Artificial Intelligence
CE-417, Group 1
Computer Eng. Department
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Courtesy: Most slides are adopted from CSE-573 (Washington U.), original slides for the textbook, and CS-188 (UC. Berkeley).

Uncertainty

Uncertainty

- Let action A_t = leave for airport t minutes before flight
- Will A_t get me there on time?
- Problems:
  1) partial observability (road state, other drivers' plans, etc.)
  2) noisy sensors (KCBS traffic reports)
  3) uncertainty in action outcomes (flat tire, etc.)
  4) immense complexity of modelling and predicting traffic
- Hence a purely logical approach either
  1) risks falsehood: "A_{25} will get me there on time" or
  2) leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc. etc."

- (A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport . . .)

Methods for handling uncertainty

- **Default** or **nonmonotonic** logic:

  Assume my car does not have a flat tire

  Assume A_{25} works unless contradicted by evidence

- Issues: What assumptions are reasonable? How to handle contradiction?

- **Rules with fudge factors**:

$$A_{25} \mapsto_{0.3} AtAirportOnTime$$
$$Sprinkler \mapsto_{0.99} WetGrass$$
$$WetGrass \mapsto_{0.7} Rain$$

- Issues: Problems with combination, e.g., Sprinkler causes Rain??

Methods for handling uncertainty (cont.)

- **Probability**
  - Given the available evidence,

    A_{25} will get me there on time with probability 0.04
- Mahaviracarya (9th C.), Cardamo (1565) theory of gambling
- **Fuzzy logic** handles **degree of truth** NOT uncertainty
  - e.g., WetGrass is true to degree 0.2

Probability

- Probabilistic assertions **summarize** effects of
  - **laziness**: failure to enumerate exceptions, qualifications, etc.
  - **ignorance**: lack of relevant facts, initial conditions, etc.
- Subjective or Bayesian probability:

  Probabilities relate propositions to one's own state of knowledge
  - e.g., $P(A_{25} \mid \text{no reported accidents}) = 0.06$

    These are **not** claims of a "probabilistic tendency" in the current situation
  - (but might be learned from past experience of similar situations)
- Probabilities of propositions change with new evidence:
  - e.g., $P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

- Suppose I believe the following:
  - $P(A_{25}$ gets me there on time$|\ldots) = 0.04$
  - $P(A_{90}$ gets me there on time$|\ldots) = 0.70$
  - $P(A_{120}$ gets me there on time$|\ldots) = 0.95$
  - $P(A_{1440}$ gets me there on time$|\ldots) = 0.9999$
- Which action to choose?
- Depends on my preferences for missing flight vs. airport cuisine, etc.
- Utility theory is used to represent and infer preferences
- Decision theory = utility theory + probability theory

Probability basics

- Begin with a set Ω—the sample space
  e.g., 6 possible rolls of a die.
  $\omega \in \Omega$ is a sample point/possible world/atomic event/outcome
- A probability space or probability model is a sample space with an assignment $P(\omega)$ for every $\omega \in \Omega$ s.t.
  - $0 \leq P(\omega) \leq 1$
  - $\Sigma_{\omega} P(\omega) = 1$

e.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

- An event A is any subset of Ω
  - $P(A) = \Sigma_{\{\omega \in A\}} P(\omega)$
- e.g., $P(\text{die roll} < 4) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

Random variables

- A random variable is a function from sample points to some range, e.g., the reals or Booleans
  - e.g., Odd(1) = true.
- P induces a probability distribution for any r.v. X:
- $P(X = x_i) = \Sigma_{\{\omega:X(\omega)=x_i\}} P(\omega)$
- e.g., P(Odd = true) = P(1) + P(3) + P(5) = 1/6 + 1/6 + 1/6 = 1/2

Propositions

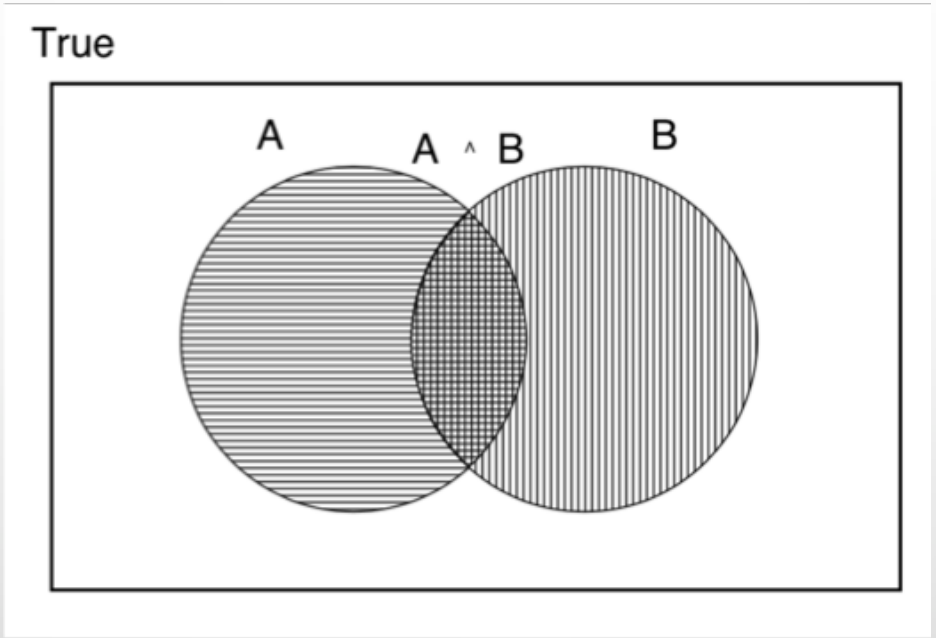
- Think of a proposition as the event (set of sample points) where the proposition is true
- Given Boolean random variables A and B:
  event a = set of sample points where $A(\omega) = true$
  event $\neg a$ = set of sample points where $A(\omega) = false$

  event $a \wedge b$ = points where $A(\omega) = true$ and $B(\omega) = true$
- Often in AI applications, the sample points are **defined** by the values of a set of random variables, i.e., the sample space is the Cartesian product of the ranges of the variables
- With Boolean variables, sample point = propositional logic **model**
  - e.g., A = true, B = false, or $a \wedge \neg b$.
- Proposition = disjunction of atomic events in which it is true
- e.g., $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

  $\Rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

Why use probability?

- The definitions imply that certain logically related events must have related probabilities
- e.g., $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$



- de Finetti (1931): an agent who bets according to probabilities that violate these axioms can be forced to bet so as to lose money regardless of outcome.

Syntax for propositions

- Propositional or Boolean random variables

  e.g., Cavity (do I have a cavity?)

  Cavity = true is a proposition, also written *cavity*

- Discrete random variables (finite or infinite)

  e.g., Weather is one of ⟨sunny, rain, cloudy, snow⟩

  Weather = rain is a proposition

  Values must be exhaustive and mutually exclusive

- Continuous random variables (bounded or unbounded)

  e.g., Temp = 21.6; also allow, e.g., Temp < 22.0.

- Arbitrary Boolean combinations of basic propositions

Prior probability

- Prior or unconditional probabilities of propositions
  e.g., P(Cavity = true) = 0.1 and P(Weather = sunny) = 0.72

  correspond to belief prior to arrival of any (new) evidence

- Probability distribution gives values for all possible assignments:

  P(Weather) = ⟨0.72, 0.1, 0.08, 0.1⟩ (normalized, i.e., sums to 1)

- Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s (i.e., every sample point)

  P(Weather, Cavity) = a 4 × 2 matrix of values:

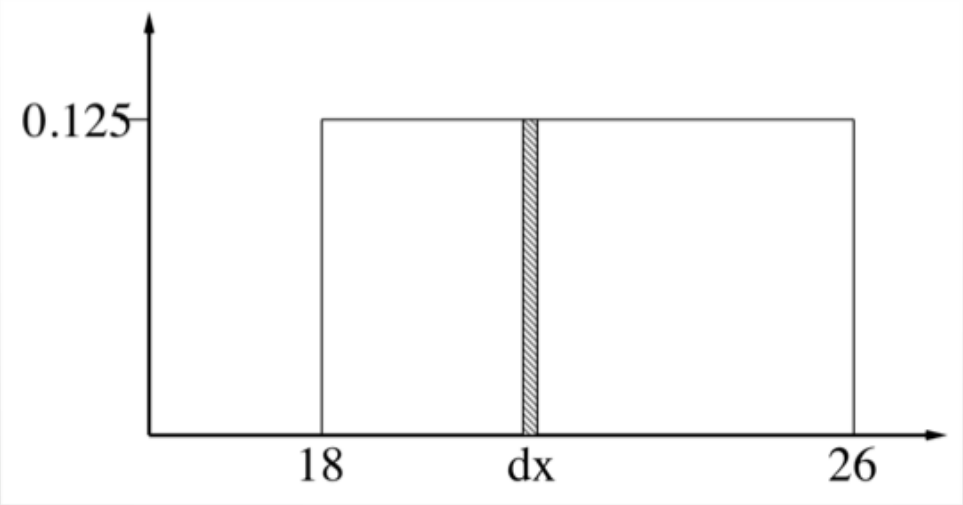
$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = true$	0.144	0.02	0.016	0.02
$Cavity = false$	0.576	0.08	0.064	0.08

Every question about a domain can be answered by the joint distribution because every event is a sum of sample points

Probability for continuous variables

- Express distribution as a parameterized function of value:

P (X = x) = U[18, 26](x) = uniform density between 18 and 26



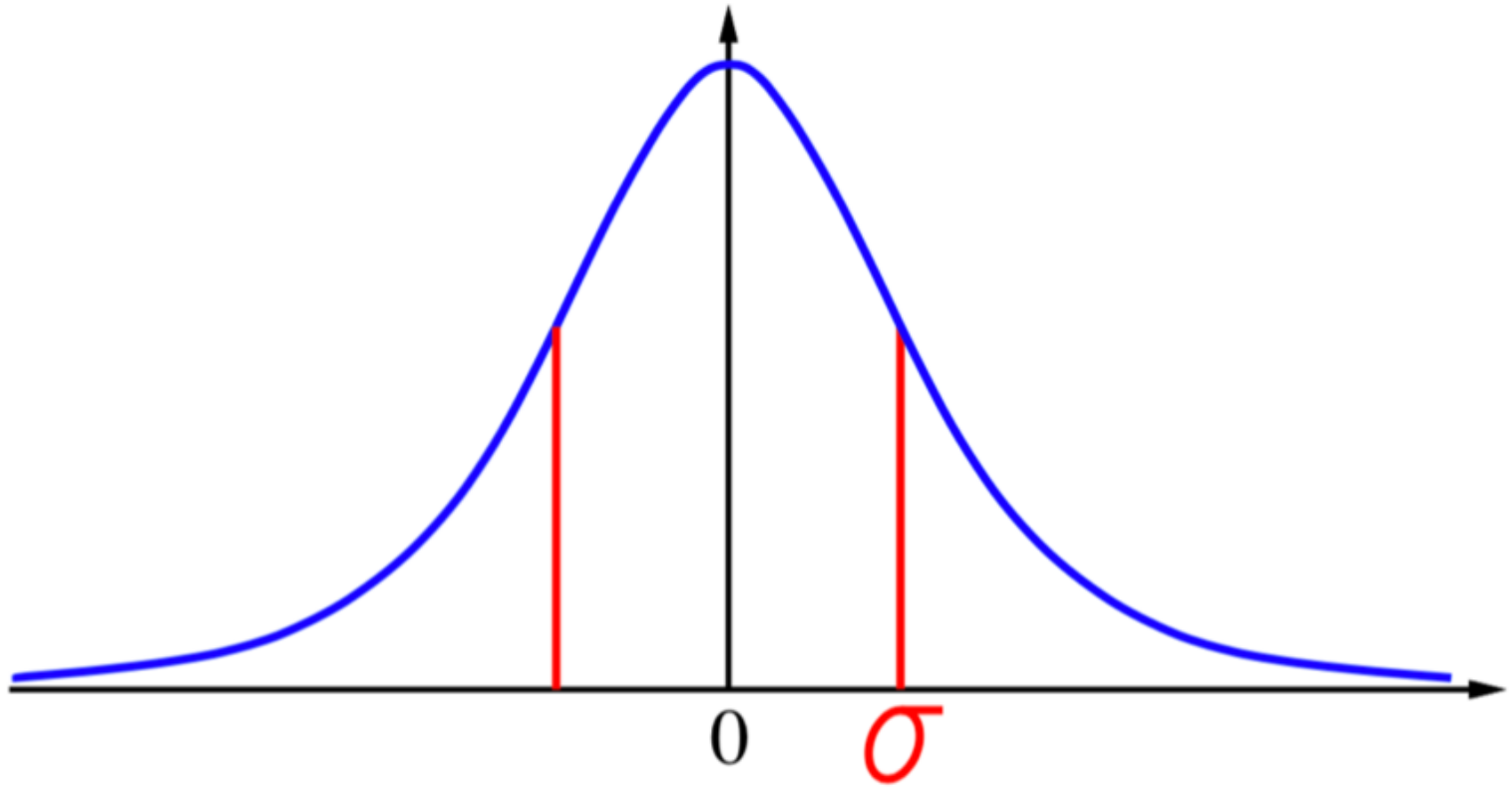
Here P is a density; integrates to 1.

P (X = 20.5) = 0.125 really means

$\lim_{dx \to 0} P(20.5 \leq X \leq 20.5 + dx)/dx = 12.5$

Gaussian density

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



Conditional probability

- Conditional or posterior probabilities
  e.g., P(cavity | toothache) = 0.8
  i.e., **given that** toothache **is all I know**

  NOT "if toothache then 80% chance of cavity"

- (Notation for conditional distributions:
  P(Cavity | Toothache) = 2-element vector of 2-element vectors)

- If we know more, e.g., cavity is also given, then we have P(cavity | toothache, cavity) = 1

- New evidence may be irrelevant, allowing simplification, e.g.,

  P(cavity | toothache, rain) = P(cavity | toothache) = 0.8

This kind of inference, sanctioned by domain knowledge, is crucial

Conditional probability (cont.)

- Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

- Product rule gives an alternative formulation:

$$P(a \wedge b) = P(a|b)\,P(b) = P(b|a)\,P(a)$$

- A general version holds for whole distributions, e.g.,

$\mathbf{P}(\text{Weather}, \text{Cavity}) = \mathbf{P}(\text{Weather}|\text{Cavity})\,\mathbf{P}(\text{Cavity})$ (View as a 4×2 set of equations, not matrix mult.)

Conditional probability (cont.)

- Chain rule is derived by successive application of product rule:

$$\begin{aligned}\mathbf{P}(X_1,\ldots,X_n) &= \mathbf{P}(X_1,\ldots,X_{n-1})\,\mathbf{P}(X_n|X_1,\ldots,X_{n-1}) \\ &= \mathbf{P}(X_1,\ldots,X_{n-2})\,\mathbf{P}(X_{n_1}|X_1,\ldots,X_{n-2})\,\mathbf{P}(X_n|X_1,\ldots,X_{n-1}) \\ &= \ldots \\ &= \textstyle\prod_{i=1}^{n}\mathbf{P}(X_i|X_1,\ldots,X_{i-1})\end{aligned}$$

Inference by enumeration

- Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- For any proposition ϕ, sum the atomic events, where it is true:

$P(\phi) = \Sigma_{\omega:\omega \models \phi} P(\omega)$

P(cavity $P(\neg cavity|toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$: 0.28

$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$

Normalization

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- Denominator can be viewed as a normalization constant α
- P(Cavity | toothache) = α P(Cavity, toothache)
  = α[P(Cavity,toothache,catch)+P(Cavity,toothache,¬catch)]
  =α[⟨0.108,0.016⟩+⟨0.012,0.064⟩]
  = α⟨0.12,0.08⟩=⟨0.6,0.4⟩
- General idea: compute distribution on query variable by fixing evidence variables and summing over hidden variables

Inference by enumeration (cont.)

- Let **X** be all the variables. Typically, we want
  the posterior joint distribution of the query variables **Y** given specific values **e**
  for the evidence variables **E**
- Let the hidden variables be $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$
- Then the required summation of joint entries is done by summing out the hidden variables:
- $\mathbf{P}(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha\, \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$

Inference by enumeration (cont.)

- The terms in the summation are joint entries because **Y**, **E**, and **H** together exhaust the set of random variables

Obvious problems:

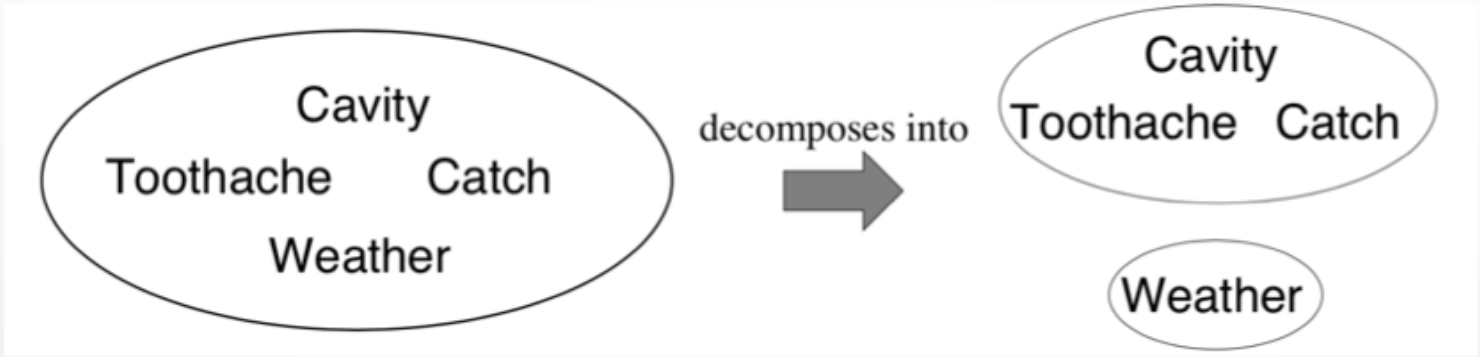
1) Worst-case time complexity $O(d^n)$ where d is the largest arity

2) Space complexity $O(d^n)$ to store the joint distribution

3) How to find the numbers for $O(d^n)$ entries???

Independence

- A and B are independent iff
  P(A|B) = **P**(A) or **P**(B|A) = **P**(B) or **P**(A, B) = **P**(A) **P**(B)



P(Toothache, Catch, Cavity, Weather) = P(Toothache, Catch, Cavity) P(Weather)

- 32 entries reduced to 12; for n independent biased coins, $2^n \rightarrow n$
- Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

- P(Toothache, Cavity, Catch) has $2^3 - 1 = 7$ independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

  (1) P(catch | toothache, cavity) = P(catch | cavity)

- The same independence holds if I haven't got a cavity:

  (2) P(catch | toothache, ¬cavity) = P(catch | ¬cavity)

- Catch is **conditionally independent** of Toothache given Cavity:

  P(Catch | Toothache, Cavity) = **P**(Catch | Cavity)

- Equivalent statements:

P(Toothache | Catch, Cavity) = **P**(Toothache | Cavity)
P(Toothache, Catch | Cavity) = **P**(Toothache | Cavity)**P**(Catch | Cavity)

Conditional independence (cont.)

- Write out full joint distribution using chain rule:
- P(Toothache, Catch, Cavity)
  = P(Toothache | Catch, Cavity) P(Catch, Cavity)
  = P(Toothache | Catch, Cavity) P(Catch | Cavity) P(Cavity)
  = P(Toothache | Cavity) P(Catch | Cavity) P(Cavity)
- i.e., 2 + 2 + 1 = 5 independent numbers (equations 1 and 2 remove 2)
- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.
- **Conditional independence is our most basic and robust form of knowledge about uncertain environments.**

Bayes' Rule

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\Rightarrow \text{ Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha\mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

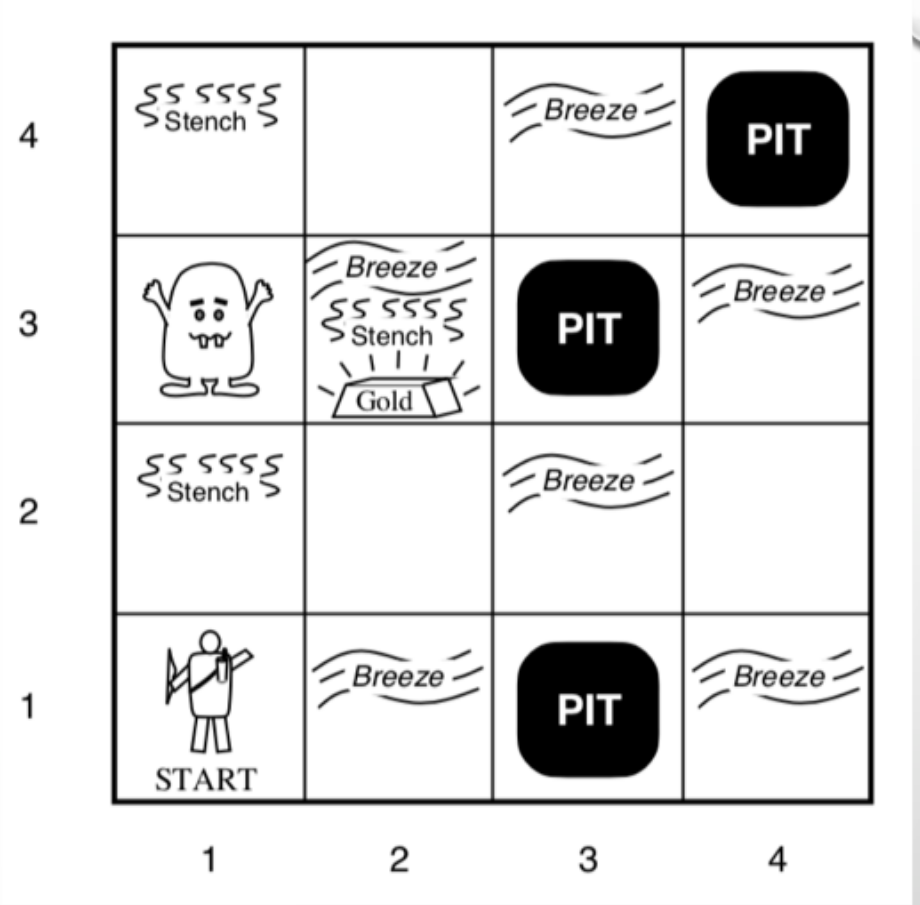
- **P**(Cavity | toothache ∧ catch)
  = α**P**(toothache ∧ catch | Cavity) **P**(Cavity)
  = α**P**(toothache | Cavity) **P**(catch | Cavity) **P**(Cavity)

- This is an example of a naive Bayes model:
  $P(\text{Cause}, \text{Effect}_1, \ldots, \text{Effect}_n) = P(\text{Cause}) \prod_i P(\text{Effect}_i | \text{Cause})$

- Total number of parameters is **linear** in n

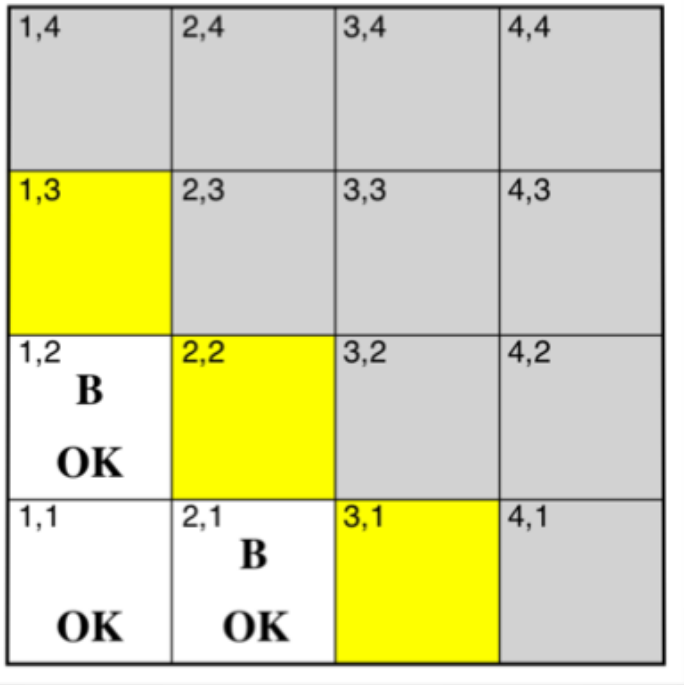
Wumpus World PEAS description

- Environment:
  - Squares adjacent to wumpus are smelly
  - Squares adjacent to pit are breezy
  - Glitter iff gold is in the same square
  - Shooting kills wumpus if you are facing it
  - Shooting uses up the only arrow
  - Grabbing picks up gold if in same square
  - Releasing drops the gold in same square



- Actuators: Left turn, Right turn, Forward, Grab, Release, Shoot
- Sensors: Breeze, Glitter, Smell

Wumpus World



- P_{ij} = true iff [i, j] contains a pit
- B_{ij} = true iff [i, j] is breezy

  Include only $\neg B_{1,1}$, $B_{1,2}$, $B_{2,1}$ in the probability model

Specifying the probability model

- The full joint distribution is $P(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$
- Apply product rule: $P(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})\, P(P_{1,1}, \ldots, P_{4,4})$ (Do it this way to get P(Effect|Cause).)
- First term: 1 if pits are adjacent to breezes, 0 otherwise
- Second term: pits are placed randomly, probability 0.2 per square:

$$\mathbf{P}(P_{1,1}, \ldots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

Observations and query

- We know the following facts:
- $b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

  $known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$
- Query is $P(P_{1,3} | known, b)$

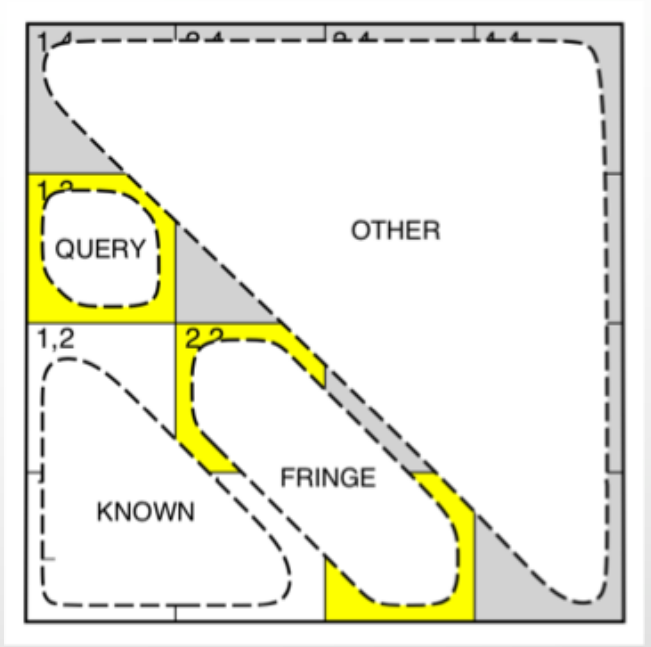
  Define $Unknown = P_{ij}$s other than $P_{1,3}$ and $Known$
- For inference by enumeration, we have

$$\mathbf{P}(P_{1,3} | known, b) = \alpha \Sigma_{unknown} P(P_{1,3}, unknown, known, b)$$

Grows exponentially with number of squares!

Using conditional independence

- Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares



- Define Unknown = Fringe ∪ Other

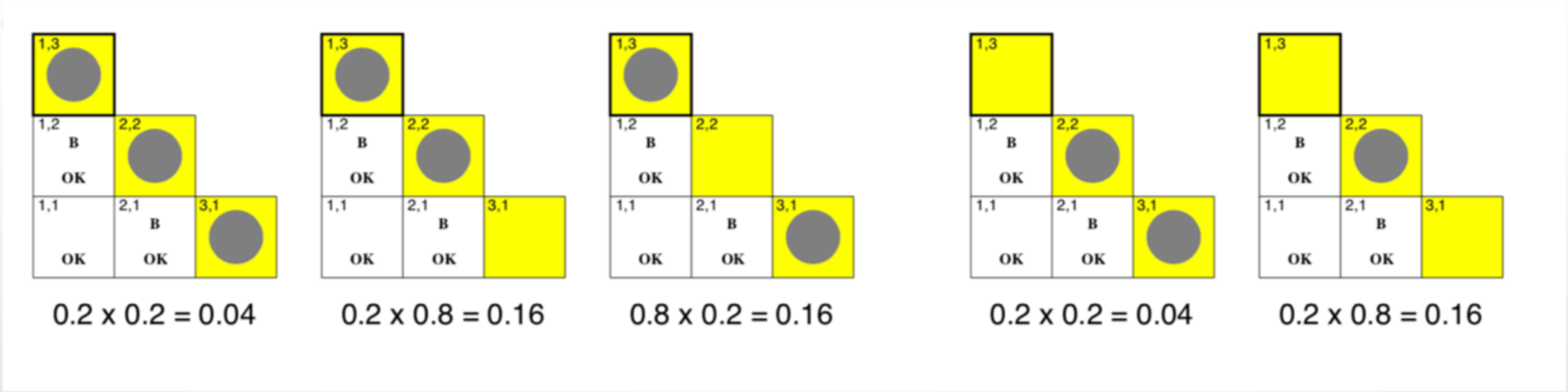
$\mathbf{P}(b \mid P_{1,3}, \text{Known}, \text{Unknown}) = \mathbf{P}(b \mid P_{1,3}, \text{Known}, \text{Fringe})$

- Manipulate query into a form where we can use this!

Using conditional independence (cont.)

$$
\begin{aligned}
\mathbf{P}(P_{1,3}|known, b) &= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b) \\
&= \alpha \sum_{unknown} \mathbf{P}(b|P_{1,3}, known, unknown)\mathbf{P}(P_{1,3}, known, unknown) \\
&= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|known, P_{1,3}, fringe, other)\mathbf{P}(P_{1,3}, known, fringe, other) \\
&= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|known, P_{1,3}, fringe)\mathbf{P}(P_{1,3}, known, fringe, other) \\
&= \alpha \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}, known, fringe, other) \\
&= \alpha \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3})P(known)P(fringe)P(other) \\
&= \alpha\, P(known)\mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe)P(fringe) \sum_{other} P(other) \\
&= \alpha' \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe)P(fringe)
\end{aligned}
$$

Using conditional independence (cont.)



- $\mathbf{P}(P_{1,3} \mid \text{known}, b) = \alpha' \langle 0.2(0.04 + 0.16 + 0.16), 0.8(0.04 + 0.16) \rangle \approx \langle 0.31, 0.69 \rangle$
- $\mathbf{P}(P_{2,2} \mid \text{known}, b) \approx \langle 0.86, 0.14 \rangle$

Summary

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- Independence and conditional independence provide the tools