Encryption using s^2-mod-n

Software Security

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Objectives of today's lecture

- ➜ Getting to know the requirements of a strong cryptographic *pseudo-random bit generator*
- ➜ Understanding the principles of the *Pseudo One-Time-Pad*, which corresponds to the symmetric variant of s^2-mod-n
- ➜ Being able to apply the *asymmetric variant* of s^2-mod-n based on calculating square roots of the residue class ring of n for decryption

Prime Factorization

Definition

The prime factorization of a natural number n is the product

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \quad \ldots \quad \cdot p_k^{e_k}$$

where $p_1, \ldots, p_k$ are different prime numbers in pairs and the exponents are positive natural numbers, i.e. $e_1, \ldots, e_k \in \mathbb{N}^+$

What means prime factorization and why is this operation so important for asymmetric encryption systems?

Algorithms

- Pollard's rho algorithm
- Quadratic sieve algorithm
- Number field sieve

➜ No polynomial algorithm for prime factor decomposition has been found yet!

Pseudo One-Time-Pad s^2-mod-n

Fundamentals of Number Theory

Factorization is hard

There is no polynomial algorithm to efficiently calculate the prime numbers p and q from a given n, so that $p \cdot q = n$ applies

There are two other operations that are based on factorization. Why these operations are important too?

Implications

There are two other algorithms that are as hard as factorization

1. Calculating a square root $\bmod n$
2. Testing for a square $\bmod n$[1]

➜ However, if you know p and q, then both tasks can be solved efficiently, e.g. root extraction using the CRA (Chinese Remainder Algorithm)!

[1] Also called the *quadratic residuacity problem*

Two Variants for s^2-mod-n

Symmetric-key und Asymmetric-key Concelation

System type / Security level		Concelation: sym. — sym. concelation system	Concelation: asym. — asym. concelation system	Authentikation: sym. — sym. authentication system	Authentikation: asym. — digital signature system
information theoretical		Vernam-Chiffre (one-time pad)		Authentication codes	
crypto-graphi-cally strong against...	active attack	Pseudo-one-time-pad with s^2-mod-n-Generator			GMR
	passive attack		System with s^2-mod-n-Generator		
well re-searched	mathe-matical		RSA		RSA
	chaos	DES/AES		DES/AES	

Source: *Andreas Pfitzmann: Security in IT-Networks*, 2013

What cryptographic assumption is the basis for s2-mod-n?

Functioning of the Symmetric-key Version

Components of the secret key

1. Product of two prime numbers p and q with $n = p \cdot q$
2. Randomly chosen initial value s with $s \in \mathbb{Z}_n^*$,

   where $\mathbb{Z}_n^* = \{a : \mathbb{Z}_n \mid gcd(a, n) = 1\}$ and
   $\mathbb{Z}_n = \{0, \ldots, n-1\}$

Calculation of the pseudo-random bit sequence

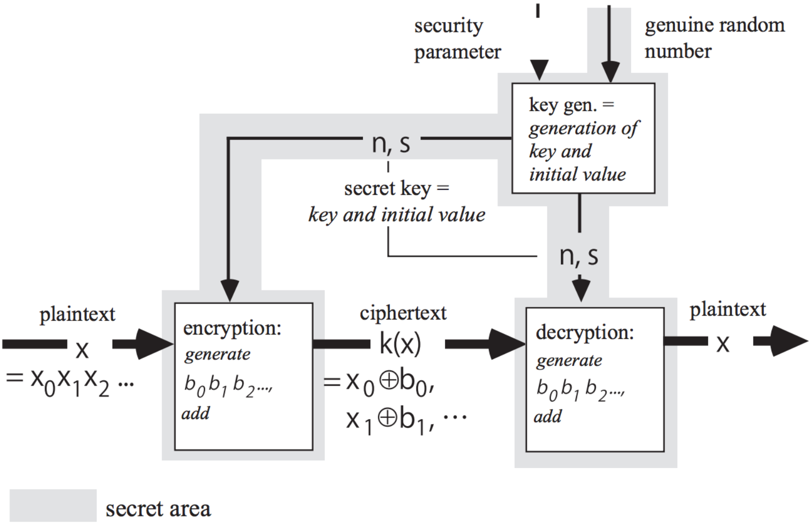
1. Calculate in each step s_{i+1} with $s_{i+1} = s_i^2 \bmod n$
2. Select the last bit in each step $b_i = s_i \bmod 2$

Encryption and decryption

➜ Add (or subtract) the pseudo-random bit sequence to the plaintext or ciphertext using the XOR-Operation simular to the One-Time Pad

Why is s2-mod-n also called Pseudo One-Time-Pad?

Symmetric-key Version of s^2-mod-n



Example Calculation Symmetric-key Variant

Given the following key

- $n = 77$ with $n = 7 \cdot 11$ and inital value $s = 64$

Calculating s-sequence

- $64^2 \equiv 15 \bmod 77$
- $15^2 \equiv 71 \bmod 77$
- $71^2 \equiv 36 \bmod 77$
- $36^2 \equiv 64 \bmod 77$

Calculating Bit-sequence

- $15 \equiv 1 \bmod 2$
- $71 \equiv 1 \bmod 2$
- $36 \equiv 0 \bmod 2$
- $64 \equiv 0 \bmod 2$

Encryption

- The pseudo-random bit sequence 1100 is added to the plaintext 0011 using XOR, so that we get the ciphertext 1111

Disadvantages of Symmetric-key Encryption

~~What are the differences between the symmetric and the~~ asymmetric variant of s2-mod-n?

Problem

Assuming n people want to communicate in pairs, so you need k different keys with $k = n \cdot (n-1)/2$

Examples

- for $n = 100$ we obtain $k = 4.950$ keys
- for $n = 1000$ we obtain $k = 499.500$ keys

➜ Quadratic increase

Solution

- Asymmetric encryption requires only $k = 2 \cdot n$ differnet keys
- Keys for symmetric encryption can afterwards be exchanged using asymmetric encryption

Functioning of the Asymmetric-key Version

Components Secret Key

- Two prime numbers p and q with $p \equiv q \equiv 3 \mod 4$

Components Public Key

- Product n with $n = p \cdot q$

Encryption

Which part of the key is public, which part is secret?

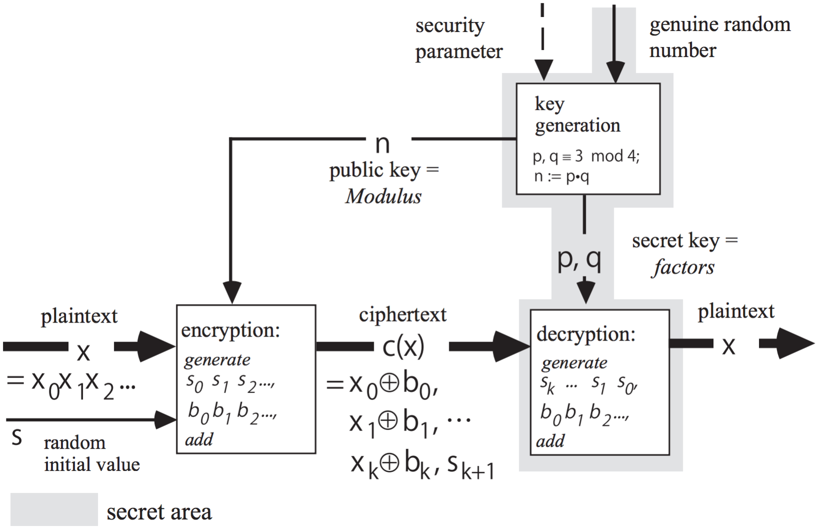
1. Create a bit sequence with s^2-mod-n-method for a randomly selected start value s
2. Add the bit sequence to the plain text to calculate the ciphertext, then send the ciphertext and a last s_{k+1} that was not used for the encryption

Decryption

1. Determine the bit sequence using p und q by successively extracting the square roots[2] from s_{k+1}
2. Calculate the plaintext using XOR operation

[2] Only use roots that are themselves squares again!

Asymmetric-key Version of s^2-mod-n



Computing Square Roots

How is it possible to compute a square root efficiently?
Illustrate the procedure using an example.

Computing a square root is hard

There is no polynomial algorithm to calculate a square root for mod n if you do not know the prime factors p und q

Procedure if you know the prime factors

1. Calculate the square roots for mod p and mod q using the following formulas
   - $y_p = y^{\frac{p+1}{4}} \bmod p$
   - $y_q = y^{\frac{q+1}{4}} \bmod q$

   ➜ Note: Formulas are only valid for $p \equiv q \equiv 3 \bmod 4$
2. Calculate the square root for mod n from y_p and y_q using the *Chinese Remainder Algorithm* (CRA)

Example s^2-mod-n Asymmetric-key Encryption

The following prime numbers are given

- $p = 3$ and $q = 7$ with $n = p \cdot q = 21$
- We assume further on that $p \equiv q \equiv 3 \bmod 4$ is fulfilled

Assumption: The following root is to be calculated

- $y = \sqrt{4} \equiv ?$

Task: How can a square root be calculated efficiently?

- $y = \sqrt{4} \equiv 2 \equiv 5 \equiv 16 \equiv 19 \bmod 21$

How to calculate the square roots for modulo p and q?

Formulas

- $y_p = y^{\frac{p+1}{4}} \mod p$
- $y_q = y^{\frac{q+1}{4}} \mod q$

Computing the square roots

- $y_3 = 4^{\frac{3+1}{4}} = 4^1 \equiv 1 \mod 3$
- $y_7 = 4^{\frac{7+1}{4}} = 4^2 \equiv 2 \mod 7$

➜ Now we have two intermediate results y_3 and y_7

Note

- The calculation rule can only be used under the condition $p \equiv q \equiv 3 \mod 4$!

How to combine the intermediate results with CRA?

Chinese Remainder Algorithm (CRA)

$$CRA\,(y_p, y_q, p, q) = u \cdot p \cdot y_q + v \cdot q \cdot y_p \mod n$$

Instantiation

$$CRA\,(1, 2, 3, 7) = u \cdot 3 \cdot 2 + v \cdot 7 \cdot 1 \mod 21,$$

How to calculate the base vectors u and v?

- The integer variables u and v must fulfill the condition
  $\gcd(3, 7) = u \cdot 3 + v \cdot 7 = 1$
- Values for u and v can be calculated using the *Extended Euclidean algorithm*

How to combine the intermediate results with CRA?

Extended Euclidean algorithm

$$7 = 2 \cdot 3 + 1 \qquad (q = s_1 \cdot p + r_1)$$

$$3 = 3 \cdot 1 + 0 \qquad (p = s_2 \cdot r_1 + r_2)$$

In reverse order, i.e. solve all equations to the rest and then insert them step by step

$$0 = 3 - 3 \cdot 1 \qquad \text{(skip this equation, because } r_2 = 0\text{)}$$

$$1 = 1 \cdot 7 - 2 \cdot 3 \qquad (r_1 = q - s_1 \cdot p)$$

➜ The base vectors are $u = -2$ and $v = 1$

➜ Results in the square root $CRA(1, 2, 3, 7) = 16$

➜ **Note**: In addition, check whether 16 is a square again

Why is s^2-mod-n cryptographically strong?

Why is s^2-mod-n cryptographically strong?

Why is s2-mod-n cryptographically strong? Note you do not need to provide a formal proof for this property, however you should be able to explain the rough idea behind this proof.

Question

Why is the generator s^2-mod-n (symmetric variant) an unpredictable (cryptographically stronger) Pseudo-Random Bit Generator (PRBG)?

Proof obligation

Under the factorizing assumption (resp. in our proof under the quadratic-residuosity-assumption), there is no polynomial algorithm that can distinguish the random sequence generated by the PRBG from a real random sequence

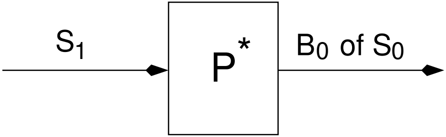
Proof by Contradiction

Assumption

There is a polynomial algorithm P which predicts the left continuation bit of a given k-bit sequence with a probability greater than $\frac{1}{2}$ (assuming that n of the residue class is also known)

Proof (First Step)

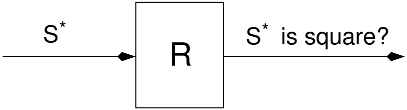
Then an algorithm P^* can be constructed from P, which calculates Bit B_0 of the initial value S_0 for a given value S_1



Continuation: Proof by Contradiction

Proof (Next Step)

Then an algorithm R can be constructed from P^*, which checks for a given value S^* with Jacoby symbol $+1$, whether S^* is a square, i.e. whether the condition $S^* \in QR_n$ holds[1]



How to implement the algorithm R ?

1. Calculate S_1 by squaring S^* with $S_1 = (S^*)^2$
2. Calculate from S_1 with P^* Bit B_0 of S_0
3. Compare B_0 with the last bit of S^*
4. If the bits are identical, than $S^* = S_0$ and therefore $S^* \in QR_n$

[1] It is important that S^* has the Jacobi symbol $+1$ (see above). If you calculate the roots of a given number in the residue class ring mod n, you obtain 4 possible roots (2 with Jacobi value $+1$ & 2 with Jacobi value -1). Note, only one of the roots (Q) is a square (always Jacobi value $+1$). Further, $-Q \bmod n$ is the other root with Jacobi value $+1$. To prove the proof obligation $(S^* = S_0) \Leftrightarrow (S^* \bmod 2 = S_0 \bmod 2)$ you still need the property that n is odd, which is derivable from $p \equiv q \equiv 3 \bmod 4$

Conclusion: Proof of Contradiction

Conclusion

The algorithm R is able to perform a square test in polynomial time without knowing p and q for any S^* with Jacobi symbol $+1$, i.e. R is able to check, whether $S^* \in QR_n$

Proof (Last Step)

The derived statement is obviously in contradiction to the quadratic residuosity assumption, which is strongly related to the factorization assumption. This means that an attacker who can efficiently predict random numbers would also be able to factorize in polynomial time[1].

[1] In other words, if an attacker is unable to factorize efficient, he will also not be able to break the s^2-mod-n algorithm