EEL 3040

Control System

Lab - 9

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Inverted Pendulum

Content

1) **Introduction** :-

a) **Objective** :-

The primary objective of this experiment is to design and implement a state-feedback control system to successfully balance a rotary pendulum in its inherently unstable upright vertical position. This involves deriving the system's mathematical model, linearizing it around the equilibrium point, and using pole placement techniques to design a stabilizing controller. A secondary objective is to implement a hybrid controller that combines a non-linear energy-based swing-up strategy with the linear balancing controller.

b) **Apparatus and Software** :-

The following equipment and software were used for this experiment:

- Quanser SRV02-ET Rotary Servo
- Quanser Rotary Pendulum Module
- Quanser VoltPAQ Power Amplifier
- Data Acquisition (DAQ) Card
- MATLAB/Simulink Software

c) **System Modeling and Theoretical Background** :-

To design a linear controller, the dynamics of the rotary inverted pendulum must first be modeled and then linearized around the desired equilibrium point (the upright position).

Model Convention :-

The rotary inverted pendulum model is shown below. The rotary arm is actuated by the servo motor, and the pendulum is free to rotate at the end of the arm. The arm angle is denoted by θ and the pendulum angle by α.

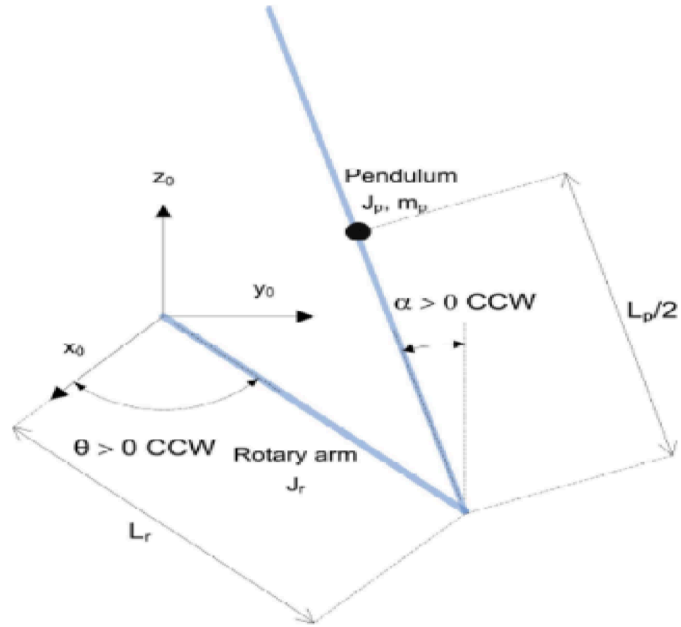

Figure 2.1: Rotary inverted pendulum conventions

Nonlinear Equations of Motion :-

Using the Euler-Lagrange method, the nonlinear equations of motion (EOMs) that describe the coupled dynamics of the system are found to be :-

For the rotary arm:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Where the state vector is defined as

$$x^\top = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}$$

$$y^\top = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

The system matrices are:

Description	Value
State-Space Matrix A	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 80.3 & -45.8 & -0.930 \\ 0 & 122 & -44.1 & -1.40 \end{bmatrix}$
State-Space Matrix B	$\begin{bmatrix} 0 \\ 0 \\ 83.4 \\ 80.3 \end{bmatrix}$
State-Space Matrix C	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
State-Space Matrix D	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Open-loop poles	{-48.42, 7.06, -5.86, and 0 }

4. Experiment 1: Balance Control

AIM :-

To design and implement a state-feedback controller using the pole placement method to balance the pendulum in its upright vertical position.

Control Design and Analysis :-

1. Stability Analysis:- The stability of the open-loop system is determined by the eigenvalues (poles) of the state matrix . The poles are calculated to be {-48.42, 7.06,-5.86, 0} . The presence of a pole in the right-half plane (+7.06) confirms that the open-loop system is unstable, as expected.

2. Controllability:- For pole placement to be possible, the system must be controllable. This is verified by constructing the controllability matrix and confirming that its rank is equal to the number of states (n = 4). The system was found to be fully controllable.

3. Pole Placement:- The goal is to design a state-feedback control law that moves the unstable open-loop poles to desired stable locations. The desired locations are chosen based on the specified performance criteria:

- Damping ratio: zeta = 0.7
- Natural frequency: (omega)n = 4 rad/s

This gives a dominant pole pair at . Two additional non-dominant poles are placed further into the left-half plane at and to ensure their response decays quickly. The final state-feedback gain matrix is calculated to place the closed-loop poles at these desired locations.

MatLab Code

```
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/MATLAB Drive/lab8_code.m

clc;
clear;
close all;

%% System Matrices
A = [0 0 1 0;
     0 0 0 1;
     0 80.3 -45.8 -0.930;
     0 122 -44.1 -1.40];
B = [0; 0; 83.4; 80.3];


% Compute [B AB A^2B A^3B]
M = [B, A*B, A^2*B, A^3*B];

% Display the result
disp('Matrix [B  AB  A^2B  A^3B] =');
disp(M);

%% Dimensions
dim_A = size(A,1);
dim_B = size(B,1);

%% Eigenvalues of A
A_eig = eig(A);

%% Controllability Check
Qc = ctrb(A,B); % Controllability matrix
k = rank(Qc);   % Rank of controllability matrix

%% Desired Closed-Loop Poles
poles_Desired = [-2.8+2.86j, -2.8-2.86j, -30, -40];
char_poly_desired = poly(poles_Desired); % Characteristic polynomial
char_poly_desired_flp = fliplr(char_poly_desired); % Flip for controller computation

%% System Characteristic Polynomial
char_poly_Sys = poly(A_eig);
char_poly_Sys2 = char_poly_Sys(:,2:5); % Remove leading 1

%% Companion (Controllable Canonical) Form
A_bar = [0 1 0 0;+
         0 0 1 0;
         0 0 0 1;
         -1*flip(char_poly_Sys2)];
B_bar = [0; 0; 0; 1];

```

```matlab
%% Difference Matrix for Controller Gain Computation
Diff_matrix = A_bar - B_bar;
updated_char_poly = -1 * Diff_matrix(dim_A,:);
updated_char_poly(1) = 0;

%% Compute K_bar
K_bar = zeros(size(char_poly_desired));
for i = 1:length(char_poly_desired)-1
    K_bar(i) = char_poly_desired_flp(i) + updated_char_poly(i);
end
K_bar = K_bar(1:end-1);

%% Controllability for Companion Form
Qc_bar = [B_bar, A_bar*B_bar, A_bar^2*B_bar, A_bar^3*B_bar];
Qc_bar_Inverse = inv(Qc_bar);

%% Transformation to Original System
W = Qc * Qc_bar_Inverse;

% Note: K_bar is manually set here
K_bar = [19200, 9843, 1707, 28.4];
W_inverse = inv(W);
K = K_bar * W_inverse;

%% Display Results
disp('Controllability Matrix Qc:');
disp(Qc);

disp('Desired Poles:');
disp(poles_Desired);

disp('Controllability Matrix (for companion matrices A_bar & B_bar) Qc_bar:');
disp(Qc_bar);

disp('Transformation Matrix W:');
disp(W);

disp('State Feedback Gain K:');
disp(K);
```

OutPut :-

```
Controllability Matrix Qc:
   1.0e+06 *

         0    0.0001   -0.0039    0.1883
         0    0.0001   -0.0038    0.1868
    0.0001   -0.0039    0.1883   -9.1039
    0.0001   -0.0038    0.1868   -9.0297

Desired Poles:
  -2.8000 + 2.8600i  -2.8000 - 2.8600i -30.0000 + 0.0000i -40.0000 + 0.0000i

Controllability Matrix (for companion matrices A_bar & B_bar) Qc_bar:
   1.0e+05 *

         0         0         0    0.0000
         0         0    0.0000   -0.0005
         0    0.0000   -0.0005    0.0233
    0.0000   -0.0005    0.0233   -1.1244

Transformation Matrix W:
   1.0e+03 *

   -3.7267    0.0421    0.0834   -0.0000
   -0.0000   -0.0002    0.0803   -0.0000
    0.0000   -3.7267    0.0421    0.0834
    0.0000   -0.0000   -0.0002    0.0803

State Feedback Gain K:
   -5.1520   28.0319   -2.7009    3.1588
```

Pre-Lab Questions :-

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Case ① : Balance Control →

Pre-lab question.

① The desired Poles are :-

$\{-48.42, +7.06, -5.86, 0\}$

The system has one Pole in the right Half Plane at +7.06, which confirms that the open-loop system is unstable.

② The characteristic eqn of A is -

$|SI - A| = 0$

Where,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 80.3 & -45.8 & -0.930 \\ 0 & 122 & -44.1 & -1.40 \end{bmatrix}$$

$$|SI - A| = \begin{vmatrix} S & 0 & -1 & 0 \\ 0 & S & 0 & -1 \\ 0 & -80.3 & S+45.8 & 0.930 \\ 0 & -122 & 44.1 & S+1.40 \end{vmatrix} = 0$$

By solving above -

$$S^4 + 47.2S^3 - 98.893S^2 - 2046.375 = 0$$

↳ characteristic eqn

③ companion matrices of A & B are →

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2046.37 & 98.893 & 47.2 \end{bmatrix}.$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

④ Given : $P_3 = -30$

$P_4 = -40$

$\Rightarrow P_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

where, $\xi = 0.7$

$\omega_n = 4$ rad/s

$P_1 = -2.8 + j\,2.86$

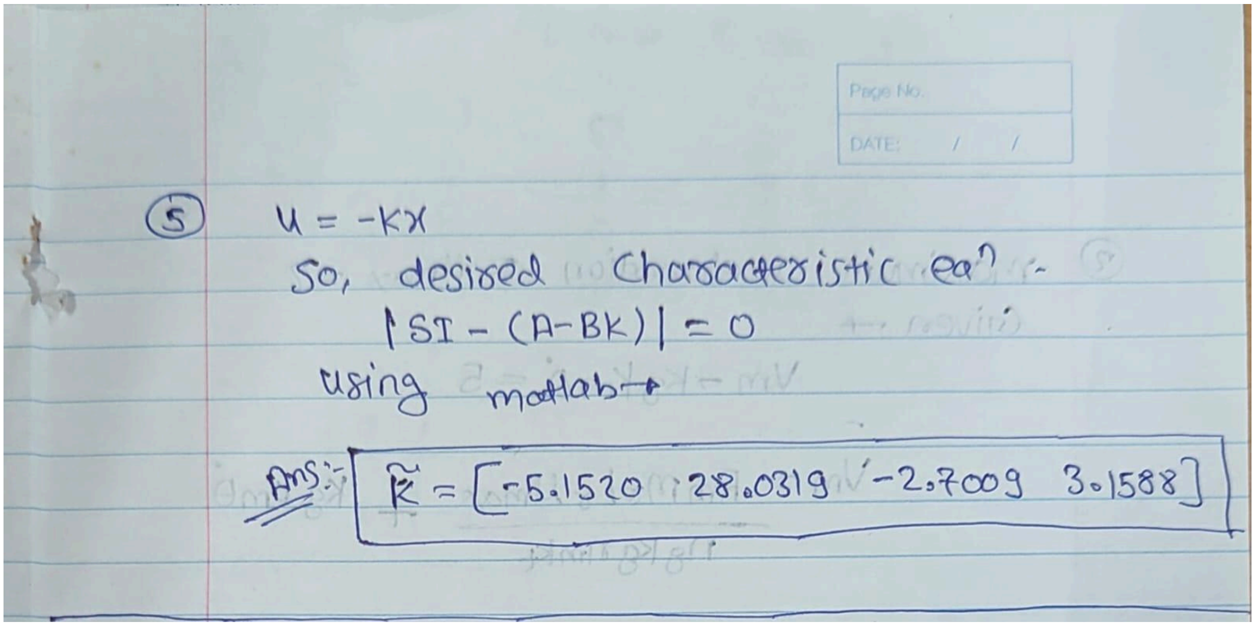
$P_2 = -2.8 - j\,2.86$

so, the desired characteristic eq^n is →

$(s-P_1)\ (s-P_2)\ (s-P_3)\ (s-P_4) = 0$

$\Rightarrow s^4 + 75.6s^3 + 1608.02s^2 + 7841.37s + 19223.52 = 0$

Ans.



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(5) $u = -Kx$

So, desired Characteristic eqⁿ :-

$|SI - (A-BK)| = 0$

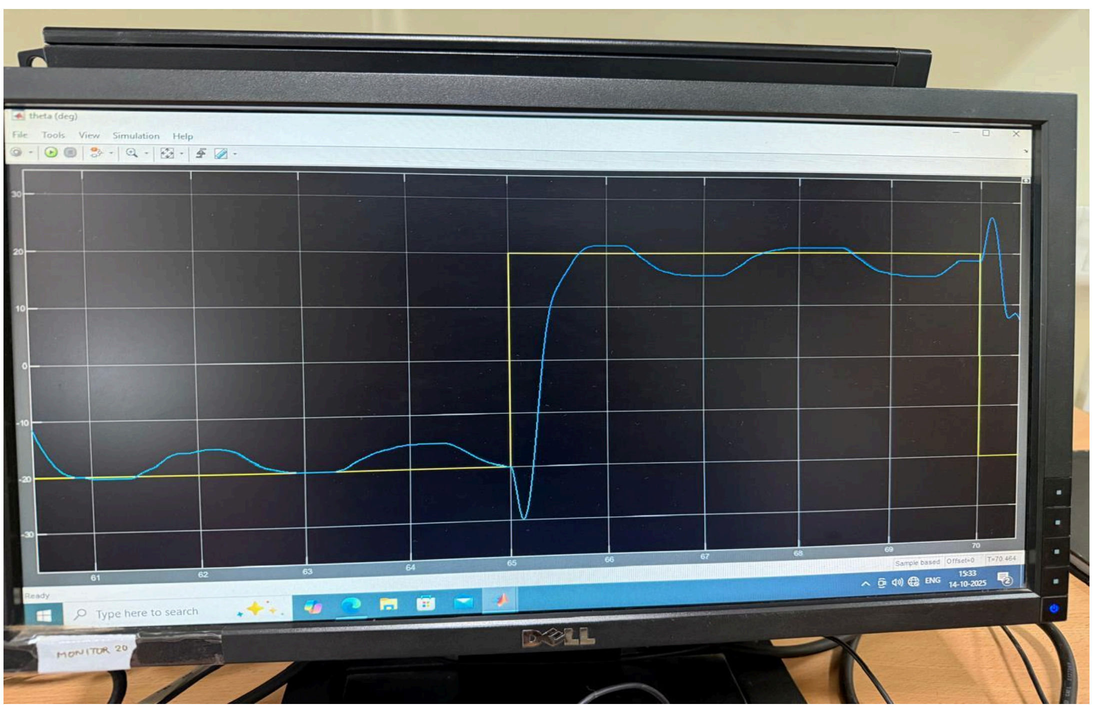
using matlab :-

Ans :- $\boxed{\tilde{K} = [-5.1520 \quad 28.0319 \quad -2.7009 \quad 3.1588]}$

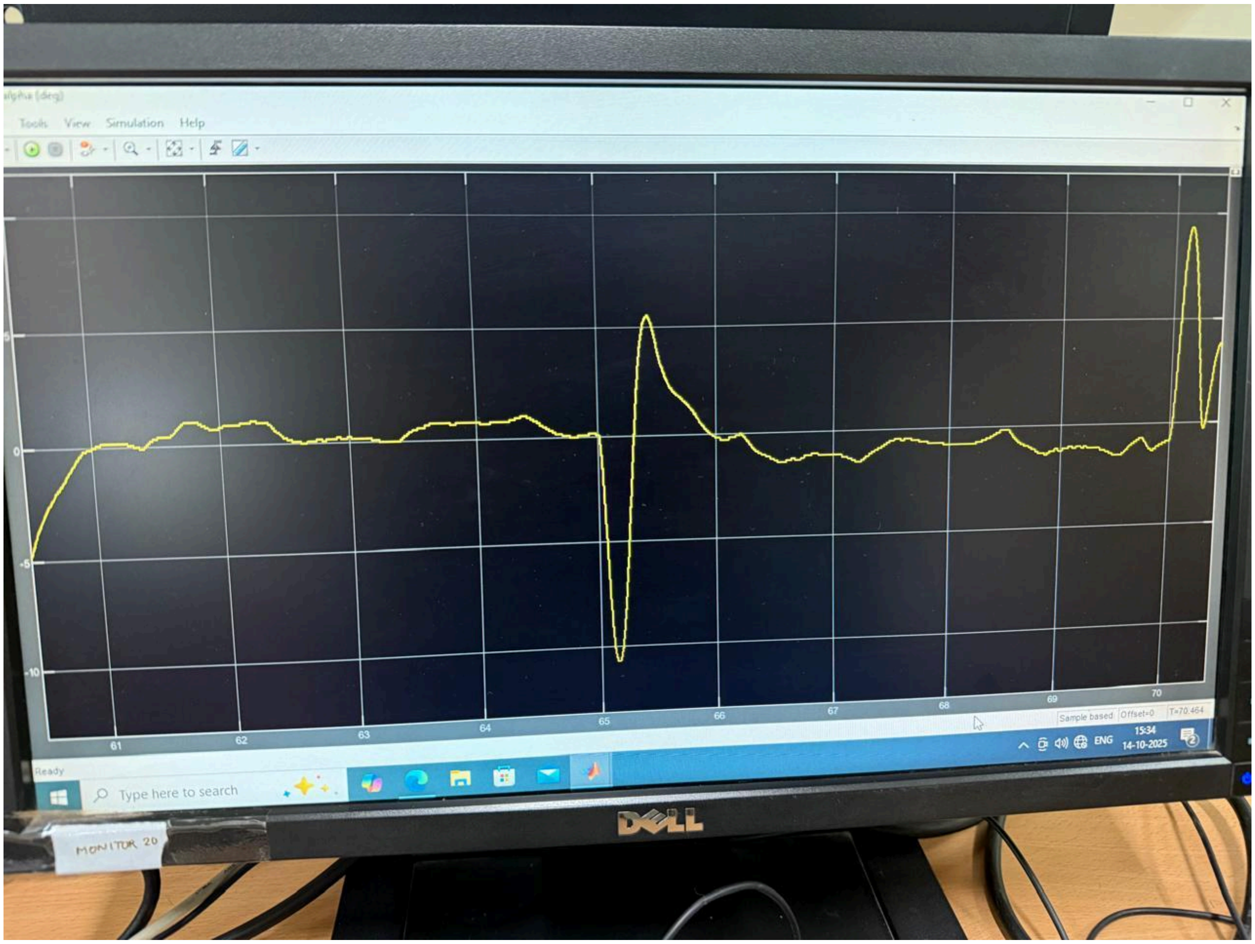
Graphs :-

The following graphs show the system's response with the balance controller active, tracking a square wave reference for the arm angle.

Desired vs. Actual Arm Angle (θ) for Balance Control.



Pendulum Angle (α) Response During Balance Control. :-



Results :-

Description	Symbol	Value
Desired Poles	DP	{-2.8+2.86i, -2.8-2.86i, -30, -40}
Companion Gain	K	{-5.120, 28.0319, -2.7009, 3.1588}

5. Experiment 2: Swing-Up Control :-

Aim :-

To implement and test a hybrid control strategy that first uses a nonlinear energy-based controller to swing the pendulum up from its hanging position and then switches to the linear state-feedback controller to balance it.

Theoretical Background

$$u = \begin{cases} K(x_d - x) & |x_2| < \epsilon \\ sat_{u_{max}}(\mu(E - E_r)\text{sign}(\dot{\alpha}\cos\alpha)) & \text{otherwise} \end{cases}$$

where ϵ defines the capture region around the upright position.

Pre-Lab Questions :-

case (2) : swing up control →

Pre lab questions →

(1) Potential Energy

i) upright → ($\alpha = 0$)

$E_p = \frac{1}{2} m_p g l_p (1 - \cos\alpha)$

$= \frac{1}{2} m_p g l_p (1 - \cos 0)$

$\boxed{E_p = 0}$

ii) Downward → ($\alpha = 180°$)

$E_p = \frac{1}{2} m_p g l_p (1 - \cos 180°)$

$= m_p g l_p$

$= 0.127 \times 9.81 \times 0.337$

$\boxed{E_p = 0.4198\ J}$

(2) maximum accelaration = u_{max} = ?

Given →

$$V_m - K_g K_m \dot{\theta} = 5$$

$$V_m = \frac{R_m M_r L_r u_{max}}{\eta_g K_g \eta_m K_t} + K_g K_m \dot{\theta}$$

$$5 = \frac{R_m M_r L_r u}{\eta_g K_g \eta_m K_t}$$

$$u_{max} = \frac{5\, \eta_g K_g K_t \eta_m}{R_m M_r L_r}$$

$$= \frac{(0.9)(70)(0.9)(0.00768)(5)}{(2.6)(0.257)(0.216)}$$

$$\boxed{u_{max} = 11.595\ m/s^2}$$

(3) controller accelaration when Pendulum hanging down = 0

So, we need an external force to swing the pendulum up.

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(ii) $u = \text{Sat}_{u_{max}} \left[\mu (\varepsilon - \varepsilon_r) \, \text{sign} (\alpha \cos \alpha) \right]$

$= 20 \, (0.42 - 0) \, (\alpha \cos \alpha)$

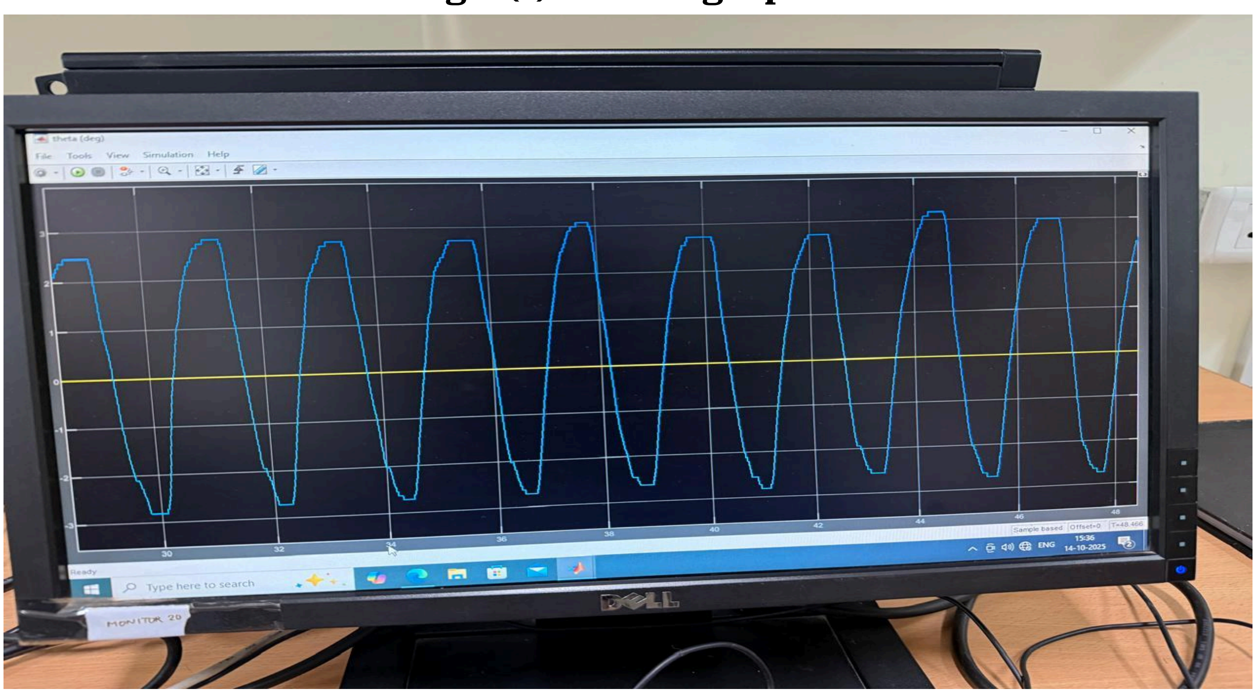
$\boxed{u = 8.4 \, m/s^2}$

maximum accelaration is 11.565, so when $\mu = 20$ & $\mu = 8.4$, it will saturate the controller.

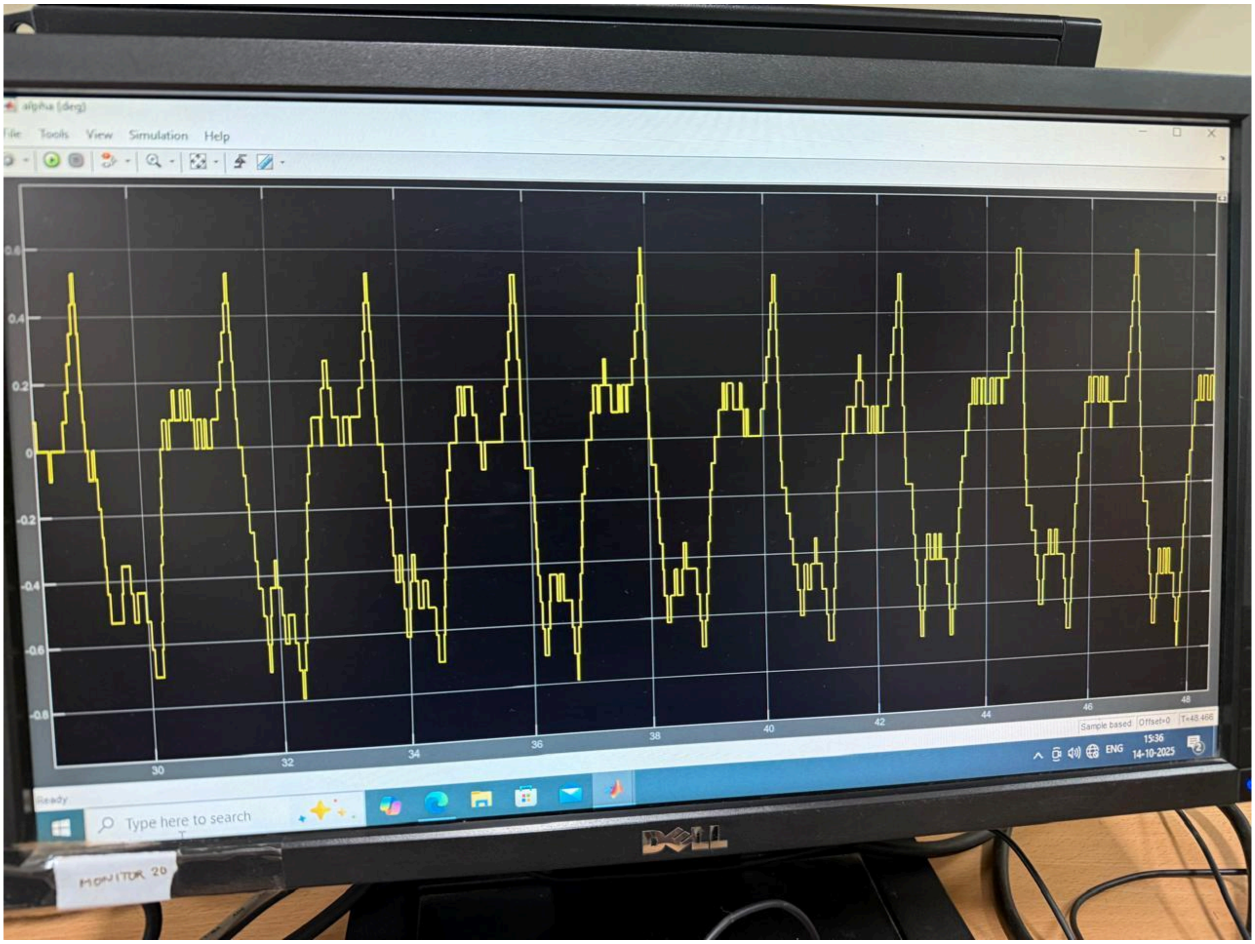
Graphs :-

The following graphs show the system response under the hybrid swing-up and balance controller.

Desired vs. Actual Arm Angle (θ) for Swing-Up and Balance Control.



Pendulum Angle (α) Response During Swing-Up and Balance Control.



Results :-

Description	Symbol	Value
Potential Energy	E_p	0.4198 J
Maximum Acceleration of Servo	u_max	11.595 m/s2
Proportional Gain	u(mu)	20

6. Observations and Discussion :-

a) The analysis of the open-loop state-space model correctly predicted the inherent instability of the inverted pendulum due to the presence of a pole in the right-half of the s-plane.

b) The state-feedback controller designed via pole placement successfully stabilized the pendulum in its upright position. The system was able to track changes in the desired arm angle while maintaining the pendulum's balance, satisfying the performance requirements.

c) The hybrid controller demonstrated effective performance. The nonlinear swing-up portion reliably brought the pendulum into the capture region, after which the linear balance controller smoothly took over to stabilize it.

d) During operation, small but continuous fluctuations were observed in the pendulum and arm angles. These reflect the controller's constant feedback adjustments required to counteract gravity and other minor disturbances, keeping the inherently unstable system balance

7. Conclusion :-

This experiment successfully demonstrated the practical application of modern control theory to stabilize an inherently unstable system. Through systematic modeling, linearization, and pole placement design, a state-feedback controller was successfully implemented to balance the rotary inverted pendulum. Furthermore, a hybrid controller combining this linear regulator with a nonlinear energy-based swing-up strategy proved effective for autonomously transitioning the pendulum from a hanging to a balanced state. The results confirm the effectiveness of state-space techniques for controlling complex real-world dynamic systems.