Курсовая работа по математическому анализу. Вариант 22.

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Содержание

1 Задание 1

$$\int \operatorname{ctg}^5 x\,dx = \int \operatorname{ctg} x\left(\frac{1}{\sin^2 x}-1\right)^2 dx = \int \operatorname{ctg} x\left(\frac{1}{\sin^4 x}-\frac{2}{\sin^2 x}+1\right) dx =$$
$$= \int \frac{\operatorname{ctg} x}{\sin^4 x}\,dx - \int \frac{2\operatorname{ctg} x}{\sin^2 x}\,dx + \int \operatorname{ctg} x\,dx = \int \frac{\cos x}{\sin^5 x}\,dx - 2\int \frac{\cos x}{\sin^3 x}\,dx + \int \frac{\cos x}{\sin x}\,dx =$$
$$= \int \frac{d\sin x}{\sin^5 x} - 2\int \frac{d\sin x}{\sin^3 x} + \int \frac{d\sin x}{\sin x} = -\frac{1}{4\sin^4 x} + \frac{1}{\sin^2 x} + \ln|\sin x| + C.$$

2 Задание 2

$$\int \sqrt{\sin 2x}\cos x\,dx = \int \sqrt{\sin 2x}\cos x\,dx = \int \sqrt{\sin 2x}\,|\cos x|\operatorname{sgn}\cos x\,dx =$$
$$= \operatorname{sgn}\cos x \int \sqrt{\frac{\sin 2x(1+\cos 2x)}{2}}\,dx = \operatorname{sgn}\cos x \int \sqrt{\frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x}\left(1+\frac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}\right)}\,dx =$$
$$= \operatorname{sgn}\cos x \int \frac{\sqrt{2\operatorname{tg} x}}{1+\operatorname{tg}^2 x}\,dx = \left|\begin{array}{l} t = \sqrt{\operatorname{tg} x} \\ t^2 = \operatorname{tg} x \\ d\operatorname{tg} x = 2t\,dt \\ \dfrac{dx}{\cos^2 x} = 2t\,dt \\ dx = 2t\cos^2 x\,dt \\ dx = \dfrac{2t}{t^4+1}\,dt \end{array}\right| = 2\sqrt{2}\operatorname{sgn}\cos x \int \frac{t^2}{(1+t^4)^2}\,dt = (*)$$

Применим метод Остроградского:

$$\int \frac{t^2}{(1+t^4)^2}\,dt = \frac{at^3+bt^2+ct+d}{t^4+1} + \int \frac{et^3+ft^2+gt+h}{t^4+1}\,dt$$

$$\frac{t^2}{(1+t^4)^2} = \left(\frac{at^3+bt^2+ct+d}{t^4+1}\right)' + \frac{et^3+ft^2+gt+h}{t^4+1}$$

$$\frac{t^2}{(1+t^4)^2} = -\frac{4t^3\left(at^3+bt^2+ct+d\right)}{\left(t^4+1\right)^2} + \frac{3at^2+2bt+c}{t^4+1} + \frac{et^3+ft^2+gt+h}{t^4+1}$$

$$\frac{t^2}{(1+t^4)^2} = \frac{t^6(f-a)+t^2(3a+f)+t^5(g-2b)+t(2b+g)+t^4(h-3c)+c+t^3(e-4d)+et^7+h}{\left(t^4+1\right)^2}$$

$$a = f = \frac{1}{4}$$

$$b = c = d = e = g = h = 0$$

$$\int \frac{t^2}{(1+t^4)^2}\,dt = \frac{t^3}{4\left(t^4+1\right)} + \frac{1}{4}\int \frac{t^2}{t^4+1}\,dt$$

$$\int \frac{t^2}{t^4+1}\,dt = \int \left(\frac{at+b}{t^2-\sqrt{2}t+1} + \frac{ct+d}{t^2+\sqrt{2}t+1}\right)dt =$$
$$= \int \frac{t^2\left(\sqrt{2}a+b-\sqrt{2}c+d\right)+t\left(a+\sqrt{2}b+c-\sqrt{2}d\right)+t^3(a+c)+b+d}{t^4+1}\,dt$$

$$a = \frac{1}{2\sqrt{2}}, b = 0, c = -\frac{1}{2\sqrt{2}}, d = 0$$

$$\int \left(\frac{t}{2\sqrt{2}\left(t^2-\sqrt{2}t+1\right)} - \frac{t}{2\sqrt{2}\left(t^2+\sqrt{2}t+1\right)}\right)dt =$$
$$= \frac{1}{2\sqrt{2}}\int \frac{t}{t^2-\sqrt{2}t+1}\,dt - \frac{1}{2\sqrt{2}}\int \frac{t}{t^2+\sqrt{2}t+1}\,dt =$$
$$= \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\int \frac{d(t^2-\sqrt{2}t+1)}{t^2-\sqrt{2}t+1} - \frac{1}{2}\int \frac{d(t^2+\sqrt{2}t+1)}{t^2+\sqrt{2}t+1} + \right.$$
$$\left. + \frac{\sqrt{2}}{2}\int \frac{dt}{t^2-\sqrt{2}t+1} + \frac{\sqrt{2}}{2}\int \frac{dt}{t^2+\sqrt{2}t+1}\right) =$$
$$= \frac{1}{4\sqrt{2}}\left(\ln\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + 2\operatorname{arctg}(\sqrt{2}t-1) + 2\operatorname{arctg}(\sqrt{2}t+1)\right)$$

$$(*) = \frac{1}{8}\operatorname{sgn}\cos x\left(\frac{4\sqrt{2}t^3}{t^4+1} + \ln\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + 2\operatorname{arctg}(\sqrt{2}t-1) + 2\operatorname{arctg}(\sqrt{2}t+1)\right) + C,$$

где $t = \sqrt{\operatorname{tg} x}$.

3 Задание 3

$$\int \frac{2x^3+7x^2+7x-1}{(x+2)^2(x^2-x+1)}\,dx = \int \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{Cx+D}{x^2-x+1}\,dx =$$
$$= \int \frac{x^3(B+C)+x^2(A+B+4C+D)+x(-A-B+4C+4D)+A+2B+4D}{(x+2)^2\,(x^2-x+1)}\,dx = (*)$$

$$\begin{cases} B+C=2 \\ A+B+4C+D=7 \\ -A-B+4C+4D=7 \\ A+2B+4D=-1 \end{cases}$$
$$A=-\frac{3}{7}, B=\frac{6}{49}, C=\frac{92}{49}, D=-\frac{10}{49}$$

$$(*) = \int \frac{2(46x-5)}{49\,(x^2-x+1)} + \frac{6}{49(x+2)} - \frac{3}{7(x+2)^2}\,dx = -\frac{3}{7}\int \frac{1}{(x+2)^2}\,dx +$$
$$+ \frac{6}{49}\int \frac{1}{x+2}\,dx + \frac{2}{49}\int \frac{46x-5}{x^2-x+1}\,dx = (**)$$

$$\int \frac{dx}{(x+2)^2} = -\frac{1}{x+2}$$
$$\int \frac{dx}{x+2} = \ln|x+2|$$

$$\int \frac{46x-5}{x^2-x+1}\,dx = \int \frac{23(2x-1)+18}{x^2-x+1}\,dx = 23\int \frac{d(x^2-x+1)}{x^2-x+1} + 18\int \frac{dx}{x^2-x+1} =$$
$$= 23\ln|x^2-x+1| + 18\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} = 23\ln|x^2-x+1| + 12\sqrt{3}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}}$$

$$(**) = \frac{3}{7}\cdot\frac{1}{x+2} + \frac{6}{49}\ln|x+2| + \frac{2}{49}\left(23\ln|x^2-x+1| + 12\sqrt{3}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}}\right) + C.$$

4 Задание 4

$$\int \frac{dx}{(\sin x + \cos x + 1)^2} = \int \frac{dx}{\left(\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} + \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} + 1\right)^2} = \frac{1}{4}\int \left(\frac{1+\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}\frac{x}{2}+1}\right)^2 dx =$$

$$= \left| \begin{array}{l} t = \operatorname{tg}\dfrac{x}{2} \\ \dfrac{x}{2} = \operatorname{arctg} t \\ \dfrac{dx}{2} = \dfrac{dt}{1+t^2} \\ dx = \dfrac{2dt}{1+t^2} \end{array} \right| = \frac{1}{4}\int \left(\frac{1+t^2}{1+t}\right)^2 \frac{2}{1+t^2}\,dt = \frac{1}{2}\int \frac{1+t^2}{(1+t)^2}\,dt = \frac{1}{2}\int \left(1 - \frac{2t}{(1+t)^2}\right) dt =$$

$$= \frac{t}{2} - \int \frac{t}{(1+t)^2}\,dt = \frac{t}{2} - \int \frac{dt}{1+t} + \int \frac{dt}{(1+t)^2} = \frac{t}{2} - \ln|1+t| - \frac{1}{1+t} =$$

$$= \frac{1}{2}\operatorname{tg}\frac{x}{2} - \ln|1+\operatorname{tg}\frac{x}{2}| - \frac{1}{1+\operatorname{tg}\frac{x}{2}} + C.$$

5 Задание 5

$$\int \frac{\sqrt[3]{1+\sqrt[5]{x^4}}}{x^2\sqrt[15]{x}}\,dx = (*)$$

Используем замену $x = t^{5/4}$, $dx = \dfrac{5}{4}t^{1/4}$.

$$(*) = \frac{5}{4}\int \frac{\sqrt[3]{1+t}}{t^{5/2}\cdot t^{1/12}}t^{1/4}\,dt = \frac{5}{4}\int \frac{\sqrt[3]{1+t}}{t^{28/12}}\,dt = \frac{5}{4}\int \sqrt[3]{\frac{1+t}{t}}t^{-2}\,dt = \left| \begin{array}{l} s = \sqrt[3]{\dfrac{1+t}{t}} \\ dt = -\dfrac{3s^2}{(s^3-1)^2}\,ds \end{array} \right| =$$

$$= -\frac{15}{4}\int s^3\,ds = -\frac{15}{4}\cdot\frac{s^4}{4} + C = -\frac{15}{16}\left(\sqrt[3]{\frac{1+t}{t}}\right)^4 + C = -\frac{15}{16}\cdot\frac{(1+x^{4/5})^{4/3}}{x^{16/15}} + C.$$

6 Задание 6

$$I = \int \frac{2x+3}{\sqrt{x^2+1}\,(x^2-2x+1)}\,dx$$

Используем подстановку Эйлера:

$$\sqrt{x^2+1} = x + t$$

$$x = \frac{1-t^2}{2t}$$

$$dx = -\frac{1+t^2}{2t^2}\,dt$$

$$I=\int\frac{8t\left(t^2-3t-1\right)}{\left(t+\frac{1}{t}\right)\left(t^2+2t-1\right)^2}\cdot\frac{1+t^2}{2t^2}\,dt=\int\frac{4t\left(t^2-1\right)\left(t^2-3t-1\right)}{\left(t^2+1\right)\left(t^2+2t-1\right)^2}\,dt=$$
$$=\frac{at+b}{t^2+2t-1}+\int\frac{ct^3+dt^2+et+f}{(t^2+2t-1)(1+t^2)}\,dt$$

$$a=11,b=-\frac{11}{2},c=4,d=-9,e=7,f=0$$

$$I=\frac{11(2t-1)}{2\left(t^2+2t-1\right)}+\int\frac{4t^3-9t^2+7t}{\left(t^2+1\right)\left(t^2+2t-1\right)}\,dt=\frac{11(2t-1)}{2\left(t^2+2t-1\right)}+I_1$$

$$I_1=\int\frac{4t^3-9t^2+7t}{\left(t^2+1\right)\left(t^2+2t-1\right)}\,dt=\int\frac{4t^3-9t^2+7t}{\left(t-\sqrt{2}+1\right)\left(t+\sqrt{2}+1\right)\left(t^2+1\right)}=$$
$$=-\int\frac{3(2t+1)}{2\left(t^2+1\right)}\,dt+\int\frac{17+14\sqrt{2}}{4\sqrt{2}\left(t+\sqrt{2}+1\right)}\,dt+\int\frac{17-14\sqrt{2}}{4\sqrt{2}\left(-t+\sqrt{2}-1\right)}\,dt$$
$$=$$
$$-\frac{3}{2}\ln\left(t^2+1\right)+\frac{1}{8}\left(28+17\sqrt{2}\right)\ln\left(t+\sqrt{2}+1\right)+\frac{1}{8}\left(28-17\sqrt{2}\right)\ln\left(8t-8\sqrt{2}+8\right)-\frac{3}{2}\operatorname{arctg}t+C$$

$$I=\frac{11(2t-1)}{2\left(t^2+2t-1\right)}+-\frac{3}{2}\ln\left(t^2+1\right)+\frac{1}{8}\left(28+17\sqrt{2}\right)\ln\left(t+\sqrt{2}+1\right)+$$
$$+\frac{1}{8}\left(28-17\sqrt{2}\right)\ln\left(8t-8\sqrt{2}+8\right)-\frac{3}{2}\operatorname{arctg}t+C,$$

где $t=\sqrt{x^2+1}-x$.

7 Задание 7

$$I=\int\operatorname{ch}^3x\operatorname{sh}^2x\,dx=\int\operatorname{ch}^2x\operatorname{sh}^2x\,d\operatorname{sh}x=\int(1+\operatorname{sh}^2x)\operatorname{sh}^2x\,d\operatorname{sh}x=\frac{\operatorname{sh}^5x}{5}+\frac{\operatorname{sh}^3x}{3}+C.$$

8 Задание 8

$$I=\int\frac{x^4}{(x^2-1)(x+2)}\,dx=\int\left(\frac{5x^2-x-2}{(x+2)\left(x^2-1\right)}+x-2\right)\,dx=\frac{x^2}{2}-2x+$$
$$+\int\frac{5x^2-x-2}{(x+2)\left(x^2-1\right)}=\frac{x^2}{2}-2x+I_1$$

$$I_1=-\int\frac{2}{x+1}\,dx+\int\frac{20}{3(x+2)}\,dx+\int\frac{1}{3(x-1)}\,dx=\frac{1}{3}\ln(1-x)-2\ln(x+1)+$$
$$+\frac{20}{3}\ln(x+2)+C$$

$$I=\frac{x^2}{2}-2x+\frac{1}{3}\ln(1-x)-2\ln(x+1)+\frac{20}{3}\ln(x+2)+C$$

9 Задание 9

$$\int (x^2+3x+5)\cos 2x\,dx = \frac{1}{2}\int (x^2+3x+5)(\sin 2x)'\,dx =$$
$$= \frac{1}{2}\left((x^2+3x+5)\sin 2x + \frac{1}{2}\int (2x+3)(\cos 2x)'\,dx\right) =$$
$$= \frac{1}{2}(x^2+3x+5)\sin 2x + \frac{1}{4}(2x+3)\cos 2x - \frac{1}{4}\sin 2x + C.$$

10 Задание 10

$$\int \operatorname{arctg}(1+\sqrt{x})\,dx = \left| \begin{array}{l} 1+\sqrt{x}=t \\ dx=(2t-2)\,dt \end{array} \right| = \int (2t-2)\operatorname{arctg} t\,dt = \int \operatorname{arctg} t(t^2-2t)'\,dt =$$
$$= (t^2-2t)\operatorname{arctg} t - \int (t^2-2t)\cdot\frac{1}{1+t^2}\,dt = (t^2-2t)\operatorname{arctg} t - t - \operatorname{arctg} t - \ln(1+t^2) + C =$$
$$= (t^2-2t-1)\operatorname{arctg} t - \ln(1+t^2) + t + C,$$

где $t = 1+\sqrt{x}$.

11 Задание 11

$$\int e^{\sqrt[3]{x}}\,dx = \left| \begin{array}{l} \sqrt[3]{x}=t \\ dx=3t\,dt \end{array} \right| = \int 3te^t\,dt = \int 3t(e^t)'\,dt = 3te^t - 3\int e^t\,dt = 3te^t - 3e^t + C =$$
$$= 3e^{\sqrt[3]{x}}(\sqrt[3]{x}-1) + C.$$

12 Задание 12

$$\int \frac{dx}{x\sqrt[3]{x-1}} = \left| \begin{array}{l} \sqrt[3]{x-1}=t \\ dx=3t^2\,dt \end{array} \right| = \int \frac{3t^2}{(t^3+1)t}\,dt = 3\int \frac{t}{t^3+1}\,dt =$$
$$= 3\left(\frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2t-1}{\sqrt{3}}\right) - \frac{1}{3}\ln(1+t) + \frac{1}{6}\ln(1-t+t^2)\right),$$

где $t = \sqrt[3]{x-1}$.

13 Задание 13

$$\int \frac{\operatorname{tg}(x+1)}{\cos^2(x+1)}\,dx = \int \operatorname{tg}(x+1)\,d\operatorname{tg}(x+1) = \frac{\operatorname{tg}^2(x+1)}{2} + C.$$