Task 3: Reliability modeling with Markov chains

Introduction

Combinatorial reliability models such as Reliability block diagrams are well suited for systems without repair, i.e. when all component failures are permanent. If a system often switches between various fault states due to the recovery from temporary faults, obtaining the reliability formula by combinatorial modeling becomes quickly intractable. In these cases, state-based models like Markov chains are better suited.

The aim of this task is to learn how Markov chains can be used for reliability modeling. Start by reading the given literature (again taken from *Fault-Tolerant Systems* by Koren and Krishna) as well as the following two hints:

- A general solution for a differential equation of the form

  $$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = a(t) \cdot y(t) + g(t) \qquad \text{is given by} \qquad y(t) = \frac{\int g(t) \cdot e^{-A(t)\mathrm{d}t} \, \mathrm{d}t + C}{e^{-A(t)\mathrm{d}t}}$$

  where $A(t) = \int a(t)\mathrm{d}t$ and C is a constant of integration that can be chosen such that the initial condition is satisfied.

- If $P_i(t)$ is the probability that a Markov model is in state i at time t, the row vector $P(t) = [P_1(t), P_2(t), ...]$ is called the *state probability vector*. The *transition rate matrix* (also known as *infinitesimal generator matrix*) Q is defined such that q_{ij} (with $i \neq j$) is the rate of transition from state i to state j and $q_{ii} = -\sum_{j \neq i} q_{ij}$. The system of differential equations that describes the behavior of the Markov model can then be written as

  $$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = P(t)Q.$$

  A general solution for the system is given by

  $$P(t) = P(0)e^{Qt}$$

  where $P(0)$ is the initial state probability vector at time $t = 0$ and $e^{Qt} = \sum_{i=0}^{\infty} \frac{(Qt)^i}{i!}$. MATLAB may be used to numerically calculate this solution as shown in the given script file.

Task 3.1

1. Build a Markov model for each system in Figure 1! Assume that all system components have the same failure rate λ and that all failures are permanent.

2. Validate the correctness of your models by comparing the resulting reliability values with those obtained from the exact formulas from Task 2! Use MATLAB to numerically solve the differential equation systems of your Markov models in order to calculate the reliabilities.

3. Give the differential equations describing the Markov models for the systems (b) and (c)!

4. Obtain a reliability formula for system (b) by solving the corresponding differential equations!

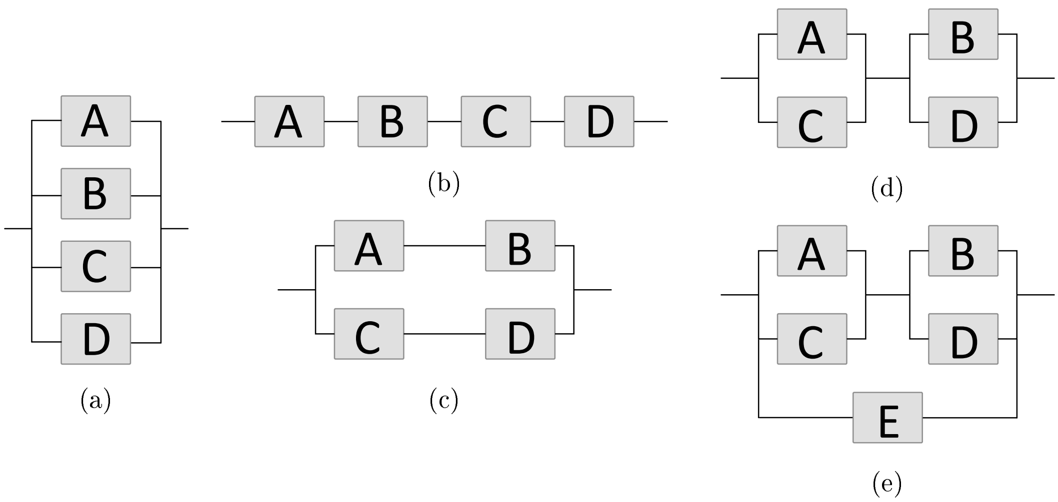

Figure 1

Task 3.2

1. Build a Markov model for a TMR system! Assume that the functional units always fail permanently at rate λ_p and that the voter never fails.

2. Obtain a reliability formula for the TMR system by solving the differential equations that describe the behavior of the Markov model!

3. How do the solutions of subtasks 1 and 2 need to be modified if the voter could fail at rate $\lambda_v > 0$?

Task 3.3

1. Build a Markov model for a TMR system that can have permanent and temporary unit failures! Use the following assumptions: permanent failures occur at rate λ_p, temporary failures occur at rate λ_t, functional units recover from a temporary failure at rate μ and the voter never fails.

2. Calculate the reliability $R(t)$ of the TMR system using the model from subtask 1! Assume that each functional unit has a total failure rate $\lambda = 2$, where a portion k of these failures is permanent and the remaining failures are temporary. Units that fail temporarily will recover at rate $\mu = 100$. Plot the reliability curves for $k \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ over the time interval $t \in [0, 1]$ into a single chart. Add the reliability curve of a single functional unit with failure rate $\lambda = 2$.

3. Repeat subtask 2 for $\mu = 100000$ and $\mu = 1$!

4. How does the variation of k and μ influence the reliability of the TMR system?