Polymorphism bond: A new design of decentralized bond exchange

Tuan Tran, PhD.

October 15, 2023

In this chapter, I propose a new design for bond exchange. This design aggregates both lending and bond trading functionalities into a unique platform, which provides leverage for asset management. The most innovative feature of the model is to introduce a new concept called polymorphism bond, which is an abstract bond that can be transformed into or absorb different bond NFTs with various maturities. With this, bonds with different maturities can be combined into a single pool without the need to managing complicated allocations, therefore it increases the liquidity and decreases computational costs. The second contribution of the model is to provide a closed-form price impact solution to the bond AMM formulas.

1 Introduction

In traditionaAMMl f model inance, fixed income markets, though attract attention mainly from institutional traders, account for about three times in size as large as global equity markets. In DeFi however, the size of the lending sector is negligible compared to the the whole crypto market. The major reason is that crypto lending requires collateral, so the TVL of any lending protocol can not surpass the total market capitalization of the collateral assets. Another reason is, lending with fixed rates are not mature enough, and bond markets even have de facto not existed yet. Therefore, fixed income markets have not attracted enough trading volume.

Generally speaking, fixed rate lending protocols in DeFi are still badly designed. Let's take an example, AAVE has fixed rate borrowing (not lending), but the rate is prohibitively expensive, so fixed rate borrowing accounts for only less than 1% of total borrowing volume. Recently, Notional Finance has become top one fixed rate lending protocol on Ethereum, but due to their bad economical design, borrowing in Notional Finance is still more expensive than in AAVE or Compound Finance, which makes it less attractive to borrowers. At the other extreme, Yield Protocol, which is another important fixed rate protocol on Ethereum, offers cheaper lending rates compared to AAVE or Compound Finance, which makes it less attractive to lenders. These two extremes lower trading volume to the existing protocols and hence, lower revenues to LPs. Not only that, these protocols force the borrowing rates equal to lending rates, which will obviously cause losses to LPs. In reality, except when capital is fully utilized, otherwise bid-ask spreads always exist due to imbalance of capital supply and demand. This is a fundamental principle in building an economically sound and solid bond protocol. From banking practice, **no banks would survive if they charge borrowers the same rates they pay to lenders**. Even if we take lending/borrowing fees into account, these fees are still not enough to cover for the lending-borrowing imbalance and for the cost of inactive capital of LPs (just for comparison purpose: AAVE clearly obtains more fees while its bid-ask spreads still exist). To cover for these impermanent/ inactive cash losses, there have to be extra trading fees, which can only come from traders who use the protocol as a means of trading/investment and not just lending/borrowing. However, with the current design where only a few short term maturities are active (3 month, 6 month and 1 year- i.e. being less flexible than AAVE/Compound and lack of leverage), these protocols clearly cannot meet investment need from traders but rather a fixed-rates add-on to AAVE or Compound.

To be more precise, bond markets are different from lending markets in that they allow investors to speculate and bet on the future fluctuation of interest rates (or, equivalently, yield curve). Long duration bonds not only help investors lock in their future gains, but also help leverage their bets (as the price of long duration bonds is more sensitive with respect to the changes in yield than shorter counterparts. As a matter of fact, **a bond of 10Y maturity offers more than 10 times leverage compared to an 1Y bond due to the compound effect, which is not the case for futures**). These are exactly features we want our bond protocol to possess.

To summarize, there are several drawbacks of current fixed rate lending protocols.

- Permanent loss for LPs even if rates remain constant.
- Inactive assets (used for liquidity purposes) causes relative losses to LPs compared to lenders.
- Rates are not better than that on AAVE or Compound, either for lenders or borrowers.
- There's a lack of maturity flexibility which will lower LPs' revenues. Therefore, it does not even provide enough borrowing/lending service for users in need. Not only that, each maturity correspond to a pool, which might cause pools to behave independently.
- They do not offer inter-maturity bond trading features.

In this note, I propose a protocol that embraces both lending and trading functionalities. This can be considered to be a decentralized bond exchange (DBX). The closest to our idea is Notional Finance (NF). However, there are several key differences:

- NF serves as a lending protocol only, while we have bond trading functionality.
- Lending maturities in NF are currently short term (restricted to 3 month, 6 month and 1 year only), which makes it difficult for investors to leverage their investment. Our protocol allows long maturities up to 25 years and can be made flexible at will.

- In NF, lending and borrowing rates are equal by design (if price slippage is ignored), which causes LPs to suffer from losses due to lack of liquidity. In our protocol, this condition is relaxed if the protocol does not connect directly to an external lending protocol.

2 Notations and concepts

- **Asset:** can be yield generating asset or not. Typical example is DAI (or cDAI). [However this does not add value]
- **Bond:** Unlike existing protocols where the underlying liability is a debt note (from which bonds are derived), we start with zero coupon bonds right off the bat. [This relates to the burning mechanism]
- **Maturity:** Unlike existing protocols that consider only short term bond maturities (3 month, 6 month and 1 year), we consider monthly-expired bonds with maturities up to 25 years. [To allow leverage]
- **Reference rate:** This is the annualized rate determined from supply and demand. If rate is r then the t-year bond price will be $P = e^{-rt}$.
- **Yield curve:** In practice, (annualized) yield curves depend on maturities and typically are increasing functions of maturities. In V1, we consider only flat yield curve, i.e. reference rate is used for all maturities.
- **Liquidity provider (LP):** LPs provide liquidity which is a pair of assets and bonds. Asset is deposited first, then bonds is minted in proportion such that the reference rate remains the same as before providing liquidity. When LPs withdraw liquidity, bonds and assets are withdrawn in proportion so that the reference rate is unchanged too.
- **Accumulated lending:** Just like AAVE, this represents the net present value of **active bonds** that have been taken out of the pool at the current time (bought-sold).
- **Accumulated borrowing:** Just like AAVE, this represents the net present value of **active bonds** that have been minted to the pool at the current time (minted-burnt).
- **Bid-Ask spread:** In current AMM models, bid-ask spread is zero but for bond trading, spreads need to exist to cover permanent losses for liquidity providers. Note that this is different from price slippage which is caused by the presence of large orders.
- **Bond AMM (AYC):** the reference rate is determined by a deterministic function of pool states. Pool states include: asset amount, bond amount, accumulated lending and accumulated borrowing. Trading rates are different from reference rates where there are slippage and bid-ask spread involved.
- **Bond virtualization:** Bonds in the pools are virtual in the sense that they can be converted from one maturity to another to meet demand from traders, or can be negative (budget deficit).

- **Bond burning:** Bonds are burnt gradually in such a way that no trading does not cause the reference to change. This is different from Notional Finance.

3 Existing models

Floating rates. The lending pool model proposed by AAVE and Compound Finance is not only the most important one in the lending space, but also the only one that relies on sound and solid economical reasoning. In this model, the borrowing interest rate is designed as an increasing, deterministic function of the capital utilization ratio, which reflects the supply and demand of the capital market. Recall that,

$$r_t^B = F(U_t) = F(\frac{B_t}{L_t}),$$

where B and L denote the total borrowing and lending volume of the pool at time t. If fees are ignored, then the conservation equation below that links lending and borrowing rates should satisfy all the time:

$$r_t^L = U_t r^B t,$$

The only drawback of this model is that rates fluctuate over time, while lenders and borrowers often requires certainty for their income inflow or expense outflow. This is the main reason of which many subsequent (fixed rate) lending protocols have born out.

Fixed rates AMM models. The major players in this space are Yield Protocol and Notional Finance. In this model, lenders are bond buyers, borrowers are bond issuers (backed by collateral). The lending pool now becomes a trading pool with an AMM that determines the interest rates in real time depending on the market suppy/demand of capital. Liquidity providers (LPs) deposit assets and bonds (actually not bonds but debt notes) and receive trading fees as rewards for their liquidity contribution.

The AMM model of Yield protocol has the form

$$x^{1-t} + y^{1-t} = C,$$

where x, y denote the total liquidity of assets and debt notes available in the trading pool, and t denote time to maturity of bonds (unique maturity). This equation is equivalent to $r = \frac{y}{x} - 1$.

The AMM model of Notional Finance has the form

$$r = R(\frac{y}{x+y}),$$

where R is a logistic-like function. We can see that the pricing mechanism of both Yield and Notional is somewhat similar to Uniswap V1. However, as we show in the sequel, **this model suffer from permanent loss** even if the interest rate stays constant due to the fact that LPs absorb all bid-ask spread. **In practice, no bank would survive if lending rates are equal to borrowing rates, therefore this model is a flaw if lenders are more populous than borrowers**.

Split principle and yield. There are many protocols in this vain: Pendle Finance, Element Finance, Sense protocol, Swivel, just to name a few. The central idea is that, given a yield-generating token (for example, debt note from Compound), one can split the token into two parts: principle and yield, and tokenize them. This means you can sell the future floating yield of your token for a fixed income, and this fixed yield is determined by supply and demand on the market, while the principle can be redeemed at maturity to receive an amount equal to the current asset value. The principle part can thus be viewed as a zero coupon bond. Due to the no arbitrage argument, the discount rate applied on the principle should be equal to the fixed rate to exchange for the floating yield. For example, if your one year floating yield is sold at 4%, then your principle will be priced at 4% discount.

Apparently, Pendle Finance et al. seem to offer a real bond trading flatform. But in fact, they don't. What they really offer is an interest rate swap service, where floating rates come from underlying lending protocols like Compound. Therefore, **they don't act as price discovery leader**. Plus, as all bonds are short term, they are essentially a means of lending/borrowing and not for trading/investment.

Structured products. There are BarnBridge and Tranche Finance in this category. If we view floating interest rates (like AAVE, Compound) as revenue flows of a company, then we can bring back the classical idea of corporate financing into the DeFi space, where investors are split into two tranches: seniors (or debt holders) and juniors (or equity holders). Seniors and Juniors deposit their money into a fund, and this fund will be deposited into AAVE or Compound. Depending on the proportion of each type of tranche in the fund, the fixed rate can be determined accordingly. Typically, debt holders enjoy low and fixed interest rates, while equity holders enjoy whatever is left on the table. From a mathematical point of view, equity holders take long position on a European option.

Capital efficiency. Morpho protocol (18M fund raising) is a successful lending app that tries to solve the capital efficiency problem. Essentially, they want borrowers and lenders to enjoy the same interest rate. To do that, they build an add-on dApp on top of Compound so that if users use Morpho, they have the chance to match each other (borrowers matching with lenders), and when they match, they enjoy the mid interest rate. If they do not find someone to match, they still enjoy the Compound rates by default.

Apparently, this sounds a smart idea. However, in the long run, it will destroy the underlying lending protocol. It is not difficult for lenders and borrowers to see the arbitrage opportunities between Morpho and Compound, so eventually users will register to Morpho as it costs virtually nothing to do so. Mathematically, if B_0, L_0 are potential borrowing and lending volume of the underlying protocol, and there is an amount of V matching volume using Morpho, then the utilization ratio will decay from $U_0 = \frac{B_0}{L_0}$ to

$$U_1 = \frac{B_0 - V}{L_0 - V} < U_0.$$

When $V \to B_0$, we have $U_1 \to 0$, i.e. the interest rate from the underlying protocol will eventually decay to zero, consequently, the mid-rate offered by

Morpho converges to zero too. This means Morpho does more harm than good for the underlying protocol. This is a typical example of bad economic design which is **essentially a sort of parasite dApp and has no intrinsic value to the DeFi space**. But no doubt, this is a clever idea. It is somewhat similar to Ponzi scheme in the lending space, as early users take advantages from late users in terms of interest rates. Generally speaking, all sort of lending protocols that rely on an underlying protocol to increase capital efficiency will more or less do harm to that underlying protocol and to themselves too.

4 Polymorphism bonds

The most crucial task in building a bond exchange is to model the reference interest rate r and the yield curve. In this Version 1, we assume that yield curve is flat, so that we can focus on the interest modelling part. Dynamic yield curve will be experimented in Version 2. The essential difference between our model and existing fixed-rate ones is that we do not consider multiple bond pools for different maturities but rather aggregate them into unique bond pool. This requires and entails other important technical differences. For example bond conversion, bond burning and bond virtualization. More importantly, we allow flexible and long maturities so that traders can leverage their bet with bonds and consequently, LPs gain more from trading fees to compensate for the risks they take.

The fundamental instrument of this protocol is bond. A bond is a token that allows the buyer the right to redeem it at maturity for one DAI. The price of a bond with time-to-maturity t and implied interest rate r is given by the formulae $B = e^{-rt}$. It is important to stress that, in this bond protocol, bonds can be bought, sold, mint (short), and burnt (early payment) by users. Buying and selling activities are performed just like normal tokens. As for minting, a borrower has to stake their collateral and there will be a mechanism that estimates how much she can mint at maximum. Borrower can wait until the maturity is due and pay off their loan (if she fails to pay the loan then her collateral will be liquidated), or she can burn, whole or partially, the loan before maturity by buying back the same amount or just a part of bonds she issued and then burn them (this is equivalent to buying back the bonds and hold it until maturity, take the pay-off from the bond and then pay back the loan to the protocol, but burning just helps simplifying the whole process).

4.1 The state space

The state space of our model is a vector (X, y, r, Dx), where

- X denotes the number of p-bond with zero maturity, or the net present value (NPV) of unrealized bonds in the pool.
- y denotes the amount of asset (DAI) in the pool.
- r denotes the reference interest rate.
- Dx denotes the net position of bonds to the pool, in terms of bond units. $Dx = (Dx_1, \ldots, Dx_n)$ is a vector of bonds with n different maturities

$T_1, \ldots, T_n$. $Dx_i > 0$ means traders sell Dx_i bond T_i into the pool and vice versa. The NPV of the net position is

$$DX_t = Dx * B = \sum_{i=1}^{n} \left(x_i \cdot e^{-r_t \cdot (T_i - t)} \right).$$

4.2 Model 1. Deterministic interest rate function

We assume that the reference rate is just a deterministic function of the bond proportion:

$$r = R(p), p = \frac{X}{X + y} \in [0, 1].$$

The function R should be non-decreasing to reflect the fact that more bond sellers will cause bonds to be more expensive, or rates to be low. We consider two simplistic options. Important remark: we want interest rate r to be a function of p, not bond price P to be a function of p to avoid arbitrage attack.

- Linear function: this function leads to finite-concentrating liquidity.

  $$R(p) = r_{\min} + (r_{\max} - r_{\min}) \cdot p.$$

  The bond proportion is also a linear function of interest rate

  $$p(r) = R^{-1}(r) = \frac{r - r_{\min}}{r_{\max} - r_{\min}}.$$

- Logit function: this function leads to infinite-concentrating liquidity.

  $$R(p) = \kappa \mathbf{logit}(p) + r^* := \kappa \ln \frac{p}{1 - p} + r^*.$$

  The inverse function:

  $$p(r) = R^{-1}(r) = \frac{e^{\kappa^{-1}(r - r^*)}}{e^{\kappa^{-1}(r - r^*)} + 1},$$

  where r^* is the equilibrium rate when the bond and the cash is balanced, and κ is a price impact constant. High kappa means the bond AMM has low capital concentration around the reference rate.

We can calculate y as a function of X, r :

$$y = X \left[\frac{1}{R^{-1}(r)} - 1 \right].$$

Now let's solve the price impact model. Assume that, at time t_0, the states vector is (X_0, y_0, r_0, Dx_0) and there is a trader who makes a swap of Δx bonds with time to maturity $\tau > 0$ for Δy DAI.

All p-bonds in the pool can be converted into real bonds with time to maturity τ :

$$X_0 = x_0 e^{-r_0 \tau}.$$

During the transaction, which is assumed to be done instantly, the following ODE has to hold:

$$\frac{dy}{dx} = -e^{-r\tau}.$$

We have to solve a system

$$\begin{cases} y = xe^{-r\tau}\Big[\frac{1}{R^{-1}(r)} - 1\Big] \\ \frac{dy}{dx} = -e^{-r\tau} \end{cases}$$

with initial condition (x_0, y_0, r_0). The idea is to solve (x, y) as functions of r.

From the first equation

$$dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial r}dr,$$

From the second equation

$$dy = -e^{-r\tau}dx.$$

Combine the two equation we get

$$\Big[\frac{\partial y}{\partial x} + e^{-r\tau}\Big]dx + \frac{\partial y}{\partial r}dr = 0.$$

For simplicity, denote $U(r) := \frac{1}{R^{-1}(r)} - 1$. We have

$$\frac{\partial y}{\partial x} = e^{-r\tau}U(r), \frac{\partial y}{\partial r} = xe^{-r\tau}\Big[U'(r) - \tau U(r)\Big].$$

Plug these into the ODE we obtain

$$\frac{dx}{x} + \frac{U'(r) - \tau U(r)}{1 + U(r)}dr = 0.$$

This ODE is separable, thus the solution is

$$x = \frac{x_0(1 + U(r_0))}{1 + U(r)} \exp\Big[\tau(r - r_0) - \tau\int_{r_0}^{r}\frac{dr}{1 + U(r)}\Big].$$

By the definition of U, we can rewrite

$$x = \frac{x_0 R^{-1}(r)}{R^{-1}(r_0)} \exp\Big[\tau(r - r_0) - \tau\int_{r_0}^{r} R^{-1}(r)dr\Big].$$

The explicit formula for y is

$$y = \frac{y_0[1 - R^{-1}(r)]}{1 - R^{-1}(r_0)} \exp\Big[-\tau\int_{r_0}^{r} R^{-1}(r)dr\Big].$$

Note that the above formula is during a given transaction. If there is no transaction between two moments t_1 and t_2, then because y, X are kept constant, the interest rate is constant.

Todos.

1. Derive explicit formulas for the linear and logit interest rate functions.
2. Derive efficient numerical schemes or numerical formulas if possible for the swap function $\Delta y = F(\Delta x)$ and $\Delta x = G(\Delta y)$. Method: eliminate r from two equations via Newton-Raphson algorithms.
3. Derive and plot the LOB density function that is equivalent to this bond AMM. Note that this function will depend on initial state (x_0, y_0, r_0) and the maturity. Method: use the composite function derivative rule

$$Q(B) = -\frac{dx}{dP} = -\frac{dx}{dr}\frac{dr}{dP}, \frac{dr}{dP} = -\frac{1}{\tau P}.$$

4. Prove that the model is free of arbitrage attack. The most common strategy is inter-maturity attack, where traders intentionally buy/sell a quantity dy of bond at very short maturity $\tau_1 \approx 0$ to push the interest rates go unbounded, then trades the opposite direction the same quantity of bond at very long maturity to enjoy the low price impact.
5. Update parameters. Interest rates change regime according to the market regime (bull, bear, side-way). Therefore, we need to update the parameters accordingly, for example $r_{\min}, r_{\max}, r^*, \kappa$. The easiest way is to use smoothing techniques such as rolling windows.

For the linear interest rate function

$$R(p) = r_{\min} + (r_{\max} - r_{\min}) \cdot p$$

The bond proportion

$$p(r) = R^{-1}(r) = \frac{r - r_{\min}}{r_{\max} - r_{\min}}$$

1. From here, we get

$$\int_{r_0}^{r} R^{-1}(r)dr = \int_{r_0}^{r} \frac{r - r_{\min}}{r_{\max} - r_{\min}}dr = \frac{(r - r_{\min})^2 - (r_0 - r_{\min})^2}{2(r_{\max} - r_{\min})}$$

$$x = x_0 \frac{r - r_{\min}}{r_0 - r_{\min}} \exp\left[\tau(r - r_0) - \tau\frac{(r - r_{\min})^2 - (r_0 - r_{\min})^2}{2(r_{\max} - r_{\min})}\right]$$

And

$$y = y_0 \frac{r_{\max} - r}{r_{\max} - r_0} \exp\left[-\tau\frac{(r - r_{\min})^2 - (r_0 - r_{\min})^2}{2(r_{\max} - r_{\min})}\right]$$

For logit interest rate function

$$R(p) = \kappa \ln\frac{p}{1-p} + r^*$$

and its inverse function

$$p(r) = R^{-1}(r) = \frac{e^{\kappa^{-1}(r-r^*)}}{e^{\kappa^{-1}(r-r^*)} + 1}$$

1. From here, we get

$$\int_{r_0}^{r} R^{-1}(r)dr = \int_{r_0}^{r} \frac{e^{\kappa^{-1}(r-r^*)}}{e^{\kappa^{-1}(r-r^*)}+1}dr = \kappa \ln \frac{e^{\kappa^{-1}(r-r^*)}+1}{e^{\kappa^{-1}(r_0-r^*)}+1}$$

Hence,

$$\begin{aligned} x &= x_0 \cdot \frac{e^{\kappa^{-1}(r-r^*)}}{e^{\kappa^{-1}(r-r^*)}+1} \cdot \frac{e^{\kappa^{-1}(r_0-r^*)}+1}{e^{\kappa^{-1}(r_0-r^*)}} \cdot \exp\left[\tau(r-r_0) - \tau\kappa \cdot \ln \frac{e^{\kappa^{-1}(r-r^*)}+1}{e^{\kappa^{-1}(r_0-r^*)}+1}\right] \\ &= x_0 \cdot \frac{e^{\kappa^{-1}(r-r^*)}}{e^{\kappa^{-1}(r-r^*)}+1} \cdot \frac{e^{\kappa^{-1}(r_0-r^*)}+1}{e^{\kappa^{-1}(r_0-r^*)}} \cdot e^{\tau(r-r_0)} \cdot \left[\frac{e^{\kappa^{-1}(r_0-r^*)}+1}{e^{\kappa^{-1}(r-r^*)}+1}\right]^{\tau\kappa} \\ &= x_0 \cdot \left[e^{\kappa^{-1}(r-r_0)} \cdot \frac{e^{\kappa^{-1}(r_0-r^*)}+1}{e^{\kappa^{-1}(r-r^*)}+1}\right]^{\tau\kappa+1} \end{aligned}$$

And

$$\begin{aligned} y &= y_0 \cdot \frac{e^{\kappa^{-1}(r_0-r^*)}+1}{e^{\kappa^{-1}(r-r^*)}+1} \cdot \exp\left[-\tau\kappa \cdot \ln \frac{e^{\kappa^{-1}(r-r^*)}+1}{e^{\kappa^{-1}(r_0-r^*)}+1}\right] \\ &= y_0 \cdot \left[\frac{e^{\kappa^{-1}(r_0-r^*)}+1}{e^{\kappa^{-1}(r-r^*)}+1}\right]^{\tau\kappa+1} \qquad (1) \end{aligned}$$

2. From

$$\begin{aligned} \frac{y}{x} &= \frac{y}{X}e^{-r\tau} = \frac{1-p}{p}e^{-r\tau} = \frac{1-R^{-1}(r)}{R^{-1}(r)}e^{-r\tau}, \\ \Rightarrow \frac{y}{x} &= e^{-\kappa^{-1}(r-r^*)-r\tau} \Rightarrow \frac{r-r^*}{\kappa} + r\tau = \ln\frac{x}{y} \\ \Rightarrow r &= \frac{r^* + \kappa \ln \frac{x}{y}}{1+\tau\kappa} \Rightarrow \frac{r-r^*}{\kappa} = \frac{-\tau r^* + \ln\frac{x}{y}}{1+\tau\kappa} \end{aligned}$$

From (1) we get

$$\begin{aligned} y\left(e^{\kappa^{-1}(r-r^*)}+1\right)^{\tau\kappa+1} &= C \\ \Leftrightarrow y\left(e^{\frac{-\tau r^*}{1+\tau\kappa}}\left(\frac{x}{y}\right)^{\frac{1}{1+\tau\kappa}}+1\right)^{\tau\kappa+1} &= C \\ \Leftrightarrow y^{\frac{1}{1+\tau\kappa}}\left(K\left(\frac{x}{y}\right)^{\frac{1}{1+\tau\kappa}}+1\right) &= C' \\ \Leftrightarrow Kx^{\frac{1}{1+\tau\kappa}} + y^{\frac{1}{1+\tau\kappa}} &= C' \end{aligned}$$

Here, $C = y_0\left[e^{\kappa^{-1}(r_0-r^*)}+1\right]^{\tau\kappa+1}, C' = C^{\frac{1}{1+\tau\kappa}}, K = e^{-\frac{\tau r^*}{1+\tau\kappa}}$ are constants.

3. Define $\alpha := \frac{1}{1+\tau\kappa}$, then

$$Kx^{\alpha} + y^{\alpha} = C' \tag{2}$$

From here, given the starting state (X_0, y_0, r_0) and the value of Δx or Δy, we can calculate the other.

To get the LOB density function, note that

$$P = -\frac{dy}{dx}$$

From (2) we get

$$\begin{aligned} K\alpha x^{\alpha-1}dx + \alpha y^{\alpha-1}dy = 0 \\ \Rightarrow P = -\frac{dy}{dx} = K\left(\frac{x}{y}\right)^{\alpha-1} \\ \Rightarrow y = K^{\frac{1}{\alpha-1}} P^{-\frac{1}{\alpha-1}} x \end{aligned}$$

Placing this into (2) we get

$$\begin{aligned} x^{\alpha}\left[K + K^{\frac{\alpha}{\alpha-1}} P^{-\frac{\alpha}{\alpha-1}}\right] = C' \Leftrightarrow x^{\frac{1}{1+\tau\kappa}}\left[K + K^{-\frac{1}{\tau\kappa}} P^{\frac{1}{\tau\kappa}}\right] = C' \\ \Rightarrow x = C\left[K + K^{-\frac{1}{\tau\kappa}} P^{\frac{1}{\tau\kappa}}\right]^{-1-\tau\kappa} \end{aligned}$$

The density LOB function is

$$\begin{aligned} -\frac{dx}{dP} &= C\left(1+\tau\kappa\right)\left[K + K^{-\frac{1}{\tau\kappa}} P^{\frac{1}{\tau\kappa}}\right]^{-2-\tau\kappa} K^{-\frac{1}{\tau\kappa}} P^{\frac{1}{\tau\kappa}-1} \frac{1}{\tau\kappa} \\ &= C\frac{1+\tau\kappa}{\tau\kappa} K^{-\frac{1}{\tau\kappa}} \left[K + K^{-\frac{1}{\tau\kappa}} P^{\frac{1}{\tau\kappa}}\right]^{-2-\tau\kappa} P^{\frac{1}{\tau\kappa}-1} \end{aligned}$$

4. Suppose that our pool is at state (X_0, y_0, r_0). At time t_1, traders come into our pool and perform a trade with short time to maturity $\tau_1 = T_1 - t_1 \approx 0$ and right after this, our pool moves to (X_1, y_1, r_1). Having done this, at time t_2, traders perform a trade with long time to maturity $\tau_2 = T_2 - t_2 \to \infty$ which moves the pool's state back to (X_2, y_2, r_2) with $y_2 = y_0$.

   Note that for constant κ, we have when $\tau \to 0, \alpha \to 1$ and when $\tau \to \infty, \alpha \to 0$, so we may setup our experiment as below:

   (a) Choose the parameters r^*, κ for the formula

   $$r = r^* + \kappa \ln \frac{p}{1-p},$$

   with $p = \frac{X}{X+y}$ here denotes for the proportion of bonds in the pool.

   (b) Setup the experiment by choosing (X_0, y_0) (Note that $r_0 = r^* + \kappa \ln \frac{X_0}{y_0}$).

   (c) i. **Test 1.** Choose $\alpha_1 \to 1^-$ and $\alpha_2 \to 0^+$, and calculate $\tau_1 = \kappa^{-1}\left(\frac{1}{\alpha_1} - 1\right), \tau_2 = \kappa^{-1}\left(\frac{1}{\alpha_2} - 1\right)$. We have $\tau_1 \to 0^+$ and $\tau_2 \to \infty$.

ii. **Test 2.** Choose $\alpha_1 \to 0^+$ and $\alpha_2 \to 1^-$, and calculate $\tau_1 = \kappa^{-1}\left(\frac{1}{\alpha_1} - 1\right), \tau_2 = \kappa^{-1}\left(\frac{1}{\alpha_2} - 1\right)$. We have $\tau_1 \to \infty$ and $\tau_2 \to 0^+$.

For each test,

i. Calculate $x_0 = X_0 \times e^{r_0\tau_1}, K_1 = e^{-\frac{\tau_1 r^*}{1+\tau_1\kappa}}$.

ii. Trade on the curve

$$K_1 x^{\alpha_1} + y^{\alpha_1} = K_1 x_0^{\alpha_1} + y_0^{\alpha_1}.$$

Assume that after the transaction, $(x, y) = (x_1, y_1)$.

iii. Calculate

$$r_1 = \frac{r^* + \kappa \ln \frac{x_1}{y_1}}{1 + \tau_1\kappa}, X_1 = x_1 \times e^{-r_1\tau_1}$$

.

iv. Calculate $x_{1'} = X_1 \times e^{r_1\tau_2}, K_2 = e^{-\frac{\tau_2 r^*}{1+\tau_2\kappa}}$.

v. Trade on the curve

$$K_2 x^{\alpha_2} + y^{\alpha_2} = K_2 x_{1'}^{\alpha_2} + y_1^{\alpha_2}.$$

Assume that after the transaction, $(x, y) = (x_2, y_2)$. It is hard to find the value of x_2 such that $X_2 = X_0$, but we can make $y_2 = y_0$. For $y_2 = y_0$, solve for x_2.

vi. Calculate

$$r_2 = \frac{r^* + \kappa \ln \frac{x_2}{y_2}}{1 + \tau_2\kappa}, X_2 = x_2 \times e^{-r_2\tau_2}$$

4.3 Model 2. Order Book based price impact

Option 1 leads to complicated formulas for swap functions, which may not be good for practical reasons. The LOB derived from the model also has complicated form.

We then offer a simple price-impact based model for bond AMM as follows. First, recall the OBMM model with zero price reversal (for buying bonds).

$$P_1 = P_0(1 + R), R = I(|dx|), dx = \frac{Dx}{x_0}.$$

David: should we have λ-reversal? maybe not, because change in bond price is small, so reversal should make minor effect, yet incurring possible liquidity discontinuity?

Here, P denotes the bond price and R denote the price impact. As the interest rate is defined in the log-space, we will replace both R and dx by the logarithmic counterpart.

$$|\tau dr| = I(|s_x|), -\tau dr = \log\left(\frac{P_1}{P_0}\right) \approx R, s_x = \log\left(\frac{x_1}{x_0}\right) \approx dx.$$

Here, $dr = r_1 - r_0$. We can choose $I = \tau I_0$ where I_0 is a deterministic function that does not depend on τ to obtain

$$-dr = I_0(-s_x).$$

Generally, for both buying and selling the formulas are symmetrical

$$dr = \begin{cases} -I_0\left(|s_x|\right), & \text{for buy orders } (s_x < 0, s_y > 0) \\ I_0\left(|s_y|\right), & \text{for sell orders } (s_x > 0, s_y < 0) \end{cases}$$

The case of buying orders leads to

$$s_x = -I_0^{-1}\left(-dr\right) \to x_1 = x_0 \exp\left(-I_0^{-1}\left(\frac{1}{\tau}\log\left(\frac{P_1}{P_0}\right)\right)\right).$$

Unlike V-shaped OBMM model, this model ensures **unbounded liquidity** distribution.

Let's consider three simple examples.

- $I_0(s) = \kappa s, \kappa > 0$. In this case, we obtain a counterpart formula of the Yield Protocol model.

$$\frac{x_1}{x_0} = \left(\frac{P_1}{P_0}\right)^{-\alpha}, \alpha = \frac{1}{\tau\kappa} > 0.$$

  The LOB density is

$$Q(P) = -x_1'(P) = \frac{\alpha x_0}{P_0^{-\alpha}} P^{-\alpha-1}.$$

  This LOB has triangle-shaped density.

- $I_0(s) = \kappa\sqrt{s}, \kappa > 0$. In this case,

$$\frac{x_1}{x_0} = \exp\left(-\alpha^2 \log^2\left(\frac{P_1}{P_0}\right)\right), \alpha = \frac{1}{\tau\kappa} > 0.$$

  The LOB density is

$$Q(P) = -x_1'(P) = \frac{\alpha^2 x_0}{P P_0} \log\left(\frac{P}{P_0}\right) \exp\left(-\alpha^2 \log^2\left(\frac{P}{P_0}\right)\right).$$

  This LOB has local V-shaped density.

- The above square root function is to ensure the local V-shape for the bond LOB. However, it leads to highly computational costs, especially when calculating the swap functions. Therefore we consider a logarithmic impact function (which is also a concave function).

$$I_0(s) = \kappa \ln(1+s), \kappa > 0.$$

  In this case

$$\frac{x_1}{x_0} = \exp\left(1 - \left[\frac{P_1}{P_0}\right]^{\alpha}\right), \alpha = \frac{1}{\tau\kappa} > 0.$$

  The LOB density is

$$Q(P) = -x_1'(P) = \frac{\alpha x_0}{P_0^{-\alpha}} P^{\alpha-1} \exp\left(1 - \left[\frac{P_1}{P_0}\right]^{\alpha}\right).$$

  This is a special case of Gamma distribution.

We have some comments.

- If we want the interest rate to be more stable, we need to choose $\kappa << 1$. This means we increase the capital efficiency of the LOB.
- If we use the square root function instead of the linear function, then price impact will be relatively higher for small orders than large orders.

For $I_0(s) = \kappa s^\beta, \kappa > 0, \beta > 0$. In this case, $I_0^{-1}(x) = \left(\frac{x}{\kappa}\right)^{\frac{1}{\beta}}$

$$\Rightarrow \frac{x_1}{x_0} = \exp\left(-\alpha^\gamma \log^\gamma\left(\frac{P_1}{P_0}\right)\right), \alpha = \frac{1}{\tau\kappa} > 0, \gamma = \frac{1}{\beta} > 0$$

The LOB density is

$$Q(P) = -x_1'(P) = \frac{\alpha^\gamma x_0}{PP_0}\log^{\gamma-1}\left(\frac{P}{P_0}\right)\exp\left(-\alpha^\gamma \log^\gamma\left(\frac{P_1}{P_0}\right)\right)$$

Having defined $I_0(s)$, how can we get the formulas for y for buying orders case?

From

$$P = -\frac{dy}{dx} \Rightarrow dy = -Pdx \Rightarrow \frac{dy}{dP} = -P\frac{dx}{dp} = PQ(P)$$

$$\Rightarrow y(P) = \int_{P_0}^{P} pQ(p)dp$$

Todos.

1. Plot the LOB density function and compare the capital efficiency of this model with other existing models (Uniswap, Notional Finance).
2. Derive efficient numerical schemes or analytical formulas if possible for the swap function $\Delta y = F(\Delta x)$ and $\Delta x = G(\Delta y)$.
3. Prove that the model is free of arbitrage attack.
4. David: Tam, also do 1) the general case $I(s) = \kappa s^\beta$ for arbitrary β. 2) formulas for rate r, average rate r per order, post-trade price, average price, impermanent loss, 3) derive impermanent loss formula for Yield protocol and Notional Finance (if possible), because in the paper we will compare to them

4.4 Add and remove liquidity

Assume that the current state of the pool is (X, y, r, Dx). An LP comes and provides liquidity Δy. Four questions arises:

- What is the share α of the LP? Should it be $\frac{\Delta y}{y}$?
- Later, when the LP withdraws liquidity, how much cash and bond will he receive? If he prefers to receives cash, is it αy?

- How much p-bond does the pool has to mint more at the moment where new liquidity Δy is provided?
- How much p-bond does the pool has to burn at the moment where certain liquidity Δy is removed?

The two last question is easy to answer. The amount of p-bond that needs to be minted or burnt should be proportional to the amount of cash provided or withdrawn to ensure that removing or adding liquidity does not change interest rate. Precisely, the amount of p-bond minted or burnt will be αX where $\alpha = \frac{Dy}{y}$.

Now, for the first question, obviously the share of the LP cannot be relied on the current level of cash reserves. Otherwise if there are two many borrowers, i.e. there is few cash left in the pool, the LP would receive artificially high share.

Let's denote the realized cash reserves to be

$$Y = y + Dx * B = y + DX.$$

This is the total cash that remains in the pool should all traders close their bond positions. For simplicity we ignore the slippage.

The share of the LP will be

$$\alpha = \frac{\Delta y}{\Delta y + Y}.$$

Now for the second question. Similarly to the first question, if the LP prefers to withdraw in cash then we would have to rely on the realized cash reserves. However, slippage needs to be taken into account. Moreover, sometimes there is not enough cash available in the pool to withdraw, hence I propose that LP withdraw two parts: cash and bonds. The amount of cash withdrawn immediately is

$$\Delta y = \alpha y.$$

The system will burn αX p-bond from the pool, and mint an amount of real bonds proportionally to the net position of the pool and give it to the LP.

$$\Delta x = \alpha Dx.$$

To cash out, the LP has two options: either sell the above bond positions Δx to the pool, or wait until the bonds are expired. However, if $\Delta x < 0$ (debt position) then the LP has to leave an amount of cash back into the pool that equalizes the debt, i.e. the true amount of cash that he receives is

$$\Delta y = \alpha y + \alpha \Sigma_{Dx_i < 0} Dx_i e^{-r\tau_i}.$$

4.5 Compound trading fees

Unlike token AMM, bond AMM charges trading fees on the yield space, not on the price space because otherwise it will cause huge impact on the short time-to-maturity bonds. As the fee f is small, we can approximate the fees $e^{f\tau} - 1 \approx f\tau$. I propose the following fees structure:

$$F = \min\{0.0001, 0.0001 * \tau\}.$$

After an epoch, fees should be compounded by throw back into the pool. If an amount of fees F is sent back to the pool, then a proportional amount of p-bond will be minted to ensure that the interest does not change.

5 Liquidity interconnection

Theoretically, the revenue of LPs and lenders are approximately equal at equilibrium. In an ideal market condition, the lenders are less populous than borrowers, hence LPs do not suffer from permanent loss due to bid-ask spread.

However, in imperfect market conditions, there might be more lenders than borrowers. Therefore we have to lend out surplus cash to other lending protocols to protect LPs. Similarly if the rate is too high, we need to borrow surplus money from other lending protocols.

Therefore, we can implement a simple lending mechanism as follows: whenever $r < r^* - s$, where r^* denotes the rate when the inventory ratio $p = 0.5$ and s is a small threshold parameter, then we will lend out money to an external lending protocol, for example, AAVE, until the rates get back to the safety region, i.e. $r = r^* + s$.

I now introduce a more symmetrical method. Let (r_b, r_l) be the smoothened borrowing and lending rates from AAVE.

- If $r < r_l - s$, we lend out money to AAVE until $r = r_l + s$.
- If $r > r_b + s$, we borrow money from AAVE until $r = r_b - s$.

Our goal is to keep interest rate within the AAVE bands to make it more attractive to borrowers and lenders.

[more thoughts are welcome!]

6 Empirical results

We conduct an experiment as follows.

- Simulate interest process using Vacicek model or CIR model. Time horizon one year, initial rate 5%, long term rate 5%, volatility 5%, reversion rate 0.00005 (to ensure rates fluctuate between 2.5% and 7.5%), trading frequency 1 minute.
- Simulate the reserves process (X_t, Y_t) assuming that the rate process is generated by periodic bond trading. For simplicity, we consider only one maturity.
- Output reserves process, pool value $V_t = X_t + Y_t$, accumulated trading fees F_t, market making PnL $V_t - V_0 + F_t$.