Part 4 Simualtion and interest rate models

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Interest rate model

Let $P(t, T)$ be the price at time t of a Zero-coupon bond with face value 1 payable at T.

- Simply-compounded interest rate from t to T: $L(t, T)$

$$1 + L(t, T) \times (T - t) = \frac{1}{P(t, T)} \rightarrow L(t, T) = \frac{1}{T - t}\left(\frac{1}{P(t, T)} - 1\right)$$

- Annually-compounded interest rate from t to T: $Y(t, T)$

$$(1 + Y(t, T))^{(T-t)} = \frac{1}{P(t, T)} \rightarrow Y(t, T) = \left(\frac{1}{P(t, T)^{1/(T-t)}} - 1\right)$$

- Continuous-compounded interest rate from t to T: $R(t, T)$

$$exp(R(t, T) \times (T - t)) = \frac{1}{P(t, T)} \rightarrow R(t, T) = -\frac{log\,(P(t, T))}{T - t}$$

Interest rate model

Define $r(t)$ as follows

$$r(t) = \lim_{T \to t^+} R(t, T)$$

$r(t)$ is called the short rate at time t. We can prove that

$$\begin{aligned} r(t) &= \lim_{T \to t^+} L(t, T) \\ &= \lim_{T \to t^+} Y(t, T) \end{aligned}$$

What is the price of a ZC bond with face value 1 payable at the end of year 1, given that $r(t)$ equals to 3%, 4%, 5% and 6% in the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} quarter, respectively?

Interest rate model

- If $r(t)$ is a deterministic and continuous function from 0 to T then price of a ZC bond pays \$1 at time s where $0 \leq s \leq T$ is

$$exp(-\int_s^T r(t)dt)$$

- Vasicek (1977) assumed that the short rate evolves as an Ornstein-Uhlenbeck process with constant coefficients

$$\begin{aligned} r(0) &= r_0 \\ dr(t) &= \kappa\left(\theta - r(t)\right) dt + \sigma dB(t) \end{aligned}$$

where r_0, κ, θ and σ are positive constants and $B(t)$ is a brownian motion (under **risk-neutral** measure)

Interest rate model

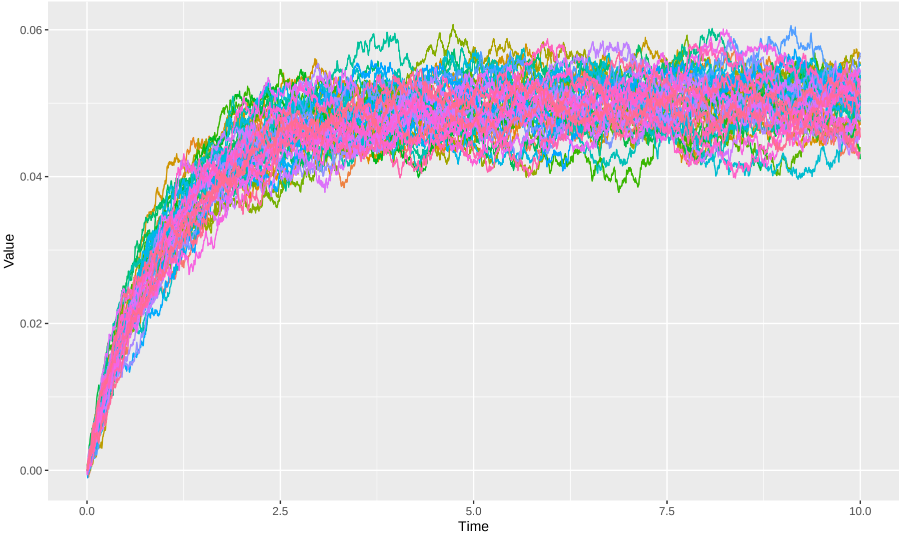
- With $r_0 = 2\%$, $\kappa = 1$, $\theta = 5\%$ and $\sigma = 0.5\%$, using Monte Carlo simulation to simulate $N = 50$ paths of instantaneous short rate from time 0 to 10.

```
T<-10
n<-2500
delta<-T/n
N<-50
kappa<-1
theta<-0.05
sigma<-0.5/100
r0<-0.02
time<-seq(0,T,length=(n+1))
```

Interest rate model

```
rt<-matrix(r0,(n+1),N)
for (i in 2:(n+1)){
  drt<-kappa*(theta-rt[(i-1),])*delta+
    sigma*sqrt(delta)*rnorm(N,0,1)
  rt[i,]<-rt[(i-1),]+drt
}
dat<-as.data.frame(rt)
names(dat)<-paste0("r",1:N)
dat%>%mutate("Time"=time)%>%
  gather("rate",Value,"r1":paste0("r",N))%>%
  ggplot(aes(Time,Value,group=rate,color=rate))+geom_line()+
  theme(legend.position="none")
```

Interest rate model



Interest rate model

- $r(t)$ satisfying this SDE is called a mean-reversion stochastic process
  - θ is the long-term mean
  - κ is the convergence rate
- Because $r(t)$ is a stochastic process, the price at time 0 of a ZC bond pays 1 at time T is

$$ZC(0) = \mathbb{E}^{\mathbb{Q}}\left(exp(-\int_0^T r(t)dt)\right)$$

- With each $r(t)$ simulated, we have a simulated value for $ZC(0)$

$$exp(-\sum_{i=1}^{n} r_{i\Delta} \times \Delta)$$

Interest rate model

Using $N = 10^4$ simulation, calculate the price at time 0 of a ZC bond pays 1 at time $T = 10$ if the instantaneous short rate follows Vasicek model with $r_0 = 2\%$, $\kappa = 1$, $\theta = 5\%$ and $\sigma = 0.5\%$

Interest rate model

Using $N = 10^4$ simulation, calculate the price at time 0 of a ZC bond pays 1 at time $T = 10$ if the instantaneous short rate follows Vasicek model with $r_0 = 2\%$, $\kappa = 1$, $\theta = 5\%$ and $\sigma = 0.5\%$

```
N<-10^5
rt<-matrix(r0,(n+1),N)
for (i in 2:(n+1)){
  drt<-kappa*(theta-rt[(i-1),])*delta+
    sigma*sqrt(delta)*rnorm(N,0,1)
  rt[i,]<-rt[(i-1),]+drt
}
ZC<-exp(-delta*apply(rt,2,sum))
mean(ZC)
```

```
## [1] 0.6249339
```

Interest rate model

- We can prove that there is a closed formula for calculating the price of ZC bond under Vasicek model:

$$ZC(0) = A(T)exp(-B(T)r_0)$$

where

$$\begin{aligned} B(T) &= \frac{1}{\kappa}(1 - exp(-\kappa T)) \\ A(T) &= exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)[B(T) - T] - \frac{\sigma^2}{4\kappa}[B(T)]^2\right\} \end{aligned}$$

- Write a function to calculate the price of ZC bond at time 0 using these formula. Check your simulation result.

Interest rate model

```r
ZC<-function(r0,kappa,theta,sigma,T){
  BT<-1/kappa*(1-exp(-kappa*T))
  AT<-exp((theta-sigma^2/(2*kappa^2))*(BT-T)-
            sigma^2/(4*kappa)*BT^2)
  ZC<-AT*exp(-BT*r0)
}
print(ZC(r0,kappa,theta,sigma,T))
```

```
## [1] 0.6250678
```

Interest rate model

In the Vasicek model, the instantaneous short rate can be negative; Cox, Ingersoll and Ross (1985) proposed the following SDE for $r(t)$

$$
\begin{aligned}
r(0) &= r_0 \\
dr(t) &= \kappa\left(\theta - r(t)\right)dt + \sigma\sqrt{r(t)}dB(t)
\end{aligned}
$$

The resulting model has been a benchmark for many years because of its analytical tractability and the fact that, the instantaneous short rate is always positive.

1. With $r_0 = 2\%$, $T = 10$, $\kappa = 1$, $\theta = 5\%$ and $\sigma = 3\%$, simulate N paths of $r(t)$ in CIR model.
2. Calculate the price of ZC bond at time 0.

Interest rate model

```
sigma<-3/100
N<-10^5
rt<-matrix(r0,(n+1),N)
for (i in 2:(n+1)){
  drt<-kappa*(theta-rt[(i-1),])*delta+
    sqrt(rt[(i-1),])*sigma*sqrt(delta)*rnorm(N,0,1)
  rt[i,]<-rt[(i-1),]+drt
}
ZC<-exp(-delta*apply(rt,2,sum))
mean(ZC)
```

```
## [1] 0.6249244
```