Fourier Analysis

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Fourier Analysis

- Discovery of the French Mathematician Joseph Fourier.

- Motivated initially from thermodynamics (heat equation).

- Fundamental to modern signal and image processing (particularly in image or signal filtering).

- Also very widely used in optics, probability theory, solutions of partial differential equations and many other areas of mathematics.

Fourier Series

- Any periodic function can be represented *accurately* as the linear combination of **sine** and **cosine** signals of different frequencies.

- This is called the **Fourier series**

- The sinusoidal signals are given as:

$$f(t) = \cos(\omega t)$$

$$g(t) = \sin(\omega t)$$

$$\omega / 2\pi = \text{frequency} = 1/\text{period}, t = \text{time}$$

Fourier Series

- The Fourier series for a signal f(t) with period *T* is given as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T} \, ; c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi nt/T} \, dt = \text{Fourier series coeff};$$

$$n = 0, \pm 1, \pm 2, \ldots = \text{frequency index}$$

$$j = \sqrt{-1}$$

$$\text{Recall Euler's formula}: e^{j2\pi nt/T} = \cos(2\pi nt/T) + j\sin(2\pi nt/T)$$
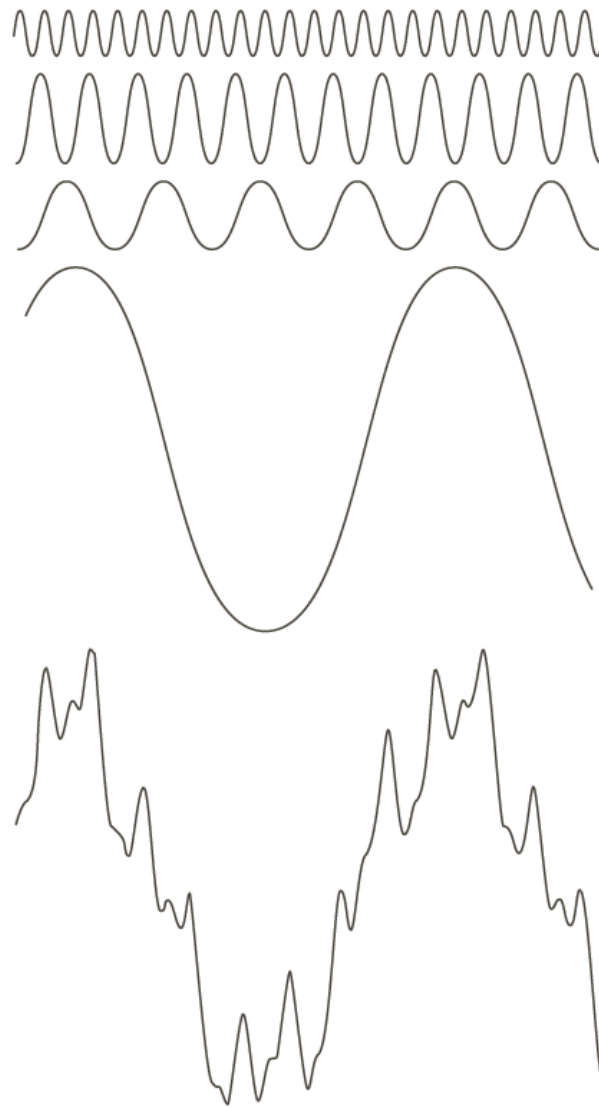
- How do you justify the formula for c_n?

FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

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Fourier Series

- It follows from the following trigonometric identity:

$$\int_{-T/2}^{T/2} e^{j2\pi nt/T} e^{-j2\pi mt/T}\, dt = \int_{-T/2}^{T/2} e^{j2\pi(n-m)t/T}\, dt; m, n \text{ are integers}$$

$$= \frac{e^{j2\pi(n-m)t/T}}{j2\pi(n-m)/T}\Big| T/2 - \frac{e^{j2\pi(n-m)t/T}}{j2\pi(n-m)/T}\Big| -T/2$$

$$= \frac{e^{j\pi(n-m)}}{j2\pi(n-m)/T} - \frac{e^{-j\pi(n-m)}}{j2\pi(n-m)/T} = \frac{\sin \pi(n-m)}{\pi(n-m)/T}$$

$$= 0 \text{ when } n \neq m$$

When $n = m$, the LHS evaluates to T

Fourier Series

- Given a signal f(t), how do we obtain its Fourier series coefficients?

- To find the coefficient at frequency 2πn/T, we compute the following integral:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi nt/T} \, dt;$$

Fourier Series

- Consider integral of the following form:

$$\int_{-T/2}^{T/2} e^{j2\pi mt/T} e^{-j2\pi mt/T} \, dt$$

- The inner product between two periodic complex valued functions f and g is given as:

$$\int_{-T/2}^{T/2} f^{*}(t) g(t) dt$$

$$f(t) = a(t)(\cos\theta(t) + j\sin\theta(t)); a(t) \in R$$

$$\rightarrow f^{*}(t) = a(t)(\cos\theta(t) - j\sin\theta(t))$$

Fourier Series

- In computing Fourier series coefficients, we are considering inner products of complex sinusoidal signals of different frequencies.

- When the frequencies are different, the inner product is 0.

- When the frequencies are the same, the inner product is T.

- Any signal can be represented using these sinusoidal signals – giving rise to an orthogonal basis.
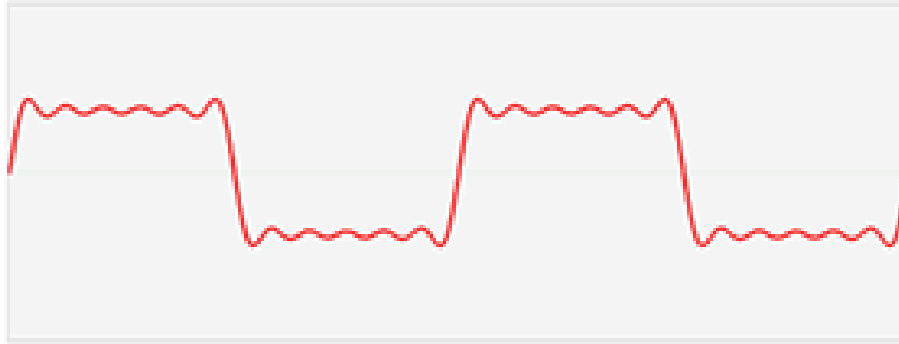
Function s(t) (in red) is a sum of six sine functions of different amplitudes and harmonically related frequencies. Their summation is called a Fourier series.

https://en.wikipedia.org/wiki/Fourier_series#/media/File:Fourier_series_and_transform.gif

Why complex exponentials?

- We need complex exponentials as they consist of both sines and cosines.

- Only sine signals cannot represent shifted sine waves or cosine waves completely.

$$\sin(\omega t + \phi) = \cos \omega t \sin \phi + \sin \omega t \cos \phi$$

Fourier transform

- Extension of Fourier series to non-periodic but absolutely integrable or square integrable signals.
- Mathematically given as follows:

$$\mathbf{F}(f(t))(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt = F(\mu)$$

$$\mu = \text{continuous frequency variable}$$

- We also have:

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

- These formulae are consistent, due to the orthogonality of the sinusoidal functions.

Fourier Transform

- F(μ) is the Fourier coefficient at frequency μ, computed by integrating out the time variable t.

- The original signal can be reconstructed accurately (without loss of information) from its Fourier coefficients by integrating out the frequency variable.

- Thus a signal and its Fourier transform (the set of all its Fourier coefficients) form a **unique** "Fourier pair".

- Convention: Time-domain signals are denoted in lower case (eg: f(t)), whereas their Fourier transforms are denoted in upper case (F(μ)).

- The Fourier operator is denoted by **F** (bold font) to distinguish it from the Fourier transform of f.

Example 1: Rect and sinc
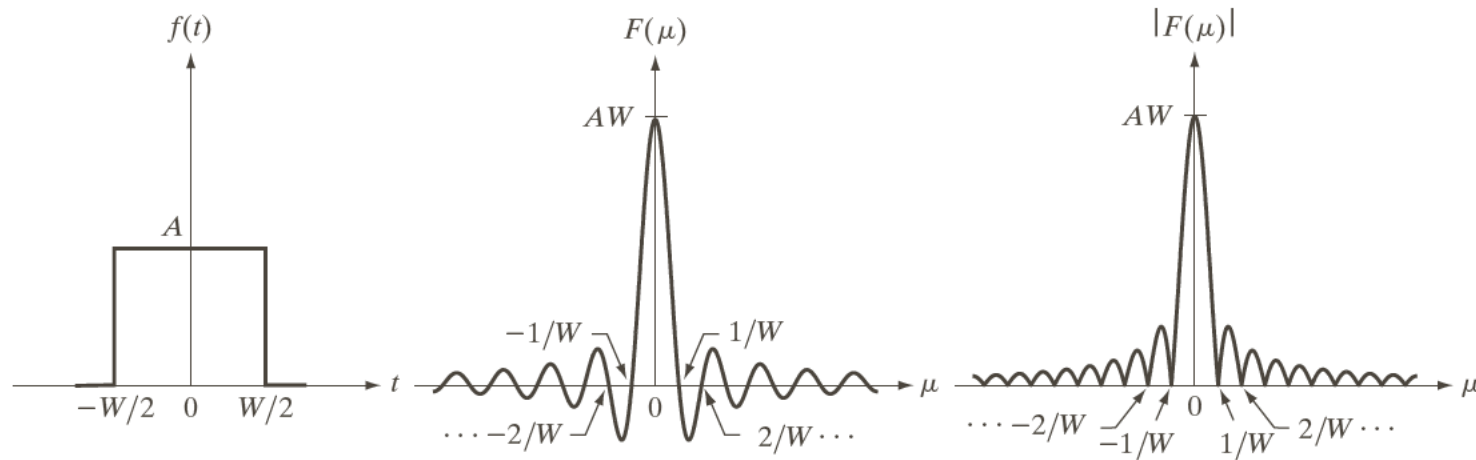
a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Example 1: Rect and sinc

$$f(t) = A, -W/2 \leq t \leq W/2$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} Ae^{-j2\pi\mu t} dt$$

$$= \frac{-A}{j2\pi\mu}[(e^{-j2\pi\mu t})_{W/2} - (e^{-j2\pi\mu t})_{-W/2}]$$

$$= \frac{A}{j2\pi\mu}[e^{j\pi\mu W} - e^{-j\pi\mu W}] = AW\frac{\sin \pi\mu W}{\pi\mu W} = AW\text{sinc}(\mu W)$$

$Note:$

$\text{sinc}(m) = \sin(\pi m)/(\pi m)$ where $\text{sinc}(0) = 1, \text{sinc}(m) = 0$ for all

other integers m

It also turns out that the Fourier transform of a sinc in the time domain is a rect in the frequency domain. This has a much more complicated proof.

Example 2: Delta Functions

- The Dirac Delta or the unit impulse signal with continuous variable t is defined as follows:

$$\delta(t) = \infty \text{ if } t = 0$$

$$= 0 \text{ otherwise}$$

$$\text{constrained to obey} \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

- Sifting property:

$$\int_{-\infty}^{\infty} \delta(t)\,f(t)\,dt = f(0)$$

Example 2: Delta Functions

- Sifting property:

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$$

- Note the Dirac delta function is defined when the time variable t is continuous.

- This is different from the Kronecker delta function defined on discrete time:

$$\delta(t) = 1 \text{ if } t = 0$$
$$= 0 \text{ otherwise}$$

Example 2: Delta Functions

- There is a sifting property for Kronecker delta functions too:

$$\sum_{t=-\infty}^{\infty} f(t)\delta(t) = f(0)$$

$$\sum_{t=-\infty}^{\infty} f(t)\delta(t-t_0) = f(t_0)$$

Example 2: Delta Functions
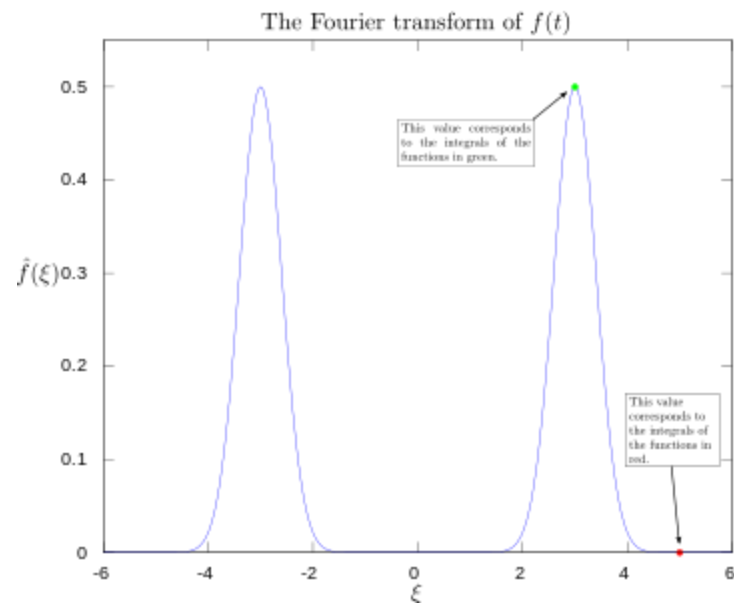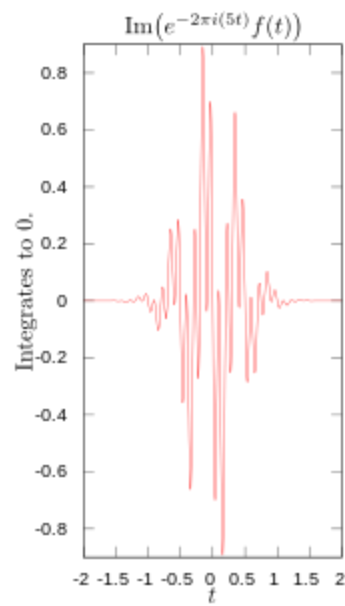
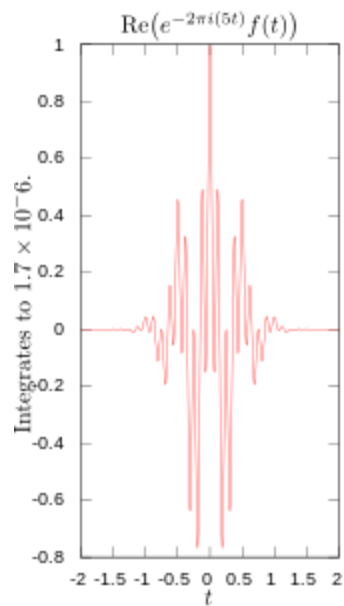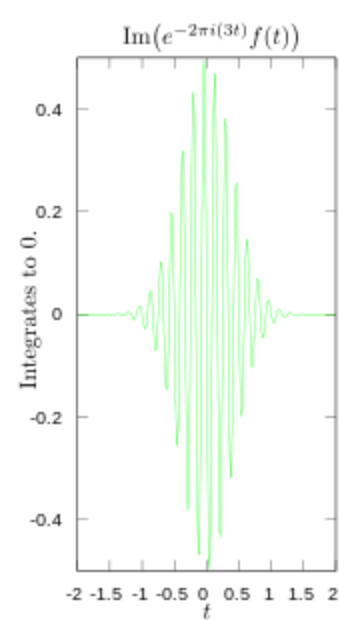- Fourier transform of the Dirac Delta Function:
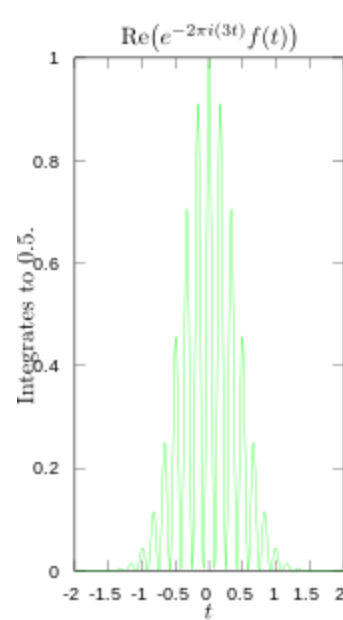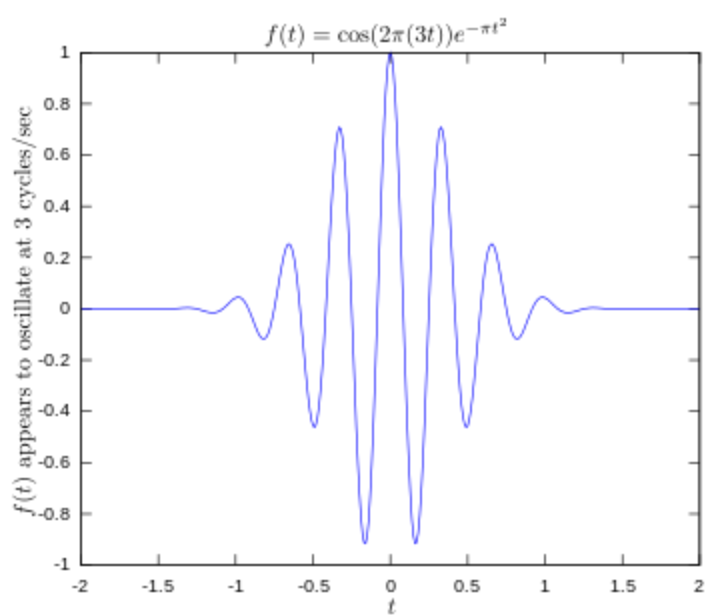
$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu(0)} = 1$$

due to the sifting property

Likewise $F(\mu) = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu(t_0)}$

Example 3

The following figures provide a visual illustration how the Fourier transform measures whether a frequency is present in a particular function. The depicted function $f(t) = \cos(6\pi t)\, e^{-\pi t^2}$ oscillates at 3 Hz (if t measures seconds) and tends quickly to 0. (The second factor in this equation is an envelope function that shapes the continuous sinusoid into a short pulse. Its general form is a Gaussian function). This function was specially chosen to have a real Fourier transform that can be easily plotted. The first image contains its graph. In order to calculate   we must integrate $e^{-2\pi i(3t)}f(t)$. The second image shows the plot of the real and imaginary parts of this function. The real part of the integrand is almost always positive, because when $f(t)$ is negative, the real part of $e^{-2\pi i(3t)}$ is negative as well. Because they oscillate at the same rate, when $f(t)$ is positive, so is the real part of $e^{-2\pi i(3t)}$. The result is that when you integrate the real part of the integrand you get a relatively large number (in this case 1/2). On the other hand, when you try to measure a frequency that is not present, as in the case when we look at   , you see that both real and imaginary component of this function vary rapidly between positive and negative values, as plotted in the third image. Therefore, in this case, the integrand oscillates fast enough so that the integral is very small and the value for the Fourier transform for that frequency is nearly zero.

https://en.wikipedia.org/wiki/Fourier_transform

$$f(t) = \cos(2\pi(3t))e^{-\pi t^2}$$

$\text{Re}\left(e^{-2\pi i(3t)}f(t)\right)$

$\text{Im}\left(e^{-2\pi i(3t)}f(t)\right)$

$\text{Re}\left(e^{-2\pi i(5t)}f(t)\right)$

$\text{Im}\left(e^{-2\pi i(5t)}f(t)\right)$

The Fourier transform of $f(t)$

This value corresponds to the integrals of the functions in green.

This value corresponds to the integrals of the functions in red.

https://en.wikipedia.org/wiki/Fourier_transform

Example 4: Cosine and Sine Waves

$$g(t) = \cos 2\pi at = \frac{e^{j2\pi at} + e^{-j2\pi at}}{2}$$

$$\mathbf{F}(g)(\mu) = \int_{-\infty}^{\infty} \frac{e^{j2\pi at} + e^{-j2\pi at}}{2} e^{-j2\pi \mu t} dt$$

$$= \frac{1}{2}(\int_{-\infty}^{\infty} e^{j2\pi at} e^{-j2\pi \mu t} dt + \int_{-\infty}^{\infty} e^{-j2\pi at} e^{-j2\pi \mu t} dt)$$

$$= \frac{1}{2}(\delta(\mu - a) + \delta(\mu + a))$$

$$h(t) = \sin 2\pi at = \frac{e^{j2\pi at} - e^{-j2\pi at}}{2j}$$

$$\mathbf{F}(h)(\mu) = \int_{-\infty}^{\infty} \frac{e^{j2\pi at} - e^{-j2\pi at}}{2j} e^{-j2\pi \mu t} dt$$

$$= \frac{1}{2j}(\int_{-\infty}^{\infty} e^{j2\pi at} e^{-j2\pi \mu t} dt - \int_{-\infty}^{\infty} e^{-j2\pi at} e^{-j2\pi \mu t} dt)$$

$$= \frac{1}{2j}(\delta(\mu - a) - \delta(\mu + a))$$

In both cases, these are Dirac delta (and not Kronecker delta) functions.

Fourier Transform

- In general, the Fourier transform of a real-valued signal yields complex-valued Fourier coefficients.

- Just like a Fourier series, it allows for complete signal recovery.

- Unlike a Fourier series, it allows for non-periodic signals and also allows for the frequencies to be non-integer (real-valued).

Fourier Transform and Music

- Audio/music signals are recorded in the time domain.

- However the different musical notes are represented by fundamental frequencies (eg: the note C#, typical scale of male vocalists, is 138.59 Hz; the scale of a typical female vocalist is 246.94 Hz).

- Fourier transform allows for identification of the musical notes within the audio signal.

Properties of the Fourier Transform

- Linearity

$$\mathbf{F}(af + g)(\mu) = a\mathbf{F}(f)(\mu) + \mathbf{F}(g)(\mu)$$

$$a = \text{complex - valued scalar}$$

- This follows from the definition of the Fourier transform.

- Is it shift invariant?

Properties of the Fourier Transform

- Is it shift invariant?

$$\mathbf{F}(f(t-t_0))(\mu) = \int_{-\infty}^{\infty} f(t-t_0)e^{-j2\pi\mu t}dt$$

$$= \int_{-\infty}^{\infty} f(t')e^{-j2\pi\mu(t'+t_0)}dt' \quad (t'=t-t_0)$$

$$= e^{-j2\pi\mu t_0}\int_{-\infty}^{\infty} f(t')e^{-j2\pi\mu t'}dt' = e^{-j2\pi\mu t_0}F(\mu)$$

The Fourier transform of a signal f shifted by time t_o is equal to the Fourier transform of the original signal f but multiplied by a phase factor dependent on t_o. This is called the **Fourier shift theorem**.

Properties of the Fourier Transform

- A variant of the Fourier shift theorem:

$$\mathbf{F}^{-1}(\mathbf{F}(\mu - \mu_0))(t) = \int_{-\infty}^{\infty} F(\mu - \mu_0) e^{j2\pi\mu t} d\mu$$

$$= \int_{-\infty}^{\infty} F(\mu') e^{j2\pi t(\mu' + \mu_0)} d\mu' \quad (\mu' = \mu - \mu_0)$$

$$= e^{j2\pi\mu_0 t} \int_{-\infty}^{\infty} F(\mu') e^{j2\pi\mu' t} d\mu' = e^{j2\pi\mu_0 t} f(t)$$

Properties of the Fourier Transform: Convolution Theorem

- Concept of convolution: we have seen the convolution in the discrete setting.

- Recall: $$(h * f)(t) = \sum_{s=-\infty}^{\infty} h(s) f(t - s)$$

- The convolution is defined in the case of continuous signals as follows:

$$(h * f)(t) = \int_{-\infty}^{\infty} h(s) f(t - s) ds$$

Properties of the Fourier Transform: Convolution Theorem

- The Fourier transform of the convolution is given as follows:

$$F[(h*f)](\mu) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(s)f(t-s)ds\right) e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} h(s)\left(\int_{-\infty}^{\infty} f(t-s)e^{-j2\pi\mu t}\,dt\right)ds$$

$$= \int_{-\infty}^{\infty} h(s)F(\mu)e^{-j2\pi\mu s}\,ds \text{ by Fourier shift theorem}$$

$$= F(\mu)\int_{-\infty}^{\infty} h(s)e^{-j2\pi\mu s}$$

$$= F(\mu)H(\mu)$$

Properties of the Fourier Transform: Convolution Theorem

- Theorem: The Fourier transform (for frequency μ) of the convolution of two signals is given by the product of their Fourier transforms at the same frequency.

- This is an extremely important theorem in signal processing.

- It will be very useful in Fourier-based filtering.

Variant of the convolution theorem

- The Fourier transform of the point-wise multiplication of two signals at frequency μ is equal to the convolution of their respective Fourier transforms, evaluated at frequency μ.

$$\mathbf{F}^{-1}[(H*F)](t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} H(\nu) F(\mu - \nu) d\nu \right) e^{j2\pi\mu t} d\mu$$

$$= \int_{-\infty}^{\infty} H(\nu) \left(\int_{-\infty}^{\infty} F(\mu - \nu) e^{j2\pi\mu t} d\mu \right) d\nu$$

$$= \int_{-\infty}^{\infty} H(\nu) \left(\int_{-\infty}^{\infty} F(\mu - \nu) e^{j2\pi\mu t} d\mu \right) d\nu$$

$$= \int_{-\infty}^{\infty} H(\nu) f(t) e^{j2\pi\nu t} d\nu$$

$$= f(t) \int_{-\infty}^{\infty} H(\nu) e^{j2\pi\nu t} d\nu$$

$$= f(t) h(t)$$

Properties of the Fourier Transform: Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\mu)|^2 d\mu$$

$$LHS = \int_{-\infty}^{\infty} f^*(t) f(t) dt = \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu \right)^* dt$$

$$= \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} F^*(\mu) e^{-j2\pi\mu t} d\mu \right) dt$$

$$= \int_{-\infty}^{\infty} F^*(\mu) \left(\int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \right) d\mu$$

$$= \int_{-\infty}^{\infty} F^*(\mu) F(\mu) d\mu = \int_{-\infty}^{\infty} |F(\mu)|^2 d\mu$$

Properties of the Fourier Transform: Parseval's theorem

- Parseval's theorem has interesting implications in image compression – which we will study later.

Properties of the Fourier Transform: Parseval's theorem (variant)

$$\int_{-\infty}^{\infty} f(t)g*(t)dt = \int_{-\infty}^{\infty} F(\mu)G*(\mu)d\mu$$

$$LHS = \int_{-\infty}^{\infty} f(t)g*(t)dt = \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} G(\mu)e^{j2\pi\mu t}d\mu \right)^* dt$$

$$= \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} G*(\mu)e^{-j2\pi\mu t}d\mu \right)dt$$

$$= \int_{-\infty}^{\infty} G*(\mu)\left(\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt \right)d\mu$$

This variant is more general than the earlier one, which assumed f(t) = g(t).

$$= \int_{-\infty}^{\infty} G*(\mu)F(\mu)d\mu = \int_{-\infty}^{\infty} F(\mu)G*(\mu)d\mu$$

Properties of the Fourier transform: Differentiation theorem

■ For a differentiable function f(t), we have:

$$\boldsymbol{F}(f'(t))(\mu) = j2\pi\mu F(\mu)$$

$$LHS = \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-j2\pi\mu t} dt = f(t)e^{-j2\pi\mu t} \mid_{\infty} -f(t)e^{-j2\pi\mu t} \mid_{-\infty} -\int_{-\infty}^{\infty} f(t)(-j2\pi\mu)e^{-j2\pi\mu t} dt$$

$$= 0 + j2\pi\mu \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t} dt = j2\pi\mu F(\mu)$$

Using integration by parts

Also $\boldsymbol{F}(f''(t))(\mu) = -4\pi^2\mu^2 F(\mu)$

Let $g(t) = f'(t)$, then $\boldsymbol{F}(g'(t))(\mu) = j2\pi\mu G(\mu)$

$= j2\pi\mu(j2\pi\mu F(\mu)) = RHS$

This assumes that the function is twice differentiable

Properties of the Fourier transform: Differentiation theorem

- Continuing this derivation recursively further for a *n*-times differentiable function f(t), we have:

$$\mathbf{F}(f^{(n)}(t))(\mu) = (j2\pi\mu)^n F(\mu)$$

Time-limited and band-limited signals

- A signal is said to be **time-limited** if it is of finite duration.

- A signal is said to be **band-limited** if for every frequency $|\mu| > B$ (where $\infty > B >= 0$), we have $F(\mu) = 0$.

- Any signal stored in a digital computer will always be time-limited (this will later extend to 2D images which are space-limited as they have a finite extent in the XY space).

Time-limited and band-limited signals

- (1) A non-zero time-limited signal cannot be band-limited.

- (2) A non-zero band-limited signal cannot be time-limited.

- To some extent, we have observed this via pairs as the rect and sinc, Dirac impulse and uniform signal. We will now prove it.

Time-limited and band-limited signals

- (1) A non-zero time-limited signal cannot be band-limited.

- Proof:
- ✓ Let us assume a time-limited signal g(t) which is zero everywhere for |t| > W/2.
- ✓ Then we can write g(t) = g(t).rect(t;W/2).
- ✓ Then taking Fourier transform on both sides, we have G(μ)=G(μ)*Wsinc(Wμ).
- ✓ As the sinc function never completely goes to zero, G cannot have finite support in the frequency domain, so g cannot be band-limited.

Time-limited and band-limited signals

- (2) A non-zero band-limited signal cannot be time-limited.

- Proof:
- ✓ Let us assume a band-limited signal h(t). Its Fourier transform H(μ) is zero everywhere for |μ| > B/2.
- ✓ Then we can write H(μ) = H(μ).rect(μ;B/2).
- ✓ Then taking inverse Fourier transform on both sides, we have h(t)=h(t)*Bsinc(Bt).
- ✓ As the sinc function never completely goes to zero, h cannot have finite support in the time domain, so h cannot be time-limited.

Time-limited and band-limited signals

- We already know that for any signal h, we must have h(t) = (h*δ)(t) where δ is the Dirac delta function.

- But the earlier proof tells us that for a **band-limited** signal h(t), we also have h(t)=h(t)*Bsinc(Bt)!

Time-limited and band-limited signals

- A band-limited signal is infinitely differentiable (converse is not true – eg: Gaussian).

Proof : Assume that $f(t)$ is bandlimited

$$\therefore f(t) = \int_{-B}^{B} F(\mu)e^{j2\pi\mu t}d\mu$$

$$f'(t) = \int_{-B}^{B} F(\mu)\frac{d}{dt}\left(e^{j2\pi\mu t}\right)d\mu = \int_{-B}^{B} j2\pi\mu F(\mu)\left(e^{j2\pi\mu t}\right)d\mu < \infty \text{ as the integrand}$$

is differentiable and the integral is over finite limits

Likewise,

$$\frac{d^n f(t)}{dt^n} = \int_{-B}^{B} (j2\pi\mu)^n F(\mu)\left(e^{j2\pi\mu t}\right)d\mu < \infty \text{ for the same reason}$$

Fourier Transforms in 2D

- Fourier Transforms and its inverse in 2D – spatial coordinates (x,y) and frequency coordinates (μ,ν).

$$F(\mu,\nu) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)e^{-j2\pi(\mu x+\nu y)}dxdy$$

$$f(x,y) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu x+\nu y)}d\mu d\nu$$

- In fact, such pairs exist in higher dimensions (3,4,…) as well.

Fourier Transforms in 2D

- All aforementioned theorems hold in higher dimensions too: shift theorem, convolution theorem, Parseval's theorem, and others.

- The higher dimensional Fourier transform has the property of **separability**.

- That is a 2D Fourier transform is obtained as follows:
- ✓ For every x, compute the 1D Fourier transform of f(x,.) integrating across y to produce a signal F(x,ν).
- ✓ Then compute the 1D Fourier transform of F(.,ν) by integrating across x to get F(μ,ν).

$$F(\mu,\nu) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)e^{-j2\pi(\mu x + \nu y)}dxdy$$

$$= \int\limits_{-\infty}^{\infty}\left(\int\limits_{-\infty}^{\infty} f(x,y)e^{-j2\pi(\nu y)}dy\right)e^{-j2\pi(\mu x)}dx = \int\limits_{-\infty}^{\infty} F(x,\nu)e^{-j2\pi(\mu x)}dx = F(\mu,\nu)$$

1D Fourier Transforms

2D Fourier transforms are said to be separable as they are computed by 1D Fourier transforms in each of the coordinates. You may verify that the separability property holds true even for inverse Fourier transforms in 2D.