$\mathbb{P}$ROBABILISTIC $\mathbb{T}$HINKING

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1 Problems

1. *Derangement Problem (5 Marks)*
   N letters are to be put in N separate envelopes. Assuming a envelope can hold only a single letter. What is the probability that at least one letter is in the correct envelope?
   Find an approximation of this probability for $N = 50$.

2. *Showman (6 Marks)*
   You have 3 identical presents. The good gift has 1000 dollars, while others have nothing. The host of the party asks you to select a present. If you select the good gift, you keep it. You select Present 1. But the host opens the second present and it has nothing. Host knows where the money is, always reveals the blank prize. (Also assume that, when we chooses the prize gift, the host chooses one of the blank with equal probability.) You are given the option to switch your guess to the third present. What are your expected winnings if you switch?

3. *True or False (4 Marks each)*
   Let A, B C and D be four events such that $\mathbb{P}(B \cap C) > 0$.
   (a) $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)$
   (b) $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$ for independent events A and B.
   (c) Given $\mathbb{P}(A|D \cap B^c) > \mathbb{P}(A|D \cap B)$ and $\mathbb{P}(A|D^c \cap B^c) > \mathbb{P}(A|D^c \cap B)$, $\mathbb{P}(A|B)$ must be greater than $\mathbb{P}(A|B^c)$

4. *Construct the following or disprove its existence (4 + 4 + 7 Marks)*

(a) A discrete random variable X for which $\mathbb{E}(X)$ is finite but $\mathbb{E}(X^2)$ is not finite.

(b) A continuous random variable X for which $\mathbb{E}(X)$ is finite but $\mathbb{E}(X^2)$ is not finite.

(c) A random variable X with $\mathbb{E}(X) = 1$, but $\mathbb{E}(e^{-X}) < 1/3$

5. *Expectation of Statistic (8 Marks)*
   From N identical lotteries with prizes, $1, 2, ..., N, n \leq N$ tickets are drawn with replacement. You are allowed to keep only the maximal prize ticket Let M = prize money obtained. Find $\mathbb{E}(M)$.

6. *Geometry of a line (5 Marks)*
   Find the probability two points taken on a line segment of length d has distance between them less than $d/3$.

7. *Gossip Man (6 + 4 Marks)*
   In a town of $(n+1)$ inhabitants, a person tells a rumor to a second person, who in turn tells it to a third one, and so on. At each step a random person is chosen to listen the rumor. Find the probability that the rumor will be told r times without

   (a) returning to the originator

   (b) being repeated to any persons

   Repeat when at each step the rumor is told to a gathering of N randomly chosen people.

8. *Bern Lee (Who?) Inequality (8 Marks)*
   Let $A_1, A_2....A_n$ be n independent events. Prove that

$$\mathbb{P}(\cap A_i^c) \leq e^{-\mathbb{P}(A_1)-\mathbb{P}(A_2)........-\mathbb{P}(A_n)}$$

9. *Convolution (6 Marks)*
   Show that the convolution of two distribution functions is also a distribution function. (Read its definition online)