EEL 3040

Control System

Lab - 5

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State Feedback Controller Design

Content

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1) Objective :-

A) AIM :-

The design objective is to find the feedback matrix k such that the closed-loop system is stable. The specific objectives of this laboratory task are:

1.) To design a feedback gain matrix K that places the closed-loop poles at desired stable locations (−1, −2, −3).

2.) To design in two scenarios: when the state matrix A is in companion form and when it is not.

3.) To verify the manually derived feedback gains using MATLAB’s built-in functions.

4.) To simulate and compare the step responses of the open-loop (unstable) system and the stabilized closed-loop system .

B) Apparatus/software required : -

MATLAB R2023a , MATLAB plotting tools .

C) Brief Theory :-

In the field of control engineering, many real-world systems exhibit instability in their natural form. The primary purpose of a control strategy is to stabilize these systems and shape their response according to specific performance requirements. The state-space method provides a versatile framework to represent and control such systems. This experiment highlights one of the fundamental approaches in modern control theory: state feedback using pole placement. By feeding back information from the system states, it is possible to modify the overall system dynamics. Specifically, the closed-loop poles which determine system stability and response can be reassigned to any desired locations in the complex plane, provided the system satisfies the controllability condition.

2) MATHEMATICAL MODEL OF THE SYSTEM :-

A linear time-invariant (LTI) system is represented as:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{(State Equation)}$$
$$y(t) = Cx(t) + Du(t) \qquad \text{(Output Equation)}$$

where:

- x(t) ∈ Rn: state vector,
- u(t) ∈ Rp: input vector,
- y(t) ∈ Rq: output vector,
- A: state matrix,
- B: input matrix,
- C: output matrix,
- D: feedthrough matrix (zero in this experiment).

System stability is determined by the eigenvalues of A. For stability, all eigenvalues must lie strictly in the left half of the complex plane.

3) Key Theoretical Formulas :-

a) State Feedback Control Law :-

The control law is:

$$u(t) = -Kx(t)$$

Where K is the feedback gain matrix.
Substituting into the state equation:

$$\dot{x}(t) = Ax(t) - BKx(t)$$
$$\dot{x}(t) = (A - BK)x(t)$$

Thus, the closed loop system matrix is:

$$A_{cl} = A - BK$$

b) Controllability Requirement :-

The controllability requirement is:

$$Q_c = [B\ AB\ A^2B\ \dots\ A^{n-1}B]$$

The system is controllable if:

$$\text{rank}(Q_c) = n$$

4) Calculations :-

1) System in Companion Form :-

we know that ,

State matrix . input matrix

$$\dot{x} = AX + BU \qquad u = -KX$$

input

state

$$Y = CX$$

output output matrix

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

$$Y = \underbrace{[1\ 0\ 0]}_{C} x$$

desired Pole = -2 , $-1 \pm j1$

desired Pole $= -2\ ,\ -1 \pm j1$

① Controllability test

$$Q_c = [B \quad AB \quad A^2B]$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

$$\therefore |Q_c| = +1(13-16)$$

$$\boxed{|Q_c| = -3}$$

② desired Poles, let select $-2, -1 \pm j$

③ $\therefore$ desired characteristic eqn is

$$(\lambda+2)(\lambda-(-1+j)(\lambda-(-1-j))=0$$

$$\Rightarrow \lambda^3 + 4\lambda^2 + 6\lambda + 4 = 0$$

$$\dot{x} = Ax + Bu \quad \text{—(I)}$$

$$\& \quad u = -Kx$$

Putting value of u in eqn (I).

$$K = [K_1, K_2, K_3]$$

$$\dot{x} = Ax + B(-Kx)$$
$$\dot{x} = [A - BK]x$$

& $$[A-BK] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 & -3-K_2 & -4-K_3 \end{bmatrix}$$

(4) characteristic eq^n at closed-loop system :-

$$|\lambda I - (A-BK)| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 & -3-K_2 & -4-K_3 \end{bmatrix} \right| = 0$$

Actual characteristic

$$\lambda^3 + (K_3+4)\lambda^2 + (K_2+3)\lambda + K_1 = 0 \quad \text{—(2)}$$

desired characteristic eq^n is :-

$$\lambda^3 + 4\lambda^2 + 6\lambda + 4 = 0 \quad \text{—(3)}$$

By comparing the coeff. of above 2 eq^n we get.

$K_1 = 4$
$K_2 = 3$
$K_3 = 0$

$\therefore$ $K = [K_1 \; K_2 \; K_3]$

$$\boxed{K = [4 \;\; 3 \;\; 0]}$$

Ans.

controller

2) System not in Companion form :-

② System Not in Companion form :-

$$\rightarrow \quad \dot{x} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

find eigan values of $A \rightarrow$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda = 1, 2, 3$$

Desired Poles $= -1, -2, -3$.

for checking controllability :-

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 1 & 3 & 9 \end{bmatrix}$$

$$|Q_c| = -1$$

$\hookrightarrow$ Controllable.

Now, $$Q_c^{-1} = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} T_1 \\ T_1 A \\ T_1 A^2 \end{bmatrix} \qquad T_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} Q_c^{-1}$$

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix} \qquad T_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Now, $\bar{x} = Tx$.

$\rightarrow \dot{\bar{x}} = T\dot{x} = T(Ax + Bu)$

$$\dot{\bar{x}} = \underbrace{TAT^{-1}}_{\bar{A}} \bar{x} + \underbrace{TB}_{\bar{B}}u$$

$$y = Cx = \underbrace{CT^{-1}}_{\bar{C}} \bar{x}$$

$$\rightarrow \bar{A} = (0\ 1\ 0) \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

similarly,

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \& \quad \bar{C} = [-1\ \ 1\ \ 0]$$

Now, it is in companion form,

Desired characteristic eqn -

$$(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0 \longrightarrow (1)$$

actual characteristic eqn -

$$|\lambda I - (\bar{A} - \bar{B}K)| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ K_1 - 6 & K_2 + 11 & \lambda + K_3 - 6 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + (K_3 - 6)\lambda^2 + (11 + K_2)\lambda + (K_1 - 6) = 0 \longrightarrow (2)$$

→ Compare eq^n ① & ② →

$K_1 = 12$

$K_2 = 0$

$K_3 = 12$

$\Rightarrow \bar{K} = [12 \;\; 0 \;\; 12]$

& since, $K = \bar{K}T$

$$= [12 \;\; 0 \;\; 12] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

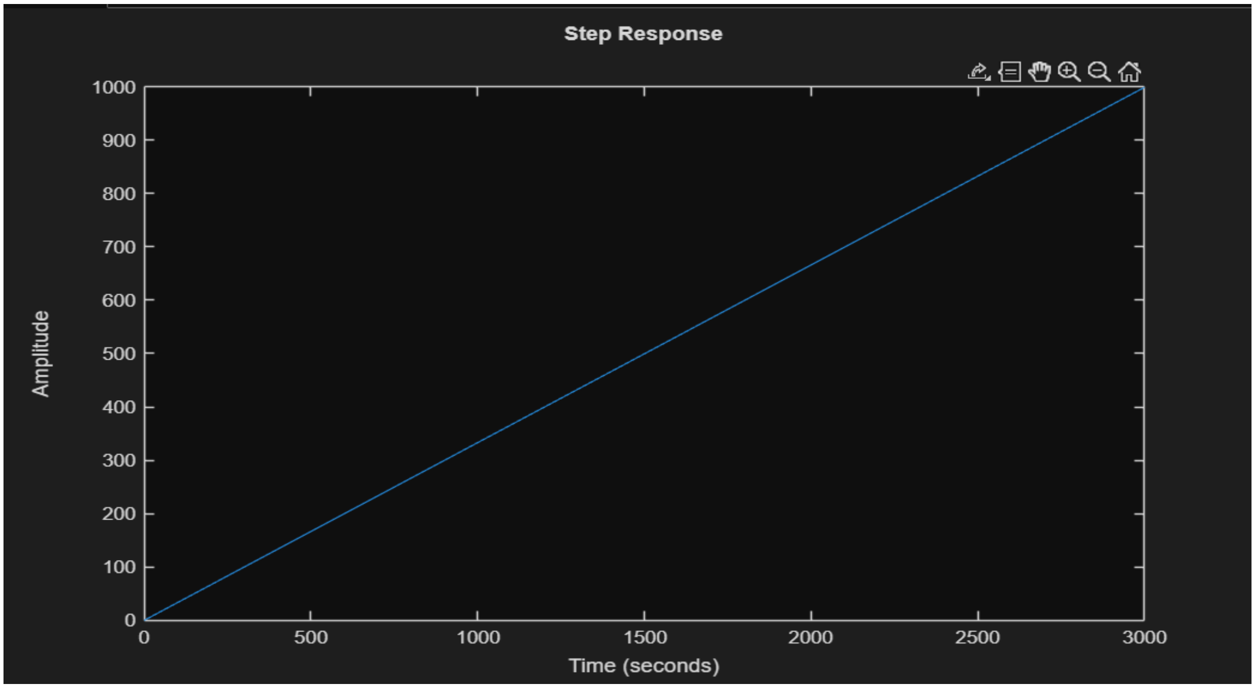
$$\boxed{K = [36 \;\; 60 \;\; 12]}$$

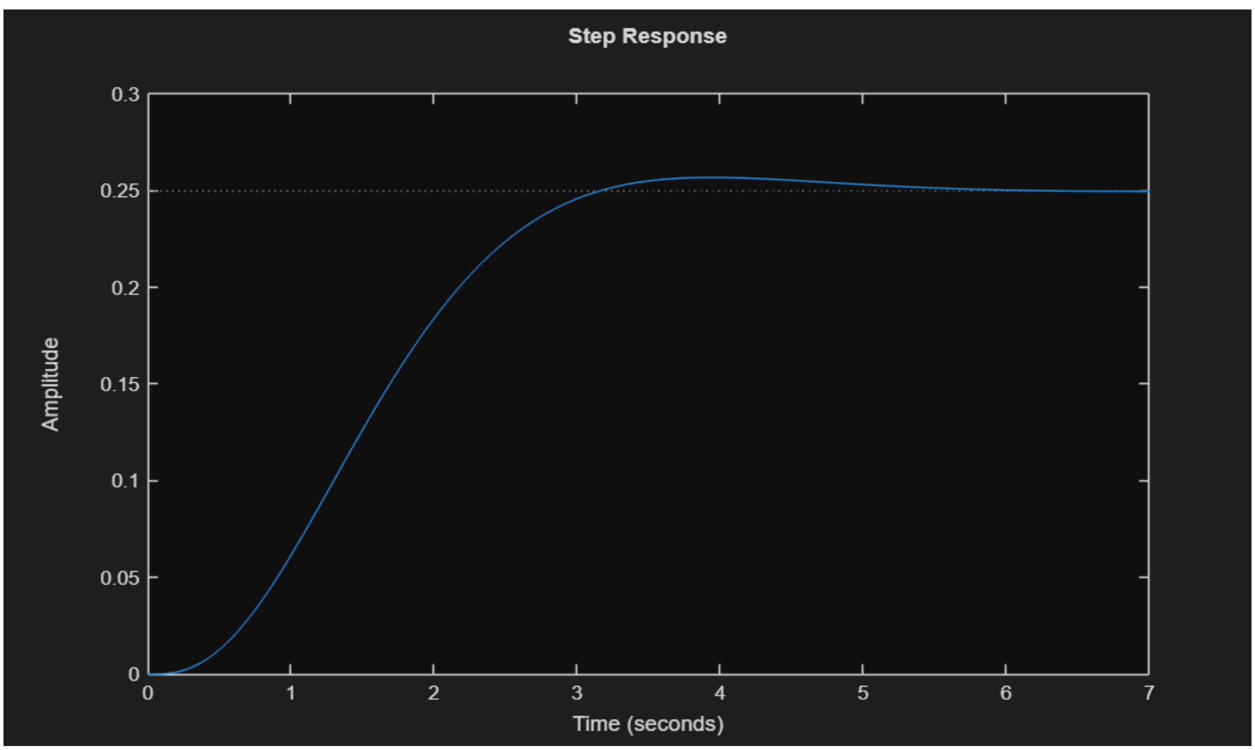
5) Matlab Code and Graphs :-

```
A = [0 1 0; 0 0 1; 0 -3 -4];
B = [0; 0; 1];
C = [1 0 0];
D = 0;

eigen_value_A = eig(A);
P = [-2 -1+1j -1-1j];
K = place(A,B,P);
display(K);

Acl = (A-B*K);
Acle = eig(Acl);
%sys = ss(A, B, C, D);
syscl = ss(Acl, B, C, D);
%step(sys);
step(syscl);
```

Graphs :-

1) Open loop step Response :-





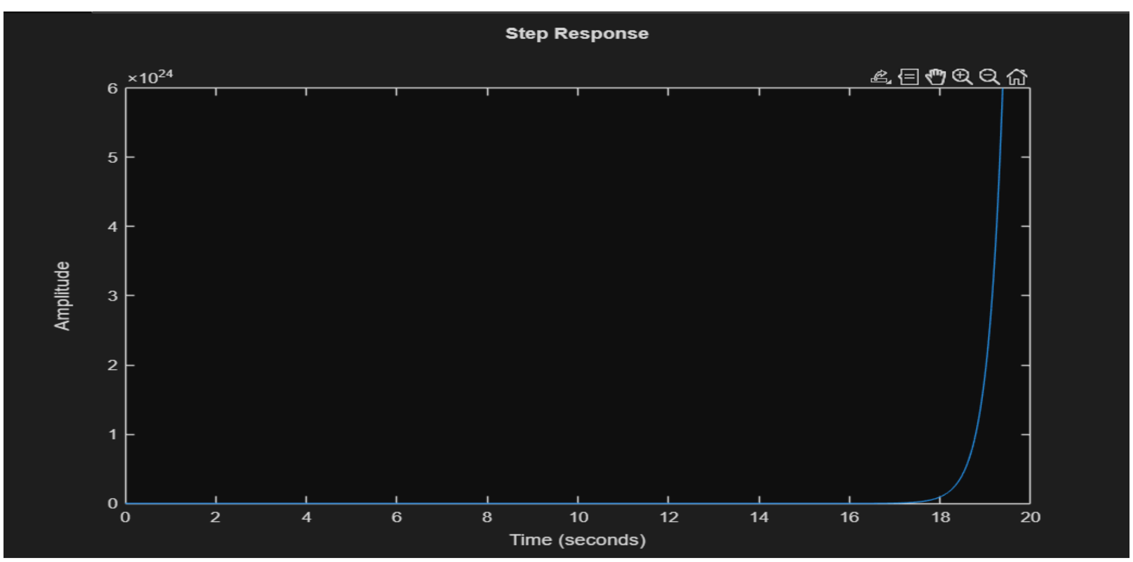
So, the Feedback gain matrix obtained for this system is:

K=[4 3 0]

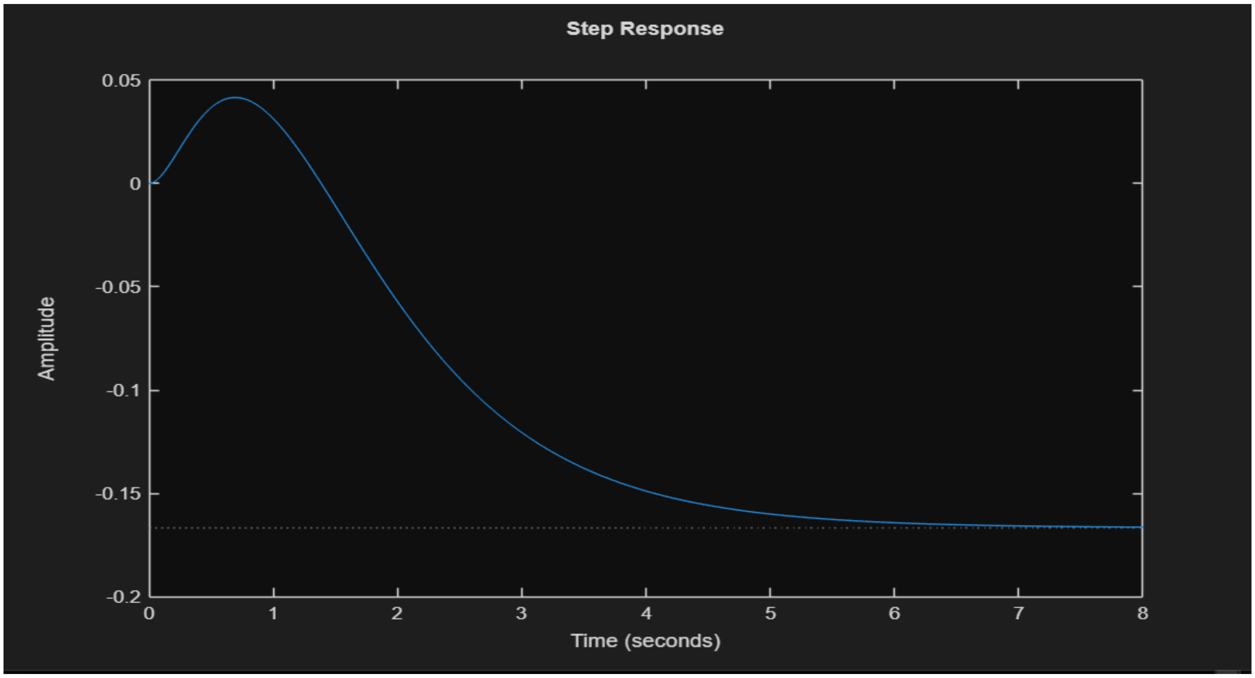
2) **System NOT in Companion Form :-**

```
A = [1 0 1; 1 2 0; 0 0 3];
B = [0; 0; 1];
C = [1 1 0];
D = 0;

eigen_value_A = eig(A);
display(eigen_value_A);
P = [-1 -2 -3];
K = place(A,B,P);
display(K);

Acl = (A-B*K);
Acle = eig(Acl);
%sys = ss(A, B, C, D);
syscl = ss(Acl, B, C, D);
%step(sys);
step(syscl);
```

Graphs :-



2) Closed Loop Step Response :-



So, the Feedback Gain Matrix for the System is:

K = [36 60 12]

6) Observation :-

1.) For both companion and non-companion forms, suitable feedback matrices were obtained:

Part 1: K = [4 3 0]

Part 2: K = [36 60 12]

2.) Open Loop poles (4,3,2) are UNSTABLE whereas Closed Loop Poles (36, 60, 12) are STABLE.

3.) Open Loop Step Response show unbounded growth. Closed Loop Response demonstrate stability, with transient decaying to steady state.

7) CONCLUSION

This experiment successfully demonstrated the effectiveness of state feedback control via pole placement in stabilizing unstable linear time-invariant systems.

Controller Design and Verification:-

State feedback gain matrices were designed and verified for both system representations.

The obtained results were: -

Companion Form: K = [4 3 0]
Non- Companion Form: K = [36 60 12]

Step Response Analysis:-

The step response comparison reinforced the theoretical results:

• The open-loop responses exhibited unbounded growth, consistent with instability.

• The closed-loop responses showed stable behaviour, with the system output settling to a steady state after a transient phase.

The study validates pole placement as a powerful and systematic technique for shaping system dynamics. By ensuring controllability, feedback gains can be designed to achieve stability and desired performance, even for inherently unstable systems.