Q6. a. $u = x\cos\theta - y\sin\theta$
$v = x\sin\theta + y\cos\theta$

$\longrightarrow$ $\frac{u}{\sin\theta} = x\cot\theta - y$

$\frac{v}{\cos\theta} = x\tan\theta + y.$

Let the image $I = f(x,y)$

$\frac{x}{\sin\theta\cos\theta} = \frac{u\cos\theta + v\sin\theta}{\sin\theta\cos\theta}$

$\therefore$ $x = u\cos\theta + v\sin\theta$
$y = v\cos\theta - u\sin\theta$

$x = u\cos\theta + v\sin\theta$

$\therefore$ $I = f(u\cos\theta + v\sin\theta,\ v\cos\theta - u\sin\theta)$

$y = \frac{x\cos\theta - u}{\sin\theta}$

$= \frac{x(\cos\theta)(u\cos\theta + v\sin\theta) - u}{\sin\theta}$

$= v\cos\theta - u\sin\theta$

$\rightarrow \frac{\partial^2 I}{\partial x^2} = \frac{\partial^2 f(x,y)}{\partial x^2} = I_{xx}$

$\rightarrow \frac{\partial^2 I}{\partial y^2} = \frac{\partial^2 f(x,y)}{\partial y^2} = I_{yy}$

$\rightarrow \frac{\partial I}{\partial u} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial u}$

$= I_x \cos\theta + I_y(-\sin\theta)$

$\rightarrow \frac{\partial^2 I}{\partial u^2} = \cos\theta\left(\frac{\partial I_x}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial I_x}{\partial y}\frac{\partial y}{\partial u}\right) - \sin\theta\left(\frac{\partial I_y}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial I_y}{\partial x}\frac{\partial x}{\partial u}\right)$

$= \cos\theta\left(I_{xx}\cos\theta + I_{yx}(-\sin\theta)\right) - \sin\theta\left(I_{yy}(-\sin\theta) + I_{xy}\cos\theta\right)$

$I_{uu} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin\theta\cos\theta - I_{yx}\sin\theta\cos\theta$

$\rightarrow I_v = \frac{\partial I}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial v}$

$= I_x\sin\theta + I_y\cos\theta$

$I_{vv} = \sin\theta\left(\frac{\partial I_x}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial I_x}{\partial y}\frac{\partial y}{\partial v}\right) + \cos\theta\left(\frac{\partial I_y}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial I_y}{\partial x}\frac{\partial x}{\partial v}\right)$

$= \sin\theta\left(I_{xx}\sin\theta + I_{yy}\cos\theta\right) + \cos\theta\left(I_{yy}\cos\theta + I_{xy}\sin\theta\right)$

$I_{vv} = I_{xx}\sin^2\theta + I_{yy}\cos^2\theta + I_{xy}\sin\theta\cos\theta + I_{xy}\sin\theta\cos\theta$

$I_{uu} + I_{vv} = I_{xx}\sin^2\theta + I_{xx}\cos^2\theta$
$+ I_{yy}\sin^2\theta + I_{yy}\sin^2\theta$
$= I_{xx} + I_{yy}.$

b. Directional directive in direction of vector v is

$$D_v I = \nabla I . v$$

$$\hat{g} = \left(\frac{I_x}{\sqrt{I_x^2+I_y^2}}, \frac{I_y}{\sqrt{I_x^2+I_y^2}} \right)$$

second derivative in direction of gradient vector is

$$D_{\hat{g}} D_{\hat{g}} I = \nabla I . \nabla D_{\hat{g}} I$$

$$D_{\hat{g}} I = \frac{I_x I_x}{\sqrt{I_x^2+I_y^2}} + \frac{I_y I_y}{\sqrt{I_x^2+I_y^2}} = \sqrt{I_x^2+I_y^2}$$

$$\nabla D_{\hat{g}} I = \left(\frac{\partial D_{\hat{g}} I}{\partial x}, \frac{\partial D_{\hat{g}} I}{\partial y} \right) = \left(\frac{I_x I_{xx} + I_y I_{xy}}{\sqrt{I_x^2+I_y^2}}, \frac{I_y I_{yy} + I_y I_{xy}}{\sqrt{I_x^2+I_y^2}} \right)$$

$$D_{\hat{g}} D_{\hat{g}} I = I_x \frac{\partial D_{\hat{g}} I}{\partial x} + I_y \frac{\partial D_{\hat{g}} I}{\partial y}$$

$$= \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{\sqrt{I_x^2+I_y^2}}$$

c. Direction perpendicular to the gradient ∇I

$$\hat{n} = \left(\frac{-I_y}{\sqrt{I_x^2+I_y^2}}, \frac{I_x}{\sqrt{I_x^2+I_y^2}} \right)$$

second directional ~~directive~~ derivative $\perp$ to gradient is

$$D_{\hat{n}} D_{\hat{n}} I = \nabla I . \nabla D_{\hat{n}} I$$

It ~~could~~ be contributing to directional derivative in direction of gradient. It can be expressed as

$$D_{\hat{n}} D_{\hat{n}} I = -\text{Laplacian}(I) = -(I_{xx} + I_{yy})$$