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NATIONAL ECONOMIC UNIVERSITY
FACULTY OF ECONOMIC MATHEMATICS (MFE)

MASTER RESEARCH PROJECT

– ACTUARY MAJOR –

Topic:

Price Process Modeling Using Quantum Mechanics

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Abstract

This study proposes a quantum modeling framework for analyzing financial price dynamics, inspired by the analogy between the inherent uncertainty in quantum mechanics and the nonlinear nature of financial markets. Instead of viewing prices as deterministic values, the model represents them as superposition states in a Hilbert space, where bid and ask states correspond to basis vectors. Hermitian operators are used to describe price and return characteristics, while the Schrödinger equation governs the evolution of the asset's wave function. Empirical results show that the quantum model estimates Value-at-Risk (VaR) more conservatively than traditional methods, effectively capturing fat tails and complex return distributions shaped by market microstructure effects. Moreover, the framework enables the extraction of Geometric Brownian Motion (GBM)-compatible segments through Kalman filtering, leading to more accurate volatility estimation. These findings highlight the model's potential for improving risk assessment in financial systems by accounting for liquidity, measurement effects, and structural nonlinearities that classical approaches often overlook.

Keywords: Quantum finance, Price operator, Superposition state, Bid-ask spread, Schrödinger equation.

Contents

1 Introduction

Throughout the journey of understanding natural laws, humanity has continuously encountered paradoxes that force us to redefine concepts once thought immutable. In classical physics, the world is described as a giant machine operating under deterministic rules, where all objects—from dust particles to galaxies—exist independently of human consciousness. This perspective, established during the times of Galileo and Newton, considers uncertainty as merely a consequence of limited information or inadequate measuring tools. For example, the moon always has a specific location and velocity, whether or not anyone is observing it; the role of science is simply to “uncover” those hidden facts.

However, the quantum revolution in the early 20th century completely shattered this worldview. Young’s double-slit experiment, Heisenberg’s uncertainty principle, and Schrödinger’s cat paradox revealed a shocking truth: at the microscopic level, particles do not have fixed states until measured. They exist as a probability cloud, a superposition of multiple potential outcomes, and only “collapse” into a definitive state upon interaction with a measurement apparatus. This not only challenges the objectivity of reality but also raises profound philosophical questions: Does consciousness play a role in “creating” the material world, or does the world exist independently of our awareness?

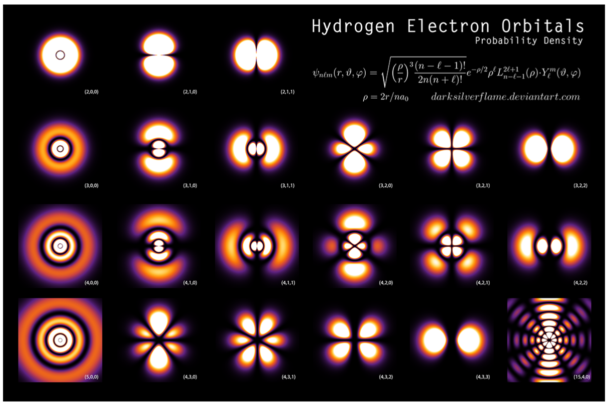


Figure 1: Quantum orbital representation

Similarly, in finance, classical economic models have long viewed prices as single numerical values reflecting the “true value” of assets based on available information. Price fluctuations were attributed to external factors: news, macroeconomic volatility, or herd psychology. However, real-world markets don’t quite operate that way. The prices we see are the result of matched buy and sell orders. Before any transaction, the price is not a fixed point but a probability field: buyers place bid prices, sellers post ask prices. The simultaneous existence of multiple prices is intrinsic to the market, where each participant brings their own expectations, strategies, and risk tolerance. The uncertainty, in this case, is intrinsic, not merely due to lack of information. Like quantum particles, prices “materialize” only upon interaction (i.e., a trade); beforehand, they exist in a fuzzy state of multiple possibilities.

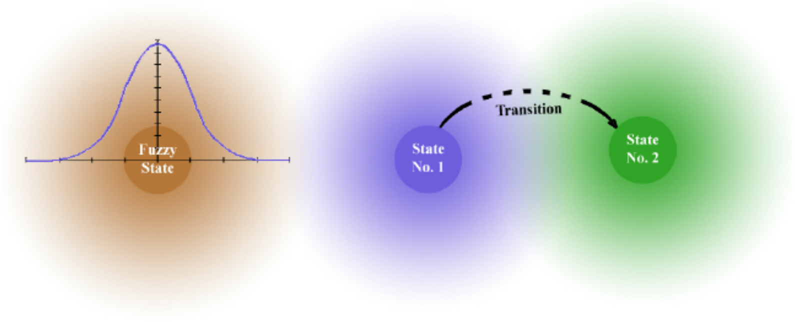


Figure 2: Fuzzy price state representation

Symbol	Ref	Ceil	Floor	Bid						Matched			Ask					
				Price 3	Vol 3	Price 2	Vol 2	Price 1	Vol 1	Price	Vol	+/-	Price 1	Vol 1	Price 2	Vol 2	Price 3	Vol 3
HDB	20.90	22.35	19.45	20.70	51,80	20.75	24,10	20.80	87,70	20.85	9,40	-0.05	20.85	5,60	20.90	52,60	20.95	155,60
HPG	25.35	27.10	23.60	24.55	193,20	24.60	159,80	24.65	30,20	24.70	60	-0.65	24.70	466,00	24.75	219,00	24.80	373,40
LPB	30.80	32.95	28.65	32.85	209,00	32.90	40,60	32.95	89,00	32.95	10	2.15						
MBB	22.65	24.20	21.10	22.20	115,70	22.25	122,10	22.30	40,80	22.35	2,00	-0.30	22.35	36,70	22.40	303,00	22.45	706,10
MSN	61.40	65.60	57.20	58.00	23,90	58.10	13,50	58.20	6,30	58.30	20	-3.10	58.30	10,90	58.40	24,30	58.50	69,70
MWG	54.70	58.50	50.90	51.40	7,40	51.50	14,10	51.60	80	51.70	1,00	-3.00	51.70	7,50	51.90	1,10	52.00	114,30
PLX	37.40	40.00	34.80					34.80	76,40	34.95	5,00	-2.45	34.90	5,00	34.95	26,00	35.30	11,00
SAB	45.85	49.05	42.65	43.60	1,00	43.65	3,80	43.70	1,90	43.75	10	-2.10	43.85	60	43.90	1,00	43.95	6,30
SHB	11.70	12.50	10.90	11.80	1,070,50	11.85	790,90	11.90	1,406,90	11.95	10,00	0.25	11.95	435,90	12.00	1,733,40	12.05	457,40
SSB	18.70	20.00	17.40	18.85	4,20	18.90	2,50	18.95	10,60	19.00	1,00	0.30	19.00	5,30	19.10	10,10	19.20	22,80
SSI	24.60	26.30	22.90	23.60	271,10	23.65	159,70	23.70	29,10	23.75	80	-0.85	23.75	90,50	23.80	238,40	23.85	123,70
STB	36.55	39.10	34.00	37.10	94,10	37.15	45,10	37.20	42,80	37.25	2,90	0.70	37.25	17,70	37.30	93,10	37.35	65,30
TCB	25.95	27.75	24.15	25.40	211,50	25.45	143,30	25.50	93,30	25.55	50	-0.40	25.55	32,70	25.60	29,50	25.65	107,40

Figure 3: Order book and bid-ask structure

In this paper, we propose a quantum modeling framework to analyze financial price dynamics. By viewing bid and ask prices as distinct eigenstates within the Hilbert space of the system, we encode the interaction between buyers and sellers using operators acting on superposed price states. To simplify, the model focuses only on the highest bid and the lowest ask. Empirical tests on stock market data show that the model not only replicates statistical characteristics of the bid-ask spread but also predicts the probability of "phase transitions" between equilibrium and disequilibrium states.

The implications of this research go beyond applying physics tools to finance. It suggests an interdisciplinary approach to analyzing social systems, where subjectivity, intrinsic uncertainty, and nonlinear interaction play central roles. At the same time, it challenges the boundaries between "objective" and "subjective" knowledge in science: Does an absolute "true value" of an asset exist, or is price merely an emergent property of interaction among sentient agents?

2 Limitations of Traditional Quantitative Financial Models

Modern financial models are predominantly developed using a bottom-up approach — starting with fundamental assumptions and gradually adding complexifying factors. However, this methodology carries two critical paradoxes:

- Accumulation of inaccurate assumptions: Each additional incorrect assumption increases the arbitrariness of the model, gradually detaching it from market realities. This "error accumulation" effect undermines the model's ability to capture the dynamic essence of financial systems.
- Lack of generalization: Tuning models to fit historical data (overfitting) often sacrifices their ability to represent the system as a whole. A robust model should converge toward a generalized equation capable of explaining multiple phenomena, rather than merely reproducing localized patterns.

The nature of financial systems, where human behavior is influenced by a variety of social and psychological factors, gives it the character of an applied social science. Due to this, standards for hypothesis testing are often loosened, resulting in a tendency to confuse statistical similarity with the discovery of a causal mechanism.

"Similarity proves nothing."

Many researchers misuse complex mathematical models to fit past market behavior (training), assuming that the future (testing) will behave similarly. However, such models are often not experimentally falsifiable, as the combination of logical reasoning and seemingly reasonable assumptions does not always equate to correctness. When the model fails to predict accurately, rather than revising its assumptions, people often blame the market for being "inefficient" to defend the model.

2.1 Limitations of Stochastic Calculus

Stochastic calculus, especially models based on Brownian motion, has become the backbone of many modern financial theories such as the Black-Scholes model, portfolio theory, and risk measurement via Value-at-Risk (VaR). However, when applied at the microstructure level of markets, these models exhibit critical limitations:

- Assumption of no bid-ask spread: Most classical models assume the bid and ask prices are identical, overlooking the ever-present spread in real markets.
- Underestimation of short-term risk: Models like VaR often assume that risk at $t = 0$ is zero, despite the reality that the bid-ask spread already reflects immediate uncertainty.
- Inability to model market impact: These models fail to explain how large orders can directly influence the market price.

2.2 Practical Challenges in Financial Markets

- Market Making Market makers earn profits by quoting higher ask prices and lower bid prices, i.e. they profit from the bid-ask spread. However, if the spread is too wide, orders are unlikely to match quickly, leading to low turnover and potentially lower total profit.

Conversely, if the bid-ask spread is too narrow, the number of executed orders may increase, but the profit per trade is smaller, which might still result in suboptimal total returns. Therefore, selecting an appropriate spread is crucial.

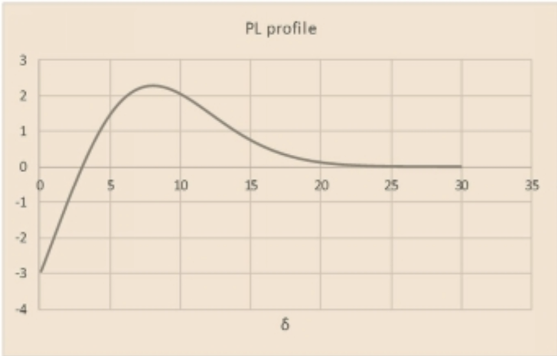


FIGURE 0.1 *Tradeoff between market maker's operating spread and P/L. Profit per cycle grows as spread increases but starts to shrink after passing the optimal spread.*

- Sales & Trading Desk When executing large orders, financial institutions often face market impact risk. The Sales & Trading desk must execute those orders while minimizing market impact, often by splitting the order over time or using more sophisticated execution strategies.

It's not just about splitting orders, one must predict market reactions, analyze market depth, identify hidden liquidity, and distinguish between genuine orders and spoofing activity.

- REPO Transactions When using stocks as collateral in a REPO agreement, determining the appropriate haircut becomes a key challenge. Models like Value-at-Risk (VaR) and Expected Shortfall are often insufficient because they ignore liquidity risk.

If the borrower defaults, the lender may be forced to liquidate a large amount of shares in an illiquid market, causing the price to drop significantly. The haircut must reflect this risk.

- Trading Example A trader sees a profitable position and sends a market order to close the entire trade. However, the position represents 10% of the average daily volume, so execution takes time.

After liquidation, the trader ends up with a loss instead of a profit, because the displayed market price reflects small trades, not the actual execution price, which was pushed down by their own order.

This shows that valuing positions based on quoted prices can be highly misleading, especially for illiquid assets. Order splitting may help but doesn't guarantee preservation of initial prices.

- Asset Management: Valuation of Large Positions Can't Rely Solely on Quotes Suppose a counterparty wants to buy 25% of the total issuance of a bond you hold. The last comparable trade occurred 2 months ago. Current liquidity only covers 3% of the required volume.

The issue is: How do you fairly price such a large block? And how do you convince the buyer that the price is fair?

Probabilistic pricing models typically output a single fair price, without accounting for trade size. Selling a $60 million block is priced the same as a $6,000 one — clearly unrealistic.

Solution: Build a quantitative framework where price is a function of order size, properly reflecting liquidity and market impact.

- Risk: Fancy Reports Don't Prevent Losses Many risk departments treat their job as calculating, analyzing, and reporting metrics. But the real mission is to prevent actual losses.

In reality, when losses occur, the numbers often far exceed any forecasts. That's because risks are assessed based on prices of small transactions, while large positions can cause market impact when liquidated.

One common technique is liquidity gap-filling: missing data for illiquid assets is simulated or merged with liquid data to compute VaR. This causes loss of critical information and does not actually help prevent losses.

- Options: Illiquid Underlying Means Double Risk If the underlying asset of an option is illiquid, things become even more complicated:

- The price dynamics of the underlying are unclear.
- Execution prices depend on order size due to liquidity.
- Uncertain prices mean Greeks become unreliable, posing major hedging risks.

We must model the bid-ask spread, understand true liquidity structures, and develop models that go beyond classical stochastic calculus.

> "Models aren't bad, they only become dangerous when we blindly trust them."

2.3 Core Limitations of the Classical Approach

In classical frameworks, asset prices at any time t are modeled as a single random variable. For instance, in the geometric Brownian motion (GBM) model:

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t,$$
$$\Rightarrow S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

The price S_t is assumed to objectively exist, independent of how it is measured, meaning there's always a "true price" that the model simply needs to estimate or sample.

However, as shown in the real-world examples above, the idea of a unique "true price" becomes questionable. Prices depend on how they are measured (i.e., how orders are executed), on trade size, on market reaction, and on liquidity structure at that moment.

Problem: Lack of a Framework for Measurement and Interaction Effects

Classical models assume that measurement has no effect on outcomes, a notion already challenged in modern physics. To more accurately capture real-world market behavior, we need a new theoretical framework where:

- Prices are not absolute values, but results of the measurement process.
- Liquidity and market interaction must be embedded in the mathematical model.
- The effect of trading behavior on the system itself must be taken into account.

This leads to a new direction, the quantum finance approach, where the act of measurement itself alters the system's state, and financial quantities are modeled not just as random variables, but as operators on a state space.

3 Theoretical Foundations of Quantum Mechanics

3.1 Vector Spaces in Quantum Mechanics

In quantum mechanics, quantum states are not represented by finite-dimensional vectors, but by wave functions that belong to an infinite-dimensional Hilbert space. These quantum states are not just simple vectors in a Euclidean space, but complex-valued functions, elements of an inner-product space where operators act on them according to quantum rules.

A function $f : \mathbb{R} \to \mathbb{C}$ is called *square-integrable* if:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty.$$

The set of such functions forms the space $L^2(\mathbb{R})$, which is a Hilbert space, a complete inner-product vector space.

Some properties in (infinite-dimensional) Hilbert space:

- A state $|\alpha\rangle$ corresponds to the wave function $\psi_\alpha(x)$.
- Inner product:
$$\langle\beta|\alpha\rangle = \int_{-\infty}^{\infty} \psi_\beta^*(x)\psi_\alpha(x) \, dx.$$
- Normalization:
$$\langle\alpha|\alpha\rangle = 1.$$
- Orthogonality:
$$\langle\alpha|\beta\rangle = 0 \Leftrightarrow \int_{-\infty}^{\infty} \psi_\alpha^*(x)\psi_\beta(x) \, dx = 0.$$
- Completeness of the basis:
$$|\psi\rangle = \sum_{n=1}^{\infty} a_n|\psi_n\rangle, \quad a_n = \langle\psi_n|\psi\rangle.$$

3.2 Postulates of Quantum Mechanics

Quantum mechanics is built on the following five fundamental Postulates:

3.2.1 Postulate 1: Quantum States Are Described by Wavefunctions

The state of a physical system is described by a wavefunction $\psi(\mathbf{r}, t)$ in a Hilbert space.

- Probability interpretation: The quantity $|\psi(\mathbf{r}, t)|^2$ represents the probability density of finding the particle at position $\mathbf{r}$ at time t.

$$\int_{\mathbb{R}^3} |\psi(\mathbf{r}, t)|^2 \, d\mathbf{r} = 1.$$

- Superposition principle: If ψ_1 and ψ_2 are valid states, then any linear combination

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle ,$$

with $c_1, c_2 \in \mathbb{C}$ is also a valid state. Upon measurement, the wavefunction collapses to an eigenstate corresponding to the measured eigenvalue with probability $|\langle\psi_i|\psi\rangle|^2 = |c_i|^2$.

3.2.2 Postulate 2: Observables Are Represented by Hermitian Operators

Every physical observable $\hat{A}$ is represented by a linear Hermitian operator acting on the Hilbert space. Hermitian operators have real eigenvalues and orthogonal eigenvectors, ensuring measurable outcomes are real numbers.

Properties of Hermitian operators

- Eigenvalues are real: If $\hat{A}\psi = a\psi$, then $a \in \mathbb{R}$.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal:

$$\langle\psi_i|\psi_j\rangle = \delta_{ij}.$$

- The set of eigenvectors forms a complete basis (for discrete spectra):

$$\sum_i |\psi_i\rangle\langle\psi_i| = \hat{I}.$$

3.2.3 Postulate 3: Measurement Outcomes Are Eigenvalues

If the system is in an eigenstate ψ_i of a Hermitian operator $\hat{A}$, measuring the corresponding physical quantity yields the eigenvalue a_i:

$$\hat{A}\psi_i = a_i\psi_i.$$

Spectrum of the operator is the set of eigenvalues a_i:

- Discrete spectrum: e.g., bound energy levels of the hydrogen atom.
- Continuous spectrum: e.g., momentum of a free particle in infinite space.

3.2.4 Postulate 4: Expectation Value

The expectation value of a physical observable $\hat{A}$ in state $\psi(\mathbf{r}, t)$ is given by:

$$\langle \hat{A} \rangle = \int_{\mathbb{R}^3} \psi^*(\mathbf{r}, t) \hat{A} \psi(\mathbf{r}, t)\, d\mathbf{r}.$$

If $\hat{A}$ is Hermitian, then $\langle \hat{A} \rangle \in \mathbb{R}$.

Definite (eigen) states: A state Ψ is called definite with respect to observable $\hat{Q}$ if

$$\hat{Q}\Psi = q\Psi,$$

i.e., Ψ is an eigenstate with eigenvalue q, and the variance is zero:

$$\langle (\hat{Q} - q)^2 \rangle = 0.$$

This means the measurement of $\hat{Q}$ yields q with 100

3.2.5 Postulate 5: Time Evolution

The wavefunction $\psi(\mathbf{r}, t)$ evolves according to the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \hat{H} \psi(\mathbf{r}, t),$$

where $\hat{H}$ is the Hamiltonian operator representing the total energy of the system:

$$\hat{H} = \hat{T} + \hat{V},$$

with $\hat{T}$ the kinetic energy operator and $\hat{V}$ the potential energy operator.

4 Quantum Mechanics in Finance

4.1 Price Operator

4.1.1 Definition

The Price Operator is a mathematical operator representing the spectrum of possible price states of an asset at a given time, rather than a single fixed value as in classical models. This concept is somewhat analogous to a random variable in probability theory but with key fundamental differences:

- Classical random variable: There is an objectively existing definite value, unknown only due to lack of information or measurement limitations.
- Quantum price operator: The price state exists as a superposition of multiple possibilities simultaneously. Only upon measurement (a transaction) does the state collapse to a single realized price. The uncertainty is intrinsic, reflecting the non-commutative and superpositional nature of price states.

4.1.2 Hermitian Matrix Representation

In a subspace $\mathcal{H}_N$ spanned by the orthonormal basis $\{|s_1\rangle, |s_2\rangle, \dots, |s_N\rangle\}$, where

$$|s_i\rangle = \begin{bmatrix} 0 \\ \vdots \\ 1 \quad \text{(at position } i) \\ \vdots \\ 0 \end{bmatrix}, \quad \langle s_i|s_j\rangle = \delta_{ij},$$

the Hermitian price operator $\hat{S}$ takes the form:

$$\hat{S} = \begin{bmatrix} s_1 & S_{12} & \cdots & S_{1N} \\ S_{21} & s_2 & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & s_N \end{bmatrix}, \quad S_{ij} = S_{ji}^*.$$

- Diagonal elements s_i: $\langle s_i|\hat{S}|s_i\rangle = s_i$ correspond to classical expected prices at state $|s_i\rangle$.
- Off-diagonal elements S_{ij} $(i \neq j)$: encode quantum correlations (entanglement and interference) between states $|s_i\rangle$ and $|s_j\rangle$.

Hermiticity ($\hat{S} = \hat{S}^\dagger$) ensures:

- Real eigenvalues: $\{s_1, \dots, s_N\} \subset \mathbb{R}$.
- Existence of a complete orthonormal eigenbasis $\{|\psi_k\rangle\}$.

State representation:

- In the computational basis:

$$|\psi\rangle = \sum_{i=1}^{N} c_i|s_i\rangle, \quad c_i = \langle s_i|\psi\rangle.$$

- In the eigenbasis of $\hat{S}$:

$$|\psi\rangle = \sum_{k=1}^{N} d_k|\psi_k\rangle, \quad d_k = \langle\psi_k|\psi\rangle.$$

4.1.3 Simple Case: Coupling Model

Consider a system with two basis states $|\text{ask}\rangle$ and $|\text{bid}\rangle$:

$$|\psi_s\rangle = c_{ask}\,|\text{ask}\rangle + c_{bid}\,|\text{bid}\rangle\,.$$

The Hermitian price operator reads:

$$\hat{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix},$$

where

- S_{11}, S_{22} are classical expected prices,
- S_{12} encodes quantum superposition.

Eigenvalues:

$$\begin{aligned} s_{\text{ask}} &= s_{\text{mid}} + \frac{\Delta}{2}, \\ s_{\text{bid}} &= s_{\text{mid}} - \frac{\Delta}{2}, \end{aligned} \quad \text{with} \quad s_{\text{mid}} = \frac{S_{11}+S_{22}}{2}, \quad \Delta = \sqrt{(S_{11}-S_{22})^2 + 4|S_{12}|^2}.$$

The price state in the new eigenbasis is

$$|\psi_s\rangle = d_{ask}\,|\psi_{\text{ask}}\rangle + d_{bid}\,|\psi_{\text{bid}}\rangle\,.$$

4.2 Return Operator

The return operator $\hat{R}_t$ describes relative price changes, defined as:

$$\hat{R}_t = \frac{\hat{S}_t}{s_{t-1}} - \hat{I} = \underbrace{z\hat{I}}_{\text{Mean shift}} + \frac{1}{2}\underbrace{\begin{bmatrix} \xi & \kappa \\ \kappa^* & -\xi \end{bmatrix}}_{\text{Difference operator } \hat{E}}, \tag{1}$$

where:

- $z = \frac{s_{\text{mid}} - s_{t-1}}{s_{t-1}}$: mean shift over time.
- $\hat{E} = \begin{bmatrix} \xi & \kappa \\ \kappa^* & -\xi \end{bmatrix}$: difference operator with
  - $\xi = \frac{S_{11}-S_{22}}{s_{t-1}}$: normalized price difference, distributed as $\mathcal{N}(0, \sigma_\xi^2)$,
  - $\kappa = \frac{2S_{12}}{s_{t-1}}$: normalized quantum coupling, distributed as $\mathcal{N}(0, \sigma_\kappa^2)$.

4.2.1 Expected Return

The expected return is computed via the quantum expectation of the return operator:

$$\mathbb{E}[R] = \langle \text{Price}|\hat{R}|\text{Price}\rangle = \frac{\Delta}{2s_{t-1}}\left(|\psi_{\text{ask}}|^2 - |\psi_{\text{bid}}|^2\right),$$

where $I = |\psi_{\text{ask}}|^2 - |\psi_{\text{bid}}|^2$ is the probability imbalance index, reflecting the asymmetry between the ask and bid states.

4.2.2 Expectation of the Difference Operator

The expectation of the difference operator $\hat{E}$ is expressed as:

$$\mathbb{E}[\hat{E}] = \langle \text{Price}|\hat{E}|\text{Price}\rangle = z + \frac{\Delta}{2s_{t-1}} I,$$

showing the aggregate expectation combining both average drift and quantum effects.

5 Random Price Process

5.1 Price Operator Dynamics

The price operator $\hat{S}(t)$ follows a stochastic differential equation:

$$\hat{S}(t+dt) = \hat{S}(t) + d\hat{S}(t), \tag{2.5.1}$$

with the matrix elements at time $t + dt$ defined as:

$$\begin{aligned} S_{11}(t+dt) &= s_{\text{mid}}(t) + \sigma\, dz + \frac{\xi}{2}, \\ S_{22}(t+dt) &= s_{\text{mid}}(t) + \sigma\, dz - \frac{\xi}{2}, \\ S_{12}(t+dt) &= \frac{\kappa}{2}, \end{aligned}$$

where:

- $dz \sim \mathcal{N}(0,1)$ is the Wiener process increment representing Gaussian white noise,
- ξ, κ are independent random variables with normal distributions:

$$\xi \sim \mathcal{N}(0, \sigma_\xi^2), \quad \kappa \sim \mathcal{N}(0, \sigma_\kappa^2),$$

- σ denotes the volatility of the mid-price.

The mid-price evolves according to Brownian motion:

$$s_{\text{mid}}(t+dt) = s_{\text{mid}}(t) + \sigma\, dz. \tag{2.5.3}$$

5.2 Quantum Probability and Transaction Price

The last traded price s_{last} is randomly chosen between s_{bid} and s_{ask} based on quantum probabilities:

$$s_{\text{last}}(t) = \begin{cases} s_{\text{ask}}(t), & \text{with probability } |\langle \psi_{\text{ask}} | \psi_s \rangle|^2, \\ s_{\text{bid}}(t), & \text{with probability } |\langle \psi_{\text{bid}} | \psi_s \rangle|^2. \end{cases}$$

6 Finding the Asset Wavefunction

6.1 Dynamical Equation Setup

In this study, instead of the price operator $\hat{S}$, we use the price difference operator $\hat{E}(t)$ to simplify calculations. The operator $\hat{E}$ is dimensionless and excludes the reference price s_0. The Schrödinger equation for the asset wavefunction $|\psi_s(t)\rangle$ is:

$$i\tau \frac{d}{dt} |\psi_s(t)\rangle = \hat{E}(t) |\psi_s(t)\rangle, \tag{1.34}$$

where:

- τ is a characteristic time parameter,

- $|\psi_s(t)\rangle = c_{\text{ask}}(t)\,|\text{ask}\rangle + c_{\text{bid}}(t)\,|\text{bid}\rangle = \begin{bmatrix} c_{\text{ask}}(t) \\ c_{\text{bid}}(t) \end{bmatrix}$ is the quantum state vector of the asset.

6.2 Linear Differential System

With the matrix form of $\hat{E}(t)$:

$$\hat{E}(t) = \frac{1}{2}\begin{bmatrix} \xi(t) & \kappa(t) \\ \kappa^*(t) & -\xi(t) \end{bmatrix},$$

the equation (**??**) splits into two linear differential equations for the probability amplitudes of ask and bid:

$$\begin{cases} 2i\tau \frac{dc_{\text{ask}}(t)}{dt} = \xi(t)c_{\text{ask}}(t) + \kappa(t)c_{\text{bid}}(t), \\ 2i\tau \frac{dc_{\text{bid}}(t)}{dt} = \kappa^*(t)c_{\text{ask}}(t) - \xi(t)c_{\text{bid}}(t). \end{cases} \tag{1.35}$$

Mathematical Interpretation:

- The factor $2i\tau$ comes from normalization of $\hat{E}$ and time scale,
- $\xi(t)$ represents normalized price difference fluctuations between ask and bid,
- $\kappa(t)$ describes quantum interference (coherence) between the two states.

6.2.1 Assumptions and Solution Method

Assumption: Over a short time interval δt, the asset price remains nearly constant, so $\xi(t)$ and $\kappa(t)$ can be treated as constants.

Solution Approach:

1. Laplace Transform: Applied to linear ODE system with constant coefficients.
2. General Solution: Define

$$\epsilon = \sqrt{\xi^2 + |\kappa|^2}, \quad \phi = \frac{\epsilon t}{2\tau}.$$

The solution is:

$$\begin{cases} c_{\text{ask}}(t) = \left[\cos(\phi) - i\frac{\xi}{\epsilon}\sin(\phi)\right] c_{\text{ask}}(0) - i\frac{\kappa}{\epsilon}\sin(\phi)\, c_{\text{bid}}(0), \\ c_{\text{bid}}(t) = \left[\cos(\phi) + i\frac{\xi}{\epsilon}\sin(\phi)\right] c_{\text{bid}}(0) - i\frac{\kappa^*}{\epsilon}\sin(\phi)\, c_{\text{ask}}(0). \end{cases}$$

Physical Interpretation:

- $\cos(\phi)$ and $\sin(\phi)$ capture quantum oscillations between ask and bid states,
- Ratios ξ/ϵ and κ/ϵ characterize energy distribution between price difference and coherence,
- Phase angle ϕ controls the state evolution speed.

7 Experiment

7.1 Calculate VaR

7.1.1 Traditional Method

In the traditional approach, the asset returns are assumed to follow a normal distribution $\mathcal{N}(\mu, \sigma^2)$, with mean μ and variance σ^2. Based on historical data, these parameters are estimated to calculate the Value-at-Risk (VaR) at the 95% confidence level. Specifically, the 95% VaR is defined as the 5th percentile of the return distribution, i.e., the value r such that $P(R \leq r) = 0.05$.

7.1.2 Quantum Method

In the quantum model, returns are described by an operator matrix R of size $N \times N$. The spectrum of returns $\{r_n\}$ is determined by solving the secular equation:

$$\det |R_{mn} - r_n \delta_{mn}| = 0 \tag{3.9}$$

When interactions between price levels are limited to adjacent levels, the matrix R is represented as a tridiagonal matrix with elements defined as follows:

- Diagonal elements R_{nn}, representing returns at price level n, follow a normal distribution $\mathcal{N}(0, \sigma_k^2)$.
- Elements just above and below the diagonal, $R_{n+1,n}$ and $R_{n-1,n}$, represent interactions between adjacent price levels and follow a complex normal distribution $\mathrm{CN}(0, \sigma_k)$.
- Elements far from the diagonal, $R_{n+m,n}$ and $R_{n-m,n}$, also follow $\mathrm{CN}(0, \sigma_k)$, representing long-range interactions between distant price levels. These interactions contribute to the fat tails observed in the return distribution.

7.1.3 Computational Procedure

The VaR calculation process using the quantum model includes the following steps:

1. Data collection: Gather daily return data of a financial asset (e.g., WMT stock) over a sufficiently long period (1–5 years) to ensure sample representativeness.
2. Construct observed distribution: Calculate the probability distribution of returns $P(r)$ from data, typically as a histogram or probability density function.
3. Initialize matrix R_{mn}:
   - Assign diagonal elements: $R_{nn} \sim \mathcal{N}(0, \sigma_k^2)$.
   - Assign elements just above and below the diagonal: $R_{n+1,n}, R_{n-1,n} \sim \mathrm{CN}(0, \sigma_k)$.
   - Assign elements far from the diagonal: $R_{n+m,n}, R_{n-m,n} \sim \mathrm{CN}(0, \sigma_k)$.
4. Calculate return spectrum: Solve the eigenvalue equation (3.9) to obtain eigenvalues $\{r_n\}$. Then build the spectral density function $\chi(r)$:

$$\chi(r) = \frac{1}{N} \sum_{n=1}^{N} \delta(r - r_n) \tag{3.10}$$

5. Parameter optimization: Use least squares to adjust parameters σ_ξ, σ_k so that $\chi(r)$ fits the observed distribution $P(r)$:

$$\sum_{i=1}^{M} w_i \left| P(r_i) - \chi(r_i|\sigma_\xi, \sigma_k, \ldots) \right|^2 \to \min \tag{3.11}$$

where w_i are appropriate weights and M is the number of data points.

6. Verification and refinement: Compare the theoretical distribution $\chi(r)$ with the observed distribution $P(r)$, paying special attention to fat-tail features. Adjust parameters or increase matrix size as needed for better modeling of interactions.

7. VaR calculation: Identify the 5% quantile of the return distribution (corresponding to 95% VaR) from the eigenvalues $\{r_n\}$.

7.1.4 Results

The 95% VaR calculation results for VSH stock are shown below:

Method	VaR 95%
Quantum method	−0.075978
Traditional method	−0.043949

Table 1: Comparison of 95% VaR results from quantum and traditional methods.

The quantum method estimates a higher risk level compared to the traditional method. This can be explained by the quantum model's ability to incorporate complex factors such as market liquidity, bid-ask spread, and long-range interactions that lead to fat tails in the return distribution.

The quantum model provides a more rigorous approach to describing the complex dynamics of asset returns, particularly in capturing non-Gaussian features of the return distribution. The results show that this method can detect higher risk levels, thereby better supporting risk management decisions in volatile financial environments.

7.2 Further development

The volatility (σ) of a stock price reflects a wide range of underlying factors, including the market's perception of the company's fundamentals, operational risks, and asset efficiency. In this section, we explore a methodology to estimate the volatility by identifying and extracting segments of the price series that are consistent with a Geometric Brownian Motion (GBM).

Our approach focuses on filtering the full price series to isolate disjoint subarrays that statistically follow the GBM pattern. Once identified, these subarrays can be merged to form a more stable and representative series for volatility estimation.

It is commonly assumed in financial theory that the price volatility (σ_{price}) is proportional to the volatility of a firm's Return on Assets (ROA), expressed as:

$$\sigma_{\text{ROA}} = \alpha \cdot \sigma_{\text{price}}$$

Here, α is a coefficient reflecting the market's amplification of firm-level asset efficiency into price movements. Although the exact value of α may eventually be determined through more advanced methods—such as those inspired by quantum inference techniques—our current focus is on estimating σ_{price} from historical price data using classical filtering.

Step 1: Bid and Ask Prices We begin by examining the raw bid and ask price series from the limit order book. This data captures the supply and demand dynamics and provides the input for identifying actual transaction prices. Below is a visualization of the bid and ask prices over time:

Step 2: Deriving the Matching Price Series From the bid and ask data, we compute the matching price series (e.g., mid-price or last traded price), which serves as the observed price process. A sample of this series is shown here:

Step 3: Extracting GBM-Compatible Segments via Kalman Filter To extract meaningful segments that follow Brownian motion dynamics, we apply the Kalman filter to smooth out market noise. The procedure includes:

- Applying a 1-dimensional Kalman filter to estimate latent price values.
- Computing the noise (residual) between the observed and filtered price.
- Testing log-returns of filtered prices using Shapiro-Wilk and other normality tests within a moving window.
- Identifying and marking subarrays that statistically follow a GBM.

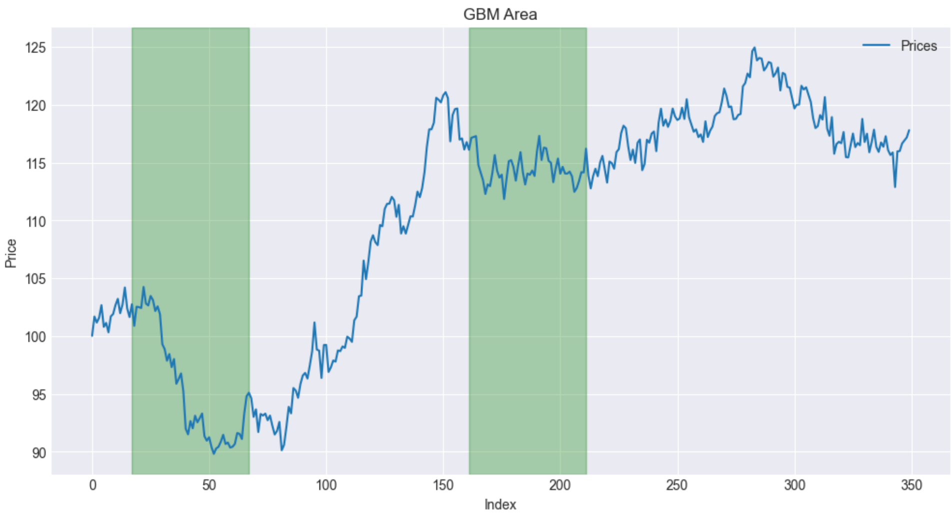
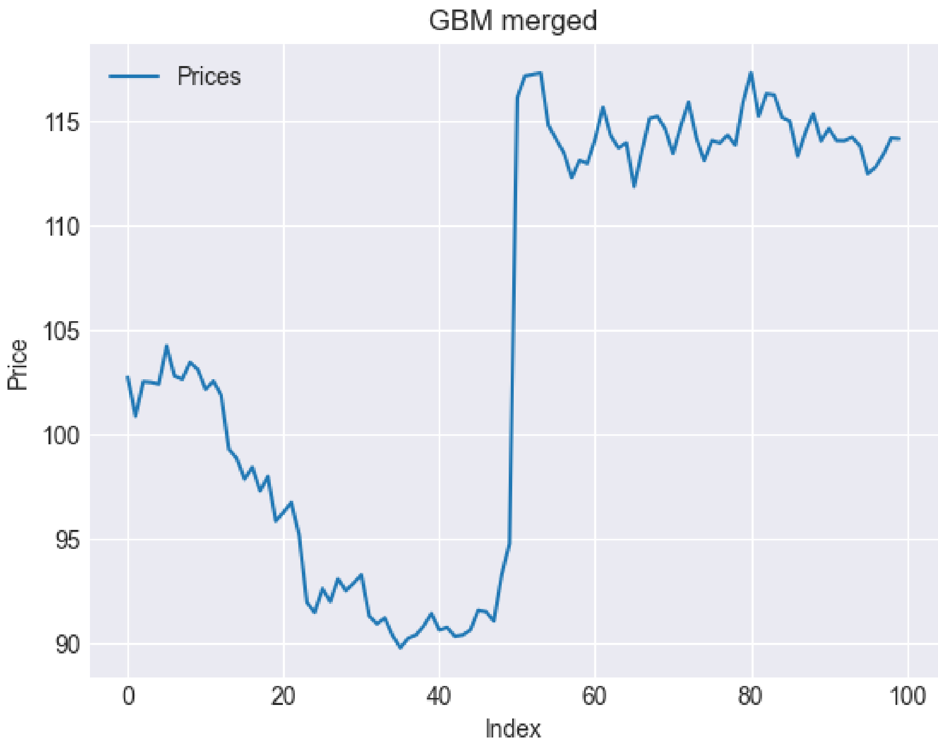


Figure 4: Identified GBM-compatible subarrays via Kalman filter

Step 4: Merging GBM Subarrays The subarrays identified as GBM-compatible may be non-continuous. We merge these disjoint segments into a composite array that captures the GBM behavior across the timeline. This results in a cleaner and more interpretable dataset for estimating volatility:



Step 5: Estimating Price Volatility While volatility is traditionally estimated via the standard deviation of log-returns under the GBM assumption, we can alternatively estimate the price volatility directly using the standard deviation of price levels. This approach is particularly useful when the model context departs from log-normal assumptions or aims to capture descriptive price-level variability.

$$\sigma_{\text{price}} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(P_t - \bar{P}\right)^2}$$

where $\bar{P} = \frac{1}{T}\sum_{t=1}^{T} P_t$ is the mean of the price series.

This estimator provides a direct measure of how much the price fluctuates per unit of time, in nominal terms, and may be useful in models where price levels (not returns) are the object of study.

Conclusion

Step 6: Interpreting the Estimated Volatility Using the merged GBM-valid price segments, the estimated price volatility is:

$$\hat{\sigma}_{\text{price, GBM}} = 10.09$$

For comparison, the estimated volatility using the entire dataset, without filtering for GBM-compatible behavior, is:

$$\hat{\sigma}_{\text{price, all}} = 10.11$$

Commentary The volatility estimates for the GBM-filtered data and the full dataset are remarkably close, differing by only 0.02. This small discrepancy suggests that, while not all parts of the time series strictly follow Geometric Brownian Motion, the overall market dynamics are sufficiently close to GBM assumptions in aggregate.

However, this does not invalidate the utility of segmentation. The Kalman-filtered and statistically validated GB

This framework forms a foundational step toward advanced financial modeling, and future research may refine the coefficient α using data-driven or quantum-inspired methods, linking asset-based fundamentals directly to market price behavior.

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