15-381 Artificial Intelligence: Representation and Problem Solving
Homework 2 - Solutions

1 [10 pts] Probability miscellany

Calvin wants to choose between his two pet activities: playing with his pet tiger Hobbes in the garden, and tormenting his mom in the kitchen. He wants to choose uniformly at random between the two activities and decides to toss a coin to decide. Since he is a kid and is broke, he gets a coin from his mom to do this. However, he isn't sure if the coin is unbiased. Can you help him "simulate" a fair coin toss by using this coin? Explain how you would do this, and show that your procedure is indeed unbiased. While you don't know the bias of the coin, you may assume that the bias remains the same at all time. Some clarifications on terminology: a coin is considered "fair" or unbiased if p, the probability of heads equals 0.5. The question asks you to describe a procedure `toss_unbiased_coin()` that returns "Heads" or "Tails" with probability 0.5. The only source of randomness that the procedure has access to, is a procedure `toss_biased_coin()` that returns a "Heads" with some unknown probability p. Note: You aren't allowed to use any source of randomness other than the coin itself, so "call rand in matlab" is not a valid answer. :)

Solution

The key idea is to recognize that the events HT and TH happen with equal probability $p(1-p)$. If we associate a HT with a H and a TH with a T, then the algorithm is as follows

1. Toss coin twice
2. If HT return "Heads", else if TH, return "Tails", else go to 1.

2 [10 pts] Representation

Suppose we have a boolean variable X. To complete describe the distribution $P(X)$, we need to specify one value: P(X=0)(since P(X=1) is simply 1-P(X=0)) Thus, we say, this distribution can be characterized with

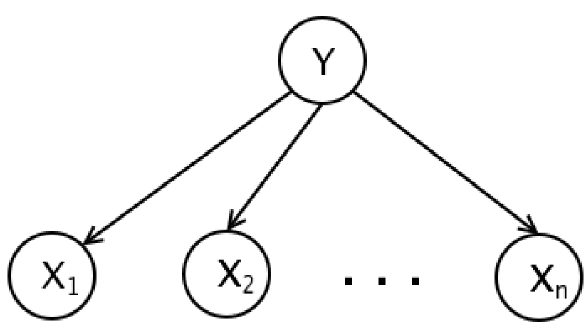


Figure 1: Bayesian network for Problem 3

one parameter. Now, consider N+1 binary random variables $X_1 \dots X_N, Y$ that factorize according to Fig. 1

1. Suppose you wish to store the joint probability distribution of these N+1 variables as a single table. How many parameters will you need to represent this table?

2. Now, suppose you were to utilize the fact the joint distribution factorizes accoording to the Bayes Network. How many parameters will you need to completely describe the distribution if you use the Bayesian Network representation? In other words, how many parameters will you need to fully specify the values of all the conditional probability tables in this Bayesian Network.

Solution

1. There are N+1 boolean variables. If we have a parameter for every possible instantiation of the variables, there will be 2^{N+1} parameters. But these parameters need to sum to one, so we can drop one parameter of the 2^{N+1}. The answer is $2^{N+1} - 1$

2. The CPT of Y will need one parameter(its a boolean variable without any parents). For each X, we'll need 2 parameters($P(X|Y), P(X|\neg Y)$ for example). Therefore, we have $2N + 1$ parameters in total.

3 [15 pts] Number of BNs

What is the maximum number of edges in a Bayesian network (BN) with n nodes? Prove that a valid BN containing this number of edges can be constructed (remember that the structure of a BN has to be a Directed *Acyclic* Graph)

Solution

$n(n-1)/2$. Proof by construction: Consider a BN over $X_1, X_2, \dots X_n$ such that there is an edge between $X_i, X_j \forall j > i$. The total number of edges in this graph is $n - 1 + (n-2) + \dots + 0 = n(n-1)/2$. To show that you cannot have a directed cycle in this graph, assume the contrary and suppose there is a cycle of the form $X_{i_1}, X_{i_2}, \dots X_{i_m}, X_{i_1}$; by construction of graph, we have $i_1 < i_2 \dots i_m < i_1$ leading to $i_1 < i_1$ which is a contradiction. Therefore no cycle exists.

Additionally,you cannot construct a BN with more than $n(n-1)/2$ edges, since any directed graph with more than $n(n-1)/2$ edges should have atleast one pair of vertices for which there is more than one edge implying that you have atleast one edge in both directions resulting in a cycle.

4 [15 pts] Conditional Independencies in Bayes Nets

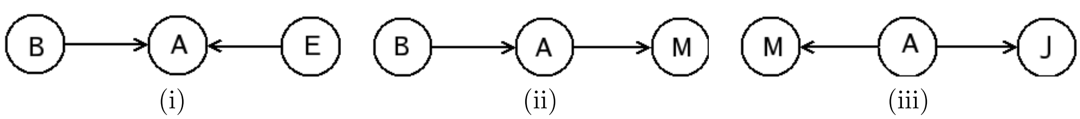


Figure 2: Parts of the alarm network

The Bayesian networks in Fig. 2 are all part of the alarm network introduced in class and in Russell and Norvig. We use the notation $X \perp Y$ to denote the variable X being independent of Y , and $X \perp Y|Z$ to denote X being independent of Y given Z.

1. For each of these three networks, write the factored joint distribution implied, in the form of $p(X, Y) = p(X)p(Y|X)$

2. Using the joint distribution you wrote down for Fig. 2(i), write down a formula for $P(B, E)$.

3. Now prove that $B \perp E$.

4. Similarly, prove that $B \perp M|A$ in the Bayesian network of Fig. 2(ii), and $M \perp J|A$ in the Bayesian network of Fig. 2(iii).

Solution

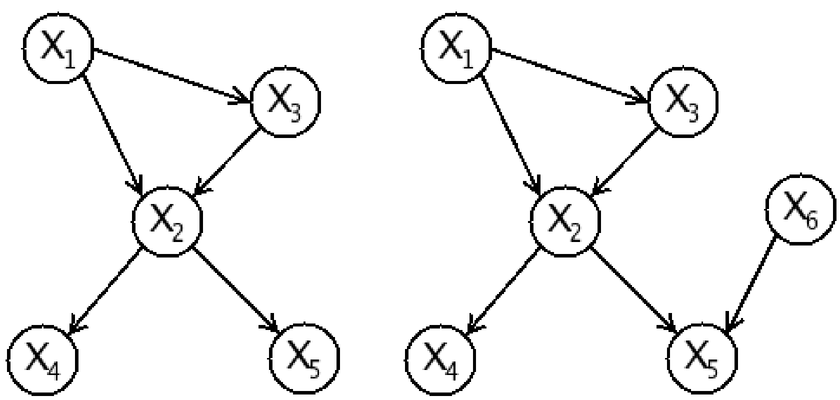
1. (i) $P(B,A,E) = P(B)P(E|B)P(A|B,E)$ (using chain rule) $= P(B)P(E)P(A|B,E)$(using fact that $B \perp E$ from BN structure). Similarly (ii) $P(B,A,M) = P(B)P(A|B)P(M|A)$ and (iii) $P(M,A,J) = P(A)P(M|A)P(J|A)$

2. P(B,E) $= \sum_a P(B,a,E) = \sum_a P(B)P(E)P(a|B,E) = P(B)P(E)\sum_a P(a|B,E) = P(B)P(E)$. Since $P(a|B,E)$ is a conditional probability distribution, the sum over all values of a is 1.

3. From solution of (ii), P(B,E) = P(B) P(E), therefore $B \perp E$ by definition of independence.

4. $P(B)P(A|B) = P(A)P(B|A)$ from Bayes rule. Therefore $P(B,A,M) = P(B)P(A|B)P(M|A) = P(A)P(B|A)P(M|A)$. $P(B,M|A) = P(B,A,M)/P(A) = P(A)P(B|A)P(M|A)/P(A) = P(B|A)P(M|A)$ Therefore $B \perp M|A$.
   $P(M,A,J) = P(A)P(M|A)P(J|A)$. Therefore $P(M,J|A) = P(M,J,A)/P(A) = P(A)P(M|A)P(J|A)/P(A) = P(M|A)P(J|A)$. Therefore $M \perp J|A$

5 [15 pts] Elimination



(i) (ii)

Figure 3: Bayesian networks for Problem 6

Consider the Bayesian Network given in Fig. 3(i). Assume that each of the variables are boolean valued. For each of the following, state the total number of operations(multiplication and addition) the variable elimination algorithm will take to compute the answer. For example, if A and B are binary variables, $\sum_a P(B)P(a|B)$ will take two multiplications($P(B) * P(A|B)$ and $P(B) * P(\neg A|B)$) and one addition(adding those two terms). Assume that the algorithm avoids unnecessary computation, so any summations that are irrelevant to the query will be avoided(see the textbook for an example of an *irrelevant* summation).

1. $P(X_3|X_4)$ i.e., the probability that X_3 is true given that X_4 is true. Assume that the variables are eliminated in the order: X_5, X_1, X_2.

2. Solution:

$$P(X_3|X_4) = P(X_3, X_4)/P(X_4) = P(X_3, X_4)/(P(X_3, X_4) + P(X_3, \neg X_4))$$
$$P(X_3, X_4) = \sum_{x_2} \sum_{x_1} \sum_{x_5} P(x_1, x_2, X_3, X_4, x_5)$$
$$P(x_1, x_2, X_3, X_4, x_5) = P(x_1)P(X_3|x_1)P(x_2|x_1, X_3)P(X_4|x_2)P(x_5|x_2)$$
$$P(X_3, X_4) = \sum_{x_2} P(X_4|x_2) \sum_{x_1} P(x_1)P(X_3|x_1)P(x_2|x_1, X_3) \sum_{x_5} P(x_5|x_2)$$

x_5 is an irrelevant variable, so we can remove it immediately

$$P(X_3, X_4) = \sum_{x_2} P(X_4|x_2) \sum_{x_1} P(x_1)P(X_3|x_1)P(x_2|x_1, X_3)$$

$$f_1(x_2) = \sum_{x_1} P(x_1)P(X_3|x_1)P(x_2|x_1, X_3)$$

Computing $f_1(x_2)$ for one choice of x_2 will take 2*2 multiplications and 1 addition = 5 operations. x_2 can take two values, so in total, this will take 10 operations. Computing $P(X_3, X_4) = \sum_{x_2} P(X_4|x_2)f_1(x_2)$ which will take another 1*2 multiplications and 1 addition = 3 operations. Therefore, to compute $P(X_3, X_4)$ will take 13 operations. Similarly $P(X_3, \neg X_4)$ will take another 13 operations, and finally computing $P(X_3, X_4)/(P(X_3, X_4) + P(X_3, \neg X_4))$ will take another 1 addition (and 1 division). In total, this will take 27 operations(28 with div)

3. $P(X_3|X_4)$ but this time, with the elimination ordering X_2, X_1, X_5

4. Solution:

$$P(X_3|X_4) = P(X_3, X_4)/P(X_4) = P(X_3, X_4)/(P(X_3, X_4) + P(X_3, \neg X_4))$$
$$P(X_3, X_4) = \sum_{x_5} \sum_{x_1} \sum_{x_2} P(x_1, x_2, X_3, X_4, x_5)$$
$$P(X_3, X_4) = \sum_{x_5} \sum_{x_1} P(x_1)P(X_3|x_1) \sum_{x_2} P(x_5|x_2)P(x_2|x_1, X_3)P(X_4|x_2)$$
$$g_1(x_5, x_1) = \sum_{x_2} P(x_5|x_2)P(x_2|x_1, X_3)P(X_4|x_2)$$

Computing $g_1(x_5, x_1)$ takes 2*2 multiplications and 1 addition = 5 operations for each choice of x_5, x_1. Since there are four such values, this will take 20 operations.

$$P(X_3, X_4) = \sum_{x_5} \sum_{x_1} P(x_1)P(X_3|x_1)g_1(x_5, x_1)$$

$g_2(x_5) = \sum_{x_1} P(x_1)P(X_3|x_1)g_1(x_5, x_1)$ will take 2*2 +1 = 5 operations for each choice of x_5. Since there are two such values of x_5, this will take 10 operations. Computing $\sum_{x_5} g_2(x_5)$ will take another 1 addition operation, leading to a total of 31 operations to compute $P(X_3, X_4)$ Computing $P(X_3|X_4)$ will therefore take 31*2+1(addition)+1(division) = 63 operations(64 including div)

5. Now suppose that you had to answer the last two questions, this time with the Bayes Net given in Fig. 3(ii). Would your answers change?(You don't have to state the number of operations; just if they will be different from previously with a brief explanation)

6. Solution:
   In general, x_6 would not be irrelevant and therefore, the answer would change.

```
Enumeration_Ask(X,e,bn) begin
    Data: X, the query variable; e, the observed values for variables E; bn, a Bayes net with variables
          X ⋃ E ⋃ Y
    Result: distribution over X
    Q(X) ← a distribution over X, initially empty;
    foreach value x_i of X do
        extend e with value x_i for X;
        Q(X_i) ← Enumerate_All(Vars[bn], e);
    end
    return Normalize(Q(X))
end
Enumerate_All(vars, e) begin
    if Empty?(vars) then
        return 1.0
    end
    Y ← First(vars);
    if Y has value y in e then
        return P(y|Parents(Y)) × Enumerate_All(REST(vars, e))
    else
        return ∑_y P(y|Parents(Y)) × Enumerate_All(Rest(vars), e_y) where e_y is e extended with Y = y
    end
end
```

Algorithm 1: The enumeration algorithm

6 [35 pts] Inference by enumeration

1. Implement exact inference by enumeration. Write a function that calculates the conditional probability distribution of one query variable given a set of evidence variables in a Bayesian network. See Algorithm 1 for the pseudo-code and Russell and Norvig chapter14.4(pp. 504) for reference. For simplicity, all variables are binary, so all distributions mentioned in the pseudocode and in Russell and Norvig, including the returned value of the function, can be represented by the probability of the variable being true. Two sample Bayesian network will be given to you in the support archive, named alarm and pedigree, details below. The following Matlab functions are provided, to access the Bayesian network data structure:

   - *create_alarm_bn*: creates the alarm Bayesian network.
   - *create_pedigree_bn*: creates the pedigree Bayesian network.
   - *bn_vars*: returns the variables in a Bayesian network, partially ordered from parents to children.
   - *bn_parents*: returns the parents of a variable.
   - *bn_cpt*: returns the conditional distribution of a variable, given the specified values of its parents. This corresponds to one row of the conditional probability table(CPT).

   See README and type help func name in Matlab (or read corresponding scripts) for documentations and examples. Note that the recursive enumeration must be performed from parents to children, i.e. the list of variables must be partially ordered such that parents are always before their children. The function bn vars will provide the ordered variables list(actually the variable index itself in the Bayesian networks given below are already ordered this way, so the ordered list is just 1,2,. . .,N). Please write your code by modifying the provided file enumeration ask.m, which contains suggested API with documentation. Matlab is STRONGLY recommended, particularly because writing necessary support code in other languages is time consuming.

2. Run your exact inference implementation on the following two Bayesian network inference problems. It should be straightforward for you to convert the question below into a function call to your code (by using appropriate variable indexes labeled on the graph below); type "help enumeration_ask" for an example. Both questions ask the conditional probability of one query variable being true given a set of evidence variables. Report run time and results.

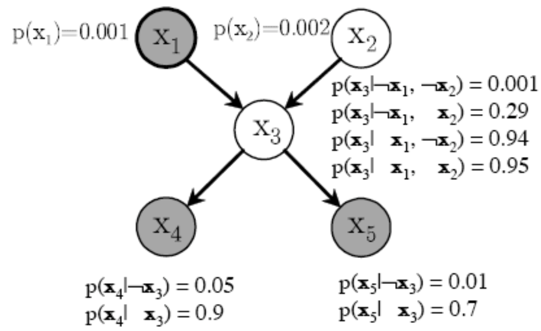


Figure 4: Bayesian network for Problem 3(a)

(a) The first Bayesian network, created by create_alarm_bn, is the alarm network in Russell and Norvig. As shown in Fig. 4, variable are indexed from 1 to 5 (denoted as X1 to X5), and CPT are the same as in Russell and Norvig figure 14.2. Evidence variables are shaded, while query variables are shaded and circled by a thick line. Calculate the conditional probability p(X1|X4,¬X5). Hint: verify with some queries that you know the answer first, e.g. p(X1|X4,X5) should be about 0.284.

(b) Solution: p(X1|X4,X5) = 0.0051

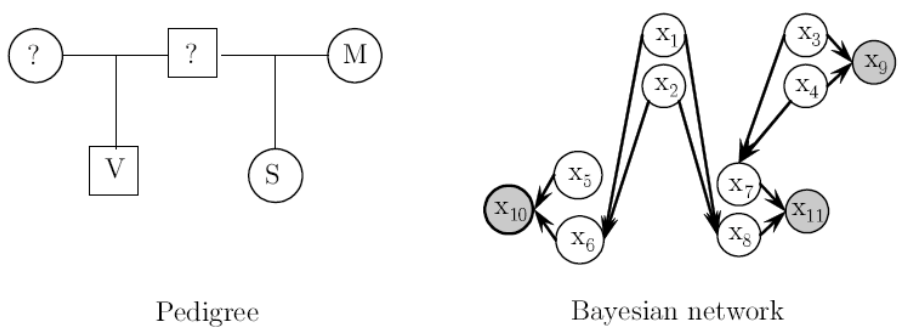


Figure 5: Bayesian network for Problem 3(b)

(c) The second Bayesian network, created by create_pedigree_bn, is about genetic inference. Consider a victim V in a plane crash, whose only family members are his half-sister S and the sister's mother M (not V's mother). Their pedigree is shown below. You need to determine whether certain remains belong to V based on genetic fingerprints of S and M. This can be solved by a Bayesian network shown in Fig. 5, indexed from 1 to 11. Evidence and query variables are shaded, while normal circles are hidden variables. You do not need to worry about the CPT if you are using Matlab; otherwise the CPT is explained in the documentation of create_pedigree_bn.m The variables (X1,X2,. . . ,X8) correspond to unobserved genetic information in the so-called Mendelian inheritance: humans have two copies of each chromosome, one from the father and one from the mother. During reproduction, one copy (chosen randomly) will be passed to the next generation. Assume you cannot determine which copy is from which parent, but can only obtain partial information in the observed variables (X9,X10,X11). However, you do not have to understand Mendelian inheritance to solve this problem. Now using the structure and CPT provided in the archive, calculate the conditional probability p(X10| $\neg$ X9,X11).

(d) Solution: p(X1|X4,X5) = 0.0407