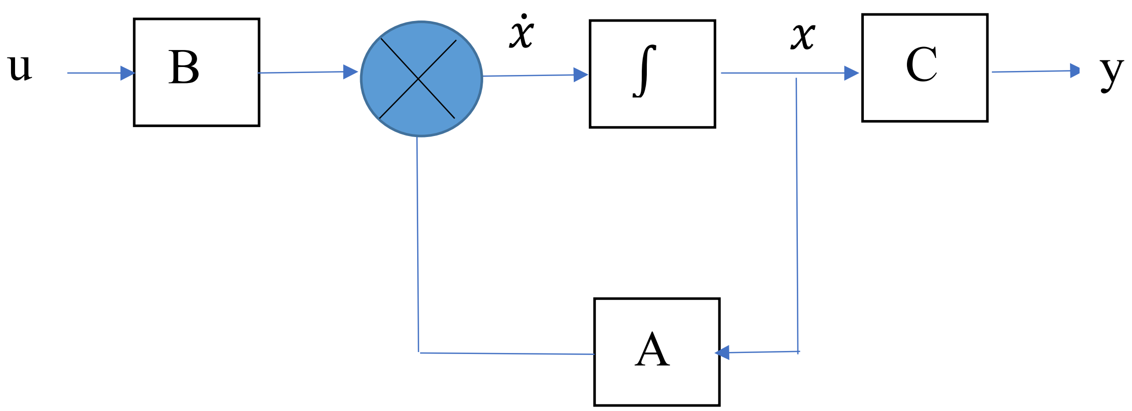
State Feedback Controller Design

Objective- The design objective is to find the feedback matrix k such that the closed-loop system is stable.

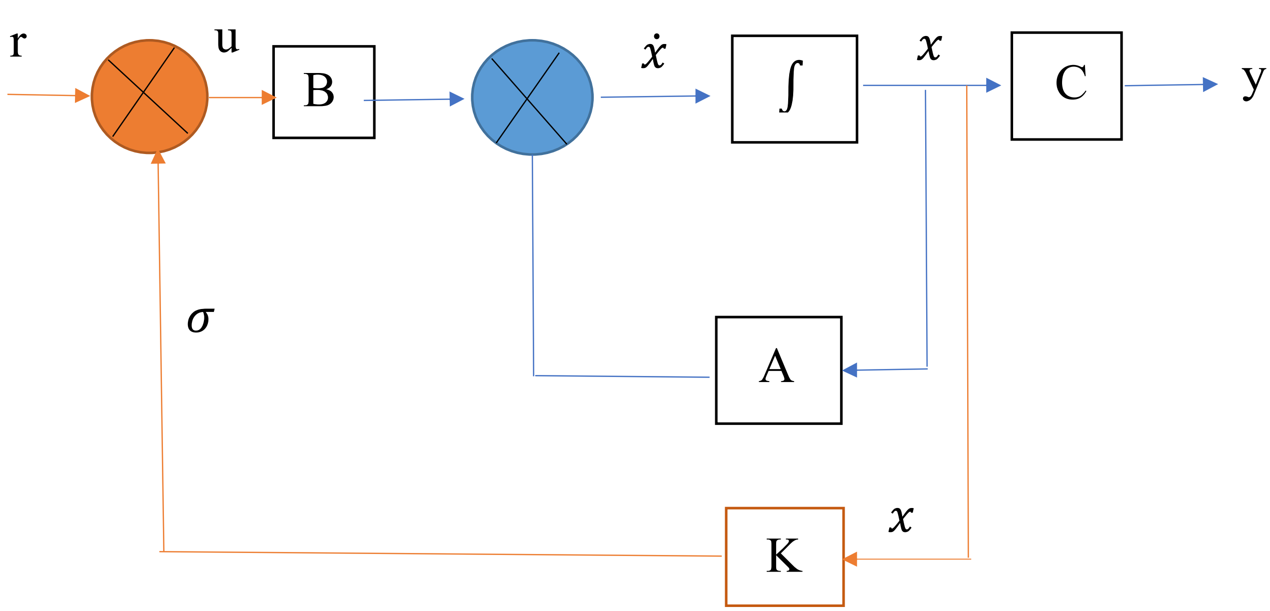
Consider a system with a state model

$$\dot{x} = Ax + Bu \tag{1}$$

$$Y = Cx \tag{2}$$



Block Diagram of the system with state feedback



$$U = r - \sigma$$
$$= r - Kx$$

$$\dot{x} = Ax + B(r - Kx)$$
$$= Ax + (Br - BKx)$$
$$= (A - BK)x + Br$$

Design of State Feedback Controller

Step-1. Check Controllability of the system

Controllability matrix Q_c is given by

$Q_c = [B\ AB\ ]$ when A is a 2× 2 matrix

$Q_c = [B\ AB\ A^2B]$ when A is a 3× 3 matrix

If the determinant of Q_c=0, then the rank of the matrix Q_c=order of the system and the system is completely state controllable

Step-2. Determine the Characteristic equation of the original system

Characteristics equation is obtained as

$|\lambda I - A \quad|$ =0

Step-3. Determine the transformation matrix P_c which converts given state model to Controllable Canonical form (CCF) (If the state model is already in CCF P =I, Identity matrix with order same as that of the system matrix A)

The transformation matrix, $P_c = \begin{bmatrix} P_1 \\ P_1A \\ \vdots \\ P_1A^{n-1} \end{bmatrix}$

And,$P_1 = [0\ 0 \dots 0 1\ ]Q_C^{-1}$

Step-1. Determine the state feedback gain matrix

In general K= $[b_n - a_n \; b_{n-1} - a_{n-1} \; \dots \dots \dots . . b_2 - a_2 \; b_1 - a_1] \, P_c$

Controllability

For the linear system given by Equation 1 and 2, if there exists an input $u_{[0,t_1]}$ which transfers the initial state x (0) $\triangleq x^0$ to the state x^1 in a finite time t_1, the state x^0 is said to be controllable. If all initial states are controllable, the system is said to be completely controllable, or simply controllable. Otherwise, the system is said to be uncontrollable.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}^0 + \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{b}u(\tau)\,d\tau$$

To study the controllability property, we may assume, without loss of generality, that $x^1 \equiv 0$. Therefore, if the system 1 is controllable, there exists an input $u_{[0,t_1]}$ such that

$$-\mathbf{x}^0 = \int_0^{t_1} e^{-\mathbf{A}\tau}\,\mathbf{b}u(\tau)\,d\tau$$

From this equation, we observe that complete controllability of a system depends on A and b, and is independent of output matrix c. The controllability of the system 1 is frequently referred to as the controllability of the pair {A, b}.

Example

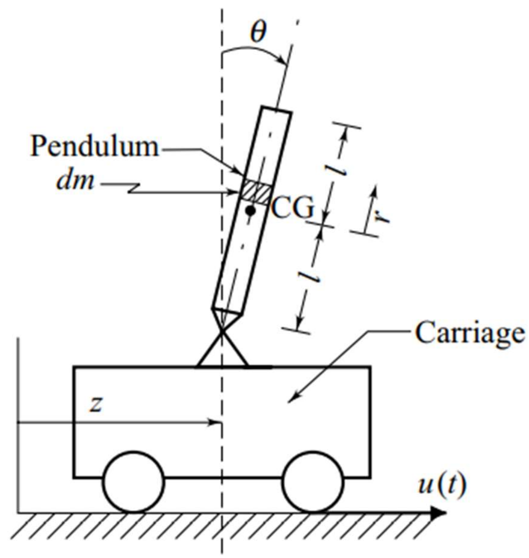


Figure. 1 Inverted pendulum system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

where $\quad \mathbf{x} = [\theta \quad \dot{\theta} \quad z \quad \dot{z}]^T$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 16.3106 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1.0637 & 0 & 0 & 0 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 0 \\ -1.4458 \\ 0 \\ 0.9639 \end{bmatrix}$$

$z(t)$ = horizontal displacement of the pivot on the cart

$\theta(t)$ = rotational angle of the pendulum