL is positive it means a gain)

$$PL = r(B - L).$$

Question is, should we force LPs to absorb all of this PL? Let's review some pros and cons of this choice.

- Pros. Reduce friction for traders, which can increase trading volume and improve revenues for LPs. This is a real trade-off for them. If PL is negligible compared to revenues coming from trading fees, it is not worth it to protect LPs.
- Cons. PL makes the game unfair for LPs. Not only they cannot enjoy interest rates like lenders (inactive capital), but also they consistently loose money due to PL. Moreover, PL makes LPs position unattractive when trading fees are low or mediocre. Consequently, LPs will withdraw their liquidity and it will make the situation worse by deepening the PL.

From the above analysis, we choose a middle way approach where LPs and users share the PL together. Consider the extreme case where LPs do not pay for the bid-ask spread, then

$$r_b r_l = r^2, r_l = U r_b = \frac{B}{L} r_b.$$

Solve this equation we have

$$r_l = r\sqrt{U}, r_b = \frac{r}{\sqrt{U}}.$$

If the reference price is the mid price, i.e. $r_b + r_l = 2r$ instead of $r_b r_l = r^2$, then the math becomes

$$r_l = \frac{2Ur}{1+U} = \frac{2B}{B+L},\ r_b = \frac{2r}{1+U} = \frac{2L}{B+L}.$$

Now if LPs absorb $\gamma \in [0,1]$ of the bid-ask spread, then

$$r_l = r(U)^{\frac{1}{2\gamma}},\ r_b = r(U)^{-\frac{1}{2\gamma}},$$

for the multiplicative case, or

$$r_l = [1 + \gamma\frac{1-U}{1+U}]r,\ r_l = [1 - \gamma\frac{1-U}{1+U}]r,$$

for the additive case.

However, if we define U to be the ratio B/L, then it might lead to an extreme cases where in the beginning this ratio can be very volatile (for example when $B << L$, hence the bid-ask spread can be very large in the beginning, which discourages bond trading. Therefore, I suggest to apply a smoothing factor as follows.

$$U = \frac{\kappa B_{LP} + B}{\kappa L_{LP} + L},$$

where we introduce LPs' liquidity contribution.

$$L_{LP} := X - (L - B),\ B_{LP} := Y + (L - B).$$

Here $\kappa := \kappa_\infty + (\kappa_0 - \kappa_\infty)e^{-\epsilon t}$ is a functional that depends on time, and will eventually decay to zero or a small quantity to reduce the role of LP's liquidity as the system gets stable (i.e. when the organic lending/borrowing volume become significant). We can replace L_{LP} and B_{LP} by an approximation $Z = \frac{X+Y}{2}$ to get

$$U = \frac{\kappa Z + B}{\kappa Z + L}.$$

Remember that the above formulas are used only if $U \leq 1$. In the case $U > 1$, we let the spread to be zero and this means LPs act as lenders because there are more borrowers than lenders (if we allow borrowing rates to be lower than lending rates, then we create arbitrage opportunities for traders and this is a waste of capital). The permanent gain (PG) for the LPs in this case is $r(B-L)$. This compensates for the PL that LPs have to suffer when $L > B$. Hence, there is another way to deal with the PL issue: **when LPs gain, their PG can be accumulated as a sort of reserves to be used to cover PL.**

Trading fees. DEXes usually charge fees that are proportional to trading volume. As for bond trading, a small fee for a close-to-maturity bond can imply a huge fee if it is converted into yield. Therefore, time to maturity should be taken into account when designing fee structure. There are two way to design bond fees:

- Fees are calculated on the yield space as fixed spread. i.e. if the theoretical rate is r, then the transaction rate will be $r - fee$ for buying bonds, and $r + fee$ for selling bond.
- Fees are proportional to trading volume and time to maturity: $fee = \epsilon VT$. This comes from the fact that the **Macaulay duration** of a zero coupon bond is equal to the time-to-maturity of the bond:

  $$P = e^{-rt} \to \frac{dP}{P} = -tdr.$$

  This means that trading a long term bond causes higher systemic risk to LPs than trading a short term bond. So long term bonds should come with higher fees, typically proportional to the time-to-maturity.

In this version, we use the fees in terms of spread for ease of calculation (though it is more expensive for long term bonds).

Bond burning. Existing fixed rate protocols such as Notional Finance or Yield Protocol use debt notes as building blocks. If Alice buys a debt note worth of 100 DAI from the pool at an interest rate of 5% per year, he will receive 105 DAI after one year. Using debt note will remove the need of using NPV in calculating the interest rate: $X = \Sigma_{i=1}^{N} x_i$. However, when traded at the second time, debt notes are treated as if they are (non-standardized) bonds. For example, if a debt note with maturity T is sold at initial rate of r_0 at $t = 0$, then at time $t > 0$, the price of the note will be $e^{r_0 T - r(T-t)}$. We prefer to use bonds as building block for sake of standardization and the convenience. Therefore, to ensure the two concepts interchangeable, bonds need to be burnt at interest rate. Indeed,

$$d(xe^{-r(t-T)}) = 0 \to \frac{dx}{x} = rdt.$$

In practice, bonds do not need to be burnt continuously. If between moments $t_0 < t_1$ there is no transaction, then the update rule is simply

$$x_{t_1} = x_{t_0} e^{-r_0(t_1 - t_0)}.$$

This rule will make sure that the interest rate remains constant when there is no transaction while time flows.

Bond virtualization. Unlike existing fixed-rate protocols where each maturity corresponds to a unique pool, we aggregate all maturities together. Initially, bonds are allocated uniformly across maturities. However, there will be some particular maturities that attract more trading volume than the others. At some point, for certain maturity the pool will run out of reserves. Of course we do not want to halt trading just because some types of bonds are exhausted. Hence there will two options.

- Convert redundant maturities into exhausted maturities using current reference rate. For example, if there is no more 1Y bond, and there still are a lot of 5Y bonds, then a minimal amount of 5Y bonds can be converted into 1Y bonds to cover the need of traders.
- Virtualization. We allow the number of bonds at some maturities to be negative, provided that the total value of the bond pool is still positive. Negative positions mean that LPs are overly borrowing from lenders (or **budget deficit**).

The first approach requires a careful allocation of bonds, while the second one is simpler to implement and will be adopted in V1 of this protocol. Note that **bond conversion is subject to interest rate risks** because traders tend to buy a lot of long term bonds during bull market, therefore the protocol has to convert short term bonds into long term bonds to sell to them. Afterwards, the yield reverses and causes LP to loose money because conversion to long term bonds is equivalent to taking the leverage on short positions.

6 Future perspectives

Interest rate smoothing. For V1, we assume that fixed rates are determined by instantaneous supply and demand. A high level of interest rate volatility can be the main driving force for bond trading. However, if that is the reason for high impermanent loss for LPs, then in V2 we can adjust the fixed rate by smoothing techniques. For instance, bond rates will be replaced by the 3 day moving average of the spot rate. Another idea is that long term rates should be smoothened with longer time windows to make them less risky compared to short term rates.

Dynamic yield curve. For sake of simplicity, we have greatly simplified the reality: yield is flat across all maturities. In practice, yield curve should be non-flat and dependent on the supply/demand on the market. This will be for V2 protocol.

Unlike debt notes, bonds are standardized and can be traded on the secondary market independently or can be borrowed or used as collateral. This means it can open new doors for asset management purposes.

Margin trading. Although our protocol allows to short or long a bond, it does not allow to leverage a trade other than trading long maturity bonds. Therefore, by using hybrid AMM model such as the one we suggest in the next chapter, bond margin trading can become reality.

Perpetual bonds. Perpetual bonds are bonds with no expiration. And because of that, it allows to tokenize the interest rate, i.e. if you buy perpetual bonds at current rate of r_0 and later, the rate moves to r_1, then the return of your investment will be $P\&L = \frac{r_1}{r_0} - 1$. Our protocol allows to consider non-zero coupon bonds as well as perpetual bonds.

Floating rate notes and swap. We can issue floating note in parallel with fixed term bonds in this protocol. The fixed rate now can be considered as swap rate to exchange for floating rates. The floating rates will be defined by a deterministic function of the state variables. This will be somewhat similar to Pendle Finance but we have our own price discovery mechanism and flexible maturities to trade bonds.