Mathland: Constructionist and immersive Mixed Reality experiences for mathematical learning in the real world

Mina Khan	Pattie Maes	3rd Author Name	4th Author Name
Affiliation	Affiliation	Affiliation	Affiliation
City, Country	City, Country	City, Country	City, Country
e-mail address	e-mail address	e-mail address	e-mail address

ABSTRACT

Mathematical experiences are intrinsic to our everyday lives, yet mathematics education is mostly confined to textbooks. Seymour Papert used the term 'Mathland' to propose a world where one would learn mathematics as naturally as one learns French while growing up in France. We built a Mixed Reality application that augments the physical world with interactive mathematical concepts to create a real-life Mathland. Using Mathland, people can collaboratively explore, experience and experiment with mathematics and physics phenomena in their real, physical environment using tangible objects. Our preliminary study with 30 participants shows that a considerable percentage of participants found Mathland to not only be engaging (83%), but also efficacious in the areas of collaborative learning (92.8%), problem solving (96.6%) and mathematics education (90%). Mathland opens up new opportunities for mathematical learning to use of Papert's constructionist principles in an immersive environment that affords situated learning, embodied interaction and playful constructionism.

Author Keywords

Virtual/Augmented Reality; Education/Learning; Play; Tangible; Wearable Computers; Embodied Interaction; Situated Learning

ACM Classification Keywords

H.5.1. Information interfaces and presentation (e.g., HCI): Multimedia Information Systems: Artificial, augmented, and virtual realities

INTRODUCTION

Galileo famously wrote "Mathematics is the language in which God has written the universe",, yet most people feel disconnected from mathematics. Mathematical anxiety (fear of numbers) is a common problem, but it is not one's actual but perceived ability that hinders one from excelling in mathematics [8,28]. The lack of interest in mathematics and mathematical anxiety leads to high dropout rates in science, technology, engineering and mathematics (STEM) fields, the so called leaky pipeline of STEM, especially for women, as women are more likely than men to drop out of STEM after a math class [14,15,47].

Mathematical anxiety can be attributed to the way mathematics is taught using abstract symbols that have no inherent meaning for students, in punitive ways such that math becomes a punishment, and as a set of rules to be memorized rather than understood [4,5,12,21,28,29]. We seek to replace the abstract symbols in mathematics with mathematical applications, the punitive teaching methods with playful experiences, and the rote learning of concepts with exploratory learning to improve mathematical understanding.

Educational environments have been gradually changing in the past few years. In particular, there has been a slow shift from education to learning; learning places more emphasis on the learner as compared to education, which is more teacher-centered. For example, the Montessori method of education engages students more actively in the learning process and gives them the freedom to explore and drive their own learning. Learning methodologies like active learning, experiential learning, and discovery-based learning are supported by the works of Jean Piaget, Jerome Bruner and Seymour Papert among others, and are rooted in the constructivist learning theory, which states that learners construct knowledge by making observations rather than just accepting facts.

Technology has played a key role in transforming, especially democratizing, education. There is a wealth of online courses, e.g. on Coursera [50] and Khan Academy [51], and learning applications, e.g. Duolingo for language learning [52]. In addition to democratizing education, technology has also innovated learning by supporting playful and project-based learning. For example, Lego Mindstorms [53] allows kids to learn coding while making programmable Lego robots, whereas Scratch [54] teaches coding as learners make creative storytelling projects.

Project-based learning was proposed by Seymour Papert as central in his constructionist learning theory [33], which extends constructivist learning to suggest that learners construct their knowledge while working on meaningful projects. Constructionist learning allows for playful, exploratory, and application-based learning, and has been considered key for mathematical learning [13,36]. Seymour Papert said, "If they grew up in France, they'd learn French perfectly well… If we all learn mathematics in mathland, we'd learn mathematics perfectly well" [20].

Our goal for Mathland is to create an immersive and constructionist world for learners to learn mathematics through explorations in the real world. Mixed Reality (MR) allows us to augment the physical world with virtual objects and annotations so that learners can explore and experiment with mathematics in the real world. As a result, Mathland serves as a mathematical lens through which the users can explore the hidden mathematical realities of our world, and also a mathematical playground where users can experiment with mathematics in applied, playful and exploratory ways.

BACKGROUND & RELATED WORK

In late 1960s, Seymour Papert developed the Logo programming language [33,35], which allowed people to learn coding and use geometry while creating drawings. Users could give movement-related commands to a robotic or digital "turtle", and the trail left behind the turtle could be used to create geometric patterns. Logo was the first constructionist application that allowed users to learn math as they engaged in creative projects.

Computer simulations have been popular in education, especially science education as many science concepts are difficult for students to understand because of the lack of real-life visuals [9]. Simulations can explain scientific phenomena in-context and without any physical limitations so that learners can visualize abstract scientific concepts [6]. Bret Victor's Kill Math [46] uses simulations and visualizations to teach math concepts, and Ken Perlin's Chalk talk [31] uses interactive simulations for teaching mathematics.

There are also computer games for learning as playful activities enhance learning [37,39], and students who learn math using games outperform those who do not [11]. Armadillo Run [44] is a 2D game that uses engineering concepts in game play. Virtual Reality (VR) has also been explored for immersive science education [27,32], e.g Project Science Space [41,42] creates 3 immersive science worlds in VR — Newton world, Maxwell world, and Pauling world. Fantastic Contraption [55] is another VR game that uses physics and engineering concepts.

Mathland integrates the constructionist, simulation-based and playful learning approaches, but instead of computer screens or virtual reality worlds, Mathland allows for mathematical learning in the real world through virtual annotations and simulations. There are some Augmented Reality (AR) applications, like Construct3D [23,24]. But instead of connecting the content to user's real physical environment and creating an immersive experience, these applications simply overlay content on top of the physical world. Also, most AR educational applications only allow learners to visualize educational/textbook content in the real world, instead of facilitating learning through new kinds of playful and constructionist activities in the real world.

Mathland creates an immersive Mixed Reality experience through 'physical presence', which connects virtual objects to the user's physical environment, body and tangible objects to allow for situated learning, embodied cognition and tangible interactions respectively. Situated learning has been shown to be helpful in mathematical learning [2,3]. Embodied cognition is important in mathematics as mathematical ideas are connected to bodily experiences [26] and students show improved math performance using embodiment [1,17]. Tangible interactions also improve understanding of abstract concepts [40,49], and physical manipulatives are considered to have cognitive and contextual advantages for mathematical learning in MR [7].

Mathland is designed to facilitate constructionist learning through applied, playful and exploratory learning activities in the real world. Constructivist learning is considered key in mathematics education [13,36], and Mathland ties learning to the user's real world as constructionist learning is intrinsically linked to people's physical, social and cultural context [3,10,18]. Seymour Papert wrote, "The practitioner of kitchen mathematics does not stop cooking and turn to math" [34]. In contrast, in Mathland, the user can learn in the real world with their peers and does not have to context-switch between the real and learning worlds.

Therefore, unlike existing computer games, simulations, and VR/AR applications for learning mathematics and physics, Mathland supports immersive and constructionist learning in the real world to foster mathematical learning.

MATHLAND

We experience mathematical phenomena in our everyday lives. Mathland provides a medium for learning mathematics in the real world using immersive and constructionist ways in MR. Immersive learning facilitates situated learning, embodied cognition and tangible interactions, whereas constructionist learning allows for playful, exploratory and applied learning in a constructionist world. We set ourselves two goals: (i) create an untethered and immersive MR experience in the real world; (ii) facilitate constructionist mathematical learning in the real world.

Mathland can be thought of as a mathematical lens that allows people to immersively experience the hidden mathematics of the objects and phenomena around them. Also, Mathland is a mathematical playground where learners not only visualize mathematics phenomena, but also tweak and play with them in a constructionist way.

Designing an immersive MR world

We wanted users to experience mathematics in their real world environment by bridging the gap between real and virtual world using MR. Immersive MR seamlessly integrates physical and virtual environments, human bodies and objects in MR and the closely couples the physical and virtual experiences in MR.

One way of accomplishing immersive MR is to integrate components of the user's physical environment in Mixed Reality so that real world can be augmented virtually in MR. This not only requires recognition of the user's static environment, but also tracking and recognition of dynamically moving physical objects, e.g a ball moving through the air, so that we can track the ball and virtually augment it with its speed and trajectory.

Real-time object detection and 3D tracking is possible through either external tracking systems, e.g. Optitrack, or through computer vision. The problem with external motion capture systems is that they require external setups and are restrictive in terms of the learning environments and objects learners can explore. The learners cannot freely walk into any space and explore it, and even in environments with the motion capture setup, objects without the trackers cannot be tracked. Thus, systems based on external tracking are not mobile and scalable for real world explorations.

Computer vision based tracking, on the other hand, does not require external setup and enables mobile use, but it is computationally expensive and state-of-the-art object recognition and 3D tracking is not quite real-time for fast moving objects. Also, room-scale 3D tracking requires 360° cameras, otherwise the objects will not be tracked outside the camera view. Currently available untethered wearable MR devices have limited computational capabilities and do not support computationally intensive computer vision algorithms, especially for 3D tracking and 360° video processing.

We believe that future advances in computer vision, 360° cameras and computational capabilities will make 3D tracking and object recognition feasible on wearable devices, but until then if we cannot track physical objects in real-time, another option is to give physical presence to virtual objects so they 'feel' real to the user. Since virtual objects already exist in the virtual space, they do not need to be tracked using external sensors.

We give 'physical presence' to virtual objects, where 'physical presence' is defined in terms of the following three characteristics:
a. Interact with the physical world, e.g. virtual objects can bounce off a physical wall,
b. Interact with people's physical bodies, e.g. people can throw a virtual ball using natural arm gestures, without the need for external controllers,
c. Physical manipulability, e.g. virtual objects can be felt, moved and rotated like physical objects.

Physical presence allows us to bridge the gap between real and virtual worlds by creating an immersive Mixed Reality world. Physical presence for virtual objects in MR facilitates better mathematical learning through situated learning using an understanding of the physical world, embodied cognition by interacting with physical bodies, and tangible interactions using physical manipulability.

Physical presence not only makes the virtual objects act 'like' physical objects, but are actually better than physical objects in the following ways:
a. virtual objects are not bound by real world laws, e.g. unlike the physical objects on Earth, they do not have to conform to the laws of gravity on Earth and we can make them experience gravity as if they were on Mars so that users can play with different possibilities.
b. unlike physical objects, virtual objects can be zoomed in or out, resized, moved back in time and then replayed, etc.
c. there are no limitations on the type and quantity of objects that can be created, e.g. we can have one or several, they can be elephants, black holes, 4D cubes, etc.

Due to the limited computer vision and computational capabilities of wearable MR devices, it is currently infeasible to fully integrate fast moving physical objects in MR. However, by giving physical presence to virtual objects, we can more than make up for this shortcoming as virtual objects are more flexible and versatile compared to physical objects. Giving physical presence to virtual objects makes virtual objects feel more like real life physical objects so we can focus on the experience design, rather than the technological challenges. Using 'physical presence', we design experiences for MR objects that afford situated learning, embodiment and tangible interactions, regardless of whether the objects are physical or virtual.

Implementing an immersive MR world using physical presence

Mathland facilitates physical presence in the following three ways:

I. Spatial mapping for interactions with the physical world

We decided to use the Microsoft Hololens for our Mixed Reality experience as it not only provides an untethered, wearable form factor so the user can freely walk around in their environment, but also spatial mapping capabilities to scan the 3D physical environment and integrate it in the Mixed Reality world. Integrating a user's physical world into the MR world allows virtual objects to interact with the physical environment like physical objects do, e.g. a virtual ball can bounce off a physical wall.

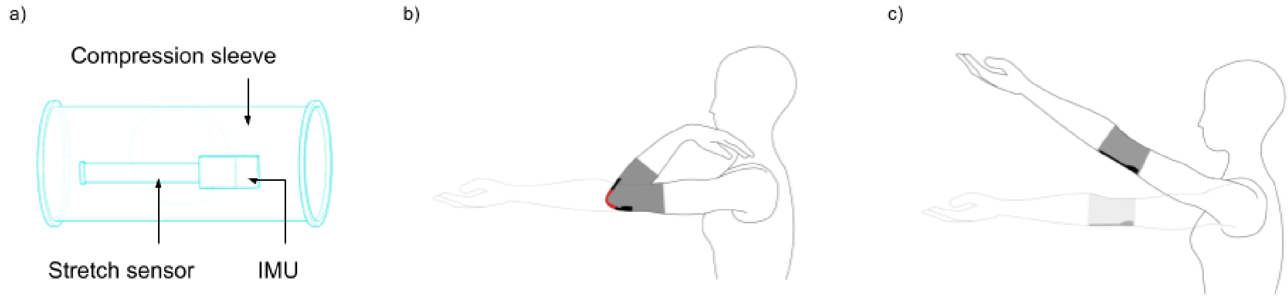

Figure 1. (a) Arm Controller comprises of a compression sleeve that can be worn by the user. There is a Figur8 sensor, which comprises of an IMU and stretch sensor. (b) and (c) Tracking user's arm movements using the Arm controller

II. Arm Controller for interactions with user's physical body

Hololens does not support full arm tracking so we developed a custom wearable Arm Controller to integrate the user's physical arm movements in MR. It allows the user to use their natural physical arm movements to interact with virtual objects in the same way they interact with physical objects. For example, users can play catch by catching and throwing the virtual ball in the same way they would play catch/throw with a physical ball. By allowing users to use their full arm movements in the virtual environment, we allow for greater embodiment through gestural congruence

[22], i.e. the user's MR gestures are in line with their physical world gestures.

We used the Figur8 [56] sensor, which consists of an IMU and a stretch sensor (Figure 1a) to implement the Arm Controller. We mounted the Figur8 sensor on a wearable sleeve, such that the IMU is slightly above the user's elbow and the stretch sensor is taut over their elbow to detect the bending and movement of the user's arms (Figure 1b, c). Using inverse kinematics on data from the Figur8 sensor, we calculate the movement and motion of the user's physical arm with respect to the user's head. Hololens already tracks the user's head position so we can use the relative position and movement of the arm to calculate the position and motion of the arm in global world coordinates. We use this information about the user's physical arm position and movement to create a virtual arm that is identical to the user's physical arms and that interacts with virtual objects to create the illusion that the user is interacting with the virtual objects using their physical arm.

The design choices for the Arm Controller were made to accommodate unrestricted and unlimited user movement, while affording accurate arm tracking. Unlike conventional outside-in trackers (like the HTC Vive), we allow for unrestricted movement of the user so that they can freely explore and play around in the real world. Also, unlike the common camera-based inside-out tracking, e.g. in Microsoft Hololens and Oculus Rift, the user's arms do not have to be in the camera view to be tracked. IMU-based inertial is not completely new, but conventional IMU trackers have significant drift. By using a stretch sensor together with an IMU, we reduce our reliance on just inertial sensing to get better accuracy and lower drift.

Moreover, our data processing model for the Figur8 sensor is based on a joint-and-hinge model and is easily extensible to other joints, e.g. placing it on the knee to track the user's leg motion, so users can freely interact with virtual objects using their physical bodies, e.g. kicking a ball, etc.

III. Object controller for physical manipulability

Virtual objects lack the tangible affordances offered by physical objects. On the Hololens, user interactions with virtual objects are limited to air-taps or other hand gestures without tangible feedback. We created a tangible Object Controller (Figure 2) for virtual objects in MR so users can interact with virtual objects using tangible interactions, In particular, the Object Controller affords tangible interactions for repositioning, rotating and resizing virtual objects.

Object Controller

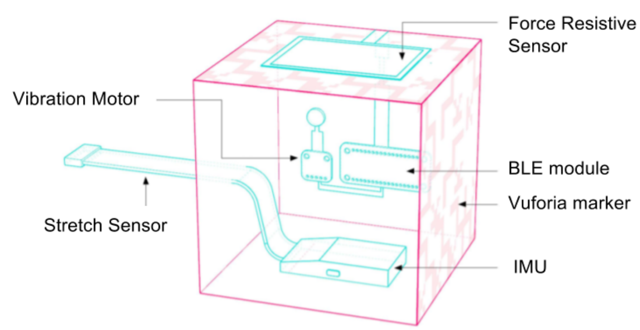

Figure 2. Object controller that enables tangible interactions with virtual objects in Mixed Reality

The Object Controller serves as a tangible placeholder or proxy for virtual objects in MR so users can reposition, rotate and resize virtual objects in the same way as they reposition, rotate and stretch physical objects. The user can gaze and tap on any of the virtual objects to connect the Object Controller to that virtual object. Any subsequent motion, i.e. a change in position and/or rotation, of the Object Controller is reflected in the position and rotation of the virtual object, and there is one-to-one mapping between the controller's and the virtual object's position and rotation. The Object Controller has a stretchable tape that the user can pull to resize the object. The stretchy tape is a stretch sensor that also gives natural haptic feedback to the user as they pull on the tape to resize a virtual object.

The Object Controller has the following components:

a. Vuforia markers for position tracking:
We put different Vuforia markers on all outer sides of the cube and use the Multi-target Vuforia Library [57] to detect the cube's position in the 3D world coordinates. Multi-target library allows us to cover the cube in markers so that no matter how the cube is turned, at least one of the markers is visible and thus, the cube can be detected regardless of its orientation.
b. Figur8's IMU for rotation tracking, and its stretch sensor for resizing with haptic feedback:
We added a Figur8 [56] sensor to the Object Controller as the Figur8 sensor provides an Inertial Measurement Unit (IMU) and a capacitive stretch sensor. We track the rotation of the cube using the IMU, and the stretch sensor (in the form of stretchable tape) to detect when the user is resizing a virtual objects by stretching the stretch sensor.
c. Force Resistive Sensor for touch input and vibration motor for haptic feedback:
We also added a Force Resistive Sensor for touch input and a vibration motor for haptic feedback to the Object Controller as we found the air-tap gesture [58] on the Hololens too tiring to use. With the Object Controller, a user can tap on the Force Resistive Sensor on the Object Controller to tap gestures, e.g. for selecting the virtual object they want to connect the Object Controller to. The vibration motor gives haptic feedback when user's tap input is detected.

Designing a constructionist world for mathematical learning

Mathematics is often taught of as a mere manipulation of abstract symbols [45], which discourages learners from engaging with mathematics. The seemingly abstract symbols and equations of mathematics have inherent meanings, e.g. pi is not just an infinitely long decimal number, but also the ratio between the circumference and radius of a circle. Mathematics also has widespread applications, e.g. calculus is applied in Newtonian mechanics. By tapping into the applications of mathematics, we aim to enable contextual and constructionist learning of mathematics so learners not only 'know' mathematical concepts, but also deeply understand them by applying and playing with those concepts in real world situations.

For the first version of Mathland, we considered different mathematical applications, but we finalized on Newtonian mechanics as people experience Newtonian physics in their everyday lives, e.g. when throwing a ball. Newtonian mechanics is often thought of as physics, but it is also applied mathematics because Newtonian physics concepts are rooted in mathematics, e.g. rate of change of displacement and velocity is linked to calculus. Mathland can not only show people the mathematics of their everyday Newtonian physics experiences, but also allow them to modify the physical laws and explore different Newtonian physics possibilities in MR.

For constructionist learning of Newtonian physics, we were inspired by the Rube Goldberg machine [25,48], which demonstrates how relatively simple objects can be used to create relatively sophisticated systems. Tangible Rube Goldberg machines have also been shown to enable constructionist learning of Newtonian Physics [30].

In order to provide creative tools that users can use to build their own creations using applied math/Newtonian physics concepts, we delved into Newtonian physics concepts. We studied the Force Concept Inventory (FCI) [19,59], which is commonly used to evaluate the understanding of key Newtonian physics concepts, and noticed that introductory Newtonian physics is about three key concepts: i. constant linear force (e.g. gravity); ii. momentary force/collision (e.g. kicking a hockey puck); and iii. force towards a point (e.g. centripetal force in circular motion).

We added three virtual items in Mathland's menu to integrate the aforementioned three key Newtonian physics concepts:
i. 'Force Field' object for creating a constant linear force;
ii. 'Velocity Vector' object for momentary force;
iii. 'Rope' for force towards a point.
We also added two more items to the Mathland menu to allow users to create more interesting set ups:
i. 'Ramp' for objects to roll down on;
ii. 'Cube' is an obstacle that objects can bounce off of.

Users can create as many instances of the virtual objects as they want, and all the instances of those objects can be independently repositioned, resized and rotated to different physics and physical properties, e.g resizing the velocity vector changes the magnitude of velocity or rotating the velocity vector and force field changes the direction of velocity and force field respectively.

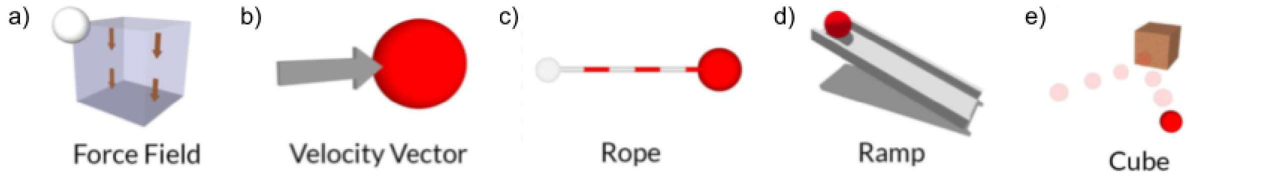

Figure 3. Five virtual items in Mathland's menu to allow the the user to experience different physical forces and interactions.

Mathland's MR starts with a virtual ball that can interact with all other virtual items and also with user's physical environment. Users can use the five Newtonian physics menu items to create new worlds, e.g. use the Rope, Force Field, and Ramp to create a virtual Rube Goldberg machine. There is no pre-existing gravitational force, but users can create their own gravity using menu items, e.g. place a global Force Field, rotate it towards the ground and change its magnitude to explore different magnitudes of gravity.

Each of the five virtual items is described in detail below:

i. Force Field (figure 3a): The Force Field is a cube that applies a constant linear force on all the objects within its boundaries, i.e. a local field. On one of its top corners, the Force Field has a white sphere, which can be used to reposition, rotate and resize the Force Field cube. By rotating the Force Field, users can change the direction of the linear force, and resizing the Force Field changes the volume impacted by the force. The Force Field has directional arrows, which reflect the direction and magnitude of the Force Field. Rotating and resizing the Force Field changes the direction and region of the force field respectively. User's can also change the local force field to a global force field by gazing at the desired force field and using a voice command "full scale". The users can also change the magnitude of the force using a slider, e.g. adjust the magnitude of a downward facing global Force Field to match the gravity on Earth, Mars, etc.

ii. Velocity Vector (figure 3b): The Velocity Vector has the visual form of a directional arrow. Users can resize and rotate the arrow to change the magnitude and direction of the Velocity Vector respectively. If a user wants to apply an instantaneous force on an object, they can move the Velocity Vector so that it slightly overlaps with that object. The Velocity Vector adds to the instantaneous velocity of every object that comes in contact with it. The user can use the velocity vector to give an initial velocity to objects or change the direction of motion of an object by superimposing one velocity on top of the other.

iii. Rope (figure 3c): On one of its ends, the Rope has a white spherical hinge, which can be fixed in the virtual space, while the other end of the Rope is unfixed. The hinge can be used to reposition, rotate and resize the rope in the MR environment, and the other end can be connected to another virtual object. The rope experiences tension (centrifugal force) toward the fixed end (or the hinge). If the user applies a velocity vector to an object attached to the rope, the rope can swing in circular or pendulum motion (depending on the other forces in the scene and the magnitude of the velocity vector).

iv. Ramp (figure 3d): The Ramp can be used by the user to direct the motion of the ball along a path.

v. Cube (figure 3e): The Cube can act as a barrier and the ball bounces off of it.

The application has two modes — Launch and Edit — and the user can switch between the two modes using voice commands. The game starts by default in the Edit mode where the user can reposition the ball in the world using the Object Controller. In the Edit mode, the ball is a static object, and does not react to any forces, collisions or velocity vectors. When the user is done editing the ball and objects in the scene, they can say "Launch ball" to enter the Launch mode. Once the user is in the Launch mode, the user cannot edit the ball or any other objects in the scene unless they say "Reset ball" to go back to the Edit mode. In the Launch mode, the ball is no longer a static, but a dynamic object and responds to the force fields, velocity vectors, etc.

We provide two visualization for the ball's physics:

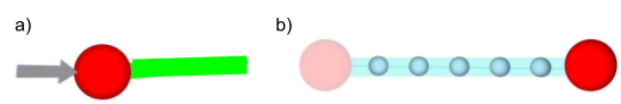

Figure 4 a) Predicted trajectory of the virtual ball; b) Trail left behind by the ball as it moves through space. The uniformly spaced spherical snapshots here indicate constant velocity.

a. Predicted trajectory (Edit mode)

In the Edit mode, the ball has a predicted trajectory visualization (Figure 4a), which helps the user observe the expected trajectory of the ball if it were launched. The predicted trajectory is based on the MR environment, i.e. the physical surrounding as well as virtual force fields, etc. As the user moves around the virtual ball or other objects in in the edit mode, the predicted trajectory of the ball changes so that the user can visualize the effects of different changes on the trajectory of the ball without having to launch it.

b. Trail and strobe effect (Launch mode)
As the ball moves in the Launch mode, it leaves a trajectory (blue line/trail) behind to allow the users to see the path that the ball has traveled through (Figure 4b). As the ball moves after it is launched, we create slightly transparent virtual snapshots of the ball at fixed intervals of time, i.e. 'strobe effect'. Using the strobe effect, the users can compare the distance traveled by the ball between fixed time intervals and get an intuition for the velocity, i.e. rate of change of distance/position, and acceleration of the ball. When the ball is traveling fast, the snapshots of the ball are spaced out more, compared to when the ball is traveling slowly. The snapshots produce an effect which is similar to the strobe light pictures commonly used in physics textbooks to help students visualize the motion of an object.

Unless the user explicitly erases them, the virtual trajectory and strobe effect persist so the learners can see the virtual ball's trajectory at any point in history. Users can also restart the simulation by saying "Reset ball" and repeat its motion with modified conditions to compare the speed and trajectory of the ball in different situations, e.g. as the ball travels in projectile motion with different initial velocities.

Moreover, in order to gamify the Mathland experience, we created a special Puzzle mode in which users use the items in Mathland's menu to create puzzles for others and solve puzzles created by others. We added another item, called a Checkpoint, which is a ring shaped object that records if the ball has passed through it. The user can place a series of numbered Checkpoints to create a MR puzzle, and to complete the level, the ball must pass through the checkpoints in the right order. Solving or creating puzzles requires users to think creatively about the Newtonian physics tools and thus encourages constructionist learning in a playful way.

Puzzles help boost social and creative learning in Mathland's constructionist world because users can create and solve puzzles with each other. Unlike VR, MR allows people to see and interact with other people in the real world for collaborative learning [16,43].

We employ the four key components of creative learning, i.e. Projects, Peers, Passion and Play [60], in Mathland. Mathland provides a collaborative environment for learners to build projects and solve mathematical puzzles as they play around with mathematical concepts in MR. Mathland also integrates the ideas of low-floor (easy to get started), high-ceiling (opportunities to increase complexity) and wide-walls (the possibility of several possible solutions) [38]. Users can use the provided virtual objects to build the simplest to the most complicates worlds and puzzles, and using the wide variety of virtual objects means that there are several possible ways to solve different puzzles.

EXPERIMENT DESIGN

We designed a preliminary experiment to evaluate constructionist learning in Mathland's immersive MR world. We particularly focused on evaluating the creative learning and problem-solving experiences in Mathland. We used the Puzzle mode of Mathland to build three puzzles, which targeted different motions that are taught in basic Newtonian physics — Circular Motion (Puzzle 1), Linear Motion (Puzzle 2) and Projectile Motion (Puzzle 3). The participants had to solve the puzzles by applying Newtonian physics concepts using the five virtual items provided in the Mathland's menu. The three puzzles and their respective solutions are shown in figure 5. We also included an optional part to the experiment for our users to design their own Mathland puzzles.

Each puzzle involved the understanding of different Newtonian physics concepts, and could be solved in multiple ways:
a. For Puzzle 1 (Circular Motion), the user had to notice that since the checkpoints are arranged in a circular orbit, the ball needs a centripetal force. This centripetal force can be created by attaching the ball to a Rope object, and giving the ball an initial velocity that is perpendicular to the rope so that the virtual ball starts swinging in a circular motion.
b. To solve Puzzle 2 (Linear Motion), the user could place inclined ramps between the checkpoints and create a full-scale downward Force Field to resemble gravity (since there is no gravity by default in the world, and without a downward force, the ball does not move down the ramps).
c. For Puzzle 3 (Projectile Motion), in addition to the downward gravity-like Force Field, the user needs a Velocity Vector to launch the ball at an angle such as that it follows a parabolic path, i.e. projectile motion.

We conducted the experiment with two participants in one trial to evaluate the collaborative learning experience in Mathland. The two participants shared a common Mixed Reality world, and had a total of 30 minutes to solve all three puzzles. We gave only one Object controller to each participant pair to encourage more collaboration and interaction between participants.

We did not include the virtual arm in the experiment as solving the puzzles mostly involves manipulating virtual objects and that was possible using the Object Controller. The goal of the experiment was not to evaluate a specific type of interaction, but to analyze the overall constructionist learning experience in Mixed Reality. Therefore, we decided that it was simpler and more efficient for users to solve the puzzles using the Object Controller since the Arm Controller is relevant in more playful scenarios, e.g. when users play catch in Mathland.

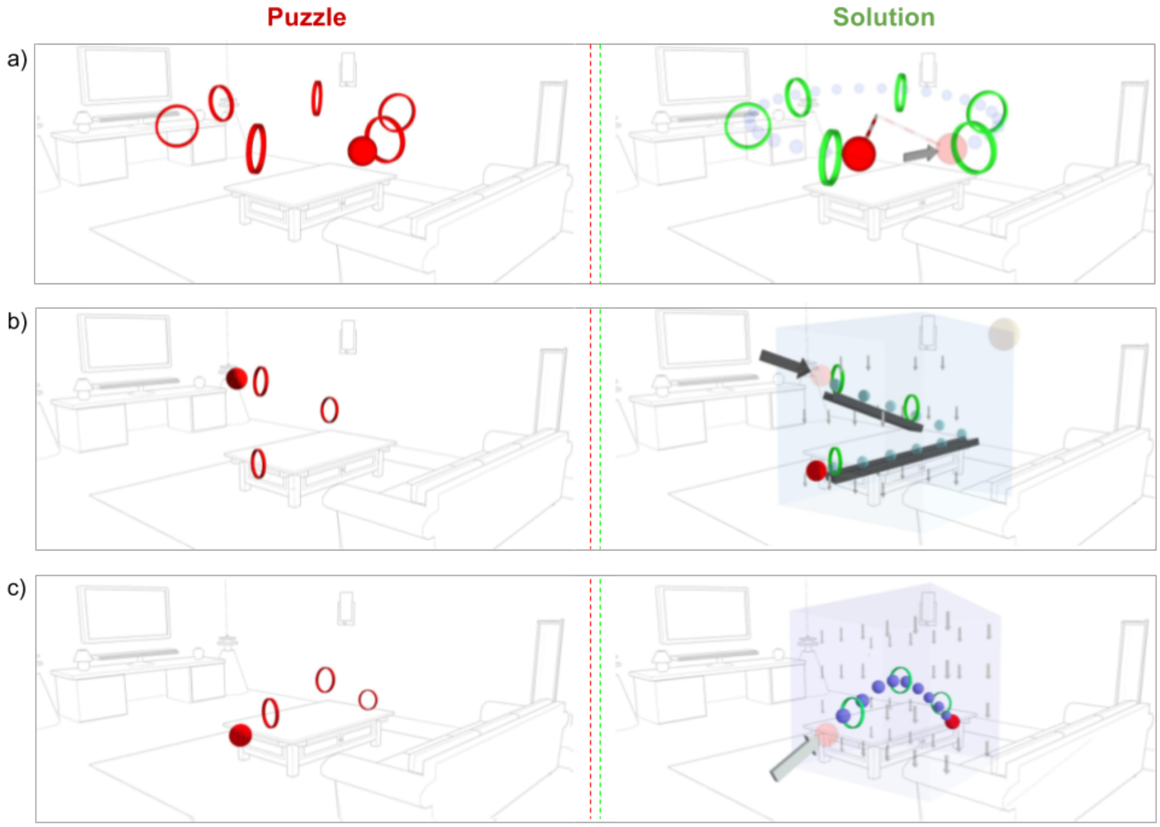

Figure 5. Puzzles created by placing three virtual checkpoints. The left side images show the puzzle whereas the right side ones show the solution. Each puzzle corresponds to a specific physics concept: a) Circular Motion b) Linear Motion c) Projectile Motion

RESULTS

We had a total of 30 participants (22 female and 8 male; 8 in the age group 16-20, 21 in the age group 20-35, and 1 above 35). 28 participants performed the study in pairs and the remaining 2 performed the study individually because of scheduling conflicts. The two people in each trial were paired based on their time availability.

On the 5-point Likert scale pre-study questions, our participants indicated that 40% of them did not find math/physics fun (Q: 'Math/physics was fun for me'), 43% did not feel confident about their math/physics skills (Q: 'I feel confident about my math/physics skills'), and 48.5% did not ace their math/physics tests (Q: 'I aced my math/physics exams'). Yet, all of our participants were able to solve the three puzzles, which were designed to evaluate the understanding of Newtonian physics concepts.

We presented our participants with a post-study survey, which had six five-point Likert scale type questions about user engagement, collaboration, creative learning, and problem-solving, which are all key to constructionism [60]. Our results (Fig 7) for each metric are as follows:

i. **Engagement**: Our users were thoroughly engaged in Mathland, 28 (93.3%) out of the 30 participants agreed (20%) or strongly agreed (73.3%) that they 'would like to use Mathland with someone else they know' (figure 6a). Moreover, 26 (86.7%) of our participants agreed (16.7%) or strongly agreed (70%) that they 'liked being in a 3D Mathland' (figure 6b).

ii. **Learning**: The participants found the puzzles to have educational value as 27 (90%) of them agreed (23.3%) or strongly agreed (66.7%) that 'people unfamiliar with much mathematics/physics can learn something in Mathland' (figure 6c).

iii. **Problem-solving:** All of our participants not only solved all three puzzles, but also enjoyed solving those puzzles. 29 (96.6%) of them agreed (13.3%) or strongly agreed (83.3%) that 'Mathland can make problem-solving fun' (figure 6d), and 25 (83.3%) agreed (6.7%) or strongly agreed (76.7%) that they 'would like to solve more puzzles in Mathland' (figure 6e).

iv. **Collaboration**: During the study we observed that partners collaborated effectively to problem-solve. In the post-study survey, out of the 28 participants who did the study with another partner, 26 (92.8%) agreed (14%) or strongly agreed (78.6%) that 'having a partner made Mathland fun' (figure 6f).

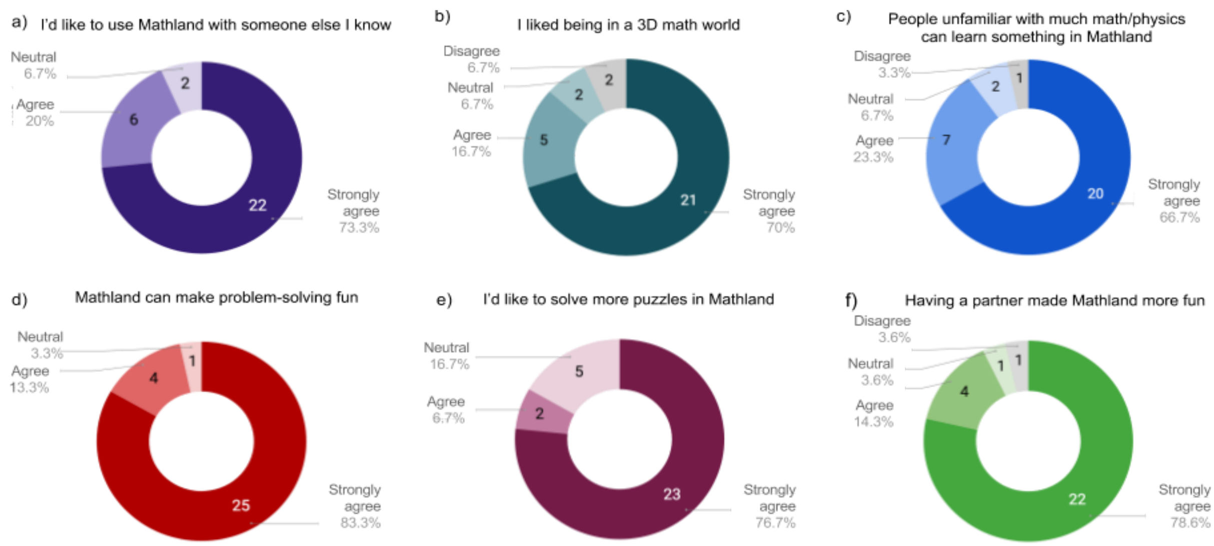

Figure 6. Our survey responses based on a 5-point Likert scale: Strongly agree, Agree, Neutral, Disagree, and Strongly Disagree.

v. **Creative Learning:** We were pleasantly surprised to see the overwhelmingly diverse responses from users. The participants suggested everything from magnetic fields to anti-gravity objects, from pong games to pool halls, and from smiley shaped checkpoint arrangement to tornado-like spirals and vortices (figure 7). We believe that these drawings are a testament that Mathland can encourage creative world building for constructionist learning. We also had two open-ended optional questions for people to mention their most and least favorite parts of Mathland. For most people, the least favorite part was the narrow field of view of the Hololens. Compared to the responses for the least favorite part, we had a lot more responses for the most favorite part. Here are the responses coded into categories:

i. Graphics: 'the previous trajectories of the ball', '3D holograms'; 'seeing the live trajectory feedback'.

ii. Simulations: 'Seeing the ball launch!'; 'Seeing it all move around'; 'controlling so many aspects of physics'; 'Being able to see trajectories and change forces'; 'seeing how all the objects worked, it made it very intuitive'; 'being able to see it all happen in real 3D was pretty cool'.

iii. Mixed Reality: 'The fact you could mix virtual and real objects'; 'I enjoyed being able to walk in an environment that was much more colorful and interactive than the original room seemed'.

iv. Collaboration: 'Working in teams', 'working w/ partner'.

v. Problem solving: 'The puzzley aspect in 3D was pretty cool'; 'planning the solution'; 'The problem-solving involved'; 'thinking about different solutions to the same puzzle'; 'having absolute freedom and being creative'.

DISCUSSION

We observed that the participants were really engaged in the experience and having fun with their partner while solving the puzzles — they were lying down on the floor to view objects from different angles, playing around with the different elements, and high-fiving each other when completing a puzzle. We had three main insights from our user study:

a. Virtual objects: It really paid off to give physical presence to virtual objects, instead of digitizing physical objects because physical objects are bounded by physical laws and for most of our users, the limitless nature of our virtual objects was their favorite part. Users were connected with the virtual objects because of their physical presence and really enjoyed seeing the trajectory of the ball in 3D, modifying the physics of objects, and experiencing the Newtonian physics phenomena unfold in front of them.

b. Problem-solving: Participants really liked going through physics concepts as they were solving puzzles and found the puzzles really engaging. This was a testament to constructionist learning as learners created new worlds using Newtonian physics tools to solve puzzles.

c. Creative learning: In response to our optional open-ended question, several users enjoyed imagining and creating new physics puzzles and explained the different ways they would solve each puzzle without being prompted to do so. Participants also enjoyed coming up with different solutions for the three Mathland puzzles, even though they were required to produce only one solution. The experiment results show that Mathland's constructionist setup triggered creative learning.

Figure 7. Selected user sketches of new Mathland puzzles in response to an optional post-survey question. These sketches signify that world-building in Mathland's constructionist environment triggers creative thinking

FUTURE WORK

Our current version of Mathland focuses more on constructionist learning using visualizations and spatial understanding rather than learning about mathematical equations. For example, as the user is rotating the virtual Ramp in Mathland, we do not show the slope or equation of the Ramp. Instead, we allow the users to observe the relative positions, angles and sizes of the objects. As our virtual ball moves through space, instead of showing the value of the ball's velocity, we show the direction of the ball's motion using the trail left behind by the ball, and the magnitude of velocity using the distances between the snapshots of the ball taken at equal time intervals.

Going forward we aim to add more support for mathematical understanding using interactive graphs and equations. Interactive equations for trajectory, velocity and acceleration are such that changing the equations for each changes the trajectory, velocity and acceleration of the ball respectively, and vice versa. Our hope is that interactive equations will allow users to understand mathematical equations with respect to their physical representations in the real world. Using interactive graphs, we will demystify the relationship between an object's distance, velocity, and acceleration, which are connected through calculus. Allowing users to visualize these relationships graphically as the area under graphs (integrals) and slope of graphs (differentials) might help users connect the real life behavior of objects to their mathematical representations.

Moreover, we plan to create an assessment mode in Mathland for use in classrooms and experimental evaluations. In the assessment mode, users will be required to answer multiple-choice questions that evaluate their mathematical and conceptual understanding of Newtonian physics. The users will answer the question and then play with the relevant concepts of the questions in Mathland, and then reanswer the question and explain their choice. Exploring different concepts in Mathland will hopefully allow users to develop a better understanding of Newtonian physics and its mathematical representations.

CONCLUSION

We live in a mathematical universe, but are unable to see the mathematics underlying our natural world. Mathland aims to unveil the hidden mathematical realities of nature. In Matland, learners not only visualize mathematics, but also use it to construct new projects, i.e. constructionist learning. Unlike traditional education, we do not confine mathematics to its symbolic representations. Instead, we show the real world applications of mathematics so learners are motivated to use mathematics to construct new worlds. In this version, we focused on Newtonian physics, and used the Rube Goldberg model of constructing complicated systems from simple parts to allow learners to construct systems and puzzles using Newtonian physics tools and visualize the physics concepts associated with these models.

We use MR to situate learning in the user's real world context, but unlike existing MR applications, which simply overly virtual content on top of the real world, we created an immersive MR experience where users can explore and play with mathematics using 'physical presence', which connects virtual objects to the user's physical environment, human body and physical objects to afford situated learning, embodied cognition and tangible interactions.

Our preliminary study shows that immersive and constructionist learning can be a promising mathematics learning experience. Our participants were fascinated by the ways they could manipulate virtual objects and use the Newtonian physics tools to visualize physics/math concepts. They also enjoyed the problem-solving aspect of Mathland where they had to build and explore their own solutions to solve physics puzzles, and the social aspect of problem-solving with a peer was an additional advantage.

Going forward, we plan to include more interactive graphs and equations in Mathland so that users also connect their mathematical intuition to the mathematical symbols and representations. We will continue to delve deeper into Newtonian physics and allow users to develop a deeper understanding of its graphs and equations.

We envision a world where people can interactively explore mathematics seamlessly in their real world, e.g. hold up a virtual prism and see the sunlight get refracted into a virtual rainbow. Mathland is both a mathematical lens and mathematical playground, and immersive and constructionist learning goes a long way in facilitating applied, playful and exploratory learning of mathematics.

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