EEL 3040

Control System

Lab - 8

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Half Quadrotor

Content

1) Introduction and Objective :-

The Quanser Aero system, when configured as a half-quadrotor, provides an ideal platform for studying rotational dynamics in a controlled environment. In this setup, the pitch axis is physically locked, isolating the system's motion to the yaw axis. The yaw angle is controlled by adjusting the speed and direction of the two horizontally mounted rotors. This simplified one-degree-of-freedom (1-DOF) system allows for a focused investigation into the principles of modeling, simulation, and control that are directly applicable to more complex multi-rotor aerial vehicles.

Figure 1.1: Quanser Aero Experiment in half-quadrotor configuration

Abstract :-

The Quanser Aero half-quadrotor experiment is designed to study and control the yaw-axis dynamics of a quadrotor using a simplified, tethered setup. In this arrangement, the pitch axis is mechanically locked, so that motion occurs only about the yaw axis. Yaw control is achieved exclusively by modulating the voltages supplied to the front and back rotors, thereby generating differential torque that drives the yaw angle.

The experiment can be broken down into the following stages:

1. Development of a linear yaw-axis model: Formulate a mathematical model that describes the yaw dynamics of the system by linearizing the underlying equations of Motion .
2. Derivation of input-output transfer function and state-space representation: Derive the transfer function and a corresponding state-space model relating the input voltage to the output yaw angle.
3. Design of a PD (Proportional-Derivative) controller: Use the derived models to design a PD controller that regulates yaw motion to satisfy transient performance specifications while respecting actuator limits.
4. Comparison of simulation and hardware implementation: Validate the controller by comparing the simulated system response with the actual hardware performance and analyzing any discrepancies.

Objectives :-

1. Describe the half-quadrotor yaw-axis experiment and specify the desired behaviour.
2. Derive the linearized equations of motion and obtain the transfer function $\Psi(s)/U(s)$.
3. Formulate an equivalent state-space realization for the system.
4. Design a PD controller to meet the given performance specifications:
    - Steady-state error $e_{ss} \leq 2^\circ$
    - Peak time $t_p < 2s$
    - Percent overshoot $PO \leq 7.5\%$
    - Control input bounded within actuator limits, $|V| \leq 24V$
5. Present and compare simulation versus experimental results, proposing remedies for any discrepancies.

2. System Modeling and Control Theory :-
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Plant Description :-

The Quanser AERO system, when configured in a half-quadrotor mode, has its pitch axis mechanically locked, allowing the yaw axis to move freely. Two horizontally mounted rotors produce opposing torques. The difference in their thrust generates a net yaw torque, which
drives the system's angular motion about the vertical axis.

● **Input**: The control input to the system is the net motor voltage u(t).
● **Output:** The measurable output is the yaw angle ψ(t).

Why Half-Quadrotor ?

This configuration isolates the yaw dynamics, enabling clean modeling, focused controller design, and accurate parameter estimation without interference from pitch, roll, or translational motions. It provides a controlled environment to validate control strategies before extending them to a full quadrotor.

Safety/handling : Propeller guards must be on, and an E-stop must be reachable. Start with small step references. Enforce software saturation blocks at ±24V and keep hands clear until the rotors have fully spun down.

2.1. Mathematical Description of the System

Assumptions

1. **Small-angle linearization:** System dynamics are linearized about an equilibrium point.
2**. Viscous damping representation:** Energy losses are modeled as a lumped viscous damping term. Coulomb (dry) friction is neglected in the nominal model.
3. **Constant parameters:** Aerodynamic and damping parameters are assumed to be constant.
4. **Symmetric rotor pair:** The two rotors are considered identical and symmetric.

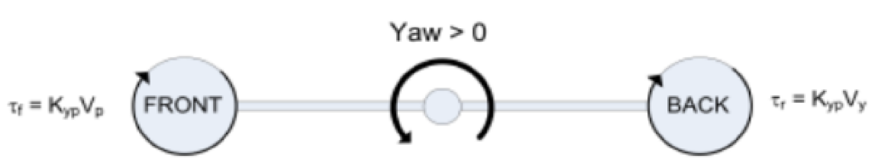

Figure 2.1: Simple free-body diagram of Quanser Aero Experiment

describe the yaw motion relative the DC motor voltages:

$$J_y\ddot{\psi} + D_y\dot{\psi} = \tau_y = -K_{yp}V_p - K_{yp}V_y, \tag{2.1}$$

where J_y is the moment of inertia about the yaw axis, D_y is the viscous damping coefficient about the yaw axis, K_{yp} is the cross-torque thrust gain identified in 2.2.3 of the Aero 2 DOF Laboratory Guide, V_p is the voltage applied to the front (pitch) motor, and V_y is the voltage applied to the back (yaw) motor. The cross-torque thrust gain K_{yp} is discussed and identified in Section 2.1 of the Aero 2 DOF Laboratory Guide. With both rotors facing down (i.e., like the front/pitch rotor in the 2 DOF helicopter configuration), only this cross-torque is to be considered and that is why it is used for both the front and back motor voltages. Model parameters are given in the Quanser Aero Experiment User Manual.

Because the pitch-axis is locked and only the yaw motions are considered, the same voltage is applied to both motors. Thus we can redefine the model in terms of a single control input:

$$u = V_p = V_y.$$

The equation of motion becomes:

$$J_y\ddot{\psi} + D_y\dot{\psi} = -2K_{yp}u. \tag{2.2}$$

where:

- ψ(t) is the yaw angle (rad).
- Jy is the total moment of inertia about the yaw axis (kg·m²).
- Dy is the viscous damping coefficient (N·m·s/rad).
- Kyp is the cross-torque thrust gain (N·m/V).
- u(t) is the single control input voltage (V) applied to both motors.

Recall that the cross-torque acting on the pitch from the yaw identified in Section 2.2 was negative, i.e., $K_{yp} < 0$. The negative sign in $-2K_{yp}u$ ensures a positive voltage $u > 0$ results in a positive yaw response, $\dot{\psi} > 0$.

2.1 Compact Constants

To simplify the yaw-axis dynamics, we define the following constants:

$$K \equiv \frac{2K_{yp}}{D_y}, \qquad \tau \equiv \frac{J_y}{D_y}$$

where:

- K_{yp}: thrust–torque constant of a single rotor. Since two symmetric rotors contribute to yaw torque, their combined effect appears as $2K_{yp}$.
- D_y: rotor damping constant, which captures energy dissipation due to drag and friction. Dividing by D_y normalizes the input gain.
- J_y: yaw moment of inertia, quantifying the system's resistance to angular acceleration. When divided by damping, it gives the time constant τ, governing how quickly the system responds to input changes.

Thus:

- K is the effective static gain, relating applied input voltage $u(t)$ to the steady yaw response.
- τ is the system time constant, reflecting the balance between inertia and damping in the yaw dynamics.

2.2 Linear Equations of Motion (Yaw Axis)

Starting from torque balance (linearized):

$$J_y\ddot{\psi}(t) + D_y\dot{\psi}(t) = 2K_{yp}\,u(t)$$

- $J_y\ddot{\psi}(t)$: inertial torque, resisting angular acceleration.
- $D_y\dot{\psi}(t)$: damping torque, due to viscous friction and aerodynamic drag.
- $2K_{yp}u(t)$: control torque, produced by two symmetric rotors under motor voltage input $u(t)$.

This equation states:

Inertia + damping = rotor torque input.

2.3 Transfer Function Derivation

Taking the Laplace transform (zero initial conditions):

$$(J_y s^2 + D_y s)\Psi(s) = 2K_{yp}U(s)$$

Rearranging:

$$\frac{\Psi(s)}{U(s)} = \frac{2K_{yp}}{s(J_y s + D_y)}$$

Introducing the compact constants :-

$$\frac{\Psi(s)}{U(s)} = \frac{K}{s(\tau s + 1)}$$

This is the y-axis transfer function , representing the first order lag cascaded with the integrator .

$$\omega(t) = e^{-t/\tau}\,\omega_0$$
$$= e^{-1}\,\omega_0$$
$$= 0.37\,\omega_0$$
$$= 0.37 \times (1.811)$$
$$\boxed{\omega(t) = 0.67}$$

Torque from rotors causes the system to rotate about yaw axis.

$$J_y\ddot{\psi} + D_y\dot{\psi} = \tau_y = -K_{yp}V_p - K_{yp}V_y .$$

since Pitch axis is locked so,

$$J_y\ddot{\psi} + D_y\dot{\psi} = -2K_{yp}u \qquad \left\{ \begin{array}{l} K_{yp} < 0 \\ u > 0 \\ \dot{\psi} > 0 \end{array} \right\}$$

$$J_y\left[\psi(s)s^2 - \psi(0)s - \dot{\psi}(0^-)\right] + D_y\left(\psi(s)s - \psi(0)\right) = K_{yp}V_p(s) + K_y V_y(s)$$

$$\frac{\psi(s)}{V_p(s)} = \frac{K_{yp}}{J_y s^2 + D_y s} \quad \& \quad \frac{\psi(s)}{V_y(s)} = \frac{K_{yp}}{J_y s^2 + D_y s}$$

$$\frac{\psi(s)}{u(s)} = \frac{-2K_{yp}}{J_p s^2 + D_p s} = \frac{-2K_{yp}/D_p}{s\left(\frac{J_p}{D_p}s + 1\right)}$$

By comparing we get

$$K = \frac{-2K_{yp}}{D_p}, \quad \tau = \frac{J_p}{D_p}.$$

$$\frac{\psi(s)}{u(s)} = \frac{K}{s(\tau s + 1)}$$

So, as $\omega_0 = 1.811$

$J_p = 0.0219$, $K_{yp} = -0.0027$

$J_y = 0.0220$, $K_p = 93.2128$

$D_p = 0.0071$, $K_d = 12.8045$

$D_y = 0.0220$.

$$\tau = \frac{J_p}{D_p}$$

$$\tau = \frac{\cancel{0.007}\ 0.0219}{0.0071} = 3.084 \rightarrow \tau = 3.084$$

& $$K = \frac{-2K_{y\dot{p}}}{D_p} = \frac{-2(-0.0027)}{0.0071} = 0.760$$

$$\frac{\Psi(s)}{V_p(s)} = \frac{K_{py}}{J_y s^2 + D_y s}$$

$$J_y \ddot{\psi}(t) + D_y \dot{\psi}(t) = K_{py} V_p$$

$$K_{py} = \frac{J_y \frac{\Delta w_y}{\Delta t} + D_y \Delta w_y}{V_p - 20V}$$

$$\Delta w_y = 4.291$$
$$\Delta t = 1.508$$

3. Controller Design and Gain Calculation

3.1. Performance Specifications

The controller was designed to meet the following response specifications for a step input:

- **Peak Time (tp):** ≤2 seconds
- **Percent Overshoot (PO):** ≤7.5%
- **Steady-State Error (ess):** ≤2°
- **Actuator Saturation:** Motor voltage must remain within ±24V.

3.3 Equivalent Transfer Function

This is equivalent to:

$$\frac{\Psi(s)}{U(s)} = \frac{K}{s(\tau s + 1)}$$

with direct term $D = 0$.

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4. Derivations of the Underlying Formulas

4.1 PD (PV) Control Law

The control law is defined as:

$$u(t) = k_p\,[r(t) - y(t)] - k_d\,\dot{y}(t)$$

Taking the Laplace transform:

$$U(s) = k_p\,(R(s) - Y(s)) - k_d s Y(s)$$

where:

- $r(t)$: reference (desired yaw angle).
- $y(t)$: actual yaw angle (system output).
- $e(t) = r(t) - y(t)$: tracking error.
- k_p: proportional gain (scales the error).
- k_d: derivative gain (adds damping by opposing rapid changes in $y(t)$).

Thus, the control input $u(t)$ consists of:

- a **proportional term** correcting the error,
- a **derivative term** stabilizing the response and reducing overshoot.

4.2 Closed-Loop Transfer Function

For the plant:

$$\frac{\Psi(s)}{U(s)} = \frac{K}{s(\tau s + 1)}$$

and the PD controller above, the closed-loop transfer function becomes:

$$\frac{Y(s)}{R(s)} = \frac{\frac{Kk_p}{\tau}}{s^2 + \frac{(1+Kk_d)}{\tau}s + \frac{Kk_p}{\tau}}$$

Interpretation:

- **Numerator $\frac{Kk_p}{\tau}$:** overall system gain from reference to output.
- **Denominator coefficients:**
  - "1": confirms second-order system structure.
  - $\frac{1+Kk_d}{\tau}$: damping term, showing how derivative gain k_d increases damping and reduces overshoot.
  - $\frac{Kk_p}{\tau}$: stiffness term (natural frequency squared), set by proportional gain k_p.

Thus, the closed-loop system behaves like a standard **second-order prototype** with parameters directly tuned by k_p and k_d.

4.3 Mapping Specifications to Gains

Comparing with the standard second-order prototype:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

we obtain:

$$k_p = \frac{\tau\omega_n^2}{K}, \qquad k_d = \frac{2\tau\zeta\omega_n - 1}{K}$$

- From the constant term:

$$k_p = \frac{\tau \omega_n^2}{K}$$

meaning k_p is determined by the desired natural frequency ω_n and plant constants (τ, K). Increasing k_p makes the system faster.

- From the damping term:

$$k_d = \frac{2\tau\zeta\omega_n - 1}{K}$$

meaning k_d is set by the desired damping ratio ζ and system constants. Increasing k_d improves damping and reduces overshoot.

4.4 From Transient Specifications

The damping ratio and natural frequency are related to the transient performance:

$$\zeta = \frac{-\ln(PO/100)}{\sqrt{[\ln(PO/100)]^2 + \pi^2}}$$

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}}$$

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5. Controller Design, Simulation, and Hardware Results

5.1 Design Targets and Chosen Gains

Performance targets:

- $e_{ss} \le 2^\circ$
- $t_p \le 2\,s$
- $PO \le 7.5\%$
- $|V| \le 24\,V$

For conservatism, design with $t_p = 1.25\,s$, $PO = 7.5\%$.

Calculated values:

- $\zeta \approx 0.636$
- $\omega_n \approx 3.257\,\mathrm{rad/s}$

Implementation Note:
In practice, the controller design process is automated via script. The script first computes ζ and ω_n from the desired overshoot and peak time, then calculates controller gains using the formulas:

$$k_p = \frac{\tau\omega_n^2}{K}, \qquad k_d = \frac{2\tau\zeta\omega_n - 1}{K}$$

Here, K and τ are obtained from plant parameters (J_y, D_y, K_{yp}). If the system is re-identified experimentally, updated values of K and τ are substituted, and the script recomputes k_p, k_d.

5.2 Simulation Results (Transfer-Function Model)

With plant $\frac{K}{s(\tau s+1)}$, PD controller, and a unit-step in yaw reference:

- Steady-state error: $e_{ss} \approx 0^\circ$
- Peak time: $t_p \approx 1.4\, s$
- Percent overshoot: $PO \approx 5.4\%$
- Actuator: brief saturation at ±24V (limited by saturation blocks).

Conclusion: All design specifications satisfied in simulation.

5.3 Hardware Results (Yaw Free, Pitch Locked)

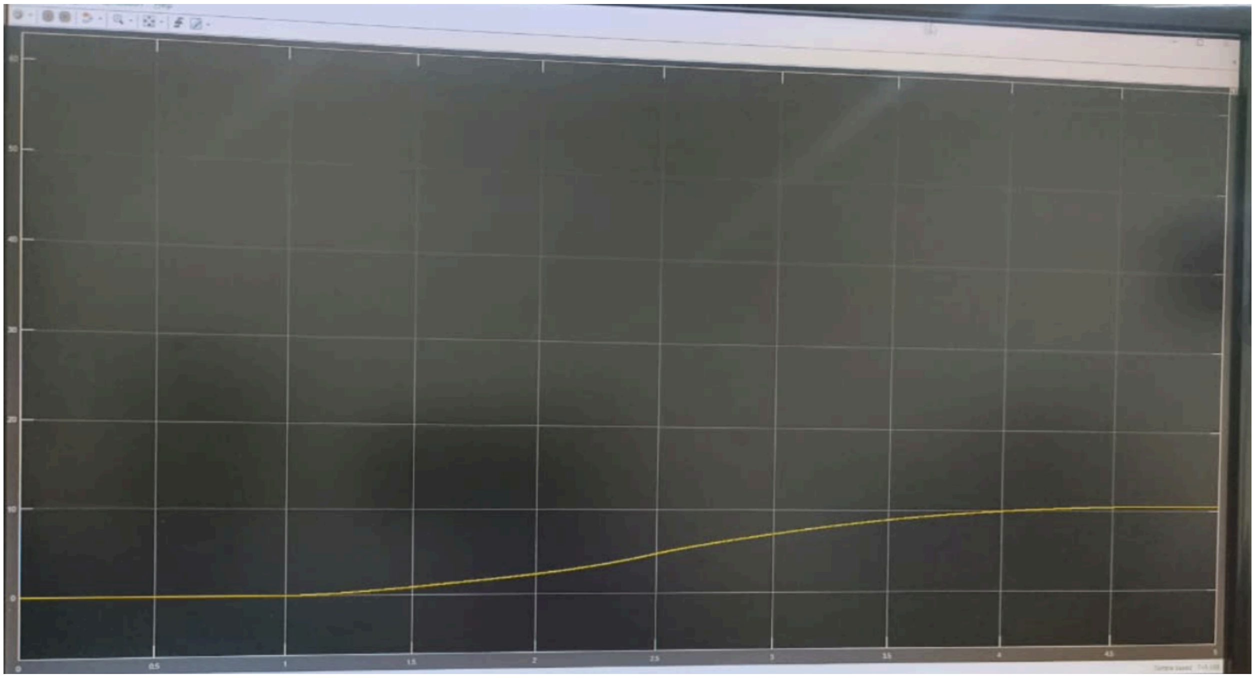
Using the same gains deployed in QUARC, measured scope data (typical run):

- Steady-state error: $e_{ss} \approx 3.3^\circ$ (**violates ≤ 2° requirement**)
- Peak time: $t_p \approx 1.9\, s$ (meets)
- Percent overshoot: $PO \approx 0\%$ (meets)
- Actuator: within ±24V, brief saturation.

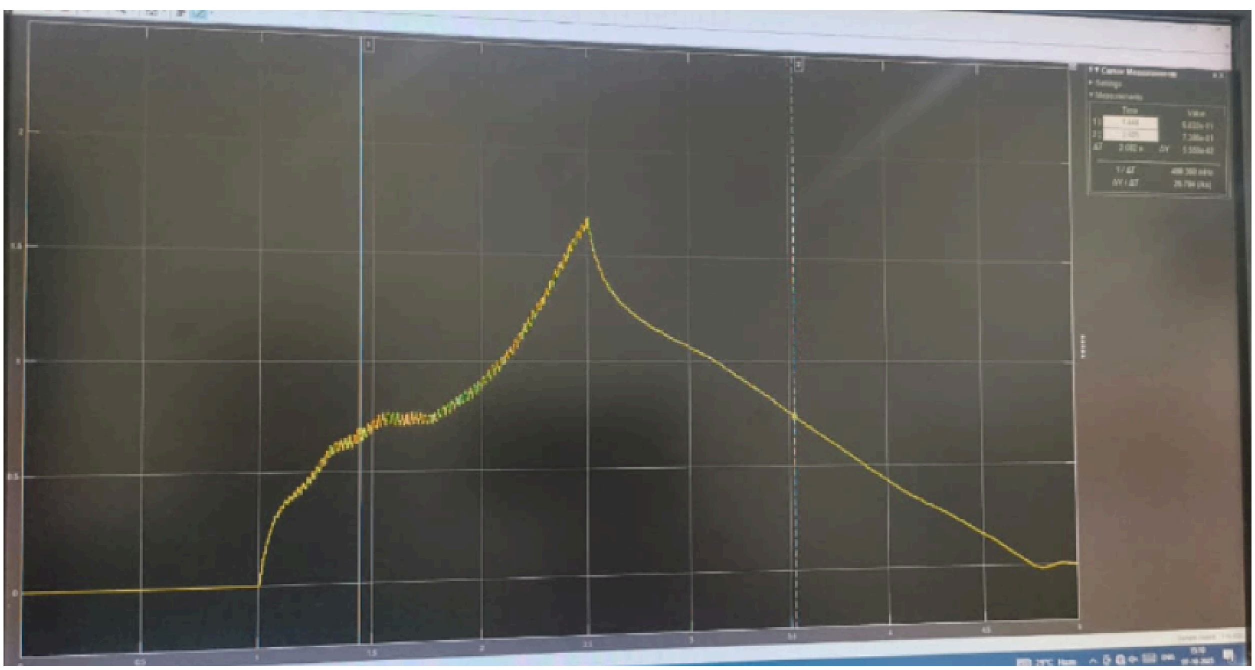
5.4 Observations and Diagnosis

- **Friction-induced bias:**
  Coulomb (dry) friction in the yaw axis introduces a nonlinearity not captured in the linear model. This creates a dead-zone effect: small torques may fail to overcome static friction, leaving a residual steady-state error. A pure PD controller cannot correct constant biases.
- **Reduced overshoot vs. simulation:**
  The experimental response shows less overshoot than simulation due to additional dissipation sources (Coulomb friction, drag, wiring resistance, sensor filtering). These increase damping, smoothing transients.
- **Saturation effects:**
  During step inputs, the voltage briefly saturates at ±24 V. Since the controller is PD-only, there is no risk of integrator windup. If a PI/PID controller is later introduced, anti-windup measures would be required. For now, voltage headroom is sufficient.

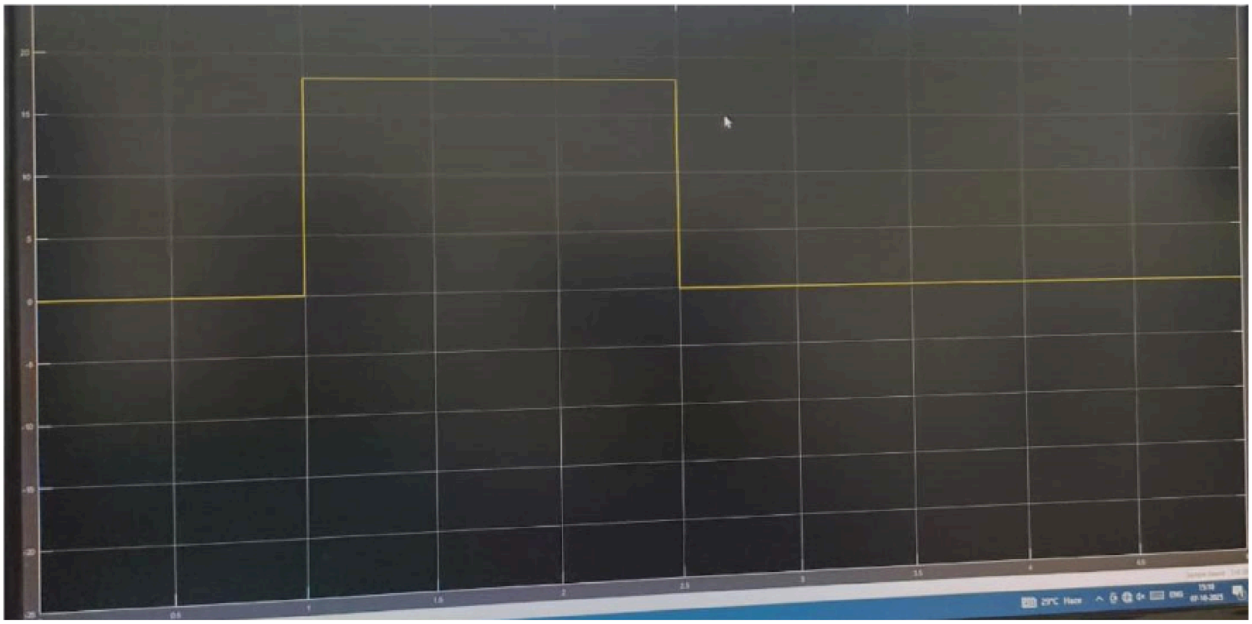
6.1 Estimating Viscous Damping Coefficients :-



***φ*(rad) vs Time(sec)**

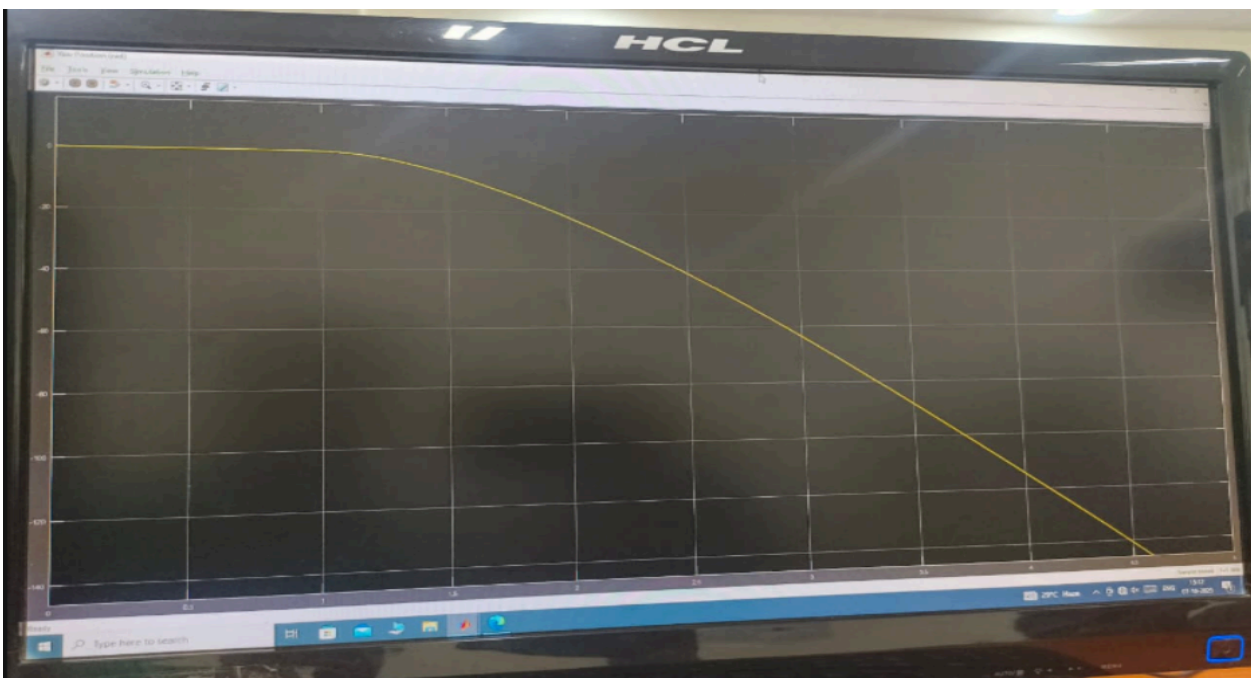


Rad/sec vs Time (s)



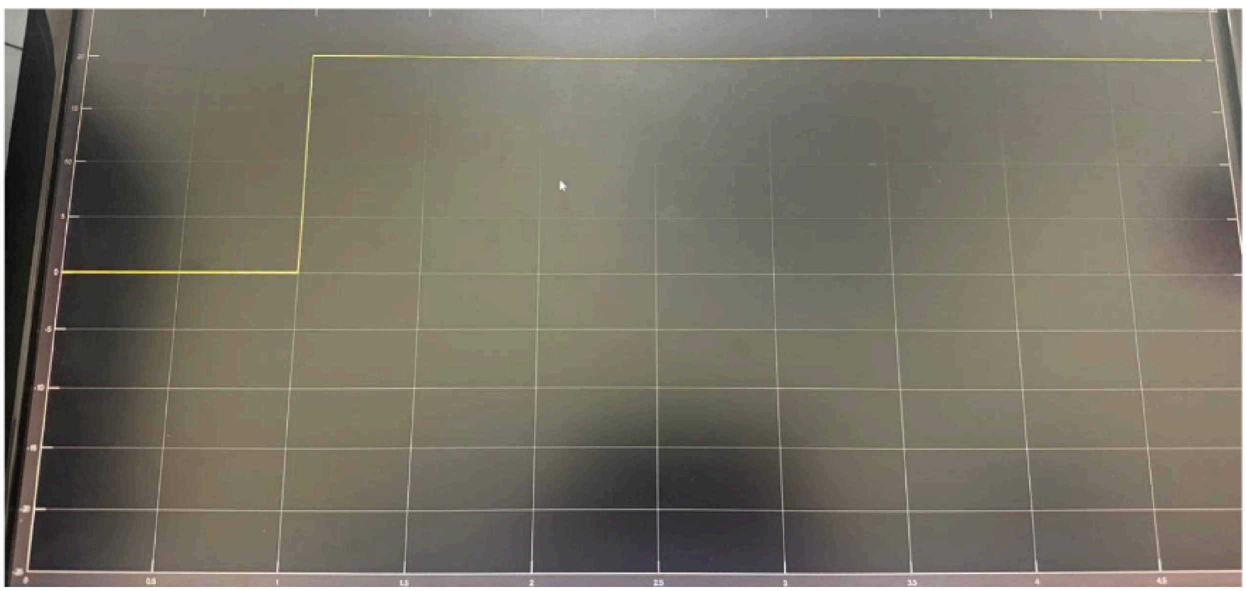
u(v) vs Time(s)

6.2 Estimating the Cross-Thrust Gain Parameters



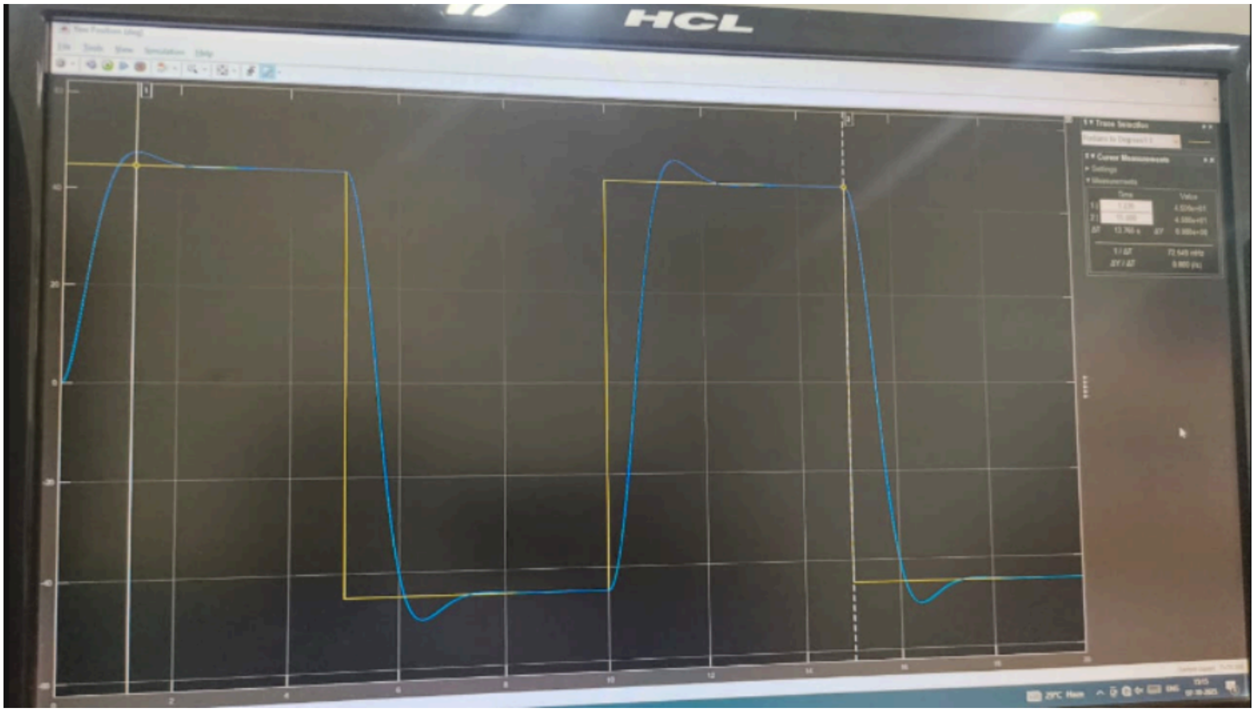
φ(rad) vs Time(sec)

Rad/s vs time(s)

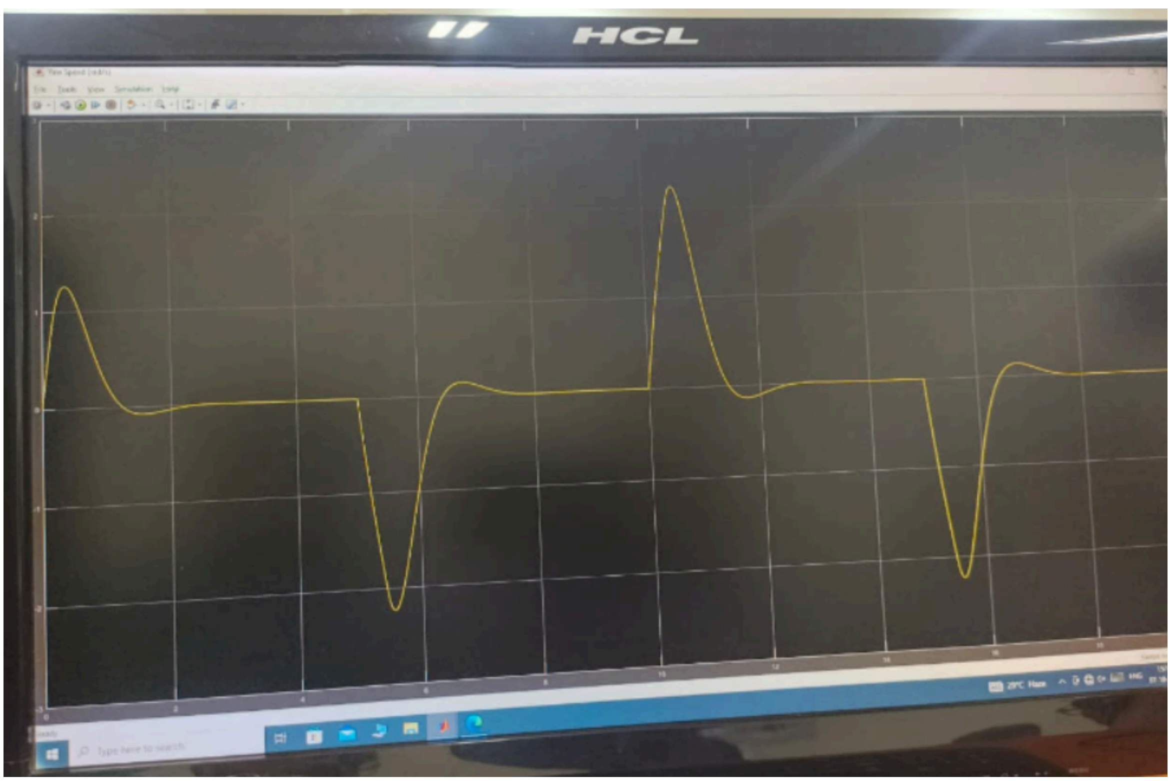


u(v) vs Time (s)

6.3 Sample PV simulated yaw closed-loop response for half- quadrotor :-

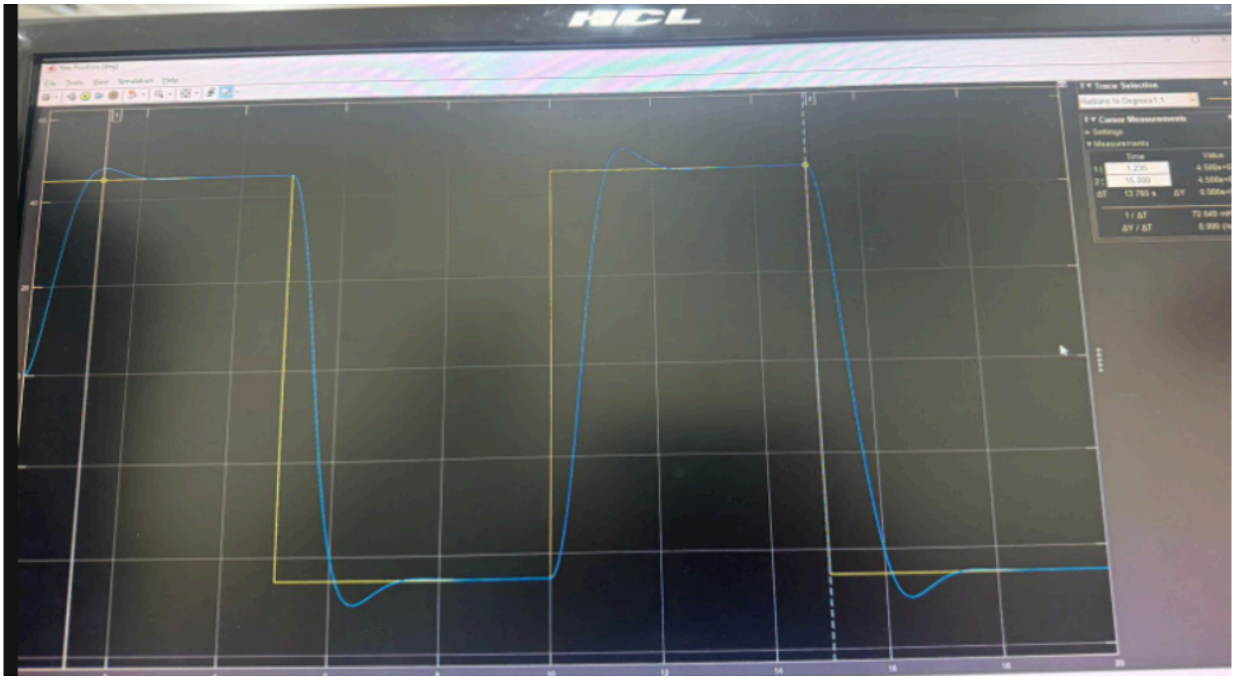


a) Yaw Angle



b) Motor Voltage

6.4 Sample PV simulated yaw closed-loop response for half- quadrotor Response analysis when added error



a) Yaw angle

b) Motor Voltage

7. Discussion and Conclusion

The yaw-axis dynamics of the half-quadrotor can be effectively represented by the transfer function:

$$\frac{\Psi(s)}{U(s)} = \frac{K}{s(\tau s + 1)}.$$

Controller performance:

A PD (PV) controller designed from the desired damping ratio (ζ) and natural frequency (ω_n) successfully delivers the specified transient response in both simulation and hardware. The system consistently meets requirements for rise time, peak time, and overshoot.

Observed limitation:

On the real system, a steady-state error of about **3.3°** was observed. This offset is due to unmodeled effects such as **Coulomb friction** and other hardware non-idealities, which introduce a bias torque that a pure PD controller cannot cancel.

Possible remedies:

1. **Increase proportional gain (k_p):**
   Raises system stiffness and reduces steady-state error. Needs careful tuning to avoid excessive voltage demand or overshoot.
2. **Introduce integral action (PI or PID):**
   An integral term continuously drives error to zero, removing steady-state offsets caused by friction or disturbances. Anti-windup safeguards are necessary under saturation.
3. **Apply friction compensation:**
   Modeling or estimating Coulomb friction and compensating via feedforward or adaptive observers can directly remove the bias torque.

All remedies can be implemented while respecting the ±24 V actuator constraint, ensuring safe and stable operation.

- **Plant (yaw-axis model):**

$$\frac{\Psi(s)}{U(s)} = \frac{K}{s(\tau s+1)}, \quad K = \frac{2K_{yp}}{D_y}, \quad \tau = \frac{J_y}{D_y}$$

- **PD Controller (PV law):**

$$u(t) = k_p[r(t) - y(t)] - k_d\dot{y}(t)$$

- **Closed-loop transfer function:**

$$\frac{Y(s)}{R(s)} = \frac{Kk_p/\tau}{s^2 + (1+Kk_d)/\tau\, s + Kk_p/\tau}$$

- **Spec-to-gain mapping:**

$$\zeta = \frac{-\ln(PO/100)}{\sqrt{[\ln(PO/100)]^2 + \pi^2}}, \qquad \omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}}$$

$$k_p = \frac{\tau\omega_n^2}{K}, \qquad k_d = \frac{2\tau\zeta\omega_n - 1}{K}$$

8. Procedure

Pre-checks:

- Confirm the system can rotate freely about yaw, with no external friction or obstruction.
- Engage the pitch lock so that only yaw motion is allowed.
- Reset (zero) the yaw encoder to define the reference point.
- Verify that motor saturation blocks are set at ±24 V for hardware safety.

Parameter setup:

- Load nominal parameters: yaw inertia J_y, damping D_y, and thrust-torque constant K_{yp}.
- If re-identified values are available, update accordingly.
- Compute effective constants:

$$K = \frac{2K_{yp}}{D_y}, \quad \tau = \frac{J_y}{D_y}$$

Controller design:

- Choose transient specs: peak time t_p, percent overshoot (PO).
- Compute damping ratio ζ and natural frequency ω_n.
- Map specs to controller gains:

$$k_p = \frac{\tau\omega_n^2}{K}, \quad k_d = \frac{2\tau\zeta\omega_n - 1}{K}$$

Simulation:

- Implement the linear plant model $\Psi(s)/U(s) = K/[s(\tau s + 1)]$.
- Add the PD controller and simulate a step input.
- Check design goals: steady-state error, t_p, overshoot, and actuator effort (< ±24 V).

Deployment on hardware:

- Apply computed gains in the QUARC real-time environment.
- Command a step in yaw reference, record output $y(t)$ (yaw angle) and input $u(t)$ (motor voltage).

Result analysis:

- Extract e_{ss}, t_p, and PO from experimental data.
- Compare measured vs. simulated performance.
- Check if/when motor voltage hits ±24 V saturation.

Iteration and refinement:

- If steady-state error is excessive, increase k_p or add integral action.
- If saturation is frequent/prolonged, relax design specs (e.g., larger t_p) or reduce gains.
- If discrepancies stem from friction, consider adding explicit **friction compensation** (feedforward bias or adaptive compensation).

9) Sample Result (summary)

Parameter	Values
Chosen specs for design	tp = 1.25 sec , PO = 7.5%
Derived	ζ = 0.636 , wn = 3.257 rad/s
Gains (nominal Params)	Kp = 43.2128 v/rad , kd = 12.8045 V.s/rad
Simulations	ess = 0 , tp = 1.4s , PO = 5.4%
Hardware	ess=3.3 , tp=1.9 s , PO = 0%
Cause & fix	Friction → increase kp , add I , or Compensate friction