Finite Difference Method for Heat Conduction PDE for a 2-D Plate

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For a more detailed analysis refer to chapter 29 of *Numerical Methods for Engineers by Steven C. Chapra and Raymond P. Canale.*

The equation of the plate is:

$$\frac{d}{dx}k\frac{dT}{dx}+\frac{d}{dy}k\frac{dT}{dy}=0$$

Where T is the temperature of the plate, x and y are spatial dimensions, and k is the thermal conductivity of the plate which can be a constant, or a function of temperature. For this problem k is considered to be a function of T:

$$k=\frac{(120-T)^{1.25}}{15}$$

The surface of the plate is separated into a grid consisting of nodes that represent the temperature, T, of the plate at that node which is described by i and j. For example: the bottom left corner node is described by i = 0 and j = 0, while the bottom right by i = 1+m and j = 0, where m is the number of internal nodes. The method employs centered finite-divided difference formulas:

$$f'(x)=\frac{f\left(x_{i+1}\right)-f\left(x_{i-1}\right)}{2\Delta x}$$

$$f''(x)=\frac{f\left(x_{i+1}\right)-2f\left(x_{i}\right)+f\left(x_{i-1}\right)}{(\Delta x)^2}$$

The above equations can be used with the equation of the plate, to derive equations describing the temperature at each node in terms of the temperature of the surrounding nodes. The following is such a derivation.

For the general node T(i, j):

$$\frac{d}{dx}(k\frac{dT}{dx})=\frac{(k\frac{dT}{dx})_{i+1,j}-(k\frac{dT}{dx})_{i-1,j}}{2\Delta x}$$

$$(\frac{dT}{dx})_{i+1,j}=\frac{T_{i+1,j}-T_{i,j}}{\Delta x},(\frac{dT}{dx})_{i-1,j}=\frac{T_{i,j}-T_{i-1,j}}{\Delta x}$$

$$\frac{d}{dx}(k\frac{dT}{dx})=\frac{\frac{\mathrm{k}_{i+1,j}(T_{i+1,j}-T_{i,j})}{\Delta x}-\frac{\mathrm{k}_{i-1,j}(T_{i,j}-T_{i-1,j})}{\Delta x}}{2\Delta x}$$

$$\frac{d}{dx}(k\frac{dT}{dx})=\frac{\mathrm{k}_{i+1,j}T_{i+1,j}-\mathrm{k}_{i+1,j}T_{i,j}-\mathrm{k}_{i-1,j}T_{i,j}+\mathrm{k}_{i-1,j}T_{i-1,j}}{2(\Delta x)^2}$$

$$\frac{d}{dy}(k\frac{dT}{dy})=\frac{\mathrm{k}_{i,j+1}T_{i,j+1}-\mathrm{k}_{i,j+1}T_{i,j}-\mathrm{k}_{i,j-1}T_{i,j}+\mathrm{k}_{i,j-1}T_{i,j-1}}{2(\Delta y)^2}$$

$$\frac{d}{dx}(k\frac{dT}{dx})+\frac{d}{dy}(k\frac{dT}{dy})=0$$

$$k_{i+1,j}T_{i+1,j} - k_{i+1,j}T_{i,j} - k_{i-1,j}T_{i,j} + k_{i-1,j}T_{i-1,j} + k_{i,j+1}T_{i,j+1} - k_{i,j+1}T_{i,j} - k_{i,j-1}T_{i,j} + k_{i,j-1}T_{i,j-1} = 0$$
$$k_{i+1,j}T_{i+1,j} + k_{i-1,j}T_{i-1,j} + k_{i,j+1}T_{i,j+1} + k_{i,j-1}T_{i,j-1} - T_{i,j}(k_{i+1,j} + k_{i-1,j} + k_{i,j+1} + k_{i,j-1}) = 0$$

$$T_{i,j} = \frac{k_{i+1,j}T_{i+1,j} + k_{i-1,j}T_{i-1,j} + k_{i,j+1}T_{i,j+1} + k_{i,j-1}T_{i,j-1}}{k_{i+1,j} + k_{i-1,j} + k_{i,j+1} + k_{i,j-1}}$$

For bottom left corner node:

$$T_{-1,0} = T_{1,0} - 2\Delta x\frac{dT}{dx} = T_{1,0}$$
$$T_{0,-1} = T_{0,1} - 2\Delta y\frac{dT}{dy} = T_{0,1}$$
$$k_{-1,0} = k_{1,0}, k_{0,-1} = k_{0,1}$$
$$T_{0,0} = \frac{2k_{1,0}T_{1,0} + 2k_{0,1}T_{0,1}}{2k_{1,0} + 2k_{0,1}}$$

For left side nodes:

$$T_{0,j} = \frac{2k_{1,j}T_{1,j} + k_{0,j+1}T_{0,j+1} + k_{0,j-1}T_{0,j-1}}{2k_{1,j} + k_{0,j+1} + k_{0,j-1}}$$

For bottom row of nodes:

$$T_{i,0} = \frac{k_{i+1,0}T_{i+1,0} + k_{i-1,0}T_{i-1,0} + 2k_{i,1}T_{i,1}}{k_{i+1,0} + k_{i-1,0} + 2k_{i,1}}$$

The rest of the procedure is simple. Boundary conditions must be described for nodes at the extremes in order to solve the PDE. An initial 'guess' is taken for the values of the interior nodes, in this case 25 °C will be used as an initial guess. The boundary condition are:

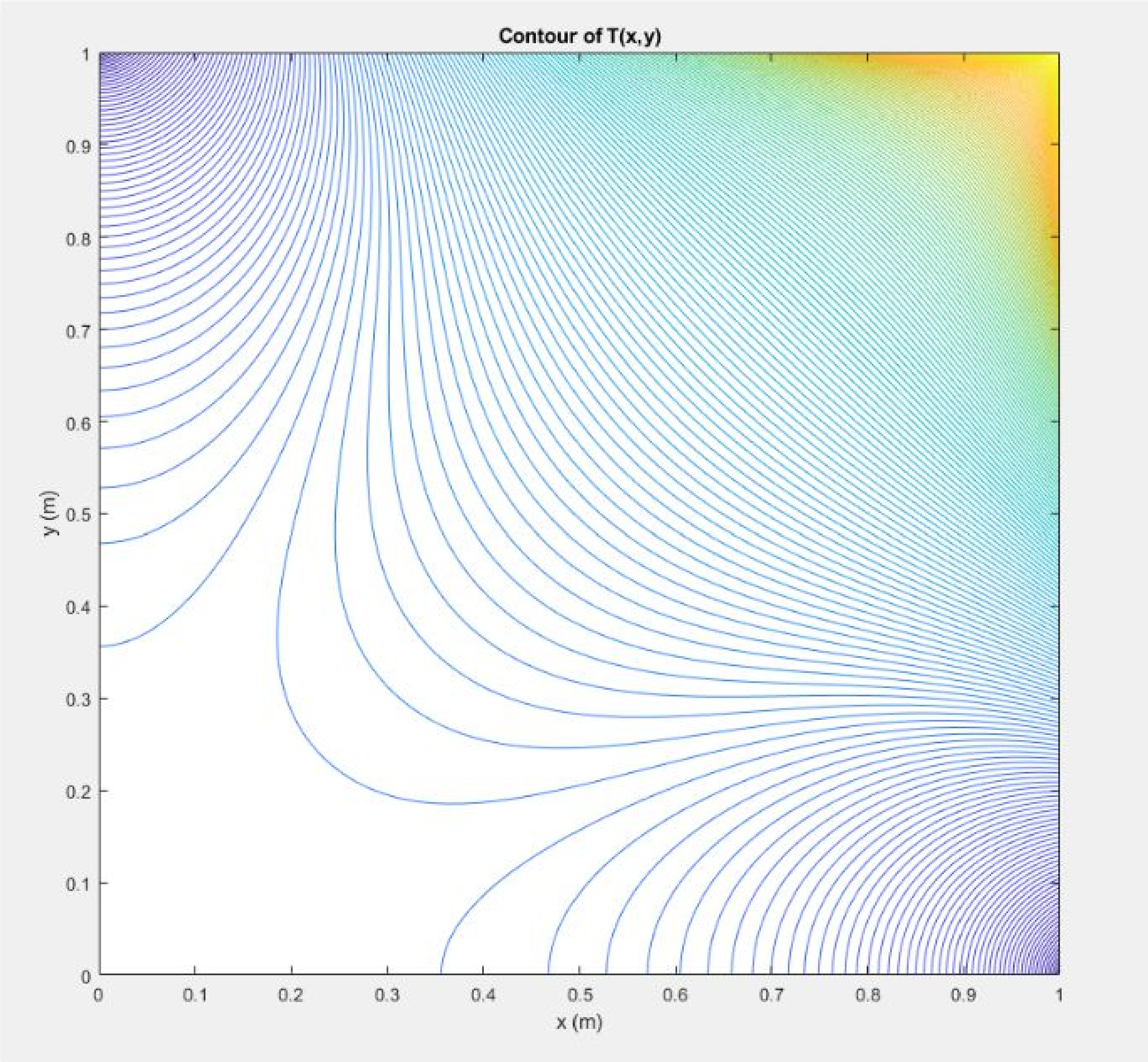
$$T'(0,y) = 0 \qquad T'(x,0) = 0 \qquad T(1,y) = 100y \qquad T(x,1) = 100x$$

The equation describing the nodes are iterated over and over until the values converge. To aid in stability of convergence, a relaxation constant is used. In this case a relaxation constant of 1.2 is used. The equation can be iterated until a desired value of error is reached. The approximate error is described by:

$$\left|(\varepsilon_a)_{i,j}\right| = \left|\frac{T_{i,j}^{old} - T_{i,j}^{new}}{T_{i,j}^{new}}\right| \times 100$$

Increasing the number of nodes or the number of iterations reduces this error.

The above method is used to make the Matlab script and is able to solve the PDE. The script starts with 4 nodes and doubles the number of nodes until the number of nodes are 256x256. The program produces the following contour that describes the temperature distribution on the surface of the plate:



Contour of T(x,y)
y (m)
x (m)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1