Reconstruction of a Sparse Image from DFT Coefficients

Problem Statement

Consider an $n \times n$ image $f(x, y)$ such that only $k \ll n^2$ elements in it are non-zero, where k is known and the locations of the non-zero elements are also known. Answer the following:

(a) How will you reconstruct such an image from a set of only m different Discrete Fourier Transform (DFT) coefficients of known frequencies, where $m < n^2$?

(b) What is the minimum value of m that your method will allow?

(c) Will your method work if k is known, but the locations of the non-zero elements are unknown? Why (not)?

Solution

(a) Reconstruction Method

Since the image $f(x, y)$ is sparse, containing only k non-zero elements at known locations, we can exploit this sparsity for reconstruction.

Let the coordinates of the non-zero pixels be given by the set $S = \{(x_i, y_i)\}_{i=1}^{k}$. The DFT of $f(x, y)$ at frequency (u, v) is given by:

$$F(u, v) = \sum_{(x,y) \in S} f(x, y) e^{-j2\pi\left(\frac{ux}{n} + \frac{vy}{n}\right)}$$

Because we know the locations of the non-zero elements, the above equation reduces to a system of m equations with k unknowns $f(x_i, y_i)$ (the values of the non-zero pixels). Therefore, we can solve this linear system by choosing m linearly independent DFT coefficients at known frequencies (u, v).

This setup allows us to reconstruct the image by solving the following system of linear equations:

$$\mathbf{A}\mathbf{f} = \mathbf{F}$$

where: - $\mathbf{A}$ is an $m \times k$ matrix with entries $A_{ij} = e^{-j2\pi\left(\frac{u_i x_j}{n} + \frac{v_i y_j}{n}\right)}$, - $\mathbf{f}$ is a $k \times 1$ vector containing the values $f(x_i, y_i)$ of the non-zero pixels, - $\mathbf{F}$ is an $m \times 1$ vector of the selected DFT coefficients $F(u_i, v_i)$.

As long as $m \geq k$ and the matrix $\mathbf{A}$ has full rank, we can solve for $\mathbf{f}$ using linear algebra methods (e.g., least squares if $m > k$).

(b) Minimum Value of m

The minimum number of DFT coefficients m needed to reconstruct $f(x, y)$ is $m = k$. This is because with k non-zero elements in known locations, we need at least k equations to uniquely determine their values. Hence, $m = k$ DFT coefficients are sufficient if they are chosen such that the resulting system matrix $\mathbf{A}$ is invertible.

(c) Reconstruction with Unknown Locations

If the locations of the non-zero elements are unknown, this method will not work. Without knowledge of the locations, we lose the ability to construct the exact matrix $\mathbf{A}$ required for the linear system. In that case, the problem becomes underdetermined, as we now need to solve not only for the values of the non-zero elements but also for their positions within the image. This would require a combinatorial search over possible locations or additional information, which makes the problem significantly harder to solve directly using only the DFT coefficients.