Similar derivations can be made for M-of-N systems in which failing processors are identified and replaced from an infinite pool of spares. This is left for the reader as an exercise. The extension to the case with only a finite set of spares is simple: the summation in the reliability expression is capped at that number of spares, rather than going to infinity.

2.4.2 Markov Models

In complex systems in which constant failure rates are assumed but combinatorial arguments are insufficient for analyzing the reliability of the system, we can use *Markov models* for deriving expressions for the system reliability. In addition, Markov models provide a structured approach for the derivation of reliabilities of systems that may include coverage factors and a repair process.

A *Markov chain* is a special type of a stochastic process. In general, a stochastic process $X(t)$ is an infinite number of random variables, indexed by time t. Consider now a stochastic process $X(t)$ that must take values from a set (called the *state space*) of discrete quantities, say the integers $0, 1, 2, \ldots$. The process $X(t)$ is called a *Markov chain* if

$$\text{Prob}\{X(t_n) = j \mid X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_{n-1}) = i_{n-1}\} = \text{Prob}\{X(t_n) = j \mid X(t_{n-1}) = i_{n-1}\}$$
$$\text{for every } t_0 < t_1 < \cdots < t_{n-1} < t_n$$

If $X(t) = i$ for some t and i, we say that the chain is in state i at time t. We will deal only with continuous time, discrete state Markov chains, for which the time t is continuous ($0 \leqslant t < \infty$) but the state $X(t)$ is discrete and integer valued. For convenience, we will use as states the integers $0, 1, 2, \ldots$. The Markov property implies that in order to predict the future trajectory of a Markov chain, it is sufficient to know its present state. This freedom from the need to store the entire history of the process is of great practical importance: it makes the problem of analyzing Markovian stochastic processes tractable in many cases.

The probabilistic behavior of a Markov chain can be described as follows. Once it moves into some state i, it stays there for a length of time that has an exponential distribution with parameter λ_i. This implies a constant *rate* λ_i of leaving state i. The probability that, when leaving state i, the chain will move to state j (with $j \neq i$) is denoted by p_{ij} ($\sum_{j \neq i} p_{ij} = 1$). The rate of transition from state i to state j is thus ??$\lambda_{ij} = p_{ij}\lambda_i$ ($\sum_{j \neq i} \lambda_{ij} = \lambda_i$).

We denote by $P_i(t)$ the probability that the process will be in state i at time t, given it started at some initial state i_0 at time 0. Based on the above notations, we can derive a set of differential equations for $P_i(t)$ ($i = 0, 1, 2, \ldots$).

For a given time instant t, a given state i, and a very small interval of time Δt, the chain can be in state i at time $t + \Delta t$ in one of the following cases:

1. It was in state i at time t and has not moved during the time interval Δt. This event has a probability of $P_i(t)(1 - \lambda_i \Delta t)$ plus terms of order Δt^2.
?? ??

2. It was at some other state j at time t ($j \neq i$) and moved from j to i during the interval Δt. This event has a probability of $P_j(t)\lambda_{ji}\Delta t$ plus terms of order Δt^2. ??

The probability of more than one transition during Δt is negligible (of order Δt^2) if Δt is small enough. Therefore, for small Δt,

$$??P_i(t+\Delta t) \approx P_i(t)(1-\lambda_i \Delta t) + \sum_{j \neq i} P_j(t)\lambda_{ji}\Delta t$$

Again, this approximation becomes more accurate as $\Delta t \to 0$, and results in

$$\boxed{\frac{\mathrm{d}P_i(t)}{\mathrm{d}t} = -\lambda_i P_i(t) + \sum_{j \neq i} \lambda_{ji} P_j(t)}$$

and, since $\lambda_i = \sum_{j \neq i} \lambda_{ij}$,

$$\boxed{\frac{\mathrm{d}P_i(t)}{\mathrm{d}t} = -\sum_{j \neq i} \lambda_{ij} P_i(t) + \sum_{j \neq i} \lambda_{ji} P_j(t)}$$

This set of differential equations (for $i = 0, 1, 2, \ldots$) can now be solved, using the initial conditions $P_{i_0}(0) = 1$ and $P_j(0) = 0$ for $j \neq i_0$ (since i_0 is the initial state).

Consider, for example, a duplex system that has a single active processor and a single standby spare that is connected only when a fault has been detected in the active unit. Let λ be the fixed failure rate of each of the processors (when active) and let c be the coverage factor. The corresponding Markov chain is shown in Figure 2.15. Note that because the integers assigned to the different states are arbitrary, we can assign them in such a way that they are meaningful and thus easier to remember. In this example, the state represents the number of good processors (0, 1, or 2, with the initial state being 2 good processors). The differential equations describing this Markov chain are:

$$?? \boxed{\begin{aligned} \frac{\mathrm{d}P_2(t)}{\mathrm{d}t} &= -\lambda P_2(t) \\ \frac{\mathrm{d}P_1(t)}{\mathrm{d}t} &= \lambda c P_2(t) - \lambda P_1(t) \\ \frac{\mathrm{d}P_0(t)}{\mathrm{d}t} &= \lambda(1-c)P_2(t) + \lambda P_1(t) \end{aligned}} \qquad \textbf{(2.34)}$$

Solving 2.34 with the initial conditions $P_2(0) = 1$, $P_1(0) = P_0(0) = 0$ yields

$$\boxed{P_2(t) = \mathrm{e}^{-\lambda t} \qquad P_1(t) = c\lambda t\mathrm{e}^{-\lambda t} \qquad P_0(t) = 1 - P_1(t) - P_2(t)}$$

and as a result,

$$\boxed{R_{\text{system}}(t) = 1 - P_0(t) = P_2(t) + P_1(t) = \mathrm{e}^{-\lambda t} + c\lambda t\mathrm{e}^{-\lambda t}} \qquad \textbf{(2.35)}$$

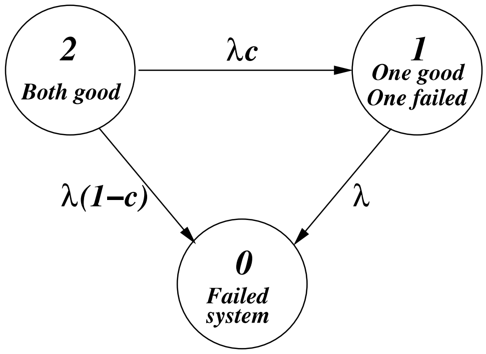

FIGURE 2.15 The Markov model for the duplex system with an inactive spare.

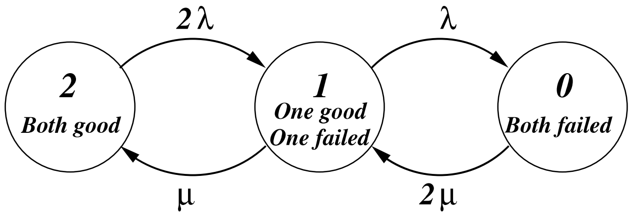

FIGURE 2.16 The Markov model for a duplex system with repair.

This expression can also be derived based on combinatorial arguments. The derivation is left to the reader as an exercise.

Our next example of a duplex system that can be analyzed using a Markov model is a system with two active processors, each with a constant failure rate of λ and a constant repair rate of μ. The Markov model for this system is depicted in Figure 2.16.

As in the previous example, the state is the number of good processors. The differential equations describing this Markov chain are

$$\begin{aligned}
\frac{\mathrm{d}P_2(t)}{\mathrm{d}t} &= -2\lambda P_2(t) + \mu P_1(t) \\
\frac{\mathrm{d}P_1(t)}{\mathrm{d}t} &= 2\lambda P_2(t) + 2\mu P_0(t) - (\lambda + \mu)P_1(t) \\
\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} &= \lambda P_1(t) - 2\mu P_0(t)
\end{aligned} \tag{2.36}$$

Solving 2.36 with the initial conditions $P_2(0) = 1$, $P_1(0) = P_0(0) = 0$ yields

$$P_2(t) = \frac{\mu^2}{(\lambda + \mu)^2} + \frac{2\lambda\mu}{(\lambda + \mu)^2}\mathrm{e}^{-(\lambda+\mu)t} + \frac{\lambda^2}{(\lambda + \mu)^2}\mathrm{e}^{-2(\lambda+\mu)t}$$

$$P_1(t) = \frac{2\lambda\mu}{(\lambda+\mu)^2} + \frac{2\lambda(\lambda-\mu)}{(\lambda+\mu)^2}\mathrm{e}^{-(\lambda+\mu)t} - \frac{2\lambda^2}{(\lambda+\mu)^2}\mathrm{e}^{-2(\lambda+\mu)t}$$
$$P_0(t) = 1 - P_1(t) - P_2(t) \tag{2.37}$$

Note that we solve only for $P_1(t)$ and $P_2(t)$; using the boundary condition that the probabilities must sum up to 1 (for every t) gives us $P_0(t)$ and reduces by one the number of differential equations to be solved.

Note also that this system does not fail completely; it is not operational while at state 0 but is then repaired and goes back into operation. For a system with repair, calculating the availability is more meaningful than calculating the reliability. The (point) availability, or the probability that the system is operational at time t, is

$$A(t) = P_1(t) + P_2(t)$$

The reliability $R(t)$, on the other hand, is the probability that the system never enters state 0 at any time during $[0,t]$ and cannot be obtained out of the above expressions. To obtain this probability, we must modify the Markov chain slightly by removing the transition out of state 0, so that state 0 becomes an *absorbing* state. This way, the probability of ever entering the state in the interval $[0,t]$ is reduced to the probability of being in state 0 at time t. This probability can be found by writing out the differential equations for this new Markov chain, solving them, and calculating the reliability as $R(t) = 1 - P_0(t)$.

Since in most applications processors are repaired when they become faulty, the long-run availability of the system, A, is a more relevant measure than the reliability. To this end, we need to calculate the long-run probabilities, $P_2(\infty)$, $P_1(\infty)$, and $P_0(\infty)$. These can be obtained either from Equation 2.37 by letting t approach ∞ or from Equation 2.36 by setting all the derivatives $\frac{\mathrm{d}P_i(t)}{\mathrm{d}t}$ $(i = 0,1,2)$ to 0 and using the relationship $P_2(\infty) + P_1(\infty) + P_0(\infty) = 1$. The availability in the long-run, A, is then

$$A = P_2(\infty) + P_1(\infty) = \frac{\mu^2}{(\lambda+\mu)^2} + \frac{2\lambda\mu}{(\lambda+\mu)^2} = \frac{\mu(\mu+2\lambda)}{(\lambda+\mu)^2} = 1 - \left(\frac{\lambda}{\lambda+\mu}\right)^2$$

2.5 Fault-Tolerance Processor-Level Techniques

All the resilient structures described so far can be applied to a wide range of modules, from very simple combinatorial logic modules to the most complex microprocessors or even complete processor boards. Still, duplicating complete processors that are not used for critical applications introduces a prohibitively large overhead and is not justified. For such cases, simpler techniques with much smaller overheads have been developed. These techniques rely on the fact that processors execute stored programs and upon an error, the program (or part of it) can be re-executed as long as the following two conditions are satisfied: the error is