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FUNDAMENTALS OF CALCULUS

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ESSENTIAL MATHEMATICS FOR ENGINEERING
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WELCOME
N. K. SUDEV

What is Calculus?

What is Calculus?

Calculus is the study of how things change.

Calculus was developed out of a need to understand continuously changing quantities. It provides a way for us to construct relatively simple quantitative models of change and to deduce their consequences.

Well known English Scientist Sir **Isaac Newton** and the German mathematician **Gottfried Wilhelm Leibnitz** deserve equal credit for independently coming up with calculus.

Leibniz provided us the current rational, notational system and some good algorithms for finding differentials.

What is Calculus?

The fundamental idea of calculus is to study change by studying **instantaneous** change, by which we mean changes over tiny intervals of time.

Calculus provides scientists and engineers the ability to model and control systems and hence the power over the material world.

Calculus completes the link of algebra and geometry by providing powerful analytical tools that allow us to understand functions through their related geometry.

The development of calculus and its applications to physical sciences and engineering is probably the most significant factor in the development of modern science.

Calculus - Different Areas

A typical course in calculus covers the following topics:

1. How to find the instantaneous change, called the **derivative**, of various functions. The process of doing so is called **differentiation**.
2. How to use derivatives to solve various kinds of problems.
3. How to go back from the derivative of a function to the function itself. This process is called **integration**.
4. Study of detailed methods for integrating functions of certain kinds.
5. How to use integration to solve various geometric problems, such as computations of areas and volumes of certain regions.

Calculus

Functions

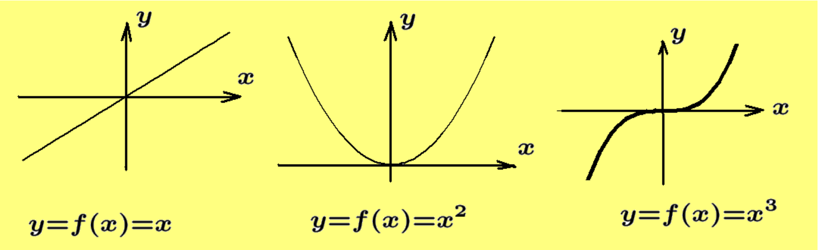
Let X and Y denote two sets of the real numbers. If a rule or relation f is given such that for each $x \in X$, there corresponds exactly one real number $y \in Y$, then y is said to be a real single-valued **function** of x.

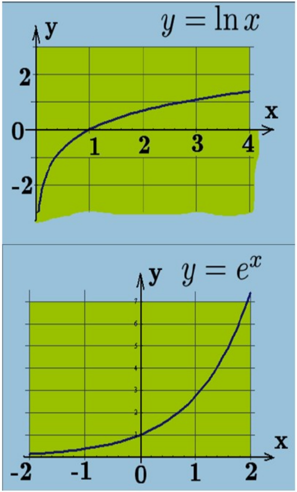
The relation between y and x is denoted $y = f(x)$ and read as "y is a function of x".

The set of ordered pairs $C = \{(x, y) : y = f(x), x \in X\}$ is called the **graph of the function** and represents a curve in the XY-plane giving a pictorial representation of the function.

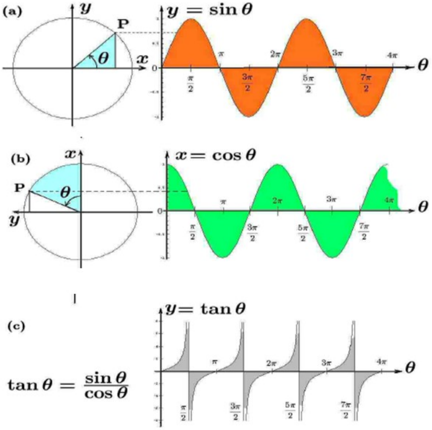
In this ordered pair (x, y), we say that x is the **independent variable** and y is the **dependent variable**.

Some Illustrations to Graphs of Functions





$y = \ln x$
y
x
2
0
-2
1
2
3
4
$y = e^x$
y
x
-2
-1
0
1
2



(a)
$y = \sin\theta$
P
θ
(b)
$x = \cos\theta$
P
θ
(c)
$y = \tan\theta$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Explicit and Implicit Functions

For a function $f : X \to Y$, written in the form $G(x, y) = 0$, if it is possible to solve for one variable in terms of another to obtain $y = f(x)$ or $x = g(y)$, then $G(x, y)$ is said to be an **explicit function**.

Example-2: Consider the equation $G(x, y) = x^2y + y - 1 = 0$. This equation can be written as $y = f(x) = \frac{1}{1+x^2}$. Therefore, $G(x, y)$ is an explicit function.

Example-2: Consider the equation $G(x, y) = x^2 + y^2 - r^2 = 0$. This equation can be written as $y = f(x) = \sqrt{r^2 - x^2}$ if $-r \leq x \leq r$ or $x = g(y) = \sqrt{r^2 - y^2}$; if $-r \leq x \leq r$. Therefore, $G(x, y)$ is an explicit function.

Explicit and Implicit Functions

If it is not possible to solve for one variable explicitly in terms of another, then the function is said to be an **implicit function**.

Example-1: Consider the expression $G(x, y) = x^2 + 2xy + y^2 + 4x + 4y - 10$. It is an implicit function.

Example-2: Consider the expression $G(x, y) = x^2\sqrt{y} + y^2\sqrt{x} + 4xy$. It is also an implicit function.

Linear Independence of Functions

A **linear combination** of a set of functions $\{f_1, f_2, \dots, f_n\}$ is the sum $y = c_1f_1 + c_2f_2 + \dots + c_nf_n$, where $c_1, c_2, \dots, c_n$ are arbitrary constants.

If there exists constants $c_1, c_2, \dots, c_n$, not all zero, such that the linear combination $c_1f_1 + c_2f_2 + \dots + c_nf_n = 0$ for all values of x, then the set of functions $\{f_1, f_2, \dots, f_n\}$ is called a **linearly dependent set of functions**.

If the linear combination $c_1f_1 + c_2f_2 + \dots + c_nf_n = 0$ hold only when $c_1 = 0, c_2 = 0, \dots, c_n = 0$, (i.e., when all arbitrary constants are simultaneously zero), then the set of functions $\{f_1, f_2, \dots, f_n\}$ is said to be **linearly independent**.

Limit of a function

Let $a < c < b$ and $f(x)$ be a function whose domain contains the interval $(a, c) \cup (c, b)$. Then, we say that $f(x)$ has the **limit** ℓ at the point $x = c$ if $f(x)$ approaches to ℓ as x approaches to c. In this case, we write $\lim\limits_{x \to c} f(x) = \ell$.

In a more mathematically accepted way, we can define the limit of a function as follows.

Let $a < c < b$ and $f(x)$ be a function whose domain contains the interval $(a, c) \cup (c, b)$. Then, we say that $f(x)$ is said to have the limit ℓ at the point $x = c$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - \ell| < \epsilon$, whenever $|x - c| < \delta$, however small ϵ, δ be.

Limit of a function

Define $f(x) = \begin{cases} 3 - x & \text{if } x < 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$

Observe that, when x is to the left of 1 and very near to 1 then $f(x) = 3 - x$ is very near to 2.
Likewise, when x is to the right of 1 and very near to 1 then $f(x) = x^2 + 1$ is very near to 2. We conclude that

$$\lim_{x \to 1} f(x) = 2.$$

Limit of a function

Define $h(x) = \begin{cases} 3 & \text{if } x \neq 7 \\ 1 & \text{if } x = 7 \end{cases}$ Calculate $\lim_{x \to 7} h(x)$.

It would be incorrect to simply plug the value 7 into the function h and thereby to conclude that the limit is 1.

In fact when x is *near to* 7 but *unequal to* 7, we see that h takes the value 3. This statement is true *no matter how close x is to* 7. We conclude that $\lim_{x \to 7} h(x) = 3$.

Limit of a function

Discuss the limits of the function at $c = 2$.

$$f(x) = \begin{cases} 2x - 4 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$$

As x approaches 2 from the left, $f(x) = 2x - 4$ approaches 0. As x approaches 2 from the right, $f(x) = x^2$ approaches 4. Thus we see that f has left limit 0 at $c = 2$, written

$$\lim_{x \to 2^-} f(x) = 0,$$

and f has right limit 4 at $c = 2$, written

$$\lim_{x \to 2^+} f(x) = 4.$$

Note that the full limit $\lim_{x \to 2} f(x)$ *does not exist* (because the left and right limits are unequal).

Properties of Limits

1. $\lim\limits_{x\to a}(c_1 f_1(x) + c_2 f_2(x)) = c_1 \cdot \lim\limits_{x\to a} f_1(x) + c_2 \cdot \lim\limits_{x\to a} f_2(x)$, where c_1 and c_2 are two arbitrary constants.

   This property is called the **Linearity Property** of limits.

2. $\lim\limits_{x\to a}(f_1(x) \cdot f_2(x)) = \lim\limits_{x\to a} f_1(x) \cdot \lim\limits_{x\to a} f_2(x)$.

3. $\lim\limits_{x\to a}\left(\dfrac{f_1(x)}{f_2(x)}\right) = \dfrac{\lim\limits_{x\to a} f_1(x)}{\lim\limits_{x\to a} f_2(x)}$.

Continuity of Functions

Let f be a function defined whose domain contains the interval (a, b). Also, let c be a point lies inside (a, b). The function f is said to be **continuous at the pont** $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

In other words, we have

Let $a < c < b$ and $f(x)$ be a function whose domain contains the interval (a, b). Then, we say that $f(x)$ is said to be **continuous at the point** $x = c$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$, whenever $|x - c| < \delta$, however small ϵ, δ be.

A function $f(x)$ is said to be **continuous** in an interval (a, b), if it is continuous at all points in (a, b).

Continuity of a Function

Is the function

$$f(x) = \begin{cases} 2x^2 - x & \text{if } x < 2 \\ 3x & \text{if } x \geq 2 \end{cases}$$

continuous at $x = 2$?

We check that $\lim_{x\to 2} f(x) = 6$. Also the actual value of f at 2, given by the second part of the formula, is equal to 6. By the definition of continuity we may conclude that f is continuous at $x = 2$.

Continuity of a Function

Where is the function

$$g(x) = \begin{cases} \dfrac{1}{x-3} & \text{if } x < 4 \\ 2x+3 & \text{if } x \geq 4 \end{cases}$$

continuous?

If $x < 3$ then the function is plainly continuous. The function is undefined at $x = 3$ so we may not even speak of continuity at $x = 3$. The function is also obviously continuous for $3 < x < 4$. At $x = 4$ the limit of g does not exist—it is 1 from the left and 11 from the right. So the function is not continuous (we sometimes say that it is *discontinuous*) at $x = 4$.

By inspection, the function is continuous for $x > 4$.

Differentiability of Functions

1. A function $y = f(x)$ is said to be **differentiable** at the point $x = c$ if the limit $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists.
2. The limit $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ is said to be the **differential coefficient** or **derivative** of the function $f(x)$ at the point $x = c$.
3. A function $y = f(x)$ is said to be **differentiable** in a an interval (region) if it is differentiable at every point in the region.
4. the derivative of a function $y = f(x)$ with respect to the independent variable x is denoted by $\frac{dy}{dx}$ or $f'(x)$ or y' etc.

Differentiability of Functions

Is the function $f(x) = x^2 + x$ differentiable at $x = 2$? If it is, calculate the derivative.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{[(2+h)^2 + (2+h)] - [2^2 + 2]}{h} \\
&= \lim_{h\to 0} \frac{[(4+4h+h^2) + (2+h)] - [6]}{h} \\
&= \lim_{h\to 0} \frac{5h + h^2}{h} \\
&= \lim_{h\to 0} 5 + h \\
&= 5.
\end{aligned}
$$

We see that the required limit exists, and that it equals 5. Thus the function $f(x) = x^2 + x$ is differentiable at $x = 2$, and the value of the derivative is 5.

Properties of Differentiation

- **The Sum Rule:** $\frac{d}{dx}\left[f_1(x) \pm f_2(x)\right] = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$.
- **The Product Rule:** $\frac{d}{dx}\left[f_1(x) \cdot f_2(x)\right] = f_1(x) \cdot \frac{d}{dx}f_2(x) + f_2(x) \cdot \frac{d}{dx}f_1(x)$.
- **The Quotient Rule:** $\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{f_2(x) \cdot \frac{d}{dx}f_1(x) - f_1(x) \cdot \frac{d}{dx}f_2(x)}{(f_2(x))^2}$.
- **The Linearity Property Rule:** $\frac{d}{dx}\left[c_1 \cdot f_1(x) + c_2 \cdot f_2(x)\right] = c_1 \cdot \frac{d}{dx}f_1(x) + c_2 \cdot \frac{d}{dx}f_2(x)$, where c_1 and c_2 are arbitrary constants.

Properties of Differentiation

- **Chain Rule:** If y is a function of u and u is a function of x; i.e., if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In other words,

$$[(f \circ g)(x)]' = f'(g(x)) \cdot g'(x).$$

The chain rule is also called **function of function rule** for differentiation.

Higher order derivatives

The second order derivative of the function $y = f(x)$ is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ and is defined as

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

In a similar way, we can define higher order derivatives as per our requirements. In general, the n-th order derivative of a function $y = f(x)$ is denoted by $\dfrac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

Applications of Differentiation

- As mentioned in the introduction, differentiation represents the **rate of change**; precisely, the rate of change of the dependent variable with respect to independent variable.
- The derivative $f'(c) = \left[\frac{dy}{dx}\right]_{x=c}$ represents the **slope of the tangent** to the curve given by $y = f(x)$ at the point $x = c$.
- The **slope of the normal** to the curve $y = f(x)$ at the point $x = c$ is given by $-\frac{1}{f'(c)}$.
- A point $x = c$ is said to be a **critical point** of the function $f(x)$ if $f(c)$ exists and if either $f'(c) = 0$ or $f'(c)$ doesn't exist.

Critical Points

Determine all the critical points for the function $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

We first need the derivative of the function in order to find the critical points and so let's get that and notice that we'll factor it as much as possible to make our life easier when we go to find the critical points.

$$\begin{aligned} f'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2\left(5x^2 + 22x - 15\right) \\ &= 6x^2(5x-3)(x+5) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere. Therefore the only critical points will be those values of x which make the derivative zero. So, we must solve.

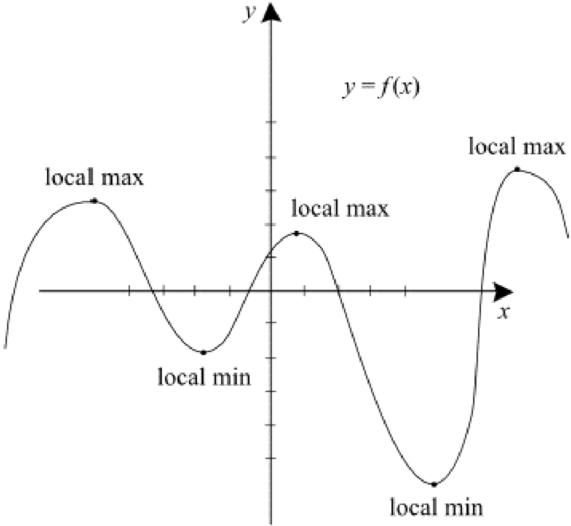
$$6x^2(5x-3)(x+5) = 0$$

Because this is the factored form of the derivative it's pretty easy to identify the three critical points. They are, $x = -5, \quad x = 0, \quad x = \frac{3}{5}$

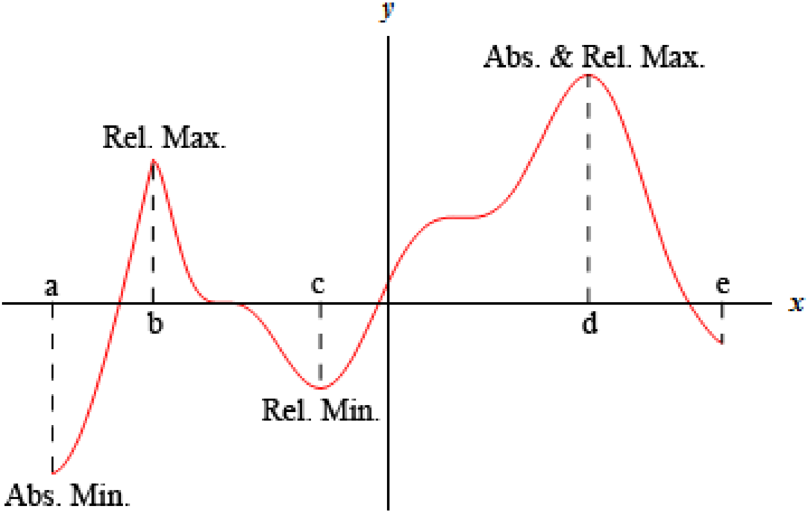
Maxima and Minima

- A function $f(x)$ is said to have an **absolute maximum** (or **global maximum**) at $x = c$ if $f(x) \leq f(c)$ for every x in the domain under consideration.
- A function $f(x)$ is said to have a **relative maximum** (or **local maximum**) at $x = c$ if $f(x) \leq f(c)$ for every x in some open interval around $x = c$.
- A function $f(x)$ is said to have an **absolute minimum** (or **global minimum**) at $x = c$ if $f(x) \geq f(c)$ for every x in the domain under consideration.
- A function $f(x)$ is said to have a **relative minimum** (or **local minimum**) at $x = c$ if $f(x) \geq f(c)$ for every x in some open interval around $x = c$.

Maxima and Minima



Maxima and Minima



Maxima and Minima

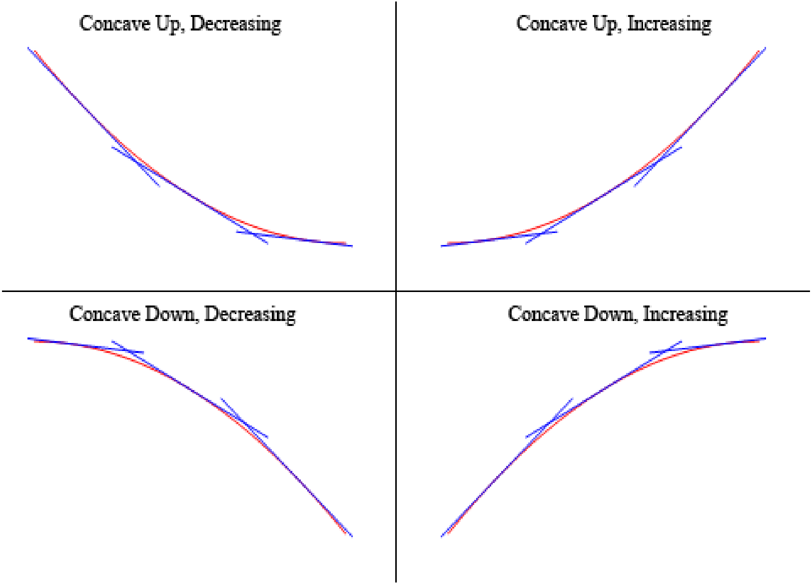
- **Extreme Value Theorem:** If a function $f(x)$ is continuous on the interval $[a, b]$, then there are two numbers $a \leq c, d \leq b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.
- **Fermat's Theorem:** If $f(x)$ has a relative extrema at $x = c$ and $f'(c)$ exists, then $x = c$ is a critical point of $f(x)$; i.e, $f'(c) = 0$.

Maxima and Minima

- Verify that the function is continuous on the interval $[a, b]$.
- Find all critical points of $f(x)$ that are in the interval $[a, b]$. Since we are only interested in what the function is doing in this interval we need not care about critical points that fall outside the interval.
- Determine the values the given function at the critical points obtained in the above step.
- Identify the absolute extrema.

Increasing and Decreasing Functions

- If $f'(x) > 0$ for every point x on some interval I, then $f(x)$ is **increasing** on the interval.
- If $f'(x) < 0$ for every point x on some interval I, then $f(x)$ is **decreasing** on the interval.
- If $f'(x) = 0$ for every x on some interval I, then $f(x)$ is **constant** on the interval.
- A function $f(x)$ is **concave up** on an interval I if all of the tangents to the curve on I are below the graph of $f(x)$ and $f(x)$ is **concave down** on I if all of the tangents to the curve on I are above the graph of $f(x)$.


Concave Up, Decreasing
Concave Up, Increasing
Concave Down, Decreasing
Concave Down, Increasing

Increasing and Decreasing Functions

- A point $x = c$ is called an **inflection point** if the function is continuous at the point and the concavity of the graph changes at that point.
- Given the function $f(x)$ then,
  - If $f''(x) > 0$ for all x in some interval I, then $f(x)$ is concave up on I.
  - If $f''(x) < 0$ for all x in some interval I, then $f(x)$ is concave down on I.
- **Second Derivative Test:** The function $f(x)$ has a relative maximum at the point $x = c$ if $f'(c) = 0$ and $f''(c) < 0$ and has a relative minimum at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$. If $f''(c) = 0$, then existence of extreme value cannot be verified precisely.

Steps to find relative extrema

- Verify that the function is continuous on the interval $[a, b]$.
- Find all critical points of $f(x)$ that are in the interval $[a, b]$. Since we are only interested in what the function is doing in this interval we need not care about critical points that fall outside the interval.
- Determine the second derivatives the function at the critical points found in the above step.
- Identify the absolute extrema, using second derivative test.

Rolle's Theorem

Suppose $f(x)$ is a function that satisfies the following conditions.

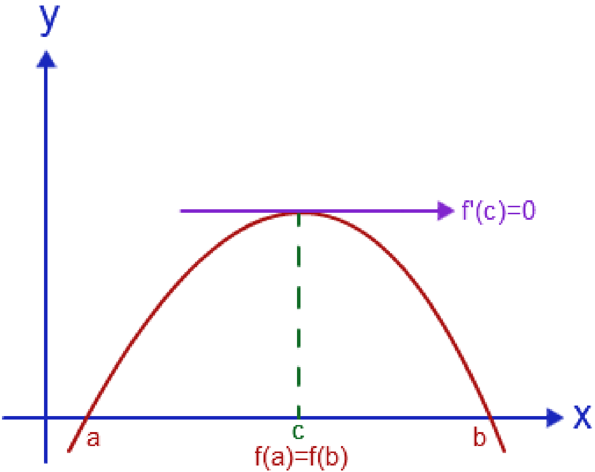
1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then, there is a number $c \in (a, b)$ such that $f'(c) = 0$.

In other words, if a function satisfies the above three conditions, we see that

1. $f(x)$ has a **critical point** in the open interval (a, b). Or,
2. there exists a point in the interval (a, b) which has a horizontal tangent.

Rolle's Theorem



Lagrange's Mean Value Theorem

Suppose $f(x)$ is a function that satisfies the following conditions.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).

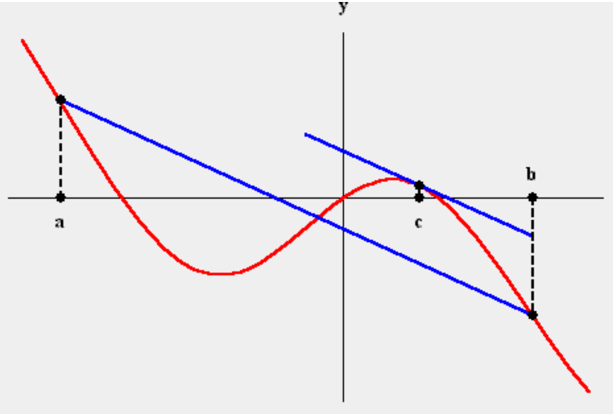
Then, there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Or, in other words, we have

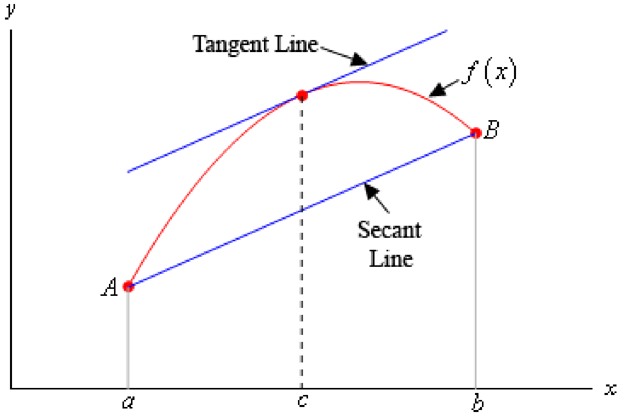
$$f(b) - f(a) = f'(c)(b - a)$$

Lagrange's Mean Value Theorem



Lagrange's Mean Value Theorem

Mean Value Theorem tells us that the secant line connecting the points $x = a$ and $x = b$ and the tangent line at $x = c$ must be parallel (see the figure).



Cauchy's Mean Value Theorem

Suppose $f(x)$ and $g(x)$ are the function which satisfy the following conditions.

1. $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$.
2. $f(x)$ and $g(x)$ are differentiable on the open interval (a, b).

Then, there is a number c such that $a < c < b$ and

$$\frac{f'(c)}{g'(c)} = \frac{(f(b) - f(a))}{(g(b) - g(a))}.$$

Or,

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Taylor's Series

1. The **Taylor series** of a real function $f(x)$, that is infinitely differentiable at a real a is the power series

$$f(x) = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+$$

2. At $x = 0$, the above series becomes

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots\ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

This series is called the **Maclaurin series** of the function $f(x)$.

Partial Derivatives

A number L is said to be the **limit** of the function $f(x,y)$ as x approaches a and as y approaches b, if $f(x,y)$ approaches to L as (x,y) approaches to (a,b). In this case, we write $\lim \lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

A function $f(x,y)$ is **continuous** at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

.

Partial Derivatives

1. Let $z = f(x, y)$ be a function of two variables. Then the **first order partial derivative** of $f(x, y)$ with respect to x, denoted by $f_x(x, y)$ or $\frac{\partial f}{\partial x}$, defined as

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The **first order partial derivative** of $f(x, y)$ with respect to y, denoted by $f_y(x, y)$ or $\frac{\partial f}{\partial y}$, is defined as

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}.$$

Partial Derivatives

The second order partial derivatives are given by

1. $f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$;
2. $f_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$;
3. $f_{yx}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$;
4. $f_{xy}(x,y) = f_{yx}(x,y)$.

In a similar way, higher order derivatives can be defined as per our requirements.

Chain Rules

1. If f is a function of two variables x and y and x and y are the functions in t, then

$$f_t = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

2. If f is a function of the variables x, y and xu and y are the functions in two variables s, t, then
   1. $f_s = f_x \cdot \frac{dx}{ds} + f_y \cdot \frac{dy}{ds}$;
   2. $f_t = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$.

Total Derivatives

If f is a function of two variables x, y and t, then the total derivative of f is denoted by df and is defined as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial t}dt.$$

The result is the differential change df in (or total differential of) the function f.

Maxima and Minima

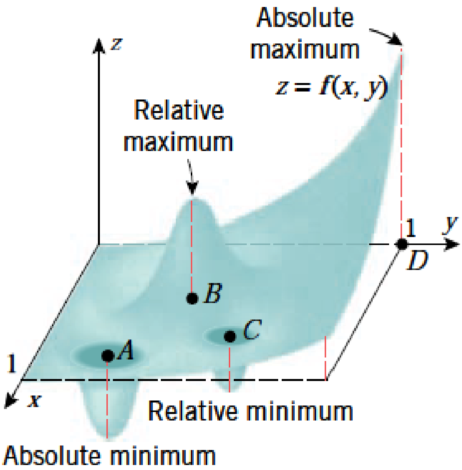
A function $f(x, y)$ of two variables is said to have a **relative maximum** at a point (x_0, y_0) if there is a disk centred at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk.

The function f has an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the domain of f.

A function $f(x, y)$ of two variables has a **relative minimum** at a point (x_0, y_0) if there is a disk centred at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk.

The function f has an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

Maxima and Minima



Method to find Absolute Extrema

Method to Find the absolute extrema of a continuous function f of two Variables on a closed and bounded set R:

1. Find the critical points of f that lie in the interior of R.
2. Find all boundary points at which the absolute extrema can occur.
3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Maxima and Minima

Extreme Value Theorem: If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.

Theorem: If f has a relative extremum at a point (x_0, y_0), and if the first order partial derivatives of f exist at this point, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a **critical point** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Maxima and Minima

The Second Partials Test: Let f be a function of two variables with continuous second-order partial derivatives in some disk centred at a critical point (x_0, y_0), and let $D = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$. Then,

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).
3. If $D < 0$, then f has a saddle point at (x_0, y_0).
4. If $D = 0$, then no conclusion can be drawn.

Indefinite Integrals

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.

If $F(x)$ is any anti-derivative of $f(x)$, then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and is denoted by $\int f(x)dx = F(x) + c$, c is any constant.

The function $f(x)$ is called the **integrand**, x is the **variable of integration** and the "c" is called the **constant of integration**.

Indefinite Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int \mathbf{e}^x\,dx = \mathbf{e}^x + c$$

$$\int k\,dx = kx + c,$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \csc^2 x\,dx = -\cot x + c$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + c$$

Indefinite Integrals

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \sinh x\,dx = \cosh x + c$$

$$\int \cosh x\,dx = \sinh x + c$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + c$$

$$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$$

$$\int \operatorname{csch}^2 x\,dx = -\coth x + c$$

$$\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + c$$

Properties of Indefinite Integrals

1. $\int k \cdot f(x)dx = k \int f(x)dx$; where k is any number. That is, we can factor multiplicative constants out of indefinite integrals.

2. $\int [c_1 f_1(x) \pm c_2 f_2(x)]\, dx = c_1 \int f_1(x)dx \pm c_2 \int f_2(x)dx$. In other words, the integral of a sum (or difference) of functions is the sum (or difference) of the individual integrals. (Linearity)

3. **Substitution Rule:** $\int f(g(x))g'(x)dx = \int f(u)du$; where $u = g(x)$.

4. $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$.

Fundamental Theorem of Calculus: Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$, by $F(x) = \int\limits_a^x f(t)dt$. Then, F is also continuous on $[a, b]$, differentiable on the open interval (a, b) and $F'(x) = f(x)$, for all x in (a, b).

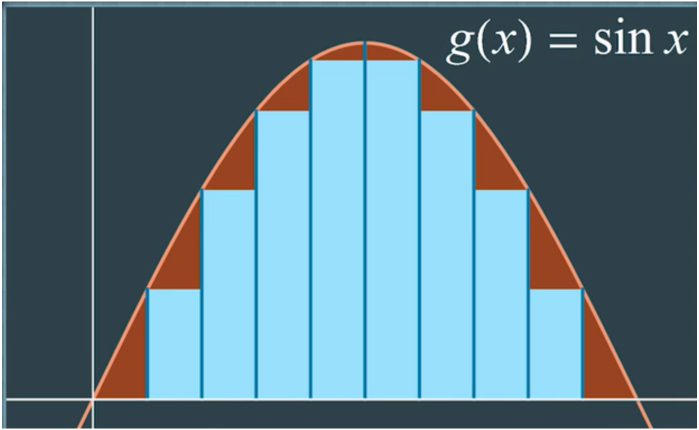
The fundamental theorem is used to compute the definite integral of a function f for which an anti-derivative F is known. If f is a real-valued continuous function on $[a, b]$, and F is an anti-derivative of f in $[a, b]$, then $\int\limits_a^b f(t)dt = F(b) - F(a)$.

Indefinite Integrals

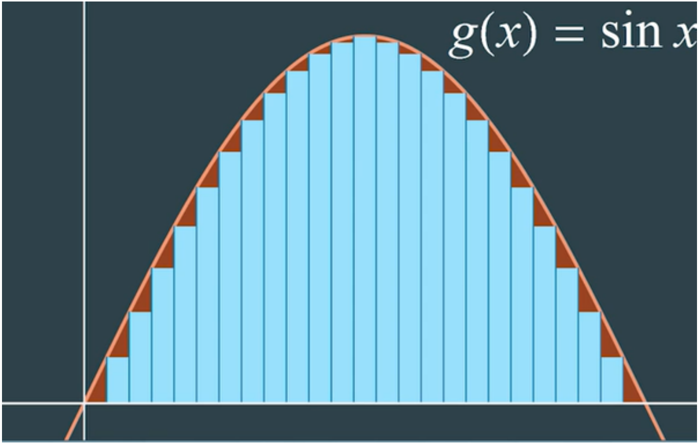
Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, δx , and from each interval choose a point, x_i^* . Then the definite integral of $f(x)$ from a to b is $\int\limits_a^b f(x)\,dx = \lim\limits_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\delta x$

That is, **a definite integral can be considered to be the limit of a sum**.

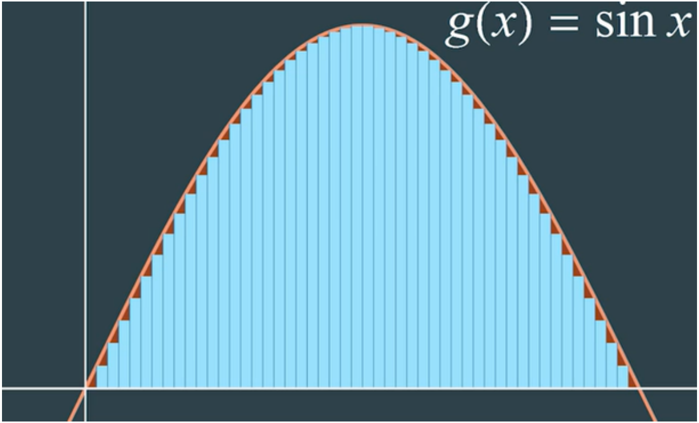
Indefinite Integrals



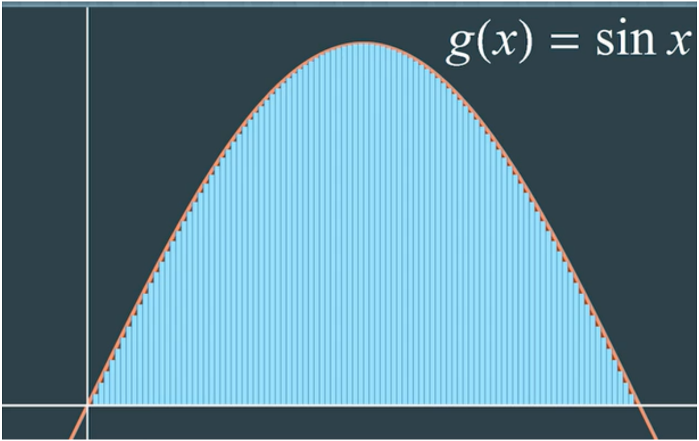
Indefinite Integrals



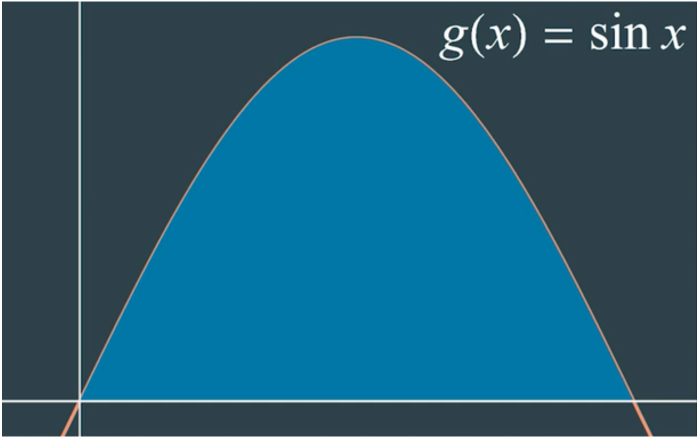
Indefinite Integrals



Indefinite Integrals



Indefinite Integrals



Properties of Definite Integrals

1. If f is integrable over $[a, b]$, then
$$\int_a^b f(x)dx = \int_a^b f(y)dy.$$
2. For any bounded function f
$$\int_a^a f(x)dx = 0.$$
3. If f is integrable over $[a, b]$, then for any real number k
$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$
4. If f is integrable over $[a, b]$,
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Properties of Definite Integrals

1. If f and g are integrable over $[a, b]$,
$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

2. If f is integrable over $[a, b]$ and over [b,c], then f is integrable over $[a, c]$ and
$$\int_a^b f(x) + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

3. If f and g are integrable over $[a, b]$, then
$f(x) \geq g(x)$ on $[a, b] \implies \int_a^b f(x)dx \geq \int_a^b g(x)dx.$
In particular, $f(x) \geq 0$ on $[a, b] \implies \int_a^b f(x)dx \geq 0.$

Properties of Definite Integrals

- If f is integrable over $[a, b]$, then for any constant c $\int_a^b c\,dx = c(b-a)$.

The **average value** of a function $f(x)$ over the interval $[a, b]$ is given by, $f_{avg} = \frac{1}{b-a} \int \lim_a^b f(x)dx$.

Mean value Theorem for Integrals: If $f(x)$ is a continuous function on $[a, b]$ then there is a number c in $[a, b]$ such that $\int\limits_a^b f(x)dx = (b-a)f(c)$.

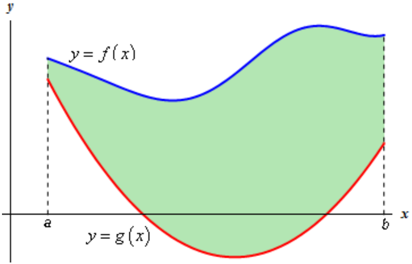
Applications of Definite Integrals

Rectification (Arc length of a curve): If f' is continuous on $[a, b]$, then the lengths of the curve $y = f(x),\ a \leq x \leq b$, is given by

$$
\begin{aligned}
&= \int_a^b \sqrt{(dx)^2 + (dy)^2} \\
&= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx \\
&= \int_a^b \sqrt{1 + (f'(x))^2}\, dx;
\end{aligned}
$$

Applications of Definite Integrals

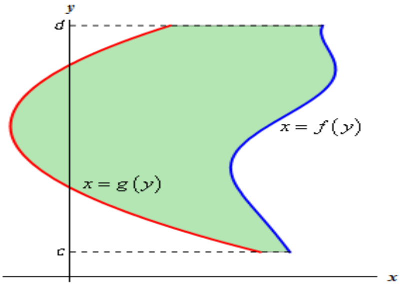
Area between Two Curves: Let $f(x)$ and $g(x)$ be two functions such that $f(x) \geq g(x)$ (see figure).



The area between the curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is given by $A = \int_a^b [f(x) - g(x)]\, dx$.

Applications of Definite Integrals

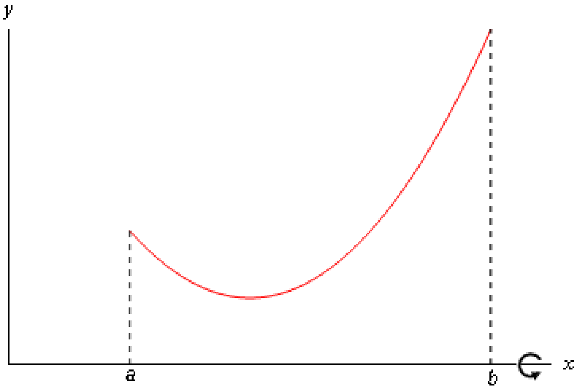
Area between Two Curves: Similarly, let $f(y)$ and $g(y)$ be two functions such that $f(y) \geq g(y)$ (see figure).



The area between the curves $x = f(y)$ and $x = g(y)$ on the interval $[a, b]$ is given by $A = \int\limits_a^b [f(y) - g(y)]\, dy$.

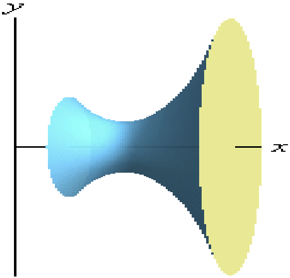
Applications of Definite Integrals

Volume of solid revolution: First consider a function $y = f(x)$ on an interval $[a, b]$ (see figure).



Applications of Definite Integrals

We then rotate this curve about X-axis to get the surface of the solid of revolution. Doing this for the curve above gives the following three dimensional region.



Applications of Definite Integrals

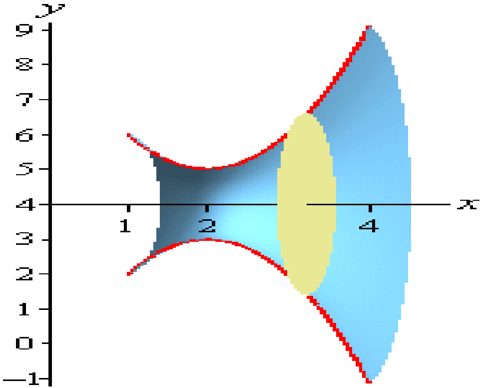
Then, the volume of this solid of revolution is obtained by $V = \int_a^b A(x)dx$ or $V = \int_c^d A(y)dy$, where, $A(x)$ and $A(y)$ is the cross-sectional area of the solid.

Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

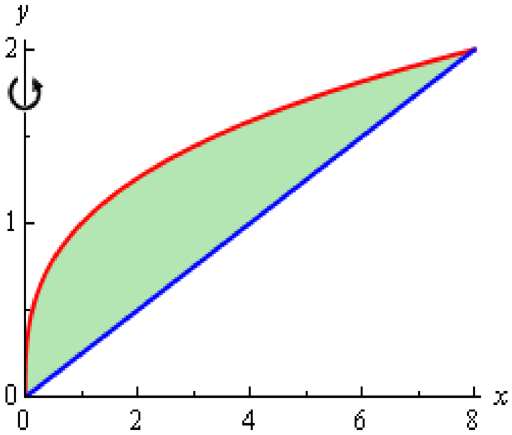
One of the easier methods for getting the cross-sectional area is to **cut the object perpendicular to the axis of rotation**.

Doing this the cross section will be either a solid disk if the object is solid or a ring if we've hollowed out a portion of the solid.

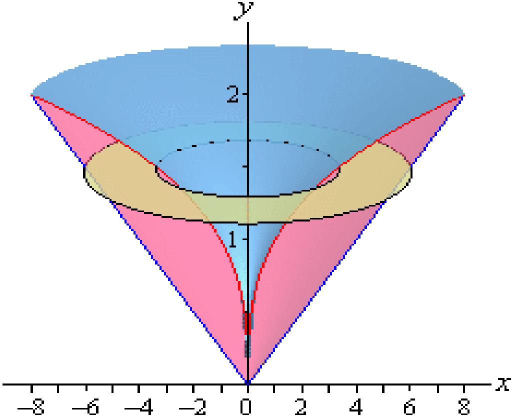
Applications of Definite Integrals

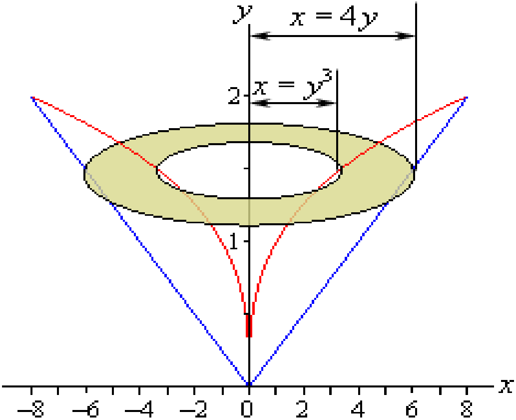


Applications of Definite Integrals



Applications of Definite Integrals





y
$x = 4y$
$x = y^3$
2
1
x
−8 −6 −4 −2 0 2 4 6 8

Applications of Definite Integrals

In the case that we get a solid disk the area is,

$$A = \pi.(radius)^2$$

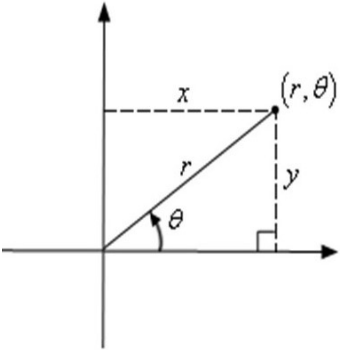
In the case that we get a ring the area is,

$$A = \pi\left[(outerradius)^2 - (innerradius)^2\right]$$

.

This method is often called the **method of disks** or **the method of rings**.

Converting to polar coordinates

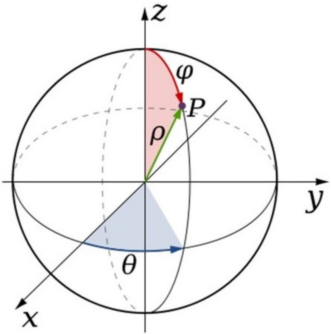


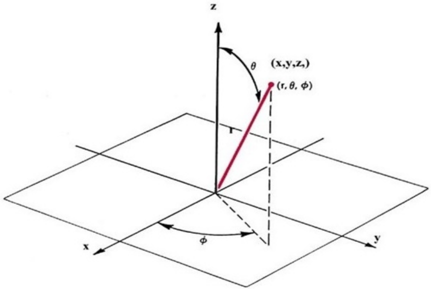
$$x = r.\cos\theta$$

$$y = r.\sin\theta$$

Conversion of Cartesian Coordinates to polar coordinates

Converting to spherical polar coordinates

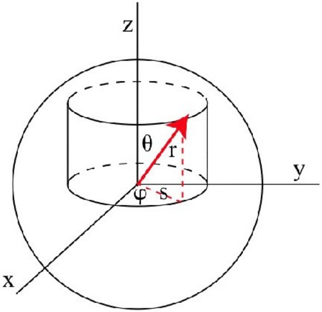




$$x = r\sin\theta\cos\emptyset;$$
$$y = r\sin\theta\sin\emptyset$$
$$z = \cos\emptyset$$

Conversion of Cartesian Coordinates to polar coordinates

Converting to cylindrical polar coordinates



$$x = r.\cos\theta$$

$$y = r.\sin\theta$$

$$z = z$$

Conversion of Cartesian Coordinates to cylindrical polar coordinates

Jacobian

If T is the transformation from the uv-plane to the xy-plane defined by the equations $u = u(x, y)$, $v = v(x, y)$, then the **Jacobian** of T is denoted by $J(x, y)$ or by $\frac{\partial(u,v)}{\partial(x,y)}$ and is defined by

$$J(x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}.$$

Example: If $x = r\cos\theta$, and $y = r\sin\theta$, then $J = r$.

Jacobian

If $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$, then the corresponding **Jacobian** is denoted by $J(x, y, z)$ or by $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ and is defined by

$$J(x, y, z) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Example: If $x = r \sin\theta \cos\phi$, and $y = r \sin\theta \sin\phi$ and $r = \cos\theta$, then $J = r^2 \sin\theta$.

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